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PPF dependent fixed point in modified Razumikhin class with applications

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Abstract

In this paper we introduce the concepts of c-$C_{\alpha\beta}$-admissible mapping, $(\alpha\beta)_c$-Θ-contraction, weak $(\alpha\beta)_c$-Θ-contraction, generalized $(\alpha\beta)_c$-Θ-contraction and establish the existence of PPF dependent fixed point theorems for such classes of contractive nonself-mappings in the Razumikhin class. We give, also, a result of existence of a PPF dependent fixed point by a condition of Suzuki type. As applications of our theorems, we deduce some PPF dependent fixed point theorems for nonself-mappings valued in a Banach space endowed with a graph or a partial order, and furnish an illustrative example to support our main theorem.

Keywords: Razumikhin class, PPF dependent fixed point, $(\alpha\beta)_c$-Θ-contraction, generalized $(\alpha\beta)_c$-Θ-contraction.
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1. Introduction

It is well known that the contraction mapping principle, formulated and proved in the PhD dissertation of Banach, has laid the foundation of metric fixed point theory for contraction mappings on complete metric

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spaces. Since then, Banach's fixed point theorem has been generalized, improved and extended in several directions, see the papers ([1, 2, 4, 6, 7, 12, 13, 15, 18, 20, 22, 24] and references therein). Bernfeld et al. [5] introduced the concept of fixed point for mappings that have different domains and ranges, which is called *PPF dependent fixed point* or the *fixed point with PPF dependence*. Also, they introduced the notion of Banach type contraction for nonself-mapping and established the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contraction mappings (see [17]). The PPF dependent fixed point theorems are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data and future consideration (see [11]). However, as proved in a recent paper by Cho et al. [9], the starting conditions [imposed by the problem setting] relative to the ambient Razumikhin class $\mathcal{R}_c$ may be converted into starting conditions relative to the constant class $\mathcal{R}_c^0$; so, ultimately, we may arrange for these PPF dependent fixed point results holding over $\mathcal{R}_c^0$.

On the other hand, Samet et al. [23] introduced and studied α-ψ-contractive mappings in complete metric spaces and provided applications of the results to ordinary differential equations. More recently, Salimi et al. [21] modified the notions of α-ψ-contractive and α-admissible mappings and established fixed point theorems to modify the results in [23].

Consistent with Jleli and Samet [19], we denote by Δ_Θ the set of all functions $\Theta : (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

(Θ_1) Θ is increasing;
(Θ_2) for all sequence $\{\alpha_n\} \subseteq (0, +\infty)$, $\lim\limits_{n\to+\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to+\infty} \Theta(\alpha_n) = 1$;
(Θ_3) there exist $0 < r < 1$ and $\ell \in (0, +\infty]$ such that $\lim\limits_{t\to 0^+} \frac{\Theta(t)-1}{t^r} = \ell$.

In this paper, motivated by the works of Hussain and Salimi, Samet et al., Cosentino et al. and Jleli and Samet, we introduce the concepts of c-$C_{\alpha\beta}$-admissible mapping, $(\alpha\beta)_c$-Θ-contraction, weak $(\alpha\beta)_c$-Θ-contraction, generalized $(\alpha\beta)_c$-Θ-contraction and establish the existence of PPF dependent fixed point theorems for such classes of contractive nonself-mappings in the Razumikhin class. We give, also, a result of existence of a PPF dependent fixed point by a condition of Suzuki type. As applications of our theorems, we deduce some PPF dependent fixed point theorems for nonself-mappings valued in a Banach space endowed with a graph or a partial order, and furnish an illustrative example to support our main theorem.

2. Preliminaries

Throughout this paper, we assume that $(E, \|\cdot\|_E)$ is a Banach space, I denotes a closed interval $[a, b]$ in $\mathbb{R}$ and $E_0 = C(I, E)$ denotes the set of all continuous E-valued functions $\phi : I \to E$ equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E.$$

Here, $\mathbb{N} = \{0, 1, ...\}$ denotes the set of all natural numbers; in addition, for each $h \in \mathbb{N}$, we put $\mathbb{N}_h = \{n \in \mathbb{N} : \ h \leq n\}$.

Definition 2.1 ([5])**.** A mapping $\phi \in E_0$ is said to be a *PPF dependent fixed point* or a *fixed point with PPF dependence* of mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

Motivated by results of Agarwal et al. [3], Ćirić et al. [8], Cosentino et al. [10] and Hussain et al. [13], we give the following notion which is suitable for our main results.

Definition 2.2. Let $c \in I$ and $T : E_0 \to E$, $\alpha, \beta : E \times E \to [0, +\infty)$ and $C_\alpha > 0$, $C_\beta \geq 0$ with $0 \leq C_\beta/C_\alpha < 1$. We say that T is a c-$C_{\alpha\beta}$-admissible mapping if the following conditions hold:

(i) $\alpha(\xi(c), \varphi(c)) \geq C_\alpha$ implies $\alpha(T\xi, T\varphi) \geq C_\alpha$, $\varphi, \xi \in E_0$;
(ii) $\beta(\xi(c), \varphi(c)) \leq C_\beta$ implies $\beta(T\xi, T\varphi) \leq C_\beta$, $\varphi, \xi \in E_0$.

Example 2.3. Let $E=\mathbb{R}$ be a real Banach space with usual norm and $I=[0,1]$. Let $c=1$, define $T:E_0\to E$ by $T\phi=\frac{1}{2}\phi(1)$ for all $\phi\in E_0$ and $\alpha,\beta:E\times E\to[0,+\infty)$ by

$$\alpha(x,y)=\begin{cases}3, & \text{if } x\geq y,\\ 2, & \text{otherwise}\end{cases} \quad \text{and } \beta(x,y)=\begin{cases}2, & \text{if } x\geq y,\\ 4, & \text{otherwise.}\end{cases}$$

Then T is an 1-$C_{\alpha\beta}$-admissible mapping, where $C_\alpha=3$ and $C_\beta=2$.

Definition 2.4. Let $T:E_0\to E$ and $\alpha,\beta:E\times E\to[0,+\infty)$ be three non-self mappings and $c\in I$.

(i) T is called an $(\alpha\beta)_c$-Θ-*contraction* if there exist $\Theta\in\Delta_\Theta$ such that, for all $\phi,\xi\in E_0$ with $\|T\phi-T\xi\|_E>0$, we have

$$\Big[\Theta(\|T\phi-T\xi\|_E)\Big]^{\alpha(\phi(c),\xi(c))}\leq\Big[\Theta(\|\phi-\xi\|_{E_0})\Big]^{\beta(\phi(c),\xi(c))}.$$

(ii) T is called a *weak* $(\alpha\beta)_c$-Θ-*contraction* if there exist $\Theta\in\Delta_\Theta$ such that, for all $\phi,\xi\in E_0$ with $\|T\phi-T\xi\|_E>0$, we have

$$\Big[\Theta(\|T\phi-T\xi\|_E)\Big]^{\alpha(\phi(c),\xi(c))}\leq\Big[\Theta(\max\{\|\phi-\xi\|_{E_0},\|\phi(c)-T\phi\|_E,\|\xi(c)-T\xi\|_E\})\Big]^{\beta(\phi(c),\xi(c))}.$$

(iii) T is called a *generalized* $(\alpha\beta)_c$-Θ-*contraction* if there exist $\Theta\in\Delta_\Theta$ such that, for all $\phi,\xi\in E_0$ with $\|T\phi-T\xi\|_E>0$, we have

$$\Big[\Theta(\|T\phi-T\xi\|_E)\Big]^{\alpha(\phi(c),\xi(c))}\leq\Big[\Theta\Big(\max\Big\{\|\phi-\xi\|_{E_0},\|\phi(c)-T\phi\|_E,\|\xi(c)-T\xi\|_E,\frac{\|\phi(c)-T\xi\|_E+\|\xi(c)-T\phi\|_E}{2}\Big\}\Big)\Big]^{\beta(\phi(c),\xi(c))}.$$

Definition 2.5 ([9])**.** The *Razumikhin* or *minimal class* (attached to c) is defined as

$$\mathcal{R}_c=\{\phi\in E_0:\ \|\phi\|_{E_0}=\|\phi(c)\|_E\}.$$

Also, denote, for simplicity

$$\mathcal{R}_c^0=\{\phi\in\mathcal{R}_c:\quad \phi \text{ is a constant function}\}.$$

It will be referred as the *constant Razumikhin class.* To get a useful representation for this subclass, we need a lot of preliminary facts. For each $u\in E$, let $H[u]$ denote the constant function of E_0, defined as

$$H[u](t)=u,\quad \text{for all } t\in I.$$

Note that, by this definition,

$$||H[u]||_{E_0}=||u||_E,\ H[u](c)=u;$$

hence, $H[u]\in\mathcal{R}_c$. We now claim that

$$\mathcal{R}_c^0=\{H[u]:\ u\in E\};$$

or, in other words, the constant Razumikhin class $\mathcal{R}_c^0$ is just the subclass of all constant functions in E_0. In fact, the right to left inclusion is clear. For the left to right inclusion, it will suffice noting that any constant function ψ in $\mathcal{R}_c$ may be written as

$$\psi=H[u],\ \text{for some } u\in E;$$

and this ends our argument.

The following properties of this subclass are almost immediate; so, we do not give details.

Proposition 2.6 ([9])**.** *Under the above conventions, the following conditions hold:*

(i) $H[u+v] = H[u] + H[v]$, $\forall u, v \in E$;
(ii) $H[\lambda u] = \lambda H[u]$, $\forall \lambda \in \mathbb{R}$, $\forall u \in E$;
(iii) $||u||_E = ||H[u]||_{E_0}$, $\forall u \in E$;
(iv) *the mapping* $u \mapsto H[u]$ *is an algebraic and topological isomorphism between* $(E, \|\cdot\|_E)$ *and* $(\mathcal{R}_c^0, \|\cdot\|_{E_0})$.

Definition 2.7. Let $c \in I$ and let $T : E_0 \to E$, $\alpha, \beta : E \times E \to [0, +\infty)$ be two mappings.

(i) T is called $(\mathcal{R}_c, \alpha\beta)$-*starting*, if there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$;
(ii) T is called $(\mathcal{R}_c^0, \alpha\beta)$-*starting*, if there exists $\phi_0 \in \mathcal{R}_c^0$ such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$.

Evidently, if T is $(\mathcal{R}_c^0, \alpha\beta)$-starting, then it is also $(\mathcal{R}_c, \alpha\beta)$-starting. The reciprocal assertion is also true, under certain regularity conditions upon T. Precisely, we have

Proposition 2.8. *Let* $c \in I$ *and let* $T : E_0 \to E$, $\alpha, \beta : E \times E \to [0, +\infty)$ *be nonself-mappings such that*

(i) T *is a c-*$C_{\alpha\beta}$*-admissible mapping;*
(ii) T *is* $(\mathcal{R}_c, \alpha\beta)$*-starting.*

Then T *is* $(\mathcal{R}_c^0, \alpha\beta)$*-starting.*

Proof. By (ii) there exist $\phi_0 \in \mathcal{R}_c$, such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$. Since $T\phi_0 \in E$, we may consider the element $\xi_0 = T\phi_0 = H[T\phi_0]$ from the constant Razumikhin class $\mathcal{R}_c^0$; this, by definition, means $\xi_0(t) = T\phi_0$, for all $t \in I$ and hence $\xi_0(c) = T\phi_0$. The condition upon ϕ_0 becomes $\alpha(\phi_0(c), \xi_0(c)) \geq C_\alpha$ and $\beta(\phi_0(c), \xi_0(c)) \leq C_\beta$. Since T is a c-$C_{\alpha\beta}$-admissible mapping, this yields

$$\alpha(T\phi_0, T\xi_0) \geq C_\alpha \quad \text{and} \quad \beta(T\phi_0, T\xi_0) \leq C_\beta$$

or, equivalently,

$$\alpha(\xi_0(c), T\xi_0) \geq C_\alpha \quad \text{and} \quad \beta(\xi_0(c), T\xi_0) \leq C_\beta.$$

This ends the proof. □

3. Main Results

We start with the following proposition which will be crucial to our main results.

Proposition 3.1. *Let* $c \in I$, $T : E_0 \to E$, $\alpha, \beta : E \times E \to [0, +\infty)$ *and* $\Theta \in \Delta_\Theta$ *be such that*

(i) T *is a c-*$C_{\alpha\beta}$*-admissible mapping;*
(ii) T *is a generalized* $(\alpha\beta)_c$*-*Θ*-contraction;*
(iii) there exists $\phi_0 \in \mathcal{R}_c$ *such that* $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ *and* $\beta(\phi_0(c), T\phi_0) \leq C_\beta$.

In addition, assume that

(Ze) T *has no PPF dependent fixed points in* $\mathcal{R}_c^0$ ($T\phi \neq \phi(c)$, *for all* $\phi \in \mathcal{R}_c^0$).

Then, there exist a sequence $\{\phi_n\}$ *in* $\mathcal{R}_c^0$, *a function* $\phi^* \in \mathcal{R}_c^0$ *and* $h \in \mathbb{N}$ *such that*

(c1) $T\phi_n = \phi_{n+1}(c)$ *and* $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq C_\alpha$ *and* $\beta(\phi_n(c), \phi_{n+1}(c)) \leq C_\beta$ *for all* $n \in \mathbb{N}$;
(c2) $\phi_n \to \phi^*$ *as* $n \to +\infty$;
(c3) $T\phi_n \neq T\phi^*$ *(hence,* $\phi_n \neq \phi^*$*), for all* $n \in \mathbb{N}_h$.

Proof. By Proposition 2.8, conditions (i) and (iii) ensure that there exists $\phi_0 \in \mathcal{R}_c^0$ such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$. Since $T\phi_0 \in E$, we may consider the element $\phi_1 = H[T\phi_0]$ from the constant Razumikhin class $\mathcal{R}_c^0$; this, by definition, means $\phi_1(t) = T\phi_0$, for all $t \in I$ and hence $\phi_1(c) = T\phi_0$. Further, since $T\phi_1 \in E$, we may consider the element $\phi_2 = H[T\phi_1]$ from the constant Razumikhin class $\mathcal{R}_c^0$; this, by definition, means $\phi_2(t) = T\phi_1$ for all $t \in I$ and hence $\phi_2(c) = T\phi_1$. The process may continue indefinitely; it gives us a sequence $\{\phi_n\}$ in the constant Razumikhin class $\mathcal{R}_c^0$, with

$$(\forall n \in \mathbb{N}_1): \quad \phi_n(t) = T\phi_{n-1}, \qquad \text{for all } t \in I; \quad \text{hence} \quad \phi_n(c) = T\phi_{n-1}. \tag{3.1}$$

Since $\phi_{n-1} - \phi_n \in \mathcal{R}_c^0$ for all $n \in \mathbb{N}_1$, it follows that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}_1$. Since T is a c-$C_{\alpha\beta}$-admissible mapping and

$$\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq C_\alpha \quad \text{and} \quad \beta(\phi_0(c), \phi_1(c)) = \beta(\phi_0(c), T\phi_0) \leq C_\beta,$$

then

$$\alpha(\phi_1(c), \phi_2(c)) = \alpha(T\phi_0, T\phi_1) \geq C_\alpha \quad \text{and} \quad \beta(\phi_1(c), \phi_2(c)) = \beta(T\phi_0, T\phi_1) \leq C_\beta.$$

Again since, T is c-$C_{\alpha\beta}$-admissible, then $\alpha(\phi_2(c), \phi_3(c)) \geq C_\alpha$ and $\beta(\phi_2(c), \phi_3(c)) \leq C_\beta$. By continuing this process, we have $\alpha(\phi_{n-1}(c), \phi_n(c)) \geq C_\alpha$ and $\beta(\phi_{n-1}(c), \phi_n(c)) \leq C_\beta$ for all $n \in \mathbb{N}_1$; and this proves the conclusion (c1).

Now, from condition (Ze), we deduce that

$$T\phi_n \neq T\phi_{n+1} \text{ (hence, } \phi_n \neq \phi_{n+1}), \text{ for all } n \in \mathbb{N}.$$

Since T is a generalized $(\alpha\beta)_c$-Θ-contraction, we have

$$\begin{aligned} &\Big[\Theta(\|T\phi_{n-1} - T\phi_n\|_E))\Big]^{\alpha(\phi_{n-1}(c), \phi_n(c))} \\ &\leq \Big[\Theta\Big(\max\Big\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - T\phi_{n-1}\|_E, \|\phi_n(c) - T\phi_n\|_E, \\ &\qquad \frac{\|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_{n-1}\|_E}{2}\Big\}\Big)\Big]^{\beta(\phi_{n-1}(c), \phi_n(c))}. \end{aligned} \tag{3.2}$$

If we use

$$\alpha(\phi_{n-1}(c), \phi_n(c)) \geq C_\alpha \text{ and } \beta(\phi_{n-1}(c), \phi_n(c)) \leq C_\beta,$$

then we obtain

$$\begin{aligned} \Theta(\|T\phi_{n-1} - T\phi_n\|_E)) \leq \Big[\Theta\Big(\max\Big\{&\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - T\phi_{n-1}\|_E, \\ &\|\phi_n(c) - T\phi_n\|_E, \frac{\|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_{n-1}\|_E}{2}\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})}. \end{aligned}$$

Therefore

$$\begin{aligned} \Theta(\|\phi_n - \phi_{n+1}\|_{E_0}) &= \Theta(\|\phi_n(c) - \phi_{n+1}(c)\|_E) = \Theta(\|T\phi_{n-1} - T\phi_n\|_E)) \\ &\leq \Big[\Theta\Big(\max\Big\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - T\phi_{n-1}\|_E, \\ &\qquad \|\phi_n(c) - T\phi_n\|_E, \frac{\|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_{n-1}\|_E}{2}\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})} \end{aligned}$$

$$
\begin{aligned}
&= \Big[\Theta\Big(\max\Big\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_{n-1}(c)-\phi_n(c)\|_E,\\
&\quad \|\phi_n(c)-\phi_{n+1}(c)\|_E, \frac{\|\phi_{n-1}(c)-\phi_{n+1}(c)\|_E+\|\phi_n(c)-\phi_n(c)\|_E}{2}\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})}\\
&= \Big[\Theta\Big(\max\Big\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_{n-1}-\phi_n\|_{E_0},\\
&\quad \|\phi_n-\phi_{n+1}\|_{E_0}, \frac{\|\phi_{n-1}-\phi_{n+1}\|_{E_0}+\|\phi_n-\phi_n\|_{E_0}}{2}\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})}\\
&\le \Big[\Theta\Big(\max\Big\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_n-\phi_{n+1}\|_{E_0}, \frac{\|\phi_{n-1}-\phi_n\|_{E_0}+\|\phi_n-\phi_{n+1}\|_{E_0}}{2}\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})}\\
&= \Big[\Theta\Big(\max\Big\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_n-\phi_{n+1}\|_{E_0},\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})},
\end{aligned}
$$

which implies

$$\Theta(\|\phi_n-\phi_{n+1}\|_{E_0}) \le \Big[\Theta\Big(\max\Big\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_n-\phi_{n+1}\|_{E_0}\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})}.$$

Now, if $\max\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_n-\phi_{n+1}\|_{E_0}\} = \|\phi_n-\phi_{n+1}\|_{E_0}$, then we have

$$\Theta(\|\phi_n-\phi_{n+1}\|_{E_0}) \le \Big[\Theta(\|\phi_n-\phi_{n+1}\|_{E_0})\Big]^{(\frac{C_\beta}{C_\alpha})} < \Theta(\|\phi_n-\phi_{n+1}\|_{E_0}),$$

which is a contradiction. Therefore, we have

$$\Theta(\|\phi_n-\phi_{n+1}\|_{E_0}) \le \Big[\Theta(\|\phi_{n-1}-\phi_n\|_{E_0})\Big]^{(\frac{C_\beta}{C_\alpha})}$$

and so

$$\Theta(\|\phi_n-\phi_{n+1}\|_{E_0}) \le \Big[\Theta(\|\phi_0-\phi_1\|_{E_0})\Big]^{(\frac{C_\beta}{C_\alpha})^n}. \tag{3.3}$$

Taking the limit as $n\to+\infty$ in (3.3), we have

$$\lim_{n\to+\infty}\Theta\big(\|\phi_n-\phi_{n+1}\|_{E_0}\big)=1$$

and since $\Theta\in\Delta_\Theta$, we obtain

$$\lim_{n\to+\infty}\|\phi_n-\phi_{n+1}\|_{E_0}=0. \tag{3.4}$$

Now, from (Θ3), there exists $0<r<1$ and $0<\ell\le+\infty$ such that,

$$\lim_{n\to+\infty}\frac{\Theta\big(\|\phi_n-\phi_{n+1}\|_{E_0}\big)-1}{[\|\phi_n-\phi_{n+1}\|_{E_0}]^r}=\ell. \tag{3.5}$$

Let $B\in(0,\ell)$ be a real number. From the definition of limit there exists $n_0\in\mathbb{N}$ such that

$$\frac{\Theta\big(\|\phi_n-\phi_{n+1}\|_{E_0}\big)-1}{[\|\phi_n-\phi_{n+1}\|_{E_0}]^r}\ge B \quad \text{for all} \quad n\in\mathbb{N}_{n_0}$$

and so

$$n[\|\phi_n-\phi_{n+1}\|_{E_0}]^r \le nA[\Theta\big(\|\phi_n-\phi_{n+1}\|_{E_0}\big)-1] \quad \text{for all} \quad n\in\mathbb{N}_{n_0},$$

where $A = \frac{1}{B} > 0$. From (3.3), we have

$$n[\|\phi_n - \phi_{n+1}\|_{E_0}]^r \leq nA[[\Theta(\|\phi_0 - \phi_1\|_{E_0})]^{(\frac{C_\beta}{C_\alpha})^n} - 1] \quad \text{for all} \quad n \in \mathbb{N}_{n_0}.$$

Taking limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n \to +\infty} n[\|\phi_n - \phi_{n+1}\|_{E_0}]^r = 0. \tag{3.6}$$

It follows from (3.6) that there exists $n_1 \in \mathbb{N}$ such that

$$n[\|\phi_n - \phi_{n+1}\|_{E_0}]^r \leq 1$$

for all $n > n_1$. This implies that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq \frac{1}{n^{1/r}}$$

for all $n > n_1$. Now, for all $m > n > n_1$ we have,

$$\|\phi_n - \phi_m\|_{E_0} \leq \sum_{i=n}^{m-1} \|\phi_i - \phi_{i+1}\|_{E_0} \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Since, $0 < r < 1$, then $\sum_{i=1}^{+\infty} \frac{1}{i^{1/r}}$ converges. Therefore, $\|\phi_n - \phi_m\|_{E_0} \to 0$ as $m, n \to +\infty$. Thus, $\{\phi_n\}$ is a Cauchy sequence. Completeness of $\mathcal{R}_c^0$ ensures that there exists $\phi^* \in \mathcal{R}_c^0$ such that $\phi_n \to \phi^*$ as $n \to +\infty$.

As a consequence, conclusion (c2) holds too. Finally, assume that conclusion (c3) is not true. Then for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m > n$ such that $T\phi^* = T\phi_m = \phi_{m+1}(c)$. This tells us that there exists an infinite sequence $\{k(n)\}$ in $\mathbb{N}$, with

$$T\phi^* = \phi_{k(n)}(c), \qquad \text{for all } n \in \mathbb{N}.$$

Passing to limit as $n \to +\infty$ in the previous inequality, we get $T\phi^* = \phi^*(c)$, which is contradiction to hypothesis (Ze). Hence, the conclusion (c3) holds too and the proof is complete. □

Theorem 3.2. *Let $c \in I$, $T : E_0 \to E$, $\alpha, \beta : E \times E \to [0, +\infty)$ and $\Theta \in \Delta_\Theta$ be such that*

(i) T is a c-$C_{\alpha\beta}$-admissible mapping;
(ii) T is an $(\alpha\beta)_c$-Θ-contraction;
(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq C_\alpha$ and $\beta(\phi_n(c), \phi_{n+1}(c)) \leq C_\beta$ for all $n \in \mathbb{N}$, then $\alpha(\phi_n(c), \phi(c)) \geq C_\alpha$ and $\beta(\phi_n(c), \phi(c)) \leq C_\beta$ for all $n \in \mathbb{N}$;
(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

Proof. Since Θ is a strictly increasing function, every $(\alpha\beta)_c$-Θ-contraction is a generalized $(\alpha\beta)_c$-Θ-contraction. Thus all conditions of Proposition 3.1 hold and hence there exist a sequence $\{\phi_n\}$ in $\mathcal{R}_c^0$, a function $\phi^* \in \mathcal{R}_c^0$ and $h \in \mathbb{N}$ such that conditions (c_1)-(c_3) hold. From, condition (c_3), since T is a $(\alpha\beta)_c$-Θ-contraction, we obtain

$$\Big[\Theta(\|T\phi_n - T\phi^*\|_E)\Big]^{\alpha(\phi_n(c), \phi^*(c))} \leq \big[\Theta(\|\phi_n - \phi^*\|_{E_0})\big]^{\beta(\phi_n(c), \phi^*(c))}$$

for all $n \in \mathbb{N}_h$, which by condition (c_1) implies

$$\Theta(\|\phi_{n+1} - T\phi^*\|_E) \leq \big[\Theta(\|\phi_n - \phi^*\|_{E_0})\big]^{(\frac{C_\beta}{C_\alpha})} \leq \Theta(\|\phi_n - \phi^*\|_{E_0}).$$

Since $\Theta \in \Delta_\Theta$, we get

$$\|T\phi_n - T\phi^*\|_E \leq \|\phi_n - \phi^*\|_{E_0}$$

and hence

$$\begin{aligned}\|T\phi^* - \phi^*(c)\|_E &\leq \|T\phi^* - T\phi_n\|_E + \|T\phi_n - \phi^*(c)\|_E \\ &= \|T\phi^* - T\phi_n\|_E + \|\phi_{n+1}(c) - \phi^*(c)\|_{E_0} \\ &\leq \|\phi^* - \phi_n\|_{E_0} + \|\phi_{n+1} - \phi^*\|_{E_0}.\end{aligned}$$

Now, taking limit as $n \to +\infty$ in the above inequality, we have $\|T\phi^* - \phi^*(c)\|_E = 0$, that is, $T\phi^* = \phi^*(c)$. □

Example 3.3. Let $(E, \|.\|_E)$ be a Banach space where $E = \mathbb{R}$ and $\|x\|_E = |x|$, $c = 1$, $C_\alpha = 4$, $C_\beta = 3$ and $E_0 = C([0,1], E)$ the set of all continuous E-valued functions on $[0,1]$ equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E.$$

Define $T : E_0 \to E$, $\alpha, \beta : E \times E \to [0, +\infty)$, and $\Theta : (0, +\infty) \to (1, \infty)$ by

$$T\phi = \begin{cases} e^{[\phi(1)]^{10}} + 1, & \text{if } \phi(1) < -1 \\ \sin[\phi(1)] + \pi, & \text{if } -1 \leq \phi(1) < 0 \\ \frac{9}{32}[\phi(1)]^2, & \text{if } 0 \leq \phi(1) \leq 1 \\ 5\phi(1) & \text{if } \phi(1) > 1 \end{cases},$$

$$\alpha(x,y) = \begin{cases} 4, & \text{if } x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}, \quad \beta(x,y) = \begin{cases} 3, & \text{if } x, y \in [0,1] \\ 10, & \text{otherwise} \end{cases} \quad \text{and } \Theta(r) = e^{\sqrt{r}}.$$

Let $\alpha(\phi(1), \psi(1)) \geq C_\alpha$ and $\beta(\phi(1), \psi(1)) \leq C_\beta$. Then, $0 \leq \phi(1) \leq 1$ and $0 \leq \psi(1) \leq 1$, and so $0 \leq T\phi = \frac{9}{32}[\phi(1)]^2 \leq 1$ and $0 \leq T\psi = \frac{9}{32}[\psi(1)]^2 \leq 1$. That is, $\alpha(T\phi, T\psi) \geq C_\alpha$ and $\beta(T\phi, T\psi) \leq C_\beta$. Therefore, T is a c-$C_{\alpha\beta}$-admissible mapping. Let $\{\phi_n\}$ be a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\alpha(\phi_n(1), \phi_{n+1}(1)) \geq C_\alpha$ and $\beta(\phi_n(1), \phi_{n+1}(1)) \leq C_\beta$ for all $n \in \mathbb{N}$. Now since $0 \leq \phi_n(1) \leq 1$ for all $n \in \mathbb{N}$ and $\phi_n \to \phi$ as $n \to +\infty$, then $0 \leq \phi(1) \leq 1$. That is, $\alpha(\phi_n(1), \phi(1)) \geq C_\alpha$ and $\beta(\phi_n(1), \phi(1)) \leq C_\beta$ for all $n \in \mathbb{N}$. Clearly, if we choose $\phi_0 \in \mathcal{R}_c$ defined by $\phi_0 \equiv 0$, the $\alpha(0, T0) \geq C_\alpha$ and $\beta(0, T0) \leq C_\beta$.

If $0 \leq \phi(1) \leq 1$ and $0 \leq \psi(1) \leq 1$, then $\alpha(\phi(1), \psi(1)) = 4$, $0 \leq T\phi = \frac{9}{32}[\phi(1)]^2 \leq 1$, $0 \leq T\psi = \frac{9}{32}[\psi(1)]^2 \leq 1$ and

$$\begin{aligned}\|T\phi - T\psi\|_E &= \frac{9}{32}|[\phi(1)]^2 - [\psi(1)]^2| \\ &= \frac{9}{32}|\phi(1) - \psi(1)||\phi(1) + \psi(1)| \leq \frac{9}{16}|\phi(1) - \psi(1)| \\ &\leq \frac{9}{16} \sup_{t\in[0,1]} |\phi(t) - \psi(t)| = \frac{9}{16}\|\phi - \psi\|_{E_0}\end{aligned}$$

and so

$$16\|T\phi - T\psi\|_E \leq 9\|\phi - \psi\|_{E_0}.$$

Therefore

$$\begin{aligned}[\Theta(\|T\phi - T\psi\|_E)]^{\alpha(\phi(1),\psi(1))} &= e^{4\sqrt{\|T\phi - T\psi\|_E}} = e^{\sqrt{16\|T\phi - T\psi\|_E}} \\ &\leq e^{\sqrt{9\|\phi - \psi\|_{E_0}}} = e^{3\sqrt{\|\phi - \psi\|_{E_0}}} = [\Theta(\|\phi - \psi\|_{E_0})]^{\beta(\phi(1),\psi(1))}.\end{aligned}$$

Otherwise, $\alpha(\phi(1),\psi(1)) = 0$ which implies,

$$[\Theta(\|T\phi - T\psi\|_E)]^{\alpha(\phi(1),\psi(1))} = 1 \leq [\Theta(\|\phi - \psi\|_{E_0})]^{\beta(\phi(1),\psi(1))}.$$

Hence, T is an $(\alpha\beta)_c$-Θ-contraction and all conditions of Theorem 3.2 hold. Thus T has a PPF dependent fixed point. Here, $\phi \equiv 0$ is a PPF dependent fixed point of T.

If in Theorem 3.2 we take, $\alpha(\phi,\xi) = 1$ and $\beta(\phi,\xi) = r$ where $0 \leq r < 1$ for all $\phi,\xi \in E$, then we derive following result.

Corollary 3.4. *Let $c \in I$, $T : E_0 \to E$ be a nonself-mapping and $\Theta \in \Delta_\Theta$ such that for all $\phi,\xi \in E_0$ with $\|T\phi - T\xi\|_E > 0$, we have*

$$\Theta(\|T\phi - T\xi\|_E) \leq \Big[\Theta(\|\phi - \xi\|_{E_0})\Big]^r,$$

where $0 \leq r < 1$. Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

Theorem 3.5. *Let $c \in I$, $T : E_0 \to E$ be a nonself-mapping, $\alpha,\beta : E \times E \to [0,+\infty)$ and $\Theta \in \Delta_\Theta$ be such that*

(i) T is a c-$C_{\alpha\beta}$-admissible mapping;
(ii) T is generalized $(\alpha\beta)_c$-Θ-contraction such that Θ is continuous;
(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\alpha(\phi_n(c),\phi_{n+1}(c)) \geq C_\alpha$ and $\beta(\phi_n(c),\phi_{n+1}(c)) \leq C_\beta$ for all $n \in \mathbb{N}$, then $\alpha(\phi_n(c),\phi(c)) \geq C_\alpha$ and $\beta(\phi_n(c),\phi(c)) \leq C_\beta$ for all $n \in \mathbb{N}$;
(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c),T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c),T\phi_0) \leq C_\beta$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

Proof. Assume that T does not have a PPF dependent fixed point in $\mathcal{R}_c^0$. Thus all conditions of Proposition 3.1 hold and hence there exist a sequence $\{\phi_n\}$ in $\mathcal{R}_c^0$, a function $\phi^* \in \mathcal{R}_c^0$ and $h \in \mathbb{N}$ such that conditions (c_1)-(c_3) hold. From, condition (c_3), since T is a generalized $(\alpha\beta)_c$-Θ-contraction, we obtain,

$$\begin{aligned}\Big[\Theta(\|\phi_{n+1}(c) - T\phi^*\|_E)\Big]^{\alpha(\phi_n(c),\,\phi_{n+1}(c))} &= \Big[\Theta(\|T\phi_n - T\phi^*\|_E)\Big]^{\alpha(\phi_n(c),\,\phi_{n+1}(c))}\\ &\leq \Big[\Theta\Big(\max\Big\{\|\phi_n - \phi^*\|_{E_0}, \|\phi_n(c) - T\phi_n\|_E, \|\phi^*(c) - T\phi^*\|_E,\\ &\qquad \frac{\|\phi_n(c) - T\phi^*\|_E + \|\phi^*(c) - T\phi_n\|_E}{2}\Big\}\Big)\Big]^{\beta(\phi_n(c),\,\phi_{n+1}(c))}\\ &= \Big[\Theta\Big(\max\Big\{\|\phi_n - \phi^*\|_{E_0}, \|\phi_n(c) - \phi_{n+1}(c)\|_E, \|\phi^*(c) - T\phi^*\|_E,\\ &\qquad \frac{\|\phi_n(c) - T\phi^*\|_E + \|\phi^*(c) - \phi_{n+1}(c)\|_E}{2}\Big\}\Big)\Big]^{\beta(\phi_n(c),\,\phi_{n+1}(c))},\end{aligned}$$

which by condition (c_1) implies,

$$\begin{aligned}\Theta(\|\phi_{n+1}(c) - T\phi^*\|_E) \;&\leq\; \Big[\Theta\Big(\max\Big\{\|\phi_n - \phi^*\|_{E_0}, \|\phi_n(c) - \phi_{n+1}(c)\|_E, \|\phi^*(c) - T\phi^*\|_E,\\ &\qquad \frac{\|\phi_n(c) - T\phi^*\|_E + \|\phi^*(c) - \phi_{n+1}(c)\|_E}{2}\Big\}\Big)\Big]^{(\frac{C_\beta}{C_\alpha})}.\end{aligned}$$

Now, since Θ is continuous, by taking limit as $n \to +\infty$, in the above inequality, we obtain

$$\Theta(\|\phi^*(c) - T\phi^*\|_E) \leq \Big[\Theta(\|\phi^*(c) - T\phi^*\|_E)\Big]^{(\frac{C_\beta}{C_\alpha})},$$

which is a contradiction. Therefore, we have $\|\phi^*(c) - T\phi^*\|_E = 0$, that is, $\phi^*(c) = T\phi^*$. □

If in Theorem 3.5 we take, $\alpha(\phi,\xi) = 1$ and $\beta(\phi,\xi) = r$ where $0 \leq r < 1$ for all $\phi,\xi \in E$, then we derive following result.

Corollary 3.6. *Let $c \in I$, $T : E_0 \to E$ be a nonself-mapping and $\Theta \in \Delta_\Theta$ be such that, for all $\phi,\xi \in E_0$ with $\|T\phi - T\xi\|_E > 0$, we have*

$$\Theta(\|T\phi - T\xi\|_E) \leq \left[\Theta\Big(\max\Big\{\|\phi-\xi\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E, \frac{\|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}{2}\Big\}\Big)\right]^r$$

where $0 \leq r < 1$. Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

Similarly, we can prove the following theorem.

Theorem 3.7. *Let $c \in I$, $T : E_0 \to E$ be a nonself-mapping, $\alpha,\beta : E \times E \to [0,+\infty)$ and $\Theta \in \Delta_\Theta$ be such that*

(i) T is a c-$C_{\alpha\beta}$-admissible mapping;
(ii) T is a weak $(\alpha\beta)_c$-Θ-contraction such that Θ is continuous;
(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\alpha(\phi_n(c),\phi_{n+1}(c)) \geq C_\alpha$ and $\beta(\phi_n(c),\phi_{n+1}(c)) \leq C_\beta$ for all $n \in \mathbb{N}$, then $\alpha(\phi_n(c),\phi(c)) \geq C_\alpha$ and $\beta(\phi_n(c),\phi(c)) \leq C_\beta$ for all $n \in \mathbb{N}$;
(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

If in Theorem 3.7 we take, $\alpha(\phi,\xi) = 1$ and $\beta(\phi(c),\xi(c)) = r$ where $0 \leq r < 1$ for all $\phi,\xi \in E$, then we derive following result.

Corollary 3.8. *Let $c \in I$, $T : E_0 \to E$ be a nonself-mapping and $\Theta \in \Delta_\Theta$ be such that, for all $\phi,\xi \in E_0$ with $\|T\phi - T\xi\|_E > 0$, we have*

$$\Theta(\|T\phi - T\xi\|_E) \leq \left[\Theta\Big(\max\Big\{\|\phi-\xi\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E\Big\}\Big)\right]^r,$$

where $0 \leq r < 1$. Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

4. Suzuki type theorems

In this section we give a result of existence of a PPF dependent fixed point in $\mathcal{R}_c^0$ by a condition of Suzuki type [25].

Theorem 4.1. *Let $c \in I$, $T : E_0 \to E$ be a nonself-mapping, $\Theta \in \Delta_\Theta$ be such that*

(i) T is a c-$C_{\alpha\beta}$-admissible mapping;
(ii) for all $\phi,\xi \in E_0$ with $\|T\phi - T\xi\|_E > 0$ and $\frac{1}{2}\|\phi(c) - T\phi\|_E \leq \|\phi-\xi\|_{E_0}$, holds

$$\Big[\Theta(\|T\phi - T\xi\|_E)\Big]^{\alpha(\phi(c),\xi(c))} \leq \Big[\Theta(\|\phi-\xi\|_{E_0})\Big]^{\beta(\phi(c),\xi(c))};$$

(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq C_\alpha$ and $\beta(\phi_n(c), \phi_{n+1}(c)) \leq C_\beta$ for all $n \in \mathbb{N}$, then $\alpha(\phi_n(c), \phi(c)) \geq C_\alpha$ and $\beta(\phi_n(c), \phi(c)) \leq C_\beta$ for all $n \in \mathbb{N}$;

(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

Proof. Assume that T does not have a PPF dependent fixed point in $\mathcal{R}_c^0$. By Proposition 2.8, conditions (i) and (iv) ensure that there exists $\phi_0 \in \mathcal{R}_c^0$ such that $\alpha(\phi_0(c), T\phi_0) \geq C_\alpha$ and $\beta(\phi_0(c), T\phi_0) \leq C_\beta$. Let $\phi_0 \in \mathcal{R}_c^0$ be such an element. Since $T\phi_0 \in E$, we may consider the element $\phi_1 = H[T\phi_0]$ from the constant Razumikhin class $\mathcal{R}_c^0$; this, by definition, means $\phi_1(t) = T\phi_0$, for all $t \in I$ and hence $\phi_1(c) = T\phi_0$. Further, since $T\phi_1 \in E$, we may consider the element $\phi_2 = H[T\phi_1]$ from the constant Razumikhin class $\mathcal{R}_c^0$; this, by definition, means $\phi_2(t) = T\phi_1$, for all $t \in I$ and hence $\phi_2(c) = T\phi_1$. The process may continue indefinitely; it gives us a sequence $\{\phi_n\}$ in the constant Razumikhin class $\mathcal{R}_c^0$, with

$$(\text{ for all } n \in \mathbb{N}_1): \ \phi_n(t) = T\phi_{n-1}, \quad \text{for all } t \in I; \quad \text{hence,} \quad \phi_n(c) = T\phi_{n-1}. \tag{4.1}$$

Since $\phi_{n-1} - \phi_n \in \mathcal{R}_c^0$ for all $n \in \mathbb{N}_1$, it follows that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}_1$. Since T is c-$C_{\alpha\beta}$-admissible and

$$\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq C_\alpha \quad \text{and} \quad \beta(\phi_0(c), \phi_1(c)) = \beta(\phi_0(c), T\phi_0) \leq C_\beta,$$

then

$$\alpha(\phi_1(c), \phi_2(c)) = \alpha(T\phi_0, T\phi_1) \geq C_\alpha \quad \text{and} \quad \beta(\phi_1(c), \phi_2(c)) = \beta(T\phi_0, T\phi_1) \leq C_\beta.$$

Again since, T is c-$C_{\alpha\beta}$-admissible, then $\alpha(\phi_2(c), \phi_3(c)) \geq C_\alpha$ and $\beta(\phi_2(c), \phi_3(c)) \leq C_\beta$. By continuing this process, we have

$$\alpha(\phi_{n-1}(c), \phi_n(c)) \geq C_\alpha \quad \text{and} \quad \beta(\phi_{n-1}(c), \phi_n(c)) \leq C_\beta \quad \text{for all } n \in \mathbb{N}_1.$$

By the imposed condition about our nonself-mapping T, a relation like $T\phi_h = \phi_h(c) = T\phi_{h-1}$, for some $h \in \mathbb{N}_1$ is impossible; so that, we must have $T\phi_n \neq T\phi_{n+1}$ and hence $\phi_n \neq \phi_{n+1}$ for all $n \in \mathbb{N}$. Now, we have

$$\frac{1}{2}\|\phi_{n-1}(c) - T\phi_{n-1}\|_E = \frac{1}{2}\|\phi_{n-1}(c) - \phi_n(c)\|_E = \frac{1}{2}\|\phi_{n-1} - \phi_n\|_{E_0} \leq \|\phi_{n-1} - \phi_n\|_{E_0}.$$

So, by (ii), we have

$$\left[\Theta(\|T\phi_{n-1} - T\phi_n\|_E))\right]^{\alpha(\phi_{n-1}(c), \phi_n(c))} \leq \left[\Theta\Big(\|\phi_{n-1} - \phi_n\|_{E_0}\Big)\right]^{\beta(\phi_{n-1}(c), \phi_n(c))}$$

and then

$$\left[\Theta(\|\phi_n(c) - \phi_{n+1}(c)\|_E))\right]^{\alpha(\phi_{n-1}(c), \phi_n(c))} \leq \left[\Theta\Big(\|\phi_{n-1} - \phi_n\|_{E_0}\Big)\right]^{\beta(\phi_{n-1}(c), \phi_n(c))},$$

which implies

$$\Theta(\|\phi_n - \phi_{n+1}\|_{E_0}) \leq \left[\Theta\big(\|\phi_{n-1} - \phi_n\|_{E_0}\big)\right]^{(\frac{C_\beta}{C_\alpha})} \tag{4.2}$$

for all $n \in \mathbb{N}_1$. Again as in proof of Proposition 3.1, we can deduce that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c^0$ and that there exists $\phi^* \in \mathcal{R}_c^0$ such that $\phi_n \to \phi^*$ as $n \to +\infty$. Also, from (iii) we have, $\alpha(\phi_n(c), \phi(c)) \geq C_\alpha$ and $\beta(\phi_n(c), \phi(c)) \leq C_\beta$ for all $n \in \mathbb{N}$.

From (4.2), we get

$$\Theta(\|\phi_n - \phi_{n+1}\|_{E_0}) \leq \left[\Theta\big(\|\phi_{n-1} - \phi_n\|_{E_0}\big)\right]^{(\frac{C_\beta}{C_\alpha})} < \Theta\big(\|\phi_{n-1} - \phi_n\|_{E_0}\big).$$

Now, since, $\Theta \in \Delta_\Theta$, we have

$$\|\phi_n - \phi_{n+1}\|_{E_0} < \|\phi_{n-1} - \phi_n\|_{E_0} \tag{4.3}$$

for all $n \in \mathbb{N}_1$. Suppose that there exists $n_0 \in \mathbb{N}_1$ such that

$$\frac{1}{2}\|\phi_{n_0}(c) - T\phi_{n_0}\|_E > \|\phi_{n_0-1} - \phi^*\|_{E_0}$$

and

$$\frac{1}{2}\|\phi_{n_0+1}(c) - T\phi_{n_0+1}\|_E > \|\phi_{n_0} - \phi^*\|_{E_0}.$$

Then, from (4.3), it follows that

$$\begin{aligned}
\|\phi_{n_0-1} - \phi_{n_0}\|_{E_0} &\leq \|\phi_{n_0-1} - \phi^*\|_{E_0} + \|\phi_{n_0} - \phi^*\|_{E_0} \\
&< \frac{1}{2}\|\phi_{n_0}(c) - T\phi_{n_0}\|_E + \frac{1}{2}\|\phi_{n_0+1}(c) - T\phi_{n_0+1}\|_E \\
&= \frac{1}{2}\|\phi_{n_0}(c) - \phi_{n_0+1}(c)\|_E + \frac{1}{2}\|\phi_{n_0+1}(c) - \phi_{n_0+2}(c)\|_E \\
&= \frac{1}{2}\|\phi_{n_0} - \phi_{n_0+1}\|_{E_0} + \frac{1}{2}\|\phi_{n_0+1} - \phi_{n_0+2}\|_{E_0} \\
&\leq \frac{1}{2}\|\phi_{n_0} - \phi_{n_0-1}\|_{E_0} + \frac{1}{2}\|\phi_{n_0} - \phi_{n_0-1}\|_{E_0} = \|\phi_{n_0} - \phi_{n_0-1}\|_{E_0},
\end{aligned}$$

which is a contradiction. Hence either

$$\frac{1}{2}\|\phi_n(c) - T\phi_n\|_E \leq \|\phi_{n-1} - \phi^*\|_{E_0}$$

or

$$\frac{1}{2}\|\phi_{n+1}(c) - T\phi_{n+1}\|_E \leq \|\phi_n - \phi^*\|_{E_0}$$

for all $n \in \mathbb{N}_1$. It is not restrictive to assume that one of these inequalities holds for all $n \in \mathbb{N}_1$, for example

$$\frac{1}{2}\|\phi_n(c) - T\phi_n\|_E \leq \|\phi_{n-1} - \phi^*\|_{E_0}.$$

Therefore, from (ii), we have

$$\Big[\Theta(\|T\phi_n - T\phi^*\|_E)\Big]^{\alpha(\phi_n(c),\phi(c))} \leq \Theta(\|\phi_n - \phi^*\|_{E_0})\Big]^{\beta(\phi_n(c),\phi(c))},$$

which implies

$$\Theta(\|T\phi_n - T\phi^*\|_E) \leq \big[\Theta(\|\phi_n - \phi^*\|_{E_0})\big]^{(\frac{C_\alpha}{C_\beta})} < \Theta(\|\phi_n - \phi^*\|_{E_0}).$$

Now, since $\Theta \in \Delta_\Theta$, we get

$$\|T\phi_n - T\phi^*\|_E < \|\phi_n - \phi^*\|_{E_0}$$

and so

$$\begin{aligned}
\|T\phi^* - \phi^*(c)\|_E &\leq \|T\phi^* - T\phi_n\|_E + \|T\phi_n - \phi^*(c)\|_E \\
&= \|T\phi^* - T\phi_n\|_E + \|\phi_{n+1}(c) - \phi^*(c)\|_{E_0} \\
&\leq \|\phi^* - \phi_n\|_{E_0} + \|\phi_{n+1} - \phi^*\|_{E_0}.
\end{aligned}$$

Taking limit as $n \to +\infty$ in the above inequality, we get $\|T\phi^* - \phi^*(c)\|_E = 0$, that is, $T\phi^* = \phi^*(c)$. By the similar method, we can deduce $T\phi^* = \phi^*(c)$ when

$$\frac{1}{2}\|\phi_{n+1}(c) - T\phi_{n+1}\|_E \leq \|\phi_n - \phi^*\|_{E_0} \quad \text{for all } n \in \mathbb{N}.$$

Hence it follows that ϕ^* is a PPF dependent fixed point of T in $\mathcal{R}_c^0$. $\square$

Corollary 4.2. *Let $c \in I$, $T : E_0 \to E$ be a nonself-mapping and $\Theta \in \Delta_\Theta$ be such that*

for all $\phi, \xi \in E_0$ with $\|T\phi - T\xi\|_E > 0$ and $\frac{1}{2}\|\phi(c) - T\phi\|_E \le \|\phi - \xi\|_{E_0}$,

$$\Theta(\|T\phi - T\xi\|_E) \le \Big[\Theta(\|\phi - \xi\|_{E_0})\Big]^r,$$

where $0 \le r < 1$. Then

(I) *T has a PPF dependent fixed point in $\mathcal{R}_c^0$;*

(II) *T has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

Proof. By taking in Theorem 4.1,

$$\alpha(\phi, \xi) = 1 \quad \text{and} \quad \beta(\phi, \xi) = r \quad \text{for all} \quad \phi, \xi \in E,$$

we can deduce that T has a PPF dependent fixed point in $\mathcal{R}_c^0$. For the uniqueness, suppose that ϕ^* and ξ^* are two PPF dependent fixed points of T in $\mathcal{R}_c$ such that $\phi^* \ne \xi^*$. So, we have

$$\frac{1}{2}\|\phi^*(c) - T\phi^*\|_E = 0 \le \|\phi^* - \xi^*\|_{E_0}$$

and hence

$$\Theta((\|T\phi^* - T\varphi^*\|_E) \le \Big[\Theta(\|\phi^* - \varphi^*\|_{E_0})\Big]^r.$$

This implies

$$\Theta(\|\varphi^* - \phi^*\|_{E_0}) = \Theta(\|\varphi^*(c) - \phi^*(c)\|_E) = \Theta(\|T\phi^* - T\varphi^*\|_E) \le \Big[\Theta(\|\phi^* - \varphi^*\|_{E_0})\Big]^r,$$

which is a contradictions. Hence $\phi^* = \varphi^*$. □

5. Some results in Banach spaces endowed with a graph

Consistent with Jachymski [15], let (X, d) be a metric space and Δ denotes the diagonal of the Cartesian product of $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [16], p. 309) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected (see for more details [6, 12, 15]).

Definition 5.1 ([15])**.** Let (X, d) be a metric space endowed with a graph G. We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is,

$$\text{for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G)$$

and T decreases weights of the edges of G in the following way:

$$\exists\, \alpha \in (0, 1) \text{ such that for all } x, y \in X, \quad (x, y) \in E(G) \Longrightarrow d(Tx, Ty) \le \alpha d(x, y).$$

Definition 5.2. Let $T : E_0 \to E$ be a nonself-mapping and $c \in I$, where E is endowed with a graph G.

(i) T is called a *graphic Θ-contraction* if there exist $\Theta \in \Delta_\Theta$ and $0 \le r < 1$ such that, for all $\phi, \xi \in E_0$ with $(\phi(c), \xi(c)) \in E(G)$ and $\|T\phi - T\xi\|_E > 0$, we have

$$\Theta(\|T\phi - T\xi\|_E) \le \Big[\Theta(\|\phi - \xi\|_{E_0})\Big]^r.$$

(ii) T is called a *weak graphic Θ-contraction* if there exist $\Theta \in \Delta_\Theta$ and $0 \le r < 1$ such that, for all $\phi, \xi \in E_0$ with $(\phi(c), \xi(c)) \in E(G)$ and $\|T\phi - T\xi\|_E > 0$, we have

$$\Theta(\|T\phi - T\xi\|_E) \le \Big[\Theta(\max\{\|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E\})\Big]^r.$$

(iii) T is called a *generalized graphic Θ-contraction* if there exists $\Theta \in \Delta_\Theta$ such that, for all $\phi, \xi \in E_0$ with $(\phi(c), \xi(c)) \in E(G)$ and $\|T\phi - T\xi\|_E > 0$, we have

$$\Theta(\|T\phi - T\xi\|_E) \le \Big[\Theta\Big(\max\Big\{\|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E, \frac{\|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}{2}\Big\}\Big)\Big]^r.$$

We have the following results.

Theorem 5.3. *Let $T : E_0 \to E$ be a nonself-mapping and $c \in I$, where E is endowed with a graph G. Suppose that the following assertions hold:*

(i) if $(\phi(c), \xi(c)) \in E(G)$, then $(T\phi, T\xi) \in E(G)$;
(ii) T is a graphic Θ-contraction;
(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $(\phi_n(c), \phi_{n+1}(c)) \in E(G)$ for all $n \in \mathbb{N}$, then $(\phi_n(c), \phi(c)) \in E(G)$ for all $n \in \mathbb{N}$;
(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $(\phi_0(c), T\phi_0) \in E(G)$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

Proof. Let $0 \le r < 1$ be the real number in the definition of graphic Θ-contraction. Define $\alpha, \beta : E \times E \to [0, +\infty)$ by,

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G) \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \beta(x, y) = r \text{ for all } x, y \in E.$$

First, we prove that T is a c-$C_{\alpha\beta}$-admissible mapping with $C_\alpha = 1$ and $C_\beta = r$. Assume that $\alpha(\phi(c), \xi(c)) \ge 1$. Then we have $(\phi(c), \xi(c)) \in E(G)$. From (i), we have $(T\phi, T\xi) \in E(G)$, that is, $\alpha(T\phi, T\xi) \ge 1$. Obviously, $\beta(T\phi, T\xi) \le r$ and hence T is a c-$C_{\alpha\beta}$-admissible mapping. From (iv), there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \ge 1$. Let $\{\phi_n\}$ be a sequence in E_0 such that $\alpha(\phi_n(c), \phi_{n+1}(c)) \ge 1$ with $\phi_n \to \phi$ as $n \to +\infty$, then $(\phi_n(c), \phi_{n+1}(c)) \in E(G)$ for all $n \in \mathbb{N}$. Thus, from (iii), we get $(\phi_n(c), \phi(c)) \in E(G)$ for all $n \in \mathbb{N}$, that is, $\alpha(\phi_n(c), \phi(c)) \ge 1$ for all $n \in \mathbb{N}$.

Now, if $(\phi(c), \xi(c)) \in E(G)$, then $\alpha(\phi(c), \xi(c)) = 1$. Hence, from definition of graphic Θ-contraction, we have

$$\Big[\Theta(\|T\phi - T\xi\|_E)\Big]^{\alpha(\phi(c), \xi(c))} \le \Big[\Theta(\|\phi - \xi\|_{E_0})\Big]^{\beta(\phi(c), \xi(c))}.$$

Otherwise, if $\alpha(\phi(c), \xi(c)) = 0$, then

$$\Big[\Theta(\|T\phi - T\xi\|_E)\Big]^{\alpha(\phi(c), \xi(c))} = 1 \le \Big[\Theta(\|\phi - \xi\|_{E_0})\Big]^{\beta(\phi(c), \xi(c))}.$$

Therefore, for all $\phi, \xi \in E_0$, we have

$$\Big[\Theta(\|T\phi - T\xi\|_E)\Big]^{\alpha(\phi(c),\xi(c))} \leq \Big[\Theta(\|\phi - \xi\|_{E_0})\Big]^{\beta(\phi(c),\xi(c))}$$

and so all conditions of Theorem 3.2 hold and T has a PPF dependent fixed point. □

Similarly, we can prove the following theorem.

Theorem 5.4. *Let $T : E_0 \to E$ be a nonself-mapping and $c \in I$, where E is endowed with a graph G. Suppose that the following assertions hold:*

(i) if $(\phi(c), \xi(c)) \in E(G)$, then $(T\phi, T\xi) \in E(G)$;

(ii) T is a weak graphic Θ-contraction or a generalized graphic Θ-contraction, such that Θ is continuous;

(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $(\phi_n(c), \phi_{n+1}(c)) \in E(G)$ for all $n \in \mathbb{N}$, then $(\phi_n(c), \phi(c)) \in E(G)$ for all $n \in \mathbb{N}$;

(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $(\phi_0(c), T\phi_0) \in E(G)$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

6. Some results in Banach spaces endowed with a partially ordered

The study of existence of fixed points in partially ordered sets has been established by Ran and Reurings [20] with applications to matrix equations. Agarwal, et al. [2], Ciric et al. [7] and Hussain et al. [12, 14] obtained some new fixed point results for nonlinear contractions in partially ordered Banach and metric spaces with some applications. In this section, as an application of our results we derive some new PPF dependent fixed and coincidence point results whenever the range space is endowed with a partial order.

Definition 6.1 ([13])**.** Let $c \in I$, $T : E_0 \to E$ and E endowed with a partial order $\preceq$. We say that T is a c-increasing non-self mapping if for $\phi, \xi \in E_0$ with $\phi(c) \preceq \xi(c)$ we have $T\phi \preceq T\xi$.

Definition 6.2. Let $T : E_0 \to E$ be a nonself-mapping and $c \in I$, where E is endowed with a partially ordered $\preceq$.

(i) T is called an *ordered Θ-contraction* if there exist $\Theta \in \Delta_\Theta$ and $0 \leq r < 1$ such that, for all $\phi, \xi \in E_0$ with $\phi(c) \preceq \xi(c)$ and $\|T\phi - T\xi\|_E > 0$, we have

$$\Theta(\|T\phi - T\xi\|_E) \leq \Big[\Theta(\|\phi - \xi\|_{E_0})\Big]^r.$$

(ii) T is called an *weak ordered Θ-contraction* if there exist $\Theta \in \Delta_\Theta$ and $0 \leq r < 1$ such that, for all $\phi, \xi \in E_0$ with $\phi(c) \preceq \xi(c)$ and $\|T\phi - T\xi\|_E > 0$, we have

$$\Theta(\|T\phi - T\xi\|_E) \leq \Big[\Theta(\max\{\|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E\})\Big]^r.$$

(iii) T is an *generalized ordered Θ-contraction* if there exist $\Theta \in \Delta_\Theta$ and $0 \leq r < 1$ such that, for all $\phi, \xi \in E_0$ with $\phi(c) \preceq \xi(c)$ and $\|T\phi - T\xi\|_E > 0$, we have

$$\Theta(\|T\phi - T\xi\|_E) \leq \Big[\Theta\Big(\max\Big\{\|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E, \frac{\|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}{2}\Big\}\Big)\Big]^r.$$

Theorem 6.3. *Let $T : E_0 \to E$ be a nonself-mapping and $c \in I$, where E is endowed with a partially ordered $\preceq$. Suppose that the following conditions hold:*

(i) T is c-increasing;

(ii) T is an ordered Θ-contraction such that Θ is continuous;

(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\phi_n(c) \preceq \phi_{n+1}(c)$ for all $n \in \mathbb{N}$, then $\phi_n(c) \preceq \phi(c)$ for all $n \in \mathbb{N}$;

(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \preceq T\phi_0$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

Proof. Let $0 \leq r < 1$ be the real number in the definition of ordered Θ-contraction. Define $\alpha, \beta : E \times E \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \beta(x, y) = r \text{ for all } x, y \in X.$$

First, we prove that T is a c-$C_{\alpha\beta}$-admissible mapping with $C_\alpha = 1$ and $C_\beta = r$. Assume that $\alpha(\phi(c), \xi(c)) \geq 1$. Then we have $\phi(c) \preceq \xi(c)$. Since T is c-increasing, we get $T\phi \preceq T\xi$, that is, $\alpha(T\phi, T\xi) \geq 1$. Obviously, $\beta(T\phi, T\xi) \leq r$ and hence T is a c-$C_{\alpha\beta}$-admissible mapping. From (iv), there exists $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \preceq T\phi_0$, that is, $\alpha(\phi_0(c), T\phi_0) \geq 1$. Let $\{\phi_n\}$ be a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for all $n \in \mathbb{N}$. Then $\phi_n(c) \preceq \phi_{n+1}(c)$ for all $n \in \mathbb{N}$. Thus, from (iii), we get $\phi_n(c) \preceq \phi(c)$ for all $n \in \mathbb{N}$, that is, $\alpha(\phi_n(c), \phi(c)) > 0$ for all $n \in \mathbb{N}$. Therefore, all the conditions of Theorem 3.2 hold and T has a PPF dependent fixed point. □

Similarly, we can prove the following:

Theorem 6.4. *Let $T : E_0 \to E$ be a nonself-mapping and $c \in I$, where E is endowed with a partially ordered $\preceq$. Suppose that the following conditions hold:*

(i) T is c-increasing;
(ii) T is an weak ordered Θ-contraction or a generalized ordered Θ-contraction, such that Θ is continuous;
(iii) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to +\infty$ and $\phi_n(c) \preceq \phi_{n+1}(c)$ for all $n \in \mathbb{N}$, then $\phi_n(c) \preceq \phi(c)$ for all $n \in \mathbb{N}$;
(iv) there exists $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \preceq T\phi_0$.

Then T has a PPF dependent fixed point in $\mathcal{R}_c^0$.

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