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In this paper, the direct one-dimensional beam model introduced by one of the authors is refined to take into account nonsymmetrical beam cross-sections. Two different beam axes are considered, and the strain is described with respect to both. Two inner constraints are assumed: a vanishing shearing strain between the cross-section and one of the two axes, and a linear relationship between the warping and twisting of the cross-section. Considering a grade one mechanical theory and nonlinear hyperelastic constitutive relations, the balance of power, and standard localization and static perturbation procedures lead to field equations suitable to describe the flexural-torsional buckling. Some examples are given to determine the critical load for initially compressed beams and to evaluate their post-buckling behavior.

1. Introduction

The flexural-torsional buckling of so-called thin-walled beams is an interesting problem in the field of the elastic stability of structural elements. This phenomenon was first investigated in [Wagner 1929] and [Kappus 1937], and since the publication of these pioneering works, many further studies have appeared on the subject, including [Vlasov 1961; Epstein 1979; Reissner 1983; Simo and Vu-Quoc 1991]. More recently, [Di Egidio et al. 2003] investigated modelization aspects, and [Anderson and Trahair 1972; Casciaro et al. 1991; Lanzo and Garcea 1996] presented a search of numerical results for standard elements. It is remarkable that most of the beam models presented in the literature are derived by projection of the results of three-dimensional continuum models on shell models (as in Vlasov 1961) or beam models (as in Simo and Vu-Quoc 1991).

In recent years, the authors have considered the direct one-dimensional beam model, originally introduced in [Tatone and Rizzi 1991; Rizzi and Tatone 1996]. This model describes the kinematics of the beam through the placement of the points of the beam axis, the rotation of the beam cross-sections (which are assumed to be plane in the reference configuration), and a coarse description of the warping of the beam cross-sections. Using a mechanical theory of grade one (see, for example Germain 1973a and 1973b, and Di Carlo 1996), the interaction of the beam with the surrounding environment is defined as a linear function of the velocities and of their first-order spatial approximations. In this way, the mechanical actions naturally appear as dual to the kinematic fields. In particular, actions which spend power on the warping velocity and on its spatial derivative are interpreted as bi-shear and bi-moment, respectively. As is customary in the literature, two inner constraints are assumed to hold: the shearing

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strain between the axis and the cross-section is assumed to vanish, and the coarse warping measure is directly proportional to the measure of twist.

In previous papers [Rizzi and Tatone 1996; Pignataro and Ruta 2003; Pignataro et al. 2004], field equations in terms of the components of the displacement were obtained, and some flexural-torsional buckling cases were investigated. For this purpose, nonlinear hyperelastic constitutive relations were adopted. However, since no clear distinction was made between the position of the centroid and of the shear center in the plane of the cross-section, it turns out that the results obtained in the above mentioned papers are precise only for beams with symmetric cross-sections.

The aim of this paper is to overcome this drawback by presenting a refined beam model suitable for describing the flexural-torsional buckling of beams with generic, nonsymmetric cross-sections. We will describe the strain measures with respect to two beam axes (one passing through the centroids, and the other through the shear centers of the cross-sections). Then, we suitably decompose the power expended by inner actions, to distinguish clearly between actions applied at each of the two places. Introducing the same inner constraints as in the papers cited above generates reactive terms that must be added to the determined part of the contact actions, and those terms account for the geometry of nonsymmetric cross-sections. In this way, we are able to obtain field equations which are more general than those found by authors of previous studies. To test the validity of the proposed model, we present some simple examples of flexural-torsional buckling and post-buckling phenomena. The results coincide with those in the literature, for example [Timoshenko and Gere 1961; Grimaldi and Pignataro 1979].

2. A direct one-dimensional beam model

Let $\mathscr{E}$ be the standard three-dimensional Euclidean ambient space. The vector space of the translations of $\mathscr{E}$, here denoted $\mathscr{U}$, is assumed to be equipped with the standard Euclidean norm and scalar (dot) product. Let us consider in $\mathscr{E}$ a smooth curve $\mathscr{C}_o$ and a prototype region $\mathscr{R}$ belonging to a plane $\mathscr{P}$. We then attach a copy of $\mathscr{R}$ to each point of $\mathscr{C}_o$, always in correspondence to the same place $o \in \mathscr{P}$. The region of $\mathscr{E}$ occupied by this construction represents the beam. The curve $\mathscr{C}_o$ corresponds to the axis of the beam, and the copies of $\mathscr{R}$ are the beam cross sections. With no loss of generality, and only from the point of view of introducing constitutive relations, we can imagine that the place $o \in \mathscr{P}$ is actually the centroid of the cross-section. Hence, the axis $\mathscr{C}_o$ is a centroidal axis. Of course, the beam can be equally described using an analogous construction starting from a different curve $\mathscr{C}_c$ in $\mathscr{E}$. Again, with no loss of generality, one can imagine that the place $c \in \mathscr{P}$ is actually the shear center of the cross-sections. For the time being, we ignore the question of whether this is the shear center according to Timoshenko or Trefftz. Indeed, for open thin-walled beams the question is immaterial since either definition provides the same place (see, for example, [Andreaus and Ruta 1998; Ruta 1998]). In this way, we can call the curve $\mathscr{C}_c$ the shear axis, or the axis of the shear centers.

A change of placement is described by the placement of the axes, the rotation of each cross-section with respect to the attitude in the first placement, and a warping superposed to these two. The transformations of the axes and the rotations of the cross-sections can be described exactly, but we will limit ourselves with a coarse description of the warping by using a single scalar parameter. We can suitably define this parameter as a (possibly weighted) average of the warping over the cross-section, or as the value of the warping at a particular place of (plane of) the cross-section. A motion is naturally defined as a one-parameter family of changes of placements, $t \in [0, +\infty]$ being the evolution scalar real parameter (the time, for instance).

Let us consider a motion and let $\mathcal{S}_0$ be the shape of the beam at $t = 0$. This may be assumed, with no loss of generality, to be the reference configuration. It is always possible to assume that $\mathcal{S}_0$ is such that the copies of the prototype plane region $\mathcal{R}$ remain plane, that is, the cross-sections undergo no warping. Once a place in $\mathcal{E}$ is chosen as origin, the axes of the beam in $\mathcal{S}_0$ are described by the regular enough position vector fields

$$\mathbf{q}_o : \rho \in [0, l] \to \mathbf{q}_o(\rho) \in \mathcal{U}, \quad \mathbf{q}_c : \rho \in [0, l] \to \mathbf{q}_c(\rho) \in \mathcal{U},$$

where ρ is a curvilinear abscissa along one of the axes compatible with the Euclidean metric of the ambient space $\mathcal{E}$. The unit vector fields tangent to the axes of the beam are

$$\mathbf{q}_o'(\rho), \quad \mathbf{q}_c'(\rho),$$

where the prime denotes derivation with respect to ρ. With no loss of generality, one can assume that in $\mathcal{S}_0$ the relative position of the places o and c in the planes containing the cross-sections does not depend on ρ, so that

$$\mathbf{q}_o'(\rho) = \mathbf{q}_c'(\rho), \qquad \text{for all } \rho \in [0, l]. \tag{1}$$

Let $\mathcal{S}_t$ be the configuration assumed by the beam in the motion we are considering at the present value t of the evolution parameter. Such a configuration is described by:

- a vector field $\mathbf{p}_o(\rho, t)$ (or, equivalently, $\mathbf{p}_c(\rho, t)$), providing the position of the substantial point identified by $\mathbf{q}_o(\rho)$ (or, equivalently, $\mathbf{q}_c(\rho)$) in $\mathcal{S}_0$;
- a proper orthogonal tensor field $\mathbf{R}(\rho, t)$, providing the rotation of the cross-sections when passing from $\mathcal{S}_0$ to $\mathcal{S}_t$;
- a scalar field $\alpha(\rho, t)$, providing a coarse description of the cross-sections' warping superposed to displacements induced by the rotation $\mathbf{R}(\rho, t)$.

The tensors $\mathbf{R}(\rho, t)$ are required to be proper orthogonal to prevent reflections of the cross-sections. Henceforth, the fields $\mathbf{p}_o$, $\mathbf{p}_c$, $\mathbf{R}$, α will be assumed to be regular enough for the analytical developments we adopt. Also, for the sake of simplicity of notation, and when no confusion may arise, in the following the dependent variables of the indicated fields will be understood and hence omitted, as well as the measures of integration in the integrals.

The tangent vector fields to the axes of the beam in the present configuration are given by

$$\mathbf{p}_o'(\rho, t), \qquad \mathbf{p}_c'(\rho, t).$$

The derivative of a field with respect to t is denoted by a superimposed dot, and $\mathrm{skw}(\mathcal{U} \otimes \mathcal{U})$ denotes the space of skew tensors on $\mathcal{U}$.

The velocity fields for the beam are

$$\begin{aligned} \mathbf{w}_o &= \dot{\mathbf{p}}_o \in \mathcal{U}, & \mathbf{w}_c &= \dot{\mathbf{p}}_c \in \mathcal{U}, \\ \mathbf{W} &= \dot{\mathbf{R}}\mathbf{R}^\top \in \mathrm{skw}(\mathcal{U} \otimes \mathcal{U}), & \omega &= \dot{\alpha} \in \mathbb{R}. \end{aligned} \tag{2}$$

A rigid, or neutral, change of placement is one such that the rotation of the cross-sections is uniform with respect to ρ, any vector out of the plane of the cross-sections (for example, either $\mathbf{p}_o'$ or $\mathbf{p}_c'$) is rotated according to this uniform rotation, and the warping remains identical to zero.

Hence, in a rigid change of placement

$$\begin{aligned} &\mathbf{R} = \mathbf{R}_0, && \mathbf{p}_o' = \mathbf{R}_0\mathbf{q}_o', \\ &\mathbf{p}_c' = \mathbf{R}_0\mathbf{q}_c' = \mathbf{R}_0\mathbf{q}_o', && \alpha = 0. \end{aligned} \tag{3}$$

Deformation is the difference between the considered change of placement and a rigid one, and must vanish if evaluated for a neutral change of placement. Suitable strain measures, pulled back to $\mathcal{S}_0$, are

$$\begin{aligned} &\mathbf{E} = \mathbf{R}^\top\mathbf{R}', && \alpha, && \eta = \alpha', \\ &\mathbf{e}_o = \mathbf{R}^\top\mathbf{p}_o' - \mathbf{q}_o', && \mathbf{e}_c = \mathbf{R}^\top\mathbf{p}_c' - \mathbf{q}_c' = \mathbf{e}_o + \mathbf{E}\mathbf{c}, \end{aligned} \tag{4}$$

where the skew tensor field $\mathbf{E}$ provides the change of curvature of one of the beam axes when passing from $\mathcal{S}_0$ to $\mathcal{S}_t$. The vector fields $\mathbf{e}_o$ and $\mathbf{e}_c$ represent the differences between the tangent to the axes in $\mathcal{S}_t$, pulled back to $\mathcal{S}_0$, and the tangent to the axes in $\mathcal{S}_0$. The vector $\mathbf{c}$ defines the position of the place c with respect to o in the plane of the cross-sections in $\mathcal{S}_0$, and $\mathbf{c}$ is uniform with respect to ρ because of (1). When the change of placement is neutral, the relations given in (3) imply that the quantities defined by (4) vanish. The strain measures given by (4) are invariant under a change of observer.

With no loss of generality, let $\mathcal{S}_0$ be a set of parallel cross-sections orthogonal to the beam axes, each of which is assumed to be a rectilinear segment of length l. Let us fix a system of orthogonal Cartesian coordinates with the x_1 axis parallel to the beam axes in $\mathcal{S}_0$ ($\rho \equiv x_1$). Furthermore, let us consider an orthonormal right-handed vector basis ($\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$) for $\mathcal{U}$ adapted to the introduced coordinate system such that

$$\mathbf{i}_1 = \mathbf{q}_o' = \mathbf{q}_c'.$$

With respect to the introduced basis, consider the decompositions

$$\begin{aligned} &\mathbf{E} = \chi_1\mathbf{i}_2 \wedge \mathbf{i}_3 + \chi_2\mathbf{i}_3 \wedge \mathbf{i}_1 + \chi_3\mathbf{i}_1 \wedge \mathbf{i}_2, \\ &\mathbf{e}_o = \varepsilon_1\mathbf{i}_1 + \varepsilon_2\mathbf{i}_2 + \varepsilon_3\mathbf{i}_3, \\ &\mathbf{e}_c = \varepsilon_{1c}\mathbf{i}_1 + \varepsilon_{2c}\mathbf{i}_2 + \varepsilon_{3c}\mathbf{i}_3 = (\varepsilon_1 + \chi_2 c_3 - \chi_3 c_2)\mathbf{i}_1 + (\varepsilon_2 - \chi_1 c_3)\mathbf{i}_2 + (\varepsilon_3 + \chi_1 c_2)\mathbf{i}_3, \end{aligned} \tag{5}$$

where χ_1 is the torsion curvature (or twist), χ_2 and χ_3 are the bending curvatures, $\wedge$ is the exterior product between vectors of $\mathcal{U}$ which provide skew tensors on $\mathcal{U}$, ε_1 is the elongation of the axis of the centroids, ε_2 and ε_3 are the shearing strains between the axis of the centroids and the cross-sections, and c_2 and c_3 are the components of $\mathbf{c}$. Equation $(5)_3$ provides, with respect to the shear axis, quantities similar to those provided by $(5)_2$. In addition, it expresses these last in terms of the components of $\mathbf{E}$ and $\mathbf{e}_o$ (see $(4)_5$).

It is useful to write the strain measures in terms of the displacement field $\mathbf{u}$ of the axis of the centroids starting from $\mathcal{S}_0$ as

$$\mathbf{u} = \mathbf{p}_o - \mathbf{q}_o = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3,$$

and to decompose the rotation $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1, \tag{6}$$

where $\mathbf{R}_1$ is a rotation of amplitude φ_1 about $\mathbf{i}_1$, $\mathbf{R}_2$ is a rotation of amplitude φ_2 about $\mathbf{R}_1\mathbf{i}_2$ (that is, the φ_1-transformed of $\mathbf{i}_2$), and $\mathbf{R}_3$ is a rotation of amplitude φ_3 about $\mathbf{R}_2\mathbf{R}_1\mathbf{i}_3$ (that is, the $\varphi_2 \circ \varphi_1$-transformed of $\mathbf{i}_3$). It can be proved (see [Di Carlo and Tatone 1980]) that the matrix representation $(\mathbf{R})_0$ of $\mathbf{R}$ with respect to the basis $(\mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3)$ is

$$(\mathbf{R})_0 = (\mathbf{R}_1)_0(\mathbf{R}_2)_1(\mathbf{R}_3)_2, \tag{7}$$

where $(\mathbf{R}_1)_0$ is the matrix representation of $\mathbf{R}_1$ with respect to the reference basis $(\mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3)$, $(\mathbf{R}_2)_1$ is the matrix representation of $\mathbf{R}_2$ with respect to the φ_1-transformed of the reference basis, and $(\mathbf{R}_3)_2$ is the matrix representation of $\mathbf{R}_3$ with respect to the $\varphi_2 \circ \varphi_1$-transformed of the reference basis.

Equations (4)–(7) yield expressions of the components of the strain measures in terms of the components of the displacement of the axis of the centroids starting from the reference configuration $\mathcal{S}_0$ as

$$\begin{aligned}
\varepsilon_1 &= (1+u_1')\cos\varphi_2\cos\varphi_3 + u_2'(\cos\varphi_1\sin\varphi_3 + \sin\varphi_1\sin\varphi_2\cos\varphi_3)\\
&\qquad\qquad + u_3'(\sin\varphi_1\sin\varphi_3 - \cos\varphi_1\sin\varphi_2\cos\varphi_3) - 1,\\
\varepsilon_2 &= -(1+u_1')\cos\varphi_2\sin\varphi_3 + u_2'(\cos\varphi_1\cos\varphi_3 - \sin\varphi_1\sin\varphi_2\sin\varphi_3)\\
&\qquad\qquad + u_3'(\sin\varphi_1\cos\varphi_3 + \cos\varphi_1\sin\varphi_2\sin\varphi_3),\\
\varepsilon_3 &= (1+u_1')\sin\varphi_2 - u_2'\sin\varphi_1\cos\varphi_2 + u_3'\cos\varphi_1\cos\varphi,\\
\chi_1 &= \varphi_1'\cos\varphi_2\cos\varphi_3 + \varphi_2'\sin\varphi_3,\\
\chi_2 &= -\varphi_1'\cos\varphi_2\sin\varphi_3 + \varphi_2'\cos\varphi_3,\\
\chi_3 &= \varphi_1'\sin\varphi_2 + \varphi_3'.
\end{aligned}$$

3. Balance of power and balance equations

Let us assume that the interaction of the beam with the environment in its present configuration $\mathcal{S}_t$ is quantified, for each test velocity field attainable by the beam, using a linear functional of the velocity fields, which we will call the external power P^e. Standard representation theorems of linear forms in finite-dimensional vector spaces equipped with a scalar product then let us express external power as

$$P^e = \int_0^l (\mathbf{b}\cdot\mathbf{w}_c + \mathbf{B}\cdot\mathbf{W} + \beta\omega) + \Big|\mathbf{t}\cdot\mathbf{w}_c + \mathbf{T}\cdot\mathbf{W} + \theta\omega\Big|_{0^-} + \Big|\mathbf{t}\cdot\mathbf{w}_c + \mathbf{T}\cdot\mathbf{W} + \theta\omega\Big|_{l^+}, \tag{8}$$

where the integral term quantifies the power expended by bulk actions, and the boundary terms quantify the power expended by contact actions. The vector fields $\mathbf{b}$ and $\mathbf{t}$ represent external force density (bulk and contact, respectively), the skew tensor fields $\mathbf{B}$ and $\mathbf{T}$ represent external couple density (bulk and contact, respectively), and the scalar fields β and θ represent the bulk and contact actions density, respectively, which exert power on the warping. It is essential and will be fundamental henceforth that in the representation of P^e the velocity used is that of the points of the shear axis.

If we move in the frame of a mechanical theory of grade one (see, for example [Di Carlo 1996]), let us assume that the behavior of any substantial point on the (axis of the) beam is influenced by the substantial points contained in one of its neighborhoods. The interaction among different parts of the beam is then quantified for each test velocity field attainable by the beam, using a linear functional of the velocity fields and of their first derivatives with respect to x_1. This functional, denoted P^i, is also called the internal power. The theorems we used to represent external power P^e enable us to express P^i as

$$P^i = \int_0^l \big(\mathbf{c}_0 \cdot \mathbf{w}_c + \mathbf{C}_0 \cdot \mathbf{W} + \gamma_0 \omega + \mathbf{c}_1 \cdot \mathbf{w}_c' + \mathbf{C}_1 \cdot \mathbf{W}' + \gamma_1 \omega'\big)\,, \tag{9}$$

where the vector fields $\mathbf{c}_0$ and $\mathbf{c}_1$ represent self-force and internal force densities, respectively, the skew tensor fields $\mathbf{C}_0$ and $\mathbf{C}_1$ represent the self-couple and internal couple densities, respectively, and the scalar fields γ_0 and γ_1 represent the self-action and the internal action densities which spend power on the warping and on its spatial rate, respectively.

It is natural to assert that $P^i \equiv 0$ for any change of placement leaving the beam mechanical state unaltered, and that is the situation for a the change of placement induced by a neutral velocity field. Substituting (3) into (2) and then into (9), the generality of the kinematic fields present in the integrand leads to the following reduced expression for the internal power

$$P^i = \int_0^l \big(\mathbf{c}_1 \cdot \mathbf{w}_c' - (\mathbf{p}_c' \wedge \mathbf{c}_1) \cdot \mathbf{W} + \mathbf{C}_1 \cdot \mathbf{W}' + \gamma_0 \omega + \gamma_1 \omega'\big). \tag{10}$$

Hence, in this beam model the self-force density necessarily vanishes, while the self-couple density is actually the moment of the internal force density with respect to the shear center, as expressed by the term containing the wedge product in the integrand of Equation (10).

Let us postulate that for any test velocity field attainable by the beam the interactions of the beam with the environment and of the parts of the beam with each other are such that at any value of the evolution parameter t the total power vanishes (see [Germain 1973a; 1973b; Di Carlo 1996]), or equivalently,

$$P^e = P^i. \tag{11}$$

From (11), standard localization procedures under suitable regularity hypotheses yield the law of action and reaction

$$\Big|\mathbf{t} \cdot \mathbf{w}_c + \mathbf{T} \cdot \mathbf{W} + \theta\omega\Big|_{x_1^-} = -\Big|\mathbf{t} \cdot \mathbf{w}_c + \mathbf{T} \cdot \mathbf{W} + \theta\omega\Big|_{x_1^+}, \qquad \text{for all } x_1 \in [0, l]. \tag{12}$$

As a consequence of (12), the boundary terms for the expression of external power (8), can be compacted with respect to the actions on the positive side of the cross-sections. Applying the fundamental theorem of integral calculus, the balance of power (11) becomes

$$\int_0^l [\big(\mathbf{t}' + \mathbf{b}\big) \cdot \mathbf{w}_c + \big(\mathbf{T}' + \mathbf{p}_c' \wedge \mathbf{c}_1 + \mathbf{B}\big) \cdot \mathbf{W} + \big(\beta + \theta' - \gamma_0\big)\,\omega + (\mathbf{t} - \mathbf{c}_1) \cdot \mathbf{w}_c' + (\mathbf{T} - \mathbf{C}_1) \cdot \mathbf{W}' + (\theta - \gamma_1)\,\omega'] = 0. \tag{13}$$

For generality of the velocity fields and their first spatial derivatives contained in the integrand, (13) yields

$$\begin{aligned} &\mathbf{c}_1 = \mathbf{t}, && \mathbf{C}_1 = \mathbf{T}, \\ &\mathbf{t}' + \mathbf{b} = \mathbf{0}, && \mathbf{T}' + \mathbf{p}_c' \wedge \mathbf{t} + \mathbf{B} = \mathbf{0}, \\ &\gamma_0 = \tau = \beta + \theta', && \gamma_1 = \mu = \theta. \end{aligned} \tag{14}$$

Equations $(14)_{1,2}$ represent two identification equations and tell us that in this beam model the internal action densities $\mathbf{c}_1$ and $\mathbf{C}_1$ actually coincide with contact force and couple densities, respectively. Equations $(14)_{3,4}$ express the local balance of force and torque. By torque we mean an action which spends power on an angular velocity, and is hence expressed both by a couple and the moment of a force. Equation $(14)_5$ is an auxiliary equation for γ_0, henceforth denoted τ and interpreted as bishear [Vlasov 1961]. Equation $(14)_6$ is an auxiliary equation for γ_1, henceforth denoted μ and interpreted as a bimoment [Vlasov 1961].

The equations in (14) were obtained by the balance of power in the present configuration. In any case, neither the identities in $(14)_{1,2}$ nor the scalar auxiliary $(14)_{5,6}$ depend on the configuration. It seems advisable, however, to pull back the local balance equations $(14)_{3,4}$ to the reference configuration. Let us then propose

$$\mathbf{s} = \mathbf{R}^\top \mathbf{t}, \qquad \mathbf{S} = \mathbf{R}^\top \mathbf{T}\mathbf{R}, \qquad \mathbf{a} = \mathbf{R}^\top \mathbf{b}, \qquad \mathbf{A} = \mathbf{R}^\top \mathbf{B}\mathbf{R},$$

where $\mathbf{s}$, $\mathbf{S}$, $\mathbf{a}$, and $\mathbf{A}$ are the contact and bulk action densities pulled back to $\mathscr{S}_0$. Taking into account the definitions of $\mathbf{e}_c$, $\mathbf{E}$, and Equation (4), the local balance equations $(14)_{3,4}$, become

$$\begin{aligned} \mathbf{s}' + \mathbf{E}\mathbf{s} + \mathbf{a} &= \mathbf{0}, \\ \mathbf{S}' + \mathbf{E}\mathbf{S} - \mathbf{S}\mathbf{E} + (\mathbf{q}_c' + \mathbf{e}_c) \wedge \mathbf{s} + \mathbf{A} &= \mathbf{0}, \end{aligned} \tag{15}$$

while the reduced expression of the internal power (10), and the balance of power (11), become

$$P^i = \int_0^l \big(\mathbf{s} \cdot \dot{\mathbf{e}}_c + \mathbf{S} \cdot \dot{\mathbf{E}} + \tau\omega + \mu\omega'\big) = P^e. \tag{16}$$

We decompose the contact action densities in $\mathscr{S}_0$ as

$$\mathbf{s} = Q_1\mathbf{i}_1 + Q_2\mathbf{i}_2 + Q_3\mathbf{i}_3, \qquad \mathbf{S} = M_1\mathbf{i}_2 \wedge \mathbf{i}_3 + M_2\mathbf{i}_3 \wedge \mathbf{i}_1 + M_3\mathbf{i}_1 \wedge \mathbf{i}_2. \tag{17}$$

We now express the strain with respect to the shear axis $\mathbf{e}_c$ in terms of the strain with respect to the centroidal axis $\mathbf{e}$, according to $(4)_3$. Then, we can write the full expression of the internal power (16), taking into account the decompositions in (17), as

$$\int_0^l \big(Q_1\dot{\varepsilon}_1 + Q_2\dot{\varepsilon}_{2c} + Q_3\dot{\varepsilon}_{3c} + M_1\dot{\chi}_1 + (M_2 + c_3Q_1)\dot{\chi}_2 + (M_3 - c_2Q_1)\dot{\chi}_3 + \tau\omega + \mu\omega'\big). \tag{18}$$

Equation (18) shows that Q_1, interpreted as normal force, spends power on the speed of elongation of the centroidal axis, while Q_2 and Q_3, interpreted as shearing forces, spend power on the speed of shearing between the cross-sections and the shear axis. That is, we imagine the normal force to be applied at the centroid of the cross-section, while the shearing forces are applied at the shear center of the cross-section. The twisting couple M_1 spends power on the speed of twisting, while the bending torques $M_2 + c_3Q_1$ and $M_3 - c_2Q_1$, which are composition of a couple and of the moment of a force, respectively, spend power on the speed of bending. Also, note that the bending torques introduced are thus evaluated with respect to the centroid of the cross-section.

Substituting $(5)_3$ and (17) into (15), and assuming the bulk actions to vanish as is customary, we obtain the components of the local balance equations for contact force and torque, with all the kinematic quantities written with respect to the centroid of the cross-sections, as

$$\begin{aligned} Q_1' + \chi_2 Q_3 - \chi_3 Q_2 &= 0, \\ Q_2' + \chi_3 Q_1 - \chi_1 Q_3 &= 0, \\ Q_3' + \chi_1 Q_2 - \chi_2 Q_1 &= 0, \\ M_1' + \chi_2 M_3 + \chi_3 M_2 + (\varepsilon_2 - \chi_1 c_3) Q_3 - (\varepsilon_3 + \chi_1 c_2) Q_2 &= 0, \\ M_2' + \chi_3 M_1 + \chi_1 M_3 + (\varepsilon_3 + \chi_1 c_2) Q_1 - (1 + \varepsilon_1 + \chi_2 c_3 - \chi_3 c_2) Q_3 &= 0, \\ M_3' + \chi_1 M_2 + \chi_2 M_1 + (1 + \varepsilon_1 + \chi_2 c_3 - \chi_3 c_2) Q_2 - (\varepsilon_2 - \chi_1 c_3) Q_1 &= 0. \end{aligned}$$

4. Inner constraints and constitutive relations

If the beam is made of homogeneous and elastic material, standard axioms of the constitutive theory [Truesdell and Noll 1965] state that the most general material response function assumes the reduced expression

$$S = \hat{S}(\mathbf{e},\ \mathbf{E},\ \alpha,\ \eta), \tag{19}$$

where S represents each of the components of $\mathbf{s}$ and $\mathbf{S}$, and τ and μ.

Let us now introduce some inner constraints. The first takes into account a standard assumption in the literature, that is, that warping actually depends on the other strain measures, and vanishes when no strain is present, as

$$\alpha = \hat{\alpha}(\mathbf{e},\ \mathbf{E}), \qquad \hat{\alpha}(\mathbf{0},\ \mathbf{0}) = 0, \tag{20}$$

where $\hat{\alpha}$ is a scalar function independent of the frame of reference, since both $\mathbf{e}$ and $\mathbf{E}$ are likewise independent. On the basis of the literature, it seems natural to postulate the particular expression for Equation (20)

$$\alpha = \xi \chi_1, \qquad \xi \in \mathbb{R}, \qquad \eta = \xi \chi_1', \tag{21}$$

where ξ is a numerical constant [Vlasov 1961; Reissner 1983; Simo and Vu-Quoc 1991; Tatone and Rizzi 1991; Rizzi and Tatone 1996]. In particular, with reference to [Simo and Vu-Quoc 1991], a possible kinematic interpretation for α is that of a suitably weighted average of out-of-plane displacement of the points initially lying on the same plane cross section.

Furthermore, let us assume that the shearing strains between the cross-sections and the axis of the centroids vanishes. This implies that in a neighborhood of the centroid, the cross section remains orthogonal to the axis of the centroids, as

$$\mathbf{e} = \varepsilon_1 \mathbf{q}_o' = \varepsilon_1 \mathbf{e}_1 \Rightarrow \varepsilon_2 = \varepsilon_3 = 0. \tag{22}$$

Note that this assumption does not imply that the whole of the cross-section remains orthogonal to the beam axes. Indeed, ε_{2c} and ε_{3c} do not vanish, as is easily obtained from $(4)_3$.

According to the principle of determinism for constrained materials [Truesdell and Noll 1965], introducing constraints implies that the contact actions consist of the sum of two contributions. One of these is determined by the motion according to a constitutive relation (19), but the other is not. For this reason, the former is called active and the latter reactive. The reactive part of the contact actions is here denoted by the subscript r and is characterized, for so-called smooth constraints [Truesdell and Noll 1965], by spending no power on any velocity field compatible with the introduced constraints. In our case, the introduced constraints are those between warping and twist (21), and of vanishing shearing strain between the cross-sections and the centroidal axis (22). Characterization of smooth constraints yields

$$0=\int_0^l \big(\mathbf{s}_r\cdot\dot{\mathbf{e}}_c+\mathbf{S}_r\cdot\dot{\mathbf{E}}+\tau_r\dot{\alpha}+\mu_r\dot{\eta}\big)=\int_0^l \big(Q_{1r}\dot{\varepsilon}_1+(M_{1r}-c_3Q_{2r}+c_2Q_{3r}+\xi\tau_r)\dot{\chi}_1 \\ +(M_{2r}+c_3Q_{1r})\dot{\chi}_2+(M_{3r}-c_2Q_{1r})\dot{\chi}_3+\mu_r\xi\dot{\chi}_1'\big), \quad (23)$$

where we used the reduced expression for the internal power (16) and the decompositions of the strain measure $(4)_3$ and of the contact actions (17). For generality of the velocity fields and of the domain of integration in (23), we obtain

$$\begin{aligned} Q_{1r}&=0, & Q_{2r},\ Q_{3r}&\in\mathbb{R},\\ M_{1r}&=c_3Q_{2r}-c_2Q_{3r}-\xi\tau_r, & M_{2r}=M_{3r}&=0,\\ \tau_r&\in\mathbb{R}, & \mu_r&=0.\end{aligned} \quad (24)$$

Thus, the normal force, the bending couples and the bi-moment are entirely determined by the motion (that is, their reactive part vanishes). On the other hand, the shearing forces and the bi-shear have a reactive part entirely undetermined by the motion. If, as is standard in the literature, the shearing force is assumed to depend only on the shearing strain, the inner constraint on the shearing strain makes the shearing force purely a constraint reaction which is altogether undetermined by the motion. Analogous reasoning can be produced for the bi-shear, to yield

$$Q_2=Q_{2r},\qquad Q_3=Q_{3r},\qquad \tau=\tau_r. \quad (25)$$

Thus, some actions are purely active, others purely reactive. The exception is the twisting couple, which has an active part and a reactive one. The reactive part takes into account reactive bi-shear and shearing forces, which are considered to be applied at the shear center. The presence of bi-shear as a reactive component of the twisting couple is well known in the literature, and was first reported by [Vlasov 1961], where it arises from the inner constraint of vanishing shearing strain of the cross-section middle line in its own plane.

To study possible cases of bifurcations of elastic equilibrium, we use a static perturbation technique [Budiansky 1974]. Let us suppose that the contact actions derive from an elastic potential, thus remaining within the limits of the standard theory established in [Koiter 1945]. Let us further assume that the material of the beam is hyperelastic. Then, we adopt nonlinear constitutive relations for the active part of the contact actions, that is, those determined by the motion. These are denoted by the subscript a and are

$$\begin{aligned}
Q_{1a} &= Q_1 = a\varepsilon_1 + \frac{1}{2} d\chi_1^2,\\
M_{1a} &= (c + d\varepsilon_1 + f_2\chi_2 + f_3\chi_3 + g\eta)\chi_1,\\
M_{2a} &= M_2 = b_2\chi_2 + \frac{1}{2} f_2\chi_1^2,\\
M_{3a} &= M_3 = b_3\chi_3 + \frac{1}{2} f_3\chi_1^2,\\
\mu_a &= \mu = h\eta + \frac{1}{2} g\chi_1^2,
\end{aligned} \tag{26}$$

where the coefficients a, b_j (j=2, 3), c, h represent the rigidities in extension, bending, torsion, warping, respectively, and d, f_j (j=2, 3), and g take into account the couplings between extension and torsion, bending and torsion, warping and torsion, respectively [Truesdell and Noll 1965; Møllmann 1986]. In (26), only those contact actions which are not entirely reactive have been characterized. Furthermore, the consequences of (24) have been considered, and the coincidence of some of the contact actions with their active part has been put into evidence.

If the bulk action β vanishes, which is a standard assumption in continuum mechanics, the Equations $(14)_{5,6}$, (21), $(24)_{3,5,6}$, (25), and (26) yield

$$\mu = h\xi\chi_1' + \frac{1}{2} g\chi_1^2 \Rightarrow \tau = \mu' = h\xi\chi_1'' + g\chi_1\chi_1',$$
$$M_1 = M_{1a} + M_{1r} = (c + d\varepsilon_1 + f_2\chi_2 + f_3\chi_3)\chi_1 - h\xi^2\chi_1'' + c_3Q_2 - c_2Q_3.$$

Comparing Equation $(\mathbf{V}.1.10)_3$ in [Vlasov 1961] and our Equation $(21)_3$ we can make the following correspondences:

$$a = EA, \qquad b_j = EI_j \ (j = 2, 3), \qquad c = GI_c,$$
$$d = EI_d, \qquad f_j = EI_{f_j} \ (j = 2, 3), \qquad h\xi^2 = EI_\omega,$$

where E and G are the elasticity moduli in extension and shear, respectively, A is the area of the cross-section, I_j ($j = 1, 2$) are the centroidal, principal moments of inertia of the cross-section, I_c is the torsion inertia factor for thin-walled cross-sections, I_d is the polar moment of ineria of the cross-section with respect to the shear center, I_ω is the warping inertia (the second moment of the sectorial area with respect to the area of the cross-section), and $I_{f_j} = \int_{\mathcal{R}} x_j r^2$ ($j = 2, 3$), where x_j is the coordinate of a point of the cross-section with respect to the centroid and r the distance of the same point from the shear center.

5. Buckling of compressed beams

Let us consider a beam compressed by a dead load (that is, a force constant in magnitude, direction and sign) of magnitude λ. It is very easy to see by means of elementary considerations that one solution of the elastic static problem, called the fundamental equilibrium path, is described by the following fields, denoted by the superscript f:

$$\begin{aligned} &\mathbf{u}^{\mathrm{f}} = -\frac{\lambda}{a}x_1\mathbf{i}_1, && \mathbf{R}^{\mathrm{f}} = \mathbf{I}, && \alpha^{\mathrm{f}} = 0, \\ &\mathbf{e}^{\mathrm{f}} = -\frac{\lambda}{a}\mathbf{i}_1, && \mathbf{E}^{\mathrm{f}} = \mathbf{0}, && \eta^{\mathrm{f}} = 0, \\ &\mathbf{s}^{\mathrm{f}} = -\lambda\mathbf{i}_1, \quad \mathbf{S}^{\mathrm{f}} = \mathbf{0}, && \tau^{\mathrm{f}} = 0, && \mu^{\mathrm{f}} = 0. \end{aligned}$$

A possible different solution of the same problem, called the bifurcated path, is denoted by the superscript b, and is then described by the fields

$$\begin{aligned} &\mathbf{u}^{\mathrm{b}} = \mathbf{u} - \frac{\lambda}{a}x_1\mathbf{i}_1, && \mathbf{R}^{\mathrm{b}} = \mathbf{R} + \mathbf{I}, && \alpha^{\mathrm{b}} = \alpha, \\ &\mathbf{e}^{\mathrm{b}} = \mathbf{e} - \frac{\lambda}{a}\mathbf{i}_1, && \mathbf{E}^{\mathrm{b}} = \mathbf{E}, && \eta^{\mathrm{b}} = \eta, \\ &\mathbf{s}^{\mathrm{b}} = \mathbf{s} - \lambda\mathbf{i}_1, \quad \mathbf{S}^{\mathrm{b}} = \mathbf{S}, && \tau^{\mathrm{b}} = \tau, && \mu^{\mathrm{b}} = \mu. \end{aligned} \tag{27}$$

From Equation (27) on, symbols of fields without superscripts denote the differences between quantities as evaluated in the bifurcated and in the fundamental paths, that is,

$$(\cdot) \coloneqq (\cdot)^{\mathrm{b}} - (\cdot)^{\mathrm{f}}.$$

In other words, the strain measures, local balance and auxiliary equations and constitutive relations henceforth will be written in terms of differences. Differences will also be assumed to depend,with regularity sufficient to our scope, on a scalar parameter σ, as

$$(\cdot) = (\cdot)(\sigma), \quad \sigma \in [0, 1], \quad (\cdot)|_{\sigma=0} = 0,$$

that is, the differences are assumed to vanish in the fundamental path, where $\sigma = 0$.

Since we are using a static perturbation procedure, let us perform a formal σ-power series expansion of the fields of interest and of the load multiplier λ in a neighborhood of $\sigma = 0$. The field equations at the first order of such a formal expansion are

$$\left.\begin{aligned} a\bar{u}_1 &= 0 \\ b_3\bar{u}_2'''' + \lambda\left(\frac{a-\lambda}{a}\right)\bar{u}_2'' - \lambda c_3\bar{\varphi}_1'' &= 0 \\ b_2\bar{u}_3'''' + \lambda\left(\frac{a-\lambda}{a}\right)\bar{u}_3'' + \lambda c_2\bar{\varphi}_1'' &= 0 \\ h\xi^2\bar{\varphi}_1'''' + \left(\frac{d\lambda - ac}{a}\right)\bar{\varphi}_1'' + \lambda(c_2\bar{u}_3'' - c_3\bar{u}_2'') &= 0 \end{aligned}\right\} \equiv \quad D_{ij}\bar{v}_j = 0, \tag{28}$$

where a superimposed bar indicates the first-order increment of the indicated quantities with respect to the perturbation parameter σ. Equation (28) has been written both in extended and in compact form, with $\bar{v}_j = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{\varphi}_1)$ and D_{ij} a symbolic linear differential operator.

Equation $(28)_1$ is independent of Equations $(28)_{2-4}$, while these last constitute a coupled system of ordinary differential equations with constant coefficients. The equations in (28), along with suitable boundary conditions, form an eigenvalue problem which provides the critical values λ_{ci}, for $i = 1, 2, 3$

of the load multiplier and the mode shapes of the linearized displacement components $\bar{u}_2$, $\bar{u}_3$, $\bar{\varphi}_1$. In the following sections, we consider two benchmark examples, one for a simply supported beam, and one for a clamped beam.

5.1. ***Simply supported beam.*** In this case the linearized boundary conditions we considered are

$$\begin{aligned} x_1 = 0: &\qquad \bar{u}_i = \bar{M}_i = 0, \quad i = 1,\ 2,\ 3; \quad \bar{\mu} = 0, \\ x_1 = l: &\qquad \bar{u}_2 = \bar{u}_3 = \bar{\varphi}_1 = \bar{Q}_1 = \bar{M}_2 = \bar{M}_3 = \bar{\mu} = 0. \end{aligned} \tag{29}$$

Note that the hinge at $x_1 = 0$ does not inhibit warping, while the support at $x_1 = l$ permits axial displacement and warping and inhibits transverse displacement and torsion rotation of the cross-section.

The solution of $(28)_1$ is not interesting from the point of view of finding a linearized bifurcated path. With the relevant boundary conditions in (29) it is immediately evident that $\bar{u}_1 = 0$. On the other hand, nontrivial solutions of equations $(28)_{2-4}$, with the relevant boundary conditions of (29), of the form

$$\begin{aligned} \bar{u}_2 &= V \sin\left(\frac{\pi x_1}{l}\right), \\ \bar{u}_3 &= W \sin\left(\frac{\pi x_1}{l}\right), \\ \bar{\varphi}_1 &= \Phi \sin\left(\frac{\pi x_1}{l}\right), \end{aligned} \tag{30}$$

where V, W, and Φ are arbitrary integration constants, exist under the condition

$$\det \begin{pmatrix} b_3 \left(\frac{\pi}{l}\right)^2 - \lambda + \frac{\lambda^2}{a} & 0 & \lambda c_3 \\ 0 & b_2 \left(\frac{\pi}{l}\right)^2 - \lambda + \frac{\lambda^2}{a} & -\lambda c_2 \\ \lambda c_3 & -\lambda c_2 & h\xi^2 \left(\frac{\pi}{l}\right)^2 - \frac{d\lambda}{a} + c \end{pmatrix} = 0. \tag{31}$$

Equation (31) expresses an eigenvalue problem which provides the searched critical load multipliers. It coincides, modulo the necessary identifications of the constitutive coefficients, with the result provided in [Grimaldi and Pignataro 1979], which was obtained starting from a shell model inspired by [Vlasov 1961]. By writing (31) explicitly, we obtain the characteristic equation of the bifurcation problem, which turns out to be a third-order polynomial in terms of the load multiplier λ. The real solutions of such an equation provide the critical values of the load multiplier, and the relevant eigenmodes are the buckling modes. It is apparent from (31) that the critical loads and the buckling modes depend on the material properties of the beam, but are also strongly influenced by the shape of the cross-sections. In particular, symmetries of the cross sections are of particular importance.

When the cross-section of the beam exhibits two axes of symmetry, the centroid coincides with the shear center, that is, $c_2 = c_3 = 0$. Then, it is apparent that the left hand side of (31) reduces to the product of three distinct factors. Hence there are three real and distinct eigenvalues of the characteristic equation

$$\lambda_{c1} = \frac{ac}{d}\left(1 + \frac{h\xi^2\pi^2}{cl^2}\right),$$
$$\lambda_{c2} = \frac{a}{2}\left(1 - \sqrt{1 - \frac{4b_3\pi^2}{al^2}}\right), \tag{32}$$
$$\lambda_{c3} = \frac{a}{2}\left(1 - \sqrt{1 - \frac{4b_2\pi^2}{al^2}}\right).$$

The three pertaining buckling modes (two purely flexional, one purely torsional) occur separately. These results coincide with those in [Grimaldi and Pignataro 1979] and also those in [Timoshenko and Gere 1961; Pignataro and Ruta 2003]. When $a \to \infty$, the two flexional critical loads in (32) reduce to the standard Euler flexional buckling loads, and no torsional buckling appears. This result can be inferred from [Timoshenko and Gere 1961] as well. Indeed, in a nonextensible beam with the centroid coincident with the shear center, a compressive dead load cannot produce, even in a buckled shape, a torsion couple inducing twist. Only flexional buckling modes are possible.

When the cross-section of the beam exhibits one axis of symmetry, for instance, x_3, it necessarily follows that $c_2 = 0$. Then, the left hand side of (31) is the product of two factors. One provides the same flexional buckling load λ_{c3} as in $(32)_3$. However, the only real root of the other factor has a really complicated expression which is not reported here for the sake of simplicity. Instead we present a simplified expression, obtained from this complicated one by considering a nonextensible beam, that is, letting $a \to \infty$, yielding

$$\lambda_{c1} = \lambda_{c2} = \frac{1}{2c_3^2}\left(\sqrt{c + h\xi^2\frac{\pi^2}{l^2}}\sqrt{c + h\xi^2\frac{\pi^2}{l^2} + 4b_3c_3^2\frac{\pi^2}{l^2}} - \left(c + h\xi^2\frac{\pi^2}{l^2}\right)\right). \tag{33}$$

Hence we deduce two very interesting results for beams with cross-sections exhibiting one axis of symmetry. First, there are two only possible buckling modes. One is purely flexional and Euler-like, in the plane described by the axis of the beam and the axis of symmetry of the cross-section (in this case, $\bar{u}_3$). The other is a coupled flexural-torsional mode (in this case, a combination of $\bar{u}_2$ and $\bar{\varphi}_1$). Second, since the centroid does not coincide with the shear center, even for nonextensible beams, a torsional (or more exactly, a flexural-torsional) buckling mode is present. This result clearly coincides with the original one provided in [Grimaldi and Pignataro 1979].

When the cross-section exhibits no symmetries, the presence of both coordinates of the shear center in the characteristic equation (31), yields a single real-valued critical load, providing flexural-torsional buckling, and the three modes $\bar{u}_2$, $\bar{u}_3$, $\bar{\varphi}_1$ are coupled and occur together. That is, under the critical load the beam bends in both transverse directions and twists. The expression of the critical load, even in the case of nonextensible beams, is cumbersome and is not reported here for the sake of simplicity. This result also clearly coincides with that in [Grimaldi and Pignataro 1979].

5.2. ***Clamped beam.*** In this case the considered linearized boundary conditions are

$$x_1 = 0: \quad \bar{u}_i = \bar{\varphi}_i = 0, \quad i = 1,\ 2,\ 3; \quad \varphi_1' = 0,$$
$$x_1 = l: \quad \bar{Q}_i = \bar{M}_i = 0, \quad i = 1,\ 2,\ 3; \quad \bar{\mu} = 0. \tag{34}$$

That is, all the displacements vanish and warping is prevented at the clamped end, while all the contact actions, as well as the bi-moment, are assumed to vanish at the free end. As in the case of the simply supported beam, here as well the solution of $(28)_1$ is of no interest for determining a linearized buckling path, and easily can be proved to be identically equal to zero.

On the other hand, nontrivial solutions of equations $(28)_{2-4}$, with the relevant boundary conditions in equations (34), of the form

$$\begin{aligned}\bar{u}_2 &= V\left[1-\cos\left(\frac{\pi x_1}{2l}\right)\right],\\ \bar{u}_3 &= W\left[1-\cos\left(\frac{\pi x_1}{2l}\right)\right],\\ \bar{\varphi}_1 &= \Phi\left[1-\cos\left(\frac{\pi x_1}{2l}\right)\right],\end{aligned} \tag{35}$$

where V, W, and Φ are arbitrary integration constants, exist under the condition

$$\det\begin{pmatrix} b_3\left(\frac{\pi}{2l}\right)^2-\lambda+\frac{\lambda^2}{a} & 0 & \lambda c_3 \\ 0 & b_2\left(\frac{\pi}{2l}\right)^2-\lambda+\frac{\lambda^2}{a} & -\lambda c_2 \\ \lambda c_3 & -\lambda c_2 & h\xi^2\left(\frac{\pi}{2l}\right)^2-\frac{d\lambda}{a}+c \end{pmatrix}=0. \tag{36}$$

It is apparent that (36) provides a characteristic equation which has the same form as that provided by (31) for the simply supported beam. The only difference between the two characteristic equations is in the coefficient $(2l)^2$ instead of l^2 in the terms multiplying b_2, b_3 and $h\xi^2$. Hence, apart from this numerical difference, it is obvious that all the results obtained for the simply supported beam are the same of those for the clamped beam. It follows that our analysis and remarks for the simply supported beam can be equally reproduced for the clamped beam. The obtained results are then qualitatively the same as those in [Grimaldi and Pignataro 1979] and for symmetric cross-sections, coincide with those in [Timoshenko and Gere 1961; Pignataro and Ruta 2003].

6. Post-buckling paths

A fundamental aspect of the study of bifurcation paths is the analysis of the post-buckling behavior of the structural element under consideration. That is, it is essential to understand whether the considered element has a stable or unstable post-buckling behavior. This also can tell us if the element is sensitive to imperfection. Another interesting aspect is related to the possibility of interaction between buckling modes, which occurs when the geometrical and mechanical characteristics are such that two or more buckling modes occur under the same critical load.

To study all these aspects, we must analyze the field equations at the second order of the formal power series expansion in terms of σ in a neighborhood of the bifurcation point. These turn out to be

$$
\begin{aligned}
D_{11}\bar{\bar{u}}_1 =& -2\Big(a(\bar{u}_2'\bar{u}_2''+\bar{u}_3'\bar{u}_3'')+d\bar{\varphi}_1'\bar{\varphi}_1''+\frac{a}{a-\lambda_c}\left(\bar{u}_2''(\lambda_c c_3\bar{\varphi}_1'-b_3\bar{u}_2''')-\bar{u}_3''(\lambda_c c_2\bar{\varphi}_1'+b_2\bar{u}_3''')\right)\Big),\\
D_{22}\bar{\bar{u}}_2+D_{24}\bar{\bar{\varphi}}_1 =& \,2b_2\big(\bar{u}_2''\bar{\varphi}_1'\big)''+2f_2\big(\varphi_1'\varphi_1''\big)'+2\lambda_c c_2\big(\bar{u}_2'\bar{u}_3''\big)'+2\lambda_c c_3\big(\bar{\varphi}_1'\big)^2\\
&+2\lambda_c\big(1-\frac{\lambda_c}{a}\big)\bar{u}_2''\bar{\varphi}_1+2\big(c-\frac{d\lambda_c}{a}-b_3\big)\big(\bar{u}_2''\bar{\varphi}_1'\big)'\\
&-2b_3\bar{u}_2'''\bar{\varphi}_1'-2h\xi^2(\bar{u}_2''\bar{\varphi}_1''')'+\frac{2ac_3}{a-\lambda}\left(\lambda_c\big(c_3(\bar{u}_2''\bar{\varphi}_1')'-c_2(\bar{u}_3''\bar{\varphi}_1')'\big)-b_2(\bar{u}_3''\bar{u}_3''')'-b_3(\bar{u}_2''\bar{u}_2''')'\right)\\
&-2\bar{\lambda}\left(\left(1-\frac{\lambda_c}{a}\right)\bar{u}_3''+\frac{1}{a-\lambda_c}\left(ac_2\bar{\varphi}_1''+b_2\bar{u}_3''''\right)\right),\\
D_{32}\bar{\bar{u}}_3+D_{34}\bar{\bar{\varphi}}_1 =& -2b_3\big(\bar{u}_3''\bar{\varphi}_1'\big)''+2f_3\big(\varphi_1'\varphi_1''\big)'-2\lambda_c c_3\big(\bar{u}_2'\bar{u}_3''\big)'\\
&+2\lambda_c c_2\big(\bar{\varphi}_1'\big)^2+2b_3\bar{u}_2'''\bar{\varphi}_1'+2h\xi^2\big(\bar{u}_3''\bar{\varphi}_1'''\big)'\\
&-2\lambda_c\left(1-\frac{\lambda_c}{a}\right)\bar{u}_3''\bar{\varphi}_1-2\left(c-\frac{d\lambda_c}{a}-b_2\right)\big(\bar{u}_3''\bar{\varphi}_1'\big)'\\
&+\frac{2ac_2}{a-\lambda}\left(\lambda_c\big(c_3(\bar{u}_2''\bar{\varphi}_1')'-c_2(\bar{u}_3''\bar{\varphi}_1')'\big)-b_2(\bar{u}_3''\bar{u}_3''')'-b_3(\bar{u}_2''\bar{u}_2''')'\right)\\
&-2\bar{\lambda}\left(\left(1-\frac{\lambda_c}{a}\right)\bar{u}_2''-\frac{1}{a-\lambda_c}\left(ac_3\bar{\varphi}_1''+b_3\bar{u}_2''''\right)\right),\\
D_{42}\bar{\bar{u}}_2+D_{43}\bar{\bar{u}}_3+D_{44}\bar{\bar{\varphi}}_1 =& \,2h\xi^2\big(\bar{u}_2'\bar{u}_3''\big)'''+2(b_2-b_3)\bar{u}_2''\bar{u}_3'\\
&+2\left(\frac{d\lambda_c}{a}-c\right)(\bar{u}_2'\bar{u}_3'')'+2\lambda_c(c_3\bar{u}_3\bar{\varphi}_1+c_2\bar{u}_2''\bar{\varphi}_1)\\
&+2\Big(\bar{\varphi}_1'\big(f_3\bar{u}_2''-f_2\bar{u}_3''\big)\Big)'+2\bar{\lambda}\left(c_3\bar{u}_2''-c_2\bar{u}_3''-\frac{d}{a}\bar{\varphi}_1''\right),
\end{aligned}
\tag{37}
$$

where λ_c is one of the critical values of the load multiplier and the D_{ij} are the components of the symbolic linear differential operator introduced in (28). That is, the field equations at the second order of the formal power series expansion in terms of σ have the same formal structure of the equations at the first order. The difference is that first order equations are homogeneous while second order equations are not (see also [Budiansky 1974; Pignataro et al. 1991]). Hence, since λ_c is an eigenvalue, that is, the symbolic operator D_{ij} is singular, a condition of solvability of the system described by equations (37) must be introduced. This turns out to be given by the requirement that the right-hand side of (37) be orthogonal to any of the eigensolutions of the first order equations provided by (28). In this case, the dot product yielding an orthogonality condition is given by the integral over the length of the beam of the sum of the products between each right hand side of (37) and the corresponding eigenmode. Such a condition, supplemented by a normalization condition on the buckling modes, will provide the expressions for determining $\bar{\lambda}$, that is, the slope of the post-buckling path. Roughly speaking, when $\bar{\lambda}=0$, the considered buckling mode is symmetric, otherwise it is imperfection-sensitive (see also [Budiansky 1974; Pignataro et al. 1991]).

In the following sections, we investigate the case of the simply supported beam considered before. Indeed, since the eigenmodes (35) are qualitatively the same as the eigenmodes (32), the qualitative behaviour of the clamped beam is the same as this of the simply supported one. Since the equation for the axial component of the displacement is immaterial for the study of buckling and hence also of post-buckling, we will leave it aside, and focus attention on eqs. $(37)_{2,3,4}$.

We will limit ourselves to examine the case when the cross section exhibits two axes of symmetry, the eigenmodes are provided by equations (30) and the critical load multipliers are given by the equations of (32). Furthermore, in that case we have $c_2 = c_3 = 0$ and $f_2 = f_3 = 0$. As a consequence, the right-hand side of (37) assumes a much simpler form (which is not reported here for the sake of brevity).

For post-buckling behavior in bending, with no loss of generality, let us suppose that the buckling mode which has taken place is $\bar{u}_2 = V \sin\left(\frac{\pi x_1}{l}\right)$ (see (30)). Once we denote the right-hand side of $(37)_2$ by $r2(x_1)$, when $\bar{u}_1 = \bar{u}_3 = \bar{\varphi}_1 = 0$, the solvability condition yields

$$\int_0^l r2(x_1)\bar{u}_2(x_1) = 0 \;\Rightarrow\; \bar{\lambda} = 0.$$

That is, the post-buckling path is symmetric, a well known result of Euler-like buckling for symmetric beams.

For post-buckling behavior in torsion, the eigenmode is $\bar{\varphi}_1 = \Phi \sin\left(\frac{\pi x_1}{l}\right)$ (see (30)). Once we denote the right hand side of $(37)_4$ by $r4(x_1)$, when $\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0$, the solvability condition yields

$$\int_0^l r4(x_1)\bar{\varphi}_1(x_1) = 0 \;\Rightarrow\; \bar{\lambda} = 0.$$

Therefore, the post-buckling path in torsion is also symmetric, which is also a well known feature of symmetric beams.

7. Conclusions

In this paper we presented a direct, one-dimensional beam model suitable for describing the flexural-torsional buckling of thin-walled beams. The kinematic description makes use of two different beam axes: the centroids, and the shear centers (we left aside the definition of the latter). Using this, we can imagine the contact forces applied at different places of the planes of the (unwarped) cross sections and the contact torques evaluated with respect to different places of the planes of the (unwarped) cross sections. The constitutive relations are nonlinear and hyperelastic. Two inner constraints are assumed to hold, that is, the (unwarped) cross sections remain orthogonal to the centroidal axis, and the descriptor of the warping is proportional to the twist. This makes it possible to use a static perturbation technique to look for bifurcations. Moreover, it is possible to provide a rough description of the quality of the post-buckling behavior and of the interaction of multiple buckling modes. The qualitative description of the phenomena coincides with that obtained with other beam models, yet with all the advantages of a direct model. Further developments of this study will be related to the qualitative analysis of frames.

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