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A superfast solver for Sylvester's resultant linear systems generated by a stable and an anti-stable polynomial

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Abstract

We develop a superfast method for the solution of $(n+m) \times (n+m)$ Sylvester's resultant linear systems associated with two real polynomials $a(z)$ and $c(z)$ of degree n and m, respectively, where $a(z)$ is a stable polynomial, i.e., all its roots lie inside the unit circle, whereas $c(z)$ is an anti-stable polynomial, i.e, $z^m c(z^{-1})$ is stable. The proposed scheme proceeds by iteratively constructing a sequence of increasing approximations of the solution. It is based on a blend of ideas from structured numerical linear algebra, computational complex analysis and linear operator theory. Each iterative step can be performed in $\mathrm{O}((n+m)\log(n+m))$ arithmetic operations by combining fast polynomial arithmetic based on FFT with displacement rank theory for structured matrices. In addition, the resulting process is shown to be quadratically convergent right from the start since the approximation error at the jth iteration is $\mathrm{O}(r^{2^j})$, where $r = (\max_{a(\alpha_i)=0} |\alpha_i|)/(\min_{c(\gamma_i)=0} |\gamma_i|)$ denotes the separation ratio between the spectrum of $a(z)$ and $c(z)$. Finally, we report and discuss the results of many numerical experiments which confirm the effectiveness and the robustness of the proposed algorithm.

Keywords: Sylvester's resultant matrices; Polynomial computations; Stable polynomials; Cyclic reduction; Spectral factorization; Displacement theory

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1. Introduction

Let $a(z) = \sum_{i=0}^{n} a_{n-i} z^i$ and $c(z) = \sum_{i=0}^{m} c_{m-i} z^i$ be two real polynomials of degree n and m, respectively. The Sylvester's resultant matrix [1] R of order $n+m$ associated with $a(z)$ and $c(z)$ is defined as $R = [T_m[\boldsymbol{a}]|T_n[\boldsymbol{c}]]^{\mathrm{T}}$, where for any given polynomial $p(z) = \sum_{i=0}^{r} p_{r-i} z^i$ with coefficient vector $\boldsymbol{p} = [p_r, \ldots, p_0]^{\mathrm{T}}$ we denote by $T_j[\boldsymbol{p}]^{\mathrm{T}}$ the following $j \times (j+r)$ triangular Toeplitz matrix

$$T_j[\boldsymbol{p}]^{\mathrm{T}} = \begin{bmatrix} p_r & p_{r-1} & \ldots & \ldots & p_0 & 0 & \ldots & 0 \\ 0 & p_r & p_{r-1} & \ldots & \ldots & p_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \ldots & 0 & p_r & p_{r-1} & \ldots & \ldots & p_0 \end{bmatrix}. \tag{1}$$

Devising efficient methods for the solution of Sylvester's resultant linear systems is a relevant issue in many diverse fields like computer algebra, control theory, signal processing and data modeling [1,12,16,19]. Over the years, several fast and superfast algorithms have been proposed which are based on the recursive properties of the triangular factorization of Sylvester's resultant matrices [2,12,14,16]. In a polynomial setting, all these recursive schemes reduces to Euclidean-type recursions and thus, due to the exponential growth of the coefficients of Euclidean remainders, they generally suffer from ill-conditioning problems and numerical instabilities. On the contrary, in this paper we present a new superfast solver which, to our numerical experience, results to be quite robust and effective when it is applied in finite precision arithmetic.

Our approach works under some auxiliary restrictions on the spectrum of the polynomials associated with the initial coefficient matrix. In particular, if $a(z)$ and $c(z)$ are such two polynomials, then we assume that $a(z)$ is stable, i.e., all its roots lie inside the unit circle in the complex plane, whereas $c(z)$ is anti-stable, i.e., the reversed polynomial $z^m c(z^{-1})$ is stable. Systems of this form often arise also in many relevant applicative and industrial problems where the primary focus is on the study of process dynamics by means of numerical procedures. These typically include time series analysis, Wiener filtering, noise variance estimation, covariance matrix computations and the study of multichannel systems (see [1,11,17,18,27,36]). In all these applications, stable and anti-stable polynomials are naturally introduced as numerators and/or denominators of the transfer functions of the considered input–output models (see [26] and the references given therein on the stability investigations of polynomials and linear discrete-time systems).

Under the stability condition, we show that the matrix problem of solving a Sylvester's resultant linear system can be reduced to the functional problem of determining certain central coefficients of the Laurent series of the reciprocal of the Laurent polynomial $p(z) = z^n a(z^{-1}) c(z^{-1})$ in a given annulus around the unit circle. It is quite remarkable to note that $p(z) = z^n a(z^{-1})c(z^{-1})$ defines the spectral factorization of $p(z)$ and, therefore, our precise goal is to reciprocate a Laurent polynomial given as product of its spectral factors.

To accomplish this task, we use again the interplay between polynomial and structured matrix computations. We observe that the coefficients of the Laurent series of the reciprocal of $p(z)$ are the entries of the central column of the inverse of the bi-infinite Toeplitz matrix T whose symbol is $p(z)$. Hence, these quantities can be iteratively approximated within an arbitrarily small error ϵ by using any method for the numerical solution of bi-infinite banded Toeplitz systems. In this paper we consider the cyclic reduction process, originally introduced in [13] for the solution of partial differential equations and, more recently, adjusted in [3–7] for solving several diverse computational problems modeled by infinite and bi-infinite banded Toeplitz-like linear systems. However, differently from the previously mentioned papers where the cyclic reduction gives the computational kernel of the considered algorithms, here this factorization technique is theoretically exploited in order to devise approximation schemes whose properties are largely independent of those of cyclic reduction.

To be more specific, notice that the available coefficients of the spectral factors of $p(z)$ define a (block) triangular factorization—known as the Wiener–Hopf factorization—of the initial guess $T^{(0)} = T$. Starting from $T^{(0)} = T$ suitably partitioned into a block tridiagonal form, the cyclic reduction process generates a sequence $\{T^{(s)}\}_{s\in\mathbb{N}}$ of invertible block Toeplitz matrices in block tridiagonal form quadratically converging to a block diagonal matrix from which the sought entries can be retrieved. We provide a modification of this scheme where a block triangular factorization of $T^{(s)} = L^{(s)}D^{(s)}U^{(s)}$ is iteratively constructed starting from that one of T. The updating relations for the block entries of $L^{(s)}$, $D^{(s)}$, and $U^{(s)}$ involve the powers of the Frobenius matrices associated with the spectral factors. Moreover, if $D_0^{(s)}$ denotes the block diagonal entry of $D^{(s)}$, then a matrix formula is stated by showing that $(D_0^{(s+1)})^{-1}$ can be generated directly from $(D_0^{(s)})^{-1}$ and certain powers of the Frobenius matrices associated with the spectral factors. In this way, we find a simple rule for the iterative generation of $\{(D_0^{(s)})^{-1}\}_s$. Since $D_0^{(s)^{-1}}$ quadratically approaches the Toeplitz matrix X formed from the sought coefficients of $1/p(z)$, such rule leads to a matrix iteration for the reciprocation of factored Laurent polynomials. The convergence is quadratic depending on the separation ratio r between the spectrum of $a(z)$ and $c(z)$, $r = (\max_{a(\alpha_i)=0} |\alpha_i|)/(\min_{c(\gamma_i)=0} |\gamma_i|)$. Moreover, the iteration is well-defined whenever $D_0^{(0)}$ is nonsingular and, by a continuity argument, it converges to X even if the cyclic reduction process applied to T breaks down at some early stage.

While similar iterative schemes have already appeared in the engineering literature for the numerical treatment of Lyapunov matrix equations [25] (see also [22] for relations between the solution of Sylvester's resultant linear systems and the properties of Lyapunov matrix equations), our approach provides a unified and, hence, clarified derivation by highlighting several new connections between the computation of X and the solution of quadratic matrix equations. Moreover, we are able to show that all the matrices generated by the proposed iterative process are Toeplitz-like in the sense that their displacement rank is upper bounded by a small constant independent of n and m. Hence, our algorithm can be efficiently performed by combining fast polynomial arithmetic using FFTs with displacement representations of structured matrices, with a dramatic reduction of its computational cost. Finally, our numerical experience indicates that it has quite good stability properties.

The paper is organized in the following way. In Section 2 we describe the basic reductions of the original matrix problem to computations with Laurent polynomials. In Section 3 we introduce and analyze our modification of the cyclic reduction process in order to compute certain coefficients of the Laurent series of the reciprocal of a Laurent polynomial given by its spectral factorization. In Section 4 it is shown how such a variant can efficiently be implemented by using fast polynomial arithmetic and displacement theory for Toeplitz-like matrices. Finally, in Section 5 we report the results of the numerical experiments performed with MATLAB whereas conclusions and further developments are drawn in Section 6.

2. Polynomial counterparts of solving Sylvester's resultant linear systems

Let $a(z)$ and $c(z)$ be two real polynomials of degree n and m, respectively, such that

$$\begin{aligned} a(z) &= a_0\prod_{i=1}^{n}(z-\alpha_i) = \sum_{i=0}^{n} a_{n-i}z^i, \quad a_i \in \mathbb{R},\ a_0 \neq 0, \quad 0 \leqslant |\alpha_i| < 1, \\ c(z) &= c_0\prod_{i=1}^{m}(z-\gamma_i) = \sum_{i=0}^{m} c_{m-i}z^i, \quad c_i \in \mathbb{R},\ c_0 \neq 0, \quad |\gamma_i| > 1. \end{aligned} \tag{2}$$

For the sake of notational convenience, assume that the zeros α_i and γ_i of $a(z)$ and $c(z)$ are ordered so that

$$0 \leqslant |\alpha_1| \leqslant |\alpha_2| \leqslant \cdots \leqslant |\alpha_n| < 1, \quad 1 < |\gamma_1| \leqslant |\gamma_2| \leqslant \cdots \leqslant |\gamma_m|. \tag{3}$$

In this paper we address the problem of efficiently computing the solution $\boldsymbol{x} \in \mathbb{R}^{m+n}$ of the linear system

$$\boldsymbol{x}^{\mathrm{T}}[T_m[\boldsymbol{a}]|T_n[\boldsymbol{c}]]^{\mathrm{T}} = \boldsymbol{b}^{\mathrm{T}}, \quad \boldsymbol{b} \in \mathbb{R}^{m+n}, \tag{4}$$

where for any polynomial $p(z) = \sum_{i=0}^{r} p_{r-i}z^i$ with coefficient vector $\boldsymbol{p} = [p_r, p_{r-1}, \ldots, p_0]^{\mathrm{T}}$ the associated $(j+r) \times j$ triangular Toeplitz matrix $T_j[\boldsymbol{p}]$ is given by (1).

Remark 1. The coefficient matrix of (4) is called the Sylvester resultant matrix generated by $a(z)$ and $c(z)$ [1] and its determinant can be explicitly expressed in terms of the zeros of its polynomial generators, namely, $\det[T_m[\boldsymbol{a}]|T_n[\boldsymbol{c}]] = a_0^m c_0^n \prod_{i=1}^n \prod_{j=1}^m (\alpha_i - \gamma_j)$ [1]. Hence, the assumption (3) immediately implies that the coefficient matrix of (4) is nonsingular and, therefore, for any fixed known vector $\boldsymbol{b}$, the solution $\boldsymbol{x}$ of (4) is uniquely determined. Moreover, up to a suitable scaling of this solution vector, it can always be assumed that $a(z)$ and $c(z)$ are monic polynomials with $a_0 = c_0 = 1$.

It is well known that (4) reduces to the solution of a Bezout polynomial equation defined by $a(z)$ and $c(z)$ [1]. To see this, let us first introduce the polynomials $x(z) = \sum_{i=1}^{n+m} x_i z^{n+m-i} = x_+(z) + x_-(z)$, where $x_-(z) = \sum_{i=0}^{n-1} x_{n+m-i} z^i$, and $b(z) = \sum_{i=1}^{n+m} b_i z^{n+m-i}$ which are defined by the coefficients of the solution vector $\mathbf{x} = [x_1, \ldots, x_{n+m}]^{\mathrm{T}}$ and by the coefficients of the known vector $\mathbf{b} = [b_1, \ldots, b_{n+m}]^{\mathrm{T}}$, respectively. Since $T_j[\boldsymbol{p}]\boldsymbol{q}$ is the coefficient vector of the polynomial $q(z)p(z)$, then from (4) we obtain

$$a(z)z^{n+m-1}x_+(z^{-1}) + c(z)z^{n-1}x_-(z^{-1}) = z^{n+m-1}b(z^{-1}),$$

which, after some algebra, yields

$$(z^n a(z^{-1}))\hat{x}_+(z) + c(z^{-1})zx_-(z) = \hat{b}(z), \tag{5}$$

where $\hat{x}_+(z) = z^{1-n-m}x_+(z) \in \mathscr{L}_{1-m}^0$, $\hat{b}(z) = z^{1-m}b(z) \in \mathscr{L}_{1-m}^n$, and $\mathscr{L}_s^r$ is the vector space of real Laurent polynomials of the form $\sum_{i=s}^r p_i z^i$, $s \leqslant r$, $s, r \in \mathbb{Z}$. The recursive solution of (5) by means of the Euclidean algorithm leads to a fast but unstable algorithm for solving (4). Below we employ a different solution method based on the properties of the zeros of $a(z)$ and $c(z)$. Observe that (5) can equivalently be rewritten as

$$\frac{\hat{x}_+(z)}{c(z^{-1})} + \frac{zx_-(z)}{z^n a(z^{-1})} = \frac{\hat{b}(z)}{z^n a(z^{-1})c(z^{-1})}, \tag{6}$$

where $g_-(z) = zx_-(z)/z^n a(z^{-1})$ is analytic in the domain $G_- = \{z \in \mathbb{C}: |z| < 1/|\alpha_n|\}$ whereas $g_+(z) = \hat{x}_+(z)/c(z^{-1})$ is analytic in the domain $G_+ = \{z \in \mathbb{C}: |z| > 1/|\gamma_1|\}$. Therefore one has $g_-(z) = \sum_{i=1}^\infty g_i^{(-)} z^i$ $\forall z \in G_-$, and $g_+(z) = \sum_{i=0}^\infty g_i^{(+)} z^{-i}$ $\forall z \in G_+$. Then, by replacing these expansions into the Eq. (6) we find

$$\sum_{i=0}^{\infty} g_i^{(+)} z^{-i} + \sum_{i=1}^{\infty} g_i^{(-)} z^i = \frac{\hat{b}(z)}{z^n a(z^{-1})c(z^{-1})}, \quad \forall z \in G = G_+ \cap G_-.$$

The function $h(z) = 1/(z^n a(z^{-1})c(z^{-1}))$ also possesses a Laurent expansion in G, namely,

$$h(z) = \frac{1}{z^n a(z^{-1})c(z^{-1})} = \sum_{i \in \mathbb{Z}} h_i z^i, \quad \forall z \in G, \tag{7}$$

and, obviously the same holds for $\hat{b}(z)/(z^n a(z^{-1})c(z^{-1}))$. From the uniqueness of the Laurent series of an analytic function in a given annulus [23], we may therefore conclude that

$$\begin{bmatrix} h_0 & h_{-1} & \dots & \dots & \dots & h_{1-n-m} \\ h_1 & h_0 & h_{-1} & & & h_{2-n-m} \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ h_{n+m-2} & \dots & \dots & h_1 & h_0 & h_{-1} \\ h_{n+m-1} & \dots & \dots & \dots & h_1 & h_0 \end{bmatrix} \begin{bmatrix} b_{n+m} \\ \vdots \\ b_{n+1} \\ b_n \\ \vdots \\ b_1 \end{bmatrix} = \begin{bmatrix} g_{m-1}^{(+)} \\ \vdots \\ g_0^{(+)} \\ g_1^{(-)} \\ \vdots \\ g_n^{(-)} \end{bmatrix} \tag{8}$$

The observation that the coefficients of $x_+(z)$ and $x_-(z)$ can be retrieved from $g_0^{(+)}, \dots, g_{m-1}^{(+)}$ and from $g_1^{(-)}, \dots, g_n^{(-)}$, respectively, finally leads to the following procedure **SolveSRLS** for the solution of the *Sylvester resultant linear system* (4).

Procedure **SolveSRLS**

(1) Evaluate the central coefficients $h_{-m-n+1}, \dots, h_{m+n-1}$ of the Laurent expansion of the reciprocal of $z^n a(z^{-1})c(z^{-1})$.
(2) Compute the first m coefficients of $g_+(z)$ and the first n coefficients of $g_-(z)$ by means of (8). Set $\tilde{g}_+(z) = g_+(z) \pmod{z^{-m}}$ and $\tilde{g}_-(z) = g_-(z) \pmod{z^{n+1}}$.
(3) Determine the coefficients of $x_+(z)$ such that $\hat{x}_+(z) = z^{1-n-m} x_+(z) = c(z^{-1})\tilde{g}_+(z) \pmod{z^{-m}}$. Analogously, find the coefficients of $x_-(z)$ by $zx_-(z) = z^n a(z^{-1})\, \tilde{g}_-(z) \pmod{z^{n+1}}$.

Remark 2. In view of the relations at step 3 of Procedure **SolveSRLS**, it can be shown that the matrix (h_{i-j}), $1 \leqslant i, j \leqslant n+m$, on the left-hand side of (8) is nonsingular. In fact, if we assume that the converse holds, then there should exist a nonzero vector $\boldsymbol{b}$ belonging to the kernel of (h_{i-j}). Corresponding to this vector, one finds $\tilde{g}_+(z) = \tilde{g}_-(z) = 0$ from which it follows that $x_+(z) = x_-(z) = 0$ and, therefore, $\boldsymbol{x} = \boldsymbol{0}$. Clearly, this is in contradiction with (4).

Since the steps 2 and 3 of **SolveSRLS** essentially amount to perform polynomial multiplications, for which fast schemes based on FFTs can be applied at the cost of $O((m+n)\log(m+n))$ arithmetic operations, it is quite obvious that the most expensive computation of **SolveSRLS** is to be carried out at the step 1. Roughly speaking, this means that, from a computational point of view, the previous procedure reduces the solution of (4) to the evaluation of certain central coefficients of the Laurent series of the reciprocal of $z^n a(z^{-1})c(z^{-1})$. We state below the precise formulation of this latter computational problem:

Problem 1 [*Reciprocation of Laurent polynomials in factored form*]. Given an odd integer k and two real Laurent polynomials $z^n a(z^{-1}) \in \mathscr{L}_0^n$ and $c(z^{-1}) \in \mathscr{L}_{-m}^0$ such that $z^n a(z^{-1}) = 1 + \sum_{i=1}^n a_i z^i = \prod_{i=1}^n (1 - \alpha_i z)$ and $c(z^{-1}) = \sum_{i=0}^{m-1} c_{m-i} z^{-i} + z^{-m} = \prod_{i=1}^m (z^{-1} - \gamma_i)$, where α_i and γ_i satisfy (3), then compute the k central coefficients $h_{-(k-1)/2}, \ldots, h_{(k-1)/2}$ of the Laurent series of the reciprocal of $p(z) = z^n a(z^{-1})c(z^{-1})$ in the annulus $G = \{z \in \mathbb{C} \colon |\gamma_1|^{-1} < |z| < |\alpha_n|^{-1}\}$.

This problem is a specific instance of the more general issue of finding the central coefficients of the Laurent expansion of the reciprocal of a Laurent polynomial having no zeros on the unit circle in the complex plane. Since the zeros of $z^n a(z^{-1})$ and $c(z)$ have modulus greater than 1, then the factorization of Problem 1 is called *spectral factorization* of the Laurent polynomial $p(z)$ [35]. Factorizations where factors have zeros with modulus greater than 1 are particularly meaningful in the solution of Markov chains of the M/G/1 type that model queueing problems [3,4,32]. Moreover, spectral factors play a key role in many diverse problems of data modeling, control theory and digital signal processing (see [1,17–19,27,36]).

In Section 3 we will first introduce a matrix analogue of the problem of reciprocating Laurent polynomials based on manipulations with bi-infinite Toeplitz matrices (operators). Then, we will show that the knowledge of the spectral factorization of the given Laurent polynomial can be exploited to produce effective computational schemes for solving Problem 1. By complementing Procedure **SolveSRLS** with these algorithms, we thus obtain a family of composite methods for the efficient solution of resultant linear systems generated by a stable and an anti-stable polynomial.

3. A matrix iteration for the reciprocation of factored Laurent polynomials

This section is concerned with the problem of the reciprocation of Laurent polynomials. We first provide a solution of the general problem based on the cyclic reduction process and, then, we specialize it for the more specific case treated in Problem 1, where the spectral factorization of the input polynomial is assumed to be known.

In view of the Cauchy integral representation of the Laurent coefficients of a meromorphic function $f(z)$ [23], one has that such coefficients could be numerically evaluated by sampling $f(z)$ in sufficiently many equidistant points on a circle and then by applying a discrete Fourier transform. This approach was considered in [33] and applied in [29] for the fast evaluation of contour integrals of rational functions. An implementation of this scheme needs the preliminary selection of the number of points and of the radius of the integration circle. Both of them are crucial parameters for the convergence and for the computational performance of the resulting quadrature procedure. A large number of points slows down the computation whereas big and small radii can lead to numerical instabilities.

The approach taken here proceeds in a very different way without requiring any critical initialization. Let $p(z) = \sum_{i=-m}^{n} p_i z^i \in \mathscr{L}_{-m}^{n}$, $p_{-m} = 1$, be the real Laurent polynomial defined as in Problem 1 by $p(z) = z^n a(z^{-1})c(z^{-1})$, where $a(z)$ and $c(z)$ satisfy (2) and (3). Consider the application which associates the Laurent polynomial $p(z)$ with the bi-infinite band Toeplitz matrix $T[p(z)]$ of symbol $p(z)$, $T[p(z)] = (p_{i-j})$ with $p_k = 0$ if $k < -m$ or $k > n$. Since $p(z)$ is a continuous function with no zeros on the unit circle, it follows that $T[p(z)]$ defines an invertible bounded linear operator acting on the Hilbert space $\ell^2(\mathbb{Z})$ of real square summable sequences $\boldsymbol{w}$ with norm $\|\boldsymbol{w}\|^2 = \sum_{k\in\mathbb{Z}} w_k^2$. Moreover, the inverse of $T[p(z)]$ is the bi-infinite Toeplitz matrix $T[1/p(z)] = (h_{i-j})$, $i, j \in \mathbb{Z}$, where h_i are the coefficients of the Laurent expansion (7) of $1/p(z)$. Then, the spectral factorization of $p(z)$ induces a triangular factorization of the corresponding bi-infinite banded Toeplitz matrix $T[p(z)]$; specifically, we find that

$$T[p(z)] = T[z^n a(z^{-1})]T[c(z^{-1})] = T[c(z^{-1})]T[z^n a(z^{-1})]. \tag{9}$$

This factorization is usually referred to as the *Wiener–Hopf factorization* [10] of $T[p(z)]$ and the triangular factors $T[c(z^{-1})]$ and $T[z^n a(z^{-1})]$ are themselves invertible operators in $\ell^2(\mathbb{Z})$.

As an application of the preceding results, now consider the solution of the linear system

$$T[p(z)]X = E, \tag{10}$$

where X and E are bi-infinite vectors with $n+m$ columns $\in \ell^2(\mathbb{Z})$, $X, E : \mathbb{Z} \to \mathbb{R}^{(n+m)\times(n+m)}$, with $E(0) = I$ and, otherwise, $E(j) = 0$ for $j \neq 0$. Thus, it can easily be seen that $X(0) = (h_{i-j})$, $1 \leqslant i, j \leqslant n+m$, is the matrix on the left-hand side of (8) and, therefore, the first and the last column of $X(0)$ provide the sought coefficients of the reciprocal of the Laurent polynomial $p(z)$.

An efficient way of solving (10) is based upon the use of the cyclic reduction scheme, originally introduced in [13] for the solution of partial differential equations and, more recently, adjusted in [5–7] for solving certain queueing problems. By employing a convenient partitioning of $T^{(0)} = T[p(z)]$ into a block tridiagonal matrix with a block Toeplitz structure, $T^{(0)} = (P_{r-s})$, $r, s \in \mathbb{Z}$, where $P_k = P_k^{(0)} = (p_{i-j+k(n+m)})$ for $1 \leqslant i, j \leqslant n+m$, $k \in \mathbb{Z}$, and $p_k = 0$ if $k < -m$ or $k > n$, then (10) is reduced to a system of linear equations of the form

$$P_1^{(0)} X(j-1) + P_0^{(0)} X(j) + P_{-1}^{(0)} X(j+1) = E(j), \quad j \in \mathbb{Z}.$$

Now, for any $h = 2l$, $l \in \mathbb{Z}$, consider equations of indexes $j = h-1$, $j = h$ and $j = h+1$: if $P_0^{(0)}$ is assumed to be nonsingular, by multiplying the first equation by $P_1^{(0)}(P_0^{(0)})^{-1}$ and the last equation by $P_{-1}^{(0)}(P_0^{(0)})^{-1}$ and, then by subtracting them from the second one, we obtain

$$P_1^{(1)}X(h-2)+P_0^{(1)}X(h)+P_{-1}^{(1)}X(h+2)=E(h),\quad h=2l,\ l\in\mathbb{Z},$$

where we set

$$\begin{aligned} P_1^{(1)} &= -P_1^{(0)}(P_0^{(0)})^{-1}P_1^{(0)} \\ P_0^{(1)} &= P_0^{(0)}-P_1^{(0)}(P_0^{(0)})^{-1}P_{-1}^{(0)}-P_{-1}^{(0)}(P_0^{(0)})^{-1}P_1^{(0)}. \\ P_{-1}^{(1)} &= -P_{-1}^{(0)}(P_0^{(0)})^{-1}P_{-1}^{(0)} \end{aligned} \tag{11}$$

Hence, these formulas allow us to define a new bounded block tridiagonal operator $T^{(1)}$ given by $T^{(1)}=(P_{i-j}^{(1)})_{i,j\in\mathbb{Z}}$, $P_{i-j}^{(1)}=0$ if $|i-j|>1$, which satisfies $T^{(1)}X^{(1)}=E$, where $X^{(1)}(k)=X^{(0)}(2k)=X(2k)$, $k\in\mathbb{Z}$.

The iterative application of formulas (11) defines the cyclic reduction process for the approximation of $X(0)$. The Toeplitz-like structure properties of the matrices $P_j^{(i)}$ generated by this process as well as its convergence behavior have been widely investigated in the papers [6,7]. In what follows, we will consider a different look at the cyclic reduction algorithm which is here used as a means for the iterative construction of a block triangular factorization of $T^{(s)}=F^{(s)}D^{(s)}G^{(s)}$ rather than of its nonzero block entries $P_j^{(s)}$, $j=-1,0,1$. This has the important advantage of replacing (11) with a different set of formulas for the block entries of $F^{(s)}$, $D^{(s)}$ and $G^{(s)}$ involving terms explicitly related to the given spectral factors $z^n a(z^{-1})$ and $c(z^{-1})$ of $p(z)$. In particular, if $D_0^{(s)}$ denotes the block diagonal entry of $D^{(s)}$, then we obtain a simple iterative scheme for generating the matrix sequence $\{D_0^{(s)^{-1}}\}_s$ which quadratically approaches $X(0)$.

An initial guess for our process is provided by the Wiener–Hopf factorization (9) of $T^{(0)}$. Suppose that $T[c(z^{-1})]$ and $T[z^n a(z^{-1})]$ are partitioned commensurably with $T[p(z)]$, that is, for $r,s\in\mathbb{Z}$ we set $T[z^n a(z^{-1})]=(L_{r-s}^{(0)})$ with $L_k^{(0)}=(a_{i-j+k(n+m)})$, and $T[c(z^{-1})]=(U_{r-s}^{(0)})$ with $U_k^{(0)}=(c_{i-j+m+k(n+m)})$, where $a_i=0$ if $i<0$ or $i>n$ and, analogously, $c_i=0$ if $i<0$ or $i>m$. Since the upper triangular matrix $U_0^{(0)}$ and the lower triangular matrix $L_0^{(0)}$ are nonsingular, one finds that the Wiener–Hopf factorization (9) of $T^{(0)}$ can be rewritten into a block form as

$$\begin{aligned}&\begin{bmatrix} \ddots & & \\ \ddots & I & \\ & F_1^{(0)} & \ddots \\ & & \ddots \end{bmatrix}\begin{bmatrix} \ddots & & \\ & D_0^{(0)} & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \ddots & \ddots & & \\ & I & G_{-1}^{(0)} & \\ & & \ddots & \ddots \end{bmatrix} \\ &\quad = F^{(0)}D^{(0)}G^{(0)}, \end{aligned} \tag{12}$$

where $F_1^{(0)}=L_1^{(0)}(L_0^{(0)})^{-1}$, $D_0^{(0)}=L_0^{(0)}U_0^{(0)}$, $G_{-1}^{(0)}=(U_0^{(0)})^{-1}U_{-1}^{(0)}$, and I denotes the identity matrix of order $n+m$.

Next result relates $F_1^{(0)}$ and $G_{-1}^{(0)}$ with the Frobenius matrices of the spectral factors $z^n a(z^{-1})$ and $c(z^{-1})$ of $p(z)$ [4]. Recall that, for a given real polynomial $u(z) = \sum_{i=0}^{n+m} u_{n+m-i} z^i$ of degree $n+m$, the associated Frobenius matrix $C(u(z)) \in \mathbb{R}^{(n+m)\times(n+m)}$ is defined by

$$C(u(z)) = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -u_{n+m}/u_0 & \dots & \dots & \dots & -u_1/u_0 \end{bmatrix}.$$

Theorem 3. *The block entries of $F^{(0)}$ and $G^{(0)}$ of* (12) *are such that*

$$F_1^{(0)} = -(JC(z^m a(z))^{\mathrm{T}} J)^{n+m}, \quad G_{-1}^{(0)} = -(JC(z^{n+m} c(z^{-1}))J)^{n+m}, \tag{13}$$

where J denotes the permutation matrix of order $n+m$ with unit anti-diagonal entries. Hence, the bi-infinite triangular matrices $F^{(0)}$ and $G^{(0)}$ are invertible and their inverses are given by: $((F^{(0)})^{-1})_{i,j} = ((JC(z^m a(z))^{\mathrm{T}} J)^{m+n})^{i-j}$, $i \geqslant j$, $i, j \in \mathbb{Z}$, *and* $((G^{(0)})^{-1})_{i,j} = ((JC(z^{n+m} c(z^{-1}))J)^{m+n})^{j-i}$, $j \geqslant i$, $j, i \in \mathbb{Z}$.

Theorem 3 was used in [4] to derive a Gohberg–Semencul type formula representing the inverse of the Toeplitz matrix $X(0) = (h_{i-j})$, $1 \leqslant i, j \leqslant n+m$, of (8) in terms of the coefficients of the spectral factors of $p(z)$. Differently, here we consider a fairly inverse viewpoint by showing that the a-priori knowledge of the spectral factors makes possible to determine $X(0)$ without performing any inversion of Toeplitz-like matrices. Next result provides an iterative scheme for the construction of a block triangular factorization of the linear operators $T^{(s)}$, generated by the cyclic reduction process, starting from that one of $T^{(0)} = T[p(z)]$ factored as in (12).

Theorem 4. *Let $\{T^{(s)}\}_{s\in\mathbb{N}}$, $T^{(s)} = (P_{i,j}^{(s)})_{i,j\in\mathbb{Z}}$, be the sequence of linear operators generated by the cyclic reduction process starting from $T^{(0)} = T[p(z)]$ by means of relations* (11), *where we assume that all the matrices to be inverted are nonsingular and, therefore, the process does not break down at any step. We have that $T^{(s)}$, $s \in \mathbb{N}$, is a bi-infinite block Toeplitz matrix in block tridiagonal form and, moreover, it has a block triangular factorization of the form*

$$T^{(s)} = F^{(s)} D^{(s)} G^{(s)}, \tag{14}$$

where $F^{(s)} = (F_{i-j}^{(s)})_{i,j\in\mathbb{Z}}$ is block lower bidiagonal, $G^{(s)} = (G_{i-j}^{(s)})_{i,j\in\mathbb{Z}}$ is block upper bidiagonal and $D^{(s)} = (D_{i-j}^{(s)})_{i,j\in\mathbb{Z}}$ is block diagonal. In addition, for $s = 1, 2, \dots$, the factorization of $T^{(s)}$ can be constructed iteratively from the one of $T^{(s-1)}$ according to the following rules:

$$F_0^{(s)} = I,\ F_1^{(s)} = -(F_1^{(s-1)})^2;\quad G_0^{(s)} = I,\ G_{-1}^{(s)} = -(G_{-1}^{(s-1)})^2;$$

$$D_0^{(s)} = D_0^{(s-1)} - D_0^{(s-1)} G_{-1}^{(s-1)} (D_0^{(s-1)} + F_1^{(s-1)} D_0^{(s-1)} G_{-1}^{(s-1)})^{-1} F_1^{(s-1)} D_0^{(s-1)}.$$

Proof. The proof follows from some straightforward calculations and, without loss of generality, we may restrict ourselves to the case where $s = 1$. From the Wiener–Hopf factorization of $T^{(0)} = T[p(z)]$, one has that

$$P_{-1}^{(0)} = D_0^{(0)} G_{-1}^{(0)},\quad P_0^{(0)} = D_0^{(0)} + F_1^{(0)} D_0^{(0)} G_{-1}^{(0)},\quad P_1^{(0)} = F_1^{(0)} D_0^{(0)},\quad (15)$$

from which it follows that $D_0^{(1)}$ is well defined if and only if $P_0^{(0)}$ is nonsingular. Hence, we are able to introduce the block Toeplitz matrix in block tridiagonal form $\hat{T}$ defined by $\hat{T} = (\hat{T}_{i-j}) = F^{(1)} D^{(1)} G^{(1)}$. The theorem is thus established by first replacing (15) into the formulas (11) and, then, by showing that $T^{(1)} = \hat{T}$. For the sake of notational simplicity we omit to indicate both the superscripts and the subscripts whenever it is possible. In this way, we set $F_1^{(0)} = F$, $G_{-1}^{(0)} = G$ and $D_0^{(0)} = D$. Then, we find that

$$\begin{aligned}\hat{T}_0 - P_0^{(1)} \\ &= F^2\{D - DG(D + FDG)^{-1} FD\} G^2 - FDG + FD(D + FDG)^{-1} DG \\ &= F\{FDG - FDG(D + FDG)^{-1} FDG - D + D(D + FDG)^{-1} D\} G.\end{aligned}$$

Since we have

$$FDG - FDG(D + FDG)^{-1} FDG = D(D + FDG)^{-1} FDG,$$

one gets that

$$\begin{aligned}\hat{T}_0 - P_0^{(1)} \\ &= F\{D(D + FDG)^{-1} FDG - D + D(D + FDG)^{-1} D\} G \\ &= FD(D + FDG)^{-1}\{FDG - (D + FDG) + D\} G = 0.\end{aligned}$$

Analogously, by comparing the subdiagonal block entries, we obtain that

$$\begin{aligned}\hat{T}_1 - P_1^{(1)} &= -F^2\{D - DG(D + FDG)^{-1} FD\} + FD(D + FDG)^{-1} FD \\ &= \{-F + F^2 DG(D + FDG)^{-1} + FD(D + FDG)^{-1}\} FD \\ &= \{-F(D + FDG) + F^2 DG - FD\}(D + FDG)^{-1} FD = 0.\end{aligned}$$

The equality of the superdiagonal block entries follows in exactly the same way and this concludes the proof. □

Since the polynomials $z^m a(z)$ and $z^{m+n} c(z^{-1})$ are stable, from Theorem 3 one easily deduces that the $F_1^{(0)}$ and $G_{-1}^{(0)}$ have spectral radius less than 1 and, therefore, in view of Theorem 4, the bi-infinite matrices $F^{(s)}$ and $G^{(s)}$ approach the identity operator I_∞ as s goes to $+\infty$. Roughly speaking, this implies that the inverse of $D_0^{(s)}$ should yield a good approximation of $X(0)$ for s sufficiently large. A more precise formulation of this claim is provided by the following theorem.

Theorem 5. *Let us assume that the cyclic reduction algorithm applied to $T^{(0)} = T[p(z)]$ does not break down at any step. Then, it generates a sequence of linear operators $\{T^{(s)}\}_{s\in\mathbb{N}}$ acting on $\ell^2(\mathbb{Z})$ for which the following statements hold*:

(1) *For each $s \in \mathbb{N}$, the matrix $D^{(s)} = I_\infty \otimes D_0^{(s)}$ of* (14) *is nonsingular and, moreover, $(D_0^{(s)})^{-1}$ satisfies the following recurrence relation*:

$$(D_0^{(s)})^{-1} = (D_0^{(s-1)})^{-1} + G_{-1}^{(s-1)}(D_0^{(s-1)})^{-1}F_1^{(s-1)}, \quad s \in \mathbb{N}^+. \tag{16}$$

(2) *As a consequence, it follows that each $T^{(s)}$ is a linear invertible operator and, therefore, the bi-infinite block vector $X^{(s)}$, $s \in \mathbb{N}$, defined by $X^{(s)}(k) = X^{(s-1)}(2k)$, $k \in \mathbb{Z}$, with $X^{(0)} = X$ solution of* (10), *is the unique solution of the linear system*

$$T^{(s)}X^{(s)} = E, \quad s \in \mathbb{N}.$$

Further, for any σ with $\min\{|\alpha_n|^{n+m}, 1/|\gamma_1|^{n+m}\} < \sigma < 1$, *there exists a positive constant C such that*

$$\|X(0) - (D_0^{(s)})^{-1}\| \leqslant C\sigma^{2^s}, \tag{17}$$

where $X(0) = (h_{i-j})$, $1 \leqslant i, j \leqslant n+m$, is the matrix on the left-hand side of (8).

Proof. The first part of the theorem is established by showing that the inverse of $D^{(s)}$ can be explicitly constructed starting from the one of $D^{(s-1)}$. Again, as in the proof of Theorem 4, for the sake of notational simplicity, we consider the case $s = 1$ and we omit to indicate superscripts and subscripts whenever it is possible. Note that $D_0^{(0)}$ is invertible since it is given by the product of invertible triangular Toeplitz matrices. Moreover, if the the cyclic reduction process is applicable, then $D_0^{(1)} = D - DG(D + FDG)^{-1}FD$ is well defined. Thus, the matrix $W = D^{-1} + GD^{-1}F$ is our candidate to be the inverse of $D_0^{(1)}$. We have that:

$$\begin{aligned} D_0^{(1)}W &= I + DGD^{-1}F - DG(D + FDG)^{-1}(I + FDGD^{-1})F \\ &= I + DG(D + FDG)^{-1}\{I + DDGD^{-1} - I - FDGD^{-1}\}F = I. \end{aligned}$$

By induction, it can similarly be proved that relation (16) holds for any $s > 0$. It is worth pointing out that this relation also implies that the matrices $(D_0^{(s)})^{-1}$ are of uniformly bounded norm. In fact, from (16) it follows that there exists a suitable positive constant C_1 such that $\|(D_0^{(s)})^{-1}\| \leqslant (1 + C_1\sigma^{2^{s-1}})\|(D_0^{(s-1)})^{-1}\|$, $s \in \mathbb{N}^+$. The increasing sequence $\{d_k\}$ defined by $d_k = \prod_{i=0}^{k}(1 + C_1\sigma^{2^i})$, $k = 0, 1, \ldots,$ can be bounded from above as follows:

$$d_k = \exp(\log(d_k)) = \exp\left(\sum_{i=0}^{k} \log\left(1 + C_1\sigma^{2^i}\right)\right) \leqslant \exp\left(C_1 \sum_{i=0}^{k} \sigma^{2^i}\right) \leqslant L.$$

Hence, we easily obtain that $\|(D_0^{(s)})^{-1}\| \leqslant L\|(D_0^{(0)})^{-1}\|$, $s \geqslant 1$. By combining this property with the results of Theorem 4, one deduces the convergence of the sequence of inverse operators $(T^{(s)})^{-1} = (G^{(s)})^{-1}(D^{(s)})^{-1}\ (F^{(s)})^{-1}$. In fact, analogously with Theorem 3, it is found that the bi-infinite triangular matrices $F^{(s)}$ and $G^{(s)}$ are invertible and their inverses are given by: $((F^{(s)})^{-1})_{i,j} = (JC(z^m a(z))^{\mathrm{T}} J)^{2^s(m+n)(i-j)}$, $i \geqslant j$, $i, j \in \mathbb{Z}$, and $((G^{(s)})^{-1})_{i,j} = (JC(z^{n+m} c(z^{-1}))J^{2^s(m+n)(j-i)}$, $j \geqslant i$, $j, i \in \mathbb{Z}$. From this, then it follows that $\|(T^{(s)})^{-1} - (D^{(s)})^{-1}\| \leqslant C_2\sigma^{2^s}$, where C_2 is a given positive constant. On the other hand, for any $s \in \mathbb{N}$, the matrices $(T^{(s)})^{-1}$ have the same entries in the positions i, j with $1 \leqslant i$, $j \leqslant n + m$. Under our notations, $X(0)$ is the $(n + m) \times (n + m)$ Toeplitz matrix made up by these entries, where we recall that $X(0) = X^{(s)}(0) = (h_{i-j})$, for $s \geqslant 0$, and, moreover, that $X(0)$ is nonsingular by virtue of Remark 2. Thus, we finally obtain that the diagonal block $(D_0^{(s)})^{-1}$ of $(D^{(s)})^{-1}$ approaches $X(0)$ and the inequality $\|(D_0^{(s)})^{-1} - X(0)\| \leqslant C\sigma^{2^s}$, $s \in \mathbb{N}$, holds for a suitable positive constant C. $\square$

Theorem 5 says that the matrix iteration (16) is eligible for the task of approximating the Toeplitz matrix $X(0)$ of (8) whose entries are the central coefficients of the Laurent series of the reciprocal of $p(z)$.

Remark 6. Observe that (16) is well-defined whenever $D_0^{(0)}$ is invertible and this is always the case as $D_0^{(0)} = L_0^{(0)}U_0^{(0)}$ is the product of nonsingular triangular Toeplitz matrices. Nevertheless, (17) is derived under the auxiliary assumption that the cyclic reduction process applied to $T[p(z)]$ has no premature termination. To overcome this discrepancy, note that in view of the Szegö's strong limit theorem [10] one deduces that $(T[p(z)])_k = (p_{i-j}) \in \mathbb{R}^{k\times k}$ is nonsingular for any sufficiently large k, say $k \geqslant 2^s$. By virtue of the results of [3] (Section 4), this implies that $T^{(s+1)}$ can be straight determined from $T^{(0)}$ by a block elimination of the variables and, moreover, that the cyclic reduction algorithm applied to $T^{(s+1)}$ goes on without the occurrence of breakdowns. By a continuity argument, it can also be shown that $T^{(s+1)}$ satisfies (14) and from this we finally conclude that (17) remains still valid in the general case.

A plain implementation of the resulting iterative process can be organized as follows:

Matrix iteration

- Initialization phase: Given the coefficients of the spectral factors $z^n a(z)$ and $c(z^{-1})$ of $p(z)$, form the matrices $L_0^{(0)}$, $L_1^{(0)}$, $U_0^{(0)}$ and $U_{-1}^{(0)}$ and, then compute $F_1^{(0)} = L_1^{(0)}(L_0^{(0)})^{-1}$, $D_0^{(0)} = L_0^{(0)}U_0^{(0)}$, and $G_{-1}^{(0)} = (U_0^{(0)})^{-1}U_{-1}^{(0)}$.
- Iterative phase: For $s = 1, 2, \ldots,$
  (1) compute $(D_0^{(s)})^{-1} = (D_0^{(s-1)})^{-1} + G_{-1}^{(s-1)}(D_0^{(s-1)})^{-1}F_1^{(s-1)}$;
  (2) check the convergence of $(D_0^{(s)})^{-1}$;
  (3) set $F_1^{(s)} = -(F_1^{(s-1)})^2$ and $G_{-1}^{(s)} = -(G_{-1}^{(s-1)})^2$.

A preliminary investigation of the properties of this scheme was led in [21] by confirming its effectiveness and numerical stability. However, it is clear that the computational cost of an unstructured implementation of the above algorithm is prohibitive since it grows as $(n+m)^3$ arithmetic operations. In the next section it will be shown that both the initialization and the iterative phase can be performed in a very efficient way by exploiting the displacement structure of the matrices generated at any step, with a substantial reduction of the asymptotic cost. More specifically, a careful implementation of the *Matrix iteration* algorithm is devised which requires $\mathrm{O}((n+m)\log(n+m))$ arithmetic operations per step.

4. Efficient implementation of the matrix iteration

In this section we make use of structured numerical linear algebra techniques in order to derive a superfast implementation of the *Matrix iteration* algorithm. Let us start by recalling some well known results which are particularly interesting for dealing with our problem. Consider the displacement operator $\mathscr{F}_1 : \mathbb{R}^{(n+m)\times(n+m)} \to \mathbb{R}^{(n+m)\times(n+m)}$ defined by $\mathscr{F}_1(A) = A - (JC(z^{n+m}c(z^{-1}))J)A(JC(z^m a(z))^{\mathrm{T}}J)$. This operator can immediately be related to the more classical displacement operator $\mathscr{F}_2(A) = A - ZAZ^{\mathrm{T}}$, where Z denotes the $(n+m)\times(n+m)$ down-shift matrix, $Z = [\boldsymbol{e}_2|\ldots|\boldsymbol{e}_{n+m}|\boldsymbol{0}]$, and $\boldsymbol{e}_i$ is the ith column of the identity matrix I of order $n+m$. Specifically, one finds that

$$JC(z^{n+m}c(z^{-1}))J = Z - \boldsymbol{e}_1\hat{\boldsymbol{c}}^{\mathrm{T}}Z, \quad JC(z^{n+m}a(z))J = Z - \boldsymbol{e}_1\hat{\boldsymbol{a}}^{\mathrm{T}}Z, \tag{18}$$

where $\hat{\boldsymbol{a}} = [1, a_1, \ldots, a_n, 0, \ldots, 0]^{\mathrm{T}}$ and $\hat{\boldsymbol{c}} = [1, c_{m-1}/c_m, \ldots, c_0/c_m, 0, \ldots, 0]^{\mathrm{T}}$. The rank of $\mathscr{F}_j(A)$, $j = 1, 2$, is called the j-displacement rank of A. The following result is classical [28].

Theorem 7. *Let us assume that the $(m+n)\times(m+n)$ matrix A has 2-displacement rank l, that is, $\mathscr{F}_2(A) = A - ZAZ^{\mathrm{T}} = \sum_{i=1}^{l}\boldsymbol{x}_i\boldsymbol{y}_i^{\mathrm{T}}$. Then, we have that $A = \sum_{i=1}^{l} L(\boldsymbol{x}_i)U(\boldsymbol{y}_i)$, where $L(\boldsymbol{x})$ denotes the lower triangular Toeplitz matrix whose first column is $\boldsymbol{x}$ and $U(\boldsymbol{y}) = L(\boldsymbol{y})^{\mathrm{T}}$.*

Relation (18) implies that the 2-displacement rank of the permuted Frobenius matrices $JC(z^{n+m}c(z^{-1}))J$ and $JC(z^{n+m}a(z))J$ is 2 at most. Next result, which follows from Barnett's factorization of Bezoutians [1,30], says that the powers of a Frobenius matrix also inherit a special displacement structure.

Theorem 8. *For a given polynomial $p(z) = \sum_{i=0}^{n+m} p_i z^i$ of degree $n+m$ and for any integer k, let us denote by $q^{(k)}(z)$ and $r^{(k)}(z)$, respectively, the quotient and the remainder in the Euclidean division of z^k by $p(z)$, that is, $z^k = q^{(k)}(z)p(z) + r^{(k)}(z)$, where the degree $l(k)$ of $r^{(k)}(z) = \sum_{i=0}^{l(k)} r_i^{(k)} z^i$ is less than $n+m$. Then, the* kth *power $(C(p(z)))^k$ of the Frobenius matrix $C(p(z))$ of order $n+m$ associated with $p(z)$ admits the following displacement representation*:

$$(C(p(z)))^k = r^{(k)}(C(p(z))) = J(U(\hat{\boldsymbol{p}}))^{-1}J\left\{L(\hat{\boldsymbol{p}})U(\boldsymbol{r}^{(k)}) - L(\hat{\boldsymbol{r}}^{(k)})U(\tilde{\boldsymbol{p}})\right\},$$

where $\tilde{\boldsymbol{p}} = [p_0, \ldots, p_{n+m-1}]^{\mathrm{T}}$, $\boldsymbol{r}^{(k)} = [r_0^{(k)}, \ldots, r_{l(k)}^{(k)}, 0, \ldots, 0]^{\mathrm{T}}$, $\hat{\mathbf{p}} = [p_{n+m}, \ldots, p_1]^{\mathrm{T}}$, and $\hat{\mathbf{r}}^{(k)} = [0, \ldots, 0, r_{l(k)}^{(k)}, \ldots, r_1^{(k)}]^{\mathrm{T}}$.

From Theorem 8 it follows that there exist two suitable symmetric matrices B_a and B_c verifying

$$C(z^m a(z)) = J(L_0^{(0)})^{-\mathrm{T}} B_a, \quad C(z^{n+m}c(z^{-1})) = J(U_0^{(0)})^{-1} B_c. \tag{19}$$

By combining these representation formulas with relation (18), we thus obtain that each matrix $(D_0^{(s)})^{-1}$, $s \geqslant 0$, has 1-displacement rank bounded from above by 2, that is, the rank of $\mathscr{F}_1((D_0^{(s)})^{-1})$ is at most 2.

Theorem 9. *We have that*

$$\begin{aligned}&\mathscr{F}_1((D_0^{(0)})^{-1})\\ &\quad = \frac{1}{c_m}(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - (JC(z^{n+m}c(z^{-1}))J)^{n+m}\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}(JC(z^m a(z))^{\mathrm{T}}J)^{n+m}).\end{aligned}$$

Proof. From (19) we obtain that

$$\mathscr{F}_1((D_0^{(0)})^{-1}) = (U_0^{(0)})^{-1}(I - C(z^{n+m}c(z^{-1}))^{\mathrm{T}}C(z^m a(z)))(L_0^{(0)})^{-1}.$$

By replacing (18) into this formula, it follows that

$$\begin{aligned}\mathscr{F}_1((D_0^{(0)})^{-1}) &= (U_0^{(0)})^{-1}(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - ZJ\hat{\boldsymbol{c}}\hat{\boldsymbol{a}}^{\mathrm{T}}JZ^{\mathrm{T}})(L_0^{(0)})^{-1}\\ &= \frac{1}{c_m}(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - (U_0^{(0)})^{-1}U_{-1}^{(0)}\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}L_1^{(0)}(L_0^{(0)})^{-1})\\ &= \frac{1}{c_m}(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - G_{-1}^{(0)}\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}F_1^{(0)}),\end{aligned}$$

where $G_{-1}^{(0)}$ and $F_1^{(0)}$ are defined as in Theorem 3. □

Theorem 10. *For any $s \geqslant 0$, we find that*

$$\mathscr{F}_1((D_0^{(s)})^{-1}) = \frac{1}{c_m}(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - (JC(z^{n+m}c(z^{-1}))J)^{l_s}\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}(JC(z^m a(z))^{\mathrm{T}}J)^{l_s}),$$

where $l_s = 2^s(n+m)$.

Proof. The proof is by induction on s. The case $s = 0$ is established by Theorem 9. For $s > 0$, observe that $\mathscr{F}_1((D_0^{(s)})^{-1})$ is equal to

$$\begin{aligned}&\mathscr{F}_1((D_0^{(s-1)})^{-1}) + (JC(z^{n+m}c(z^{-1}))J)^{l_{s-1}}\mathscr{F}_1((D_0^{(s-1)})^{-1})\\ &\quad\times (JC(z^m a(z))^{\mathrm{T}}J)^{l_{s-1}}\\ &= \frac{1}{c_m}(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}} - (JC(z^{n+m}c(z^{-1}))J)^{l_s}\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}(JC(z^m a(z))^{\mathrm{T}}J)^{l_s}). \qquad \square\end{aligned}$$

As an immediate consequence of this result we find that the displacement rank of the matrices $(D_0^{(s)})^{-1}$, $s \geqslant 0$, w.r.t. the displacement operator $\mathscr{F}_2$ can also be bounded from above by a small constant integer.

Theorem 11. *For any $s \geqslant 0$, there are uniquely determined two vectors $\boldsymbol{r}^{(s)}$ and $\boldsymbol{t}^{(s)}$ of size $n+m$, where the first entry of $\boldsymbol{r}^{(s)}$ is equal to 0, such that $\mathscr{F}_2((D_0^{(s)})^{-1}) = \boldsymbol{e}_1\boldsymbol{r}^{(s)^{\mathrm{T}}} + \boldsymbol{t}^{(s)}\boldsymbol{e}_1^{\mathrm{T}} - c_m^{-1}(JC(z^{n+m}c(z^{-1}))J)^{l_s}\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}(JC(z^m a(z))^{\mathrm{T}}J)^{l_s}$.*

Proof. The proof of the existence of $\boldsymbol{r}^{(s)}$ and $\boldsymbol{t}^{(s)}$ follows by replacing (18) into the displacement equation provided by Theorem 10. Concerning the uniqueness, from Theorem 7 one finds that $(D_0^{(s)})^{-1}$ can be represented as

$$L(\boldsymbol{t}^{(s)}) + U(\boldsymbol{r}^{(s)}) - \frac{1}{c_m}L((JC(z^{n+m}c(z^{-1}))J)^{l_s}\boldsymbol{e}_1)U(JC(z^m a(z))J)^{l_s}\boldsymbol{e}_1).$$

This means that, for any $s \geqslant 0$, the $(n+m)\times(n+m)$ matrix

$$\hat{T} = (D_0^{(s)})^{-1} + c_m^{-1}L((JC(z^{n+m}c(z^{-1}))J)^{l_s}\boldsymbol{e}_1)U((JC(z^m a(z))J)^{l_s}\boldsymbol{e}_1)$$

is Toeplitz and, hence, $\boldsymbol{t}^{(s)}$ are $\boldsymbol{r}^{(s)^{\mathrm{T}}}$ are determined by the entries on its first column and its first row, respectively. $\square$

By combining the previous result with the displacement representation of the powers of Frobenius matrices stated by Theorem 8, it allows us to develop a very efficient implementation of the *Matrix iteration* algorithm. Below, we give a description of this resulting implementation.

Structured matrix iteration

- Initialization phase:

(1) Compute the first row $\tilde{\boldsymbol{r}}^{(0)^{\mathrm{T}}}$ and the first column $\tilde{\boldsymbol{t}}^{(0)}$ of $(D_0^{(0)})^{-1}$;

(2) determine $r_a^{(n+m)}(z) = z^{n+m} \pmod{z^m a(z)}$ and $r_c^{(n+m)}(z) = z^{n+m} \pmod{z^{n+m}c(z^{-1})}$;

(3) form the displacement representations of $(JC(z^m a(z))J)^{n+m}$ and $(JC(z^{n+m}c(z^{-1}))J)^{n+m}$ shown in Theorem 8 and then compute the first column $\boldsymbol{f} = [f_1, \dots, f_{n+m}]^{\mathrm{T}}$ of $(JC(z^{n+m}c(z^{-1}))J)^{n+m}$ and the first column $\boldsymbol{g} = [g_1, \dots, g_{n+m}]^{\mathrm{T}}$ of $(JC(z^m a(z))J)^{n+m}$;
(4) set $\boldsymbol{t}^{(0)} = \tilde{\boldsymbol{t}}^{(0)} + (g_1/c_m)\boldsymbol{f}$ and $\boldsymbol{r}^{(0)} = \tilde{\boldsymbol{r}}^{(0)} + (f_1/c_m)\boldsymbol{g}$, where the first entry of $\boldsymbol{r}^{(0)}$ is replaced by 0.

- Iterative phase: for $s = 1, 2, \dots,$

(1) compute the first row $\tilde{\boldsymbol{r}}^{(s)^{\mathrm{T}}}$ and the first column $\tilde{\boldsymbol{t}}^{(s)}$ of $(D_0^{(s)})^{-1}$ by using (16), where $(D_0^{(s-1)})^{-1}$ is expressed by means of its displacement representation provided by Theorem 11;
(2) determine the coefficients of the remainders $(r_a^{((n+m)2^{s-1})}(z))^2 \pmod{z^m\, a(z)}$, and $(r_c^{((n+m)2^{s-1})}(z))^2 \pmod{z^{n+m}c(z^{-1})}$.
(3) form the displacement representations of $(JC(z^m a(z))J)^{(n+m)2^s}$ and $(JC(z^{n+m}c(z^{-1}))J)^{(n+m)2^s}$ and then compute the first column $\boldsymbol{f}$ and $\boldsymbol{g}$ of $(JC(z^{n+m}c(z^{-1}))J)^{(n+m)2^s}$ and $(JC(z^m a(z))J)^{(n+m)2^s}$, respectively;
(4) set $\boldsymbol{t}^{(s)} = \tilde{\boldsymbol{t}}^{(s)} + (g_1/c_m)\boldsymbol{f}$ and $\boldsymbol{r}^{(s)} = \tilde{\boldsymbol{r}}^{(s)} + (f_1/c_m)\boldsymbol{g}$, where the first entry of $\boldsymbol{r}^{(s)}$ is replaced by 0;
(5) check the convergence of $\boldsymbol{t}^{(s)}$ and $\boldsymbol{r}^{(s)}$.

Since the matrix $D_0^{(0)}$ is the product of two triangular Toeplitz matrices whose entries are explicitly given in terms of the coefficients of $a(z)$ and $c(z)$, it is easily found that the initialization phase can be performed at the cost of $\mathrm{O}((n+m)\log(n+m))$ arithmetic operations by means of the Sieveking–Kung algorithm [8]. Similarly, steps 1 and 3 of the iterative phase essentially amount to a polynomial multiplication and, therefore, they can be carried out at the cost of $\mathrm{O}((n+m)\log(n+m))$ arithmetic operations by using FFTs. Concerning the step 2, observe that the required Euclidean divisions can also be computed in a stable way by means of convolutions at the cost of $\mathrm{O}((n+m)\log(n+m))$ arithmetic operations by using the algorithm of [15]. Hence, the proposed iterative process can be implemented at the cost of $\mathrm{O}((n+m)\log(n+m))$ arithmetic operations per step. In view of the quadratic convergence of the sequences $\{\boldsymbol{r}^{(s)}\}$ and $\{\boldsymbol{t}^{(s)}\}$ stated by Theorem 5, we find that, for a fixed precision ϵ, $\mathrm{O}(\log(\log\epsilon^{-1}) + |\log(\log\sigma^{-1})|)$ steps are sufficient to determine approximations $\boldsymbol{r}^{(s)}$ and $\boldsymbol{t}^{(s)}$ such that

$$\|\boldsymbol{t}^{(s)} - X(0)\boldsymbol{e}_1\|_\infty \leqslant \epsilon, \quad \|\boldsymbol{r}^{(s)^{\mathrm{T}}} - \boldsymbol{e}_1^{\mathrm{T}}(X(0) - h_0\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}})\|_\infty \leqslant \epsilon.$$

By Theorem 5, the parameter σ satisfies $\min\{|\alpha_n|^{n+m}, 1/|\gamma_1|^{n+m}\} < \sigma < 1$ and it gives a measure of the separation between the spectrum of $a(z)$ and $c(z)$. Indeed, it is easily seen that

$$(\min\{|\alpha_n|^{n+m}, 1/|\gamma_1|^{n+m}\})^2 \leqslant \left|\frac{\alpha_n}{\gamma_1}\right|^{n+m} = r^{n+m} \leqslant \min\{|\alpha_n|^{n+m}, 1/|\gamma_1|^{n+m}\}.$$

Therefore we may conclude that the *Structured matrix iteration* algorithm provides numerical approximations of the sought $2(n+m)-1$ central coefficients

$h_{1-n-m}, \ldots, h_{n-m+1}$ of the Laurent series of the reciprocal of $p(z) = z^n a(z^{-1}) c(z^{-1})$ within a fixed tolerance ϵ at the overall cost of $\mathrm{O}((n+m)\log(n+m)(\log(\log\epsilon^{-1}) + |\log(\log r^{-1})|))$ arithmetic operations.

5. Numerical experiments

To check the stability properties of our method numerically, we implemented the *Structured matrix iteration* algorithm by using MATLAB and, then, we carried out numerical experiments on a pentium 550 workstation with the Linux system.

The relevant computations of our program can basically be reduced to perform two matrix operations: (a) multiplication of a triangular Toeplitz matrix $L(\boldsymbol{x})$ by a vector $\boldsymbol{y}$ and (b) inversion of the triangular Toeplitz matrices $L_0^{(0)} = L(\hat{\boldsymbol{a}})$ and $U_0^{(0)} = U(c_m\hat{\boldsymbol{c}})$. The first task is solved in a customary way by computing the coefficients of the product of the polynomials $x(z)$ and $y(z)$ whose coefficients are determined by $\boldsymbol{x}$ and $\boldsymbol{y}$. In view of the convolution theorem, this can be done by means of two FFTs and one IFFT of dimension N, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are padded with zeros in such a way that N is a power of two. In our program Fourier transforms are evaluated by calling the corresponding internal functions of MATLAB which are based on a public domain library named FFTW (http://www.fftw.org). Concerning the computations of part (b), we have implemented a recursive version of the Sieveking–Kung algorithm at the cost of $\mathrm{O}(N\log N)$ arithmetic operations, where $(n+m) \leqslant N \leqslant 2(n+m)$. It is worth realizing that $(L_0^{(0)})^{-1}$ and $(U_0^{(0)})^{-1}$ are triangular Toeplitz matrices whose entries are given by the coefficients of the Taylor series of the reciprocal of $z^n a(z^{-1})$ and $c(z)$, respectively. Since all the zeros of these polynomials lie outside the unit circle, then, from the Cauchy theorem [23], one obtains that the entries of $(L_0^{(0)})^{-1}$ and $(U_0^{(0)})^{-1}$ are exponentially decaying. Hence, $L_0^{(0)}$ and $U_0^{(0)}$ are well conditioned and, therefore, the Sieveking–Kung algorithm applied for the inversion of these two matrices can be shown to be forward stable.

The computation of Euclidean divisions at the step 2 of the *Structured matrix iteration* algorithm is also performed in a stable way by means of the Cardinal's algorithm [15]. It relies upon the following remarkable fact: if $r_a(z)$ is expressed with respect to the Horner polynomial basis associated with $z^n a(z)$, then the coefficients of the polynomial $(r_a(z))^2 \pmod{z^n a(z)}$ w.r.t. the same Horner basis are found by means of three convolutions of size N. The Horner basis associated with $z^n a(z) = a_0 z^{n+m} + \cdots + a_n z^m$ is formed by the polynomials $a_i(z) = a_0 z^i + \cdots + a_{i+1}, 0 \leqslant i \leqslant N-1, a_i = 0$ if $i > n$, generated in the evaluation of $z^n a(z)$ at a point by means of the Ruffini–Horner rule. Hence, the computation of the coefficients of a polynomial w.r.t the Horner basis associated with $z^n a(z)$ given its coefficients in the standard monomial basis is equivalent to solve a linear system whose coefficient matrix is exactly $L_0^{(0)}$. Obviously, similar conclusions also hold for the computation of $(r_c(z))^2 \pmod{z^{n+m} c(z^{-1})}$ by replacing $L_0^{(0)}$ with $U_0^{(0)}$.

The iterative phase is stopped when the conditions

$$\|\boldsymbol{t}^{(s)} - \boldsymbol{t}^{(s-1)}\|_\infty \leqslant \epsilon \|\boldsymbol{t}^{(s)}\|_\infty, \quad \|\boldsymbol{r}^{(s)} - \boldsymbol{r}^{(s-1)}\|_\infty \leqslant \epsilon \|\boldsymbol{r}^{(s)}\|_\infty, \tag{20}$$

are fulfilled, where $\epsilon = 2^{-53}$ denotes the machine precision.

In our numerical experiments, we generated stable polynomials $a(z)$ and $z^m\ c(z^{-1})$ of degree $n = m, n = 64, 128, 256, 512, 1024, 2048$, by using the Kakeya–Eneström theorem [23]. It says that all the zeros of the polynomial $p(z)$ of degree n, $p(z) = z^n + p_1 z^{n-1} + \cdots + p_n$ lie inside the unit circle whenever its coefficients p_i, $1 \leqslant i \leqslant n$, satisfy $1 > p_1 > p_2 > \cdots > p_n$. For each pair $(a(z), c(z))$, we considered the solution of the associated linear system (4) of order $2n$ with $\boldsymbol{b} = \boldsymbol{e}_n + \boldsymbol{e}_{n+1}$. Our program returns the absolute residual $\|R\boldsymbol{x} - \boldsymbol{b}\|_\infty$ produced by Gaussian elimination with partial pivoting (backslash operator in MATLAB) and the absolute residual $\|R\boldsymbol{x} - \boldsymbol{b}\|_\infty$ of the approximate solution vector $\boldsymbol{x}$ found by our implementation of the *Structured matrix iteration* algorithm.

As test suite, we considered polynomials with random coefficients which are generated according to the following rules by means of the internal MATLAB function *rand* returning uniformly distributed random numbers in the interval (0,1).

(1) For $i = 1, \ldots, n$, $a_0 = 1$, $a_i = a_{i-1}/(1 + \text{rand})$, $c_n = 1$, $c_{n-i} = c_{n-i+1}/(1 + \text{rand})$. Sylvester's resultant matrices generated by this rule are generally well conditioned.
(2) For $i = 1, \ldots, n, a_0 = 1, a_i = a_{i-1}/(1 + (\text{rand})^n), c_n = 1, c_{n-i} = c_{n-i+1}/(1 + (\text{rand})^n)$. Differently from the previous set, in this case the distribution of the coefficients of $a(z)$ and $c(z)$ is quite nonuniform and this fact affects the conditioning of the associated resultant linear system. In Fig. 1 we show a typical plot obtained by evaluating the logarithm to base 10 of the spectral condition

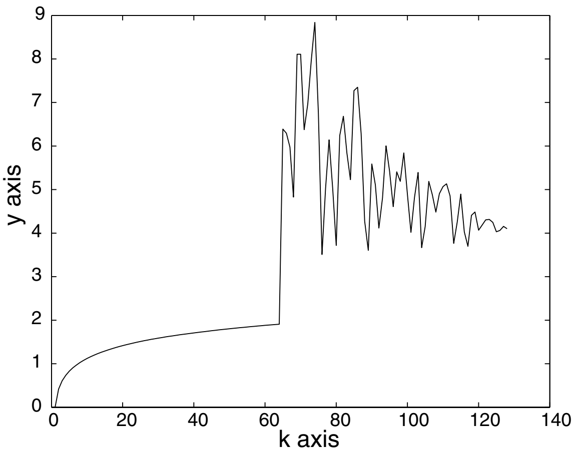

Fig. 1. Plot of the logarithm to base 10 of the conditioning of the leading principal submatrices of a coefficient matrix generated by two polynomials with a nonuniform distribution of the coefficients.

number of the leading principal submatrices of a Sylvester's resultant matrix of order 128 generated in this way. Observe that fast algorithms based on the recursive properties of the triangular factorization can produce very inaccurate results when applied to such coefficient matrices.

For any considered set of test polynomials, we generated 100 pairs $(a(z), c(z))$ and, then, we evaluated the arithmetic means of the computed residuals. Tables 1 and 2 report the results of our experiments by showing that our method is nearly accurate as Gaussian elimination with partial pivoting. In each performed experiment the number of iterations needed to satisfy (20) has never exceeded 8.

The stability properties of our method were also checked in the relevant case where $z^n a(z^{-1}) = c(z)$ and $p(z) = c(z)c(z^{-1})$ is a symmetric Laurent polynomial. In [17] it was shown that the efficient solution of Jury linear systems can be reduced to that one of Sylvester's resultant matrices generated by the spectral factors of a symmetric Laurent polynomial. To be more specific, we define the $(n+1)$st order Jury matrix $\mathscr{J}(\boldsymbol{a})$ associated with $a(z)$ by $\mathscr{J}(\boldsymbol{a}) = T(\boldsymbol{a}) + H(\boldsymbol{a})$, where $T(\boldsymbol{a})$ denotes the lower triangular Toeplitz matrix whose first column is the coefficient vector $\boldsymbol{a}$ of $a(z)$ whereas $H(\boldsymbol{a})$ is the upper triangular Hankel matrix with respect to the main antidiagonal whose first row is $\boldsymbol{a}^{\mathrm{T}}$. Jury linear systems are often encountered in problems of estimation of the transfer function of an input–output model by means of statistical methods based on the properties of cross-covariance and cross-

Table 1
Residuals for the first class of test matrices

n	$\\|R\boldsymbol{x} - \boldsymbol{b}\\|_\infty$	
	S. matrix iteration	Gaussian elimination
64	4.8e−16	6.6e−16
128	7.5e−16	4.7e−16
256	9.1e−16	1.1e−16
512	6.7e−16	2.1e−16
1024	6.0e−16	2.0e−16
2048	6.5e−16	2.4e−16

Table 2
Residuals for the second class of test matrices

n	$\\|R\boldsymbol{x} - \boldsymbol{b}\\|_\infty$	
	S. matrix iteration	Gaussian elimination
64	9.1e−14	3.3e−16
128	1.1e−13	2.8e−16
256	1.5e−13	3.7e−16
512	4.3e−13	2.3e−15
1024	1.4e−12	5.8e−15
2048	7.5e−12	9.7e−15

Table 3
Residuals for matrices generated by $z^n a(z^{-1}) = c(z) = (z^n/(n+1)) + (2z^{n-1}/(n+1)) + \cdots + 1$

n	$\|\boldsymbol{Rx} - \boldsymbol{b}\|_\infty$	
	S. matrix iteration	Gaussian elimination
64	1.5e−14	2.0e−16
128	1.2e−14	3.4e−16
256	3.2e−14	4.7e−16
512	8.1e−14	3.0e−16
1024	1.4e−13	4.4e−16
2048	4.8e−13	7.9e−16

correlation functions [11]. It is remarkable to observe that the solution of the spectral factorization problem can also lead to solving Jury linear systems. In fact, if we apply the Newton–Raphson iteration to the quadratic equation $p(z) = c(z)c(z^{-1})$, the coefficients of the spectral factor being unknown, then we obtain a linear system whose (Jacobian) matrix is a Jury matrix (see [36]).

Table 3 compares the performance of the *Structured matrix iteration* algorithm with that one of Gaussian elimination with partial pivoting on the test polynomials

$$c(z) = \frac{z^n}{n+1} + \frac{2z^{n-1}}{n+1} + \cdots + 1.$$

These polynomials were considered in [31] and provide particularly tough examples since the geometric average of the roots of $c(z)$ becomes very close to 1 as n grows.

Clearly, these results confirm again the robustness of the proposed approach.

6. Conclusions and further extensions

A novel approach to the efficient solution of Sylvester's resultant linear systems generated by a stable and an anti-stable polynomial has been presented. It is based upon the close connections between the matrix problem, its polynomial formulation and the problem of reciprocating Laurent polynomials in a given annulus in the complex plane. Moreover, it is also related to the problem of factoring polynomials with respect to the unit circle (spectral factorization problem). The resulting algorithm is an iterative one which proceeds by generating a sequence of Toeplitz-like matrices. By combining the displacement theory for structured matrices with fast polynomial arithmetic methods based on FFT, it can be implemented in a superfast way using linear storage. The experimental results produced by a MATLAB implementation of the proposed solution algorithm are also reported by showing its effectiveness and robustness.

A very interesting open question is whether or not a polynomial version of our method exists. In [4,34] the polynomial Graeffe iteration is used to compute the coefficients of the reciprocal of a Laurent polynomial. In our opinion, the relations between Graeffe's process and our iterative scheme deserve further investigations. In particular, it is not clear if Graeffe's algorithm can be modified in such a way to exploit the knowledge of the spectral factorization of the input Laurent polynomial. An extension of the polynomial Schur–Cohn algorithm was pointed out in [31] in order to efficiently perform a step of the iterative Wilson's algorithm [36] for the spectral factorization of symmetric Laurent polynomials. As we have recalled in the previous section, Wilson's method is based on Newton's iteration for a nonlinear (quadratic) equation, where at each step the generated Jacobian is a suitable Jury matrix. Therefore, the exploitation of possible connections between the Schur-Cohn algorithm and our procedure would also provide useful answers.

Finally, in this paper we have shown that spectral factorization methods can be related to superfast algorithms for the numerical treatment of a certain class of structured linear systems. A continuous analogue of the spectral factorization problem is the problem of factoring polynomials with respect to the imaginary axis in the complex plane [20] (Hurwitz factorization problem) which plays a key role in the synthesis of continuous quadratically optimal controllers [24]. The study of similar results and relations between Hurwitz factorization methods and the solution of structured linear systems should be welcome.

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