BEYOND THE CONSTRUCTION OF OPTIMAL SWITCHING SURFACES FOR AUTONOMOUS HYBRID SYSTEMS

Mauro Boccadoro [*] **Magnus Egerstedt** [**]
Paolo Valigi [*] **Yorai Wardi** [**]

[*] {`boccadoro,valigi`}`@diei.unipg.it`
Dipartimento di Ingegneria Elettronica e dell'Informazione
Università di Perugia
06125, Perugia – Italy
[**] {`magnus,ywardi`}`@ece.gatech.edu`
Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332

Abstract: In this paper we report of a technique to design optimal feedback control laws for hybrid systems with autonomous (continuous) modes. Existing techniques design the optimal switching surfaces based on a singular sample evolution of the system; hence providing a solution dependent on the initial conditions. On the other hand, the optimal switching times can be found, providing an an "open loop" control to the system, but those also are dependent on the initial conditions. The technique presented relies on a variational approach, giving the derivative of the switching times with respect to the initial conditions, thus providing a tool to design programs/algorithms generating switching surfaces which are optimal for any possible execution of the system.

Keywords: Hybrid Systems, Switching Surfaces, Optimal Control, Variational Methods

1. INTRODUCTION

Consider a switched system with autonomous continuous dynamics,

$$\dot{x}(t) = f_{q(t)}(x(t)), \tag{1}$$
$$q^+(t) = s(x(t), q(t)). \tag{2}$$

where (1) describes the continuous dynamics of the state variable $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and (2) describes the discrete event dynamics of the system. Given an initial condition $x_0 := x(t_0)$, the switching law (2) determines the switching instants t_i, $i = 1, 2, \ldots$, and thus the intervals where a certain modal function is active, as well as the initial condition for the o.d.e. which defines the evolution under the next mode. The discrete variable q is piecewise constant in time and belongs to a finite or countable set Q, hence, it can be expressed in terms of the index i as $q(i)$. In terms of such index the dynamics of a switched system is:

$$\dot{x}(t) = f_i(x(t)), \quad t \in (t_{i-1}, t_i] \tag{3}$$
$$i^+ = s(x(t), i, t). \tag{4}$$

with the understanding that $f_i := f_{q(i)}$, for a given map $q(i)$, i.e., in this case (4) only expresses the occurrence of the i^{th} switch, the specification of the next active mode being given by the map $q(i)$.

Since the continuous modes are autonomous, the evolution of the system is determined by the active modes, according to (4). When the function s does not depend by the (continuous) state variable x, the switching instants are determined as exogenous inputs, and the system is controlled in open loop (timing control); when s is dependent only on the state variables, the switching law is given in a feedback form, and it may be defined by switching surfaces in the state space.

To formulate the problem we are interested with, consider a simple execution of (3,4) with only one switch, starting at $x(t_0) = x_0$ with mode 1, switching to mode 2 at time t_1, an exogenous switch, and terminating either at a *fixed final time* t_2 or in correspondence of a *terminal manifold* defined by a function $g(x)$, so that t_2 satisfies $g(x(t_2)) = 0$. For ease of reference, denote such two sets of possible executions by χ_t and χ_g, respectively.

To fix notation, let the explicit representation of the evolution determined by mode i be given by $x(t) = \varphi_i(t, s, x(s))$, hence,

$$x(t) = \begin{cases} \varphi_1(t, t_0, x_0) & t \in [t_0, t_1] \\ \varphi_2(t, t_1, x(t_1)) & t \in (t_1, t_2] \end{cases} \tag{5}$$

Also, let $x_i := x(t_i)$, and $R := f_1(x_1) - f_2(x_1)$. In this paper the following conventions will be used: 1) vectors are column vectors; 2) the derivative of a scalar, e.g. L, w.r.t. a vector x is a row vector:

$$L_x := \frac{dL}{dx} = \left[\frac{\partial L}{\partial x_1}, \ldots, \frac{\partial L}{\partial x_n}\right]. \tag{6}$$

(hence L_x^T is a column vector). The Hessian matrix is denoted by L_{xx}. If f is a (column) vector, function of the vector x i.e.,

$$f = [f^{(1)}(x), \ldots, f^{(n)}(x)]^T$$

then

$$f_x := \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f^{(1)}}{\partial x_1} & \cdots & \frac{\partial f^{(1)}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^{(n)}}{\partial x_1} & \cdots & \frac{\partial f^{(n)}}{\partial x_n} \end{bmatrix}$$

According to this convention, for the scalars c, t and the vectors x, y, z, the usual chain rule applies to $c(x(t))$ and $c(x(y))$, i.e. $\frac{dc}{dt} = c_x \dot{x}$, $c_y = c_x x_y$ ($\dot{v}$ stays for $\frac{dv}{dt}$); also:

$$\frac{dx^T y}{dz} = y^T x_z + x^T y_z, \tag{7}$$

1.1 Problem formulation

When the optimal control problem to minimize a cost function

$$J = \int_{t_0}^{t_2} L(x(t))dt \tag{8}$$

is formulated, for some continuously differentiable function L, and such that L_{xx} is symmetric, then it is known that when $t_1 = t_1^*$, a (locally) optimal switching time, it satisfies the following condition, see e.g. (Egerstedt *et al.*, 2003):

$$c(t_1^*) := p^T(t_1^*)R(x_1^*) = 0 \tag{9}$$

where $p^T(t)$, for $t \in [t_1^*, t_2]$ is given by:

$$p^T(t) = \int_t^{t_2} L_x(x(s))\Phi_2(s,t)ds + p^T(t_2)\Phi_2(t_2,t) \tag{10}$$

with Φ_i the transition matrix of the linearized time-varying system $\dot{z}(t) = \frac{\partial f_i(x(t))}{\partial x} z(t)$, and $p^T(t_2) = 0$ for fixed final time and $p^T(t_2) = -L(x_2)g_x(x_2)/\mathcal{L}_2$, for an evolution ending at a terminal manifold, where $\mathcal{L}_2 := g_x(x_2)f(x_2)$, the Lie derivative of g along f_2 evaluated at x_2.

Assuming to start from a perturbed initial condition $\tilde{x}_0 = x_0 + \delta x_0$; it is possible to use the information of optimality of t_1^*, as a switching time, to determine $\tilde{t}_1^*$; in other words: what is the dependence of the optimal switching time on the initial conditions?

This problem is motivated by the determination of optimal switching surfaces, which tend to solve optimal control problems for autonomous system via the synthesis of feedback laws, which may be pursued for specifications of stability or optimal control. Relevant application of such technique may arise in many areas such as behavior based robotics (Arkin, 1998), or manufacturing systems (Khmelnitsky and Caramanis, 1998) to cite a few.

Computational methods exist and are based on the optimization of parametrized switching surfaces (Boccadoro *et al.*, 2005). However, the choice of the optimal values for such parameters depend on the particular trajectory chosen to run an optimization program, and thus, fundamentally, on the initial conditions (remind that the we are considering a system with no continuous inputs).

An interesting reference for this type of approach is (Giua *et al.*, 2001), which addressed a timing optimization problem, and discovered the special structure of the solution for linear quadratic problems. Indeed, in that case it is possible to identify homogeneous regions in the continuous state space, whose boundaries, when reached, determine the optimal switches, thus providing a feedback solution to a problem which is formulated in terms of an open loop strategy.

Here we explicitly investigate the relation existing between optimal switching times and initial conditions, studying how the condition of optimality (9) that switching times must satisfy, vary in dependence of the initial conditions.

This paper reports the work in progress toward this goal, which is still being pursued.

2. OPTIMAL SWITCHING TIMES V/S INITIAL CONDITIONS

It is well known that, under mild assumptions, executions of switched systems are continuous w.r.t. the initial conditions (Broucke and Arapostathis, 2002). If we assume that also the dependence of c on t_1^* as well as t_1^* on x_0 is such, we may characterize function t_1^* by deriving (9) w.r.t. x_0 and setting this derivative to zero. In fact, if starting from $\tilde{x}_0 = x_0 + \delta x_0$, it results $\tilde{t}_1^* = t_1^* + \delta t_1^*$; then, by continuity, $0 = c(\tilde{t}_1^*) = c(t_1^*) + \frac{dc}{dx_0}\delta x_0 + o(\delta x_0)$. Hence, set $\frac{dc}{dx_0} = 0$, to satisfy optimality condition for $\tilde{t}_1^*$. As we will see this yields a formula for the variational dependence of t_1^* on x_0. To go further, the superscript * will be dropped (hence assuming that t_1, x_1 etc. are relative to optimal executions) in order to reduce the notational burden.

By (7) we have that

$$\frac{dc}{dx_0} = R^T \frac{dp(t_1)}{dx_0} + p^T(t_1)\frac{dR}{dx_0} \tag{11}$$

To calculate $\frac{dp(t_1)}{dx_0}$, account for the following result, which is readily verified:

$$\frac{d}{dx}\int_{t(x)}^{a} f(s,x)ds = \int_t^a \frac{df}{dx}(s,x)ds - f(t,t)t_x \tag{12}$$

Then, considering first the simpler case of fixed final time, by (10, 7, 12)

$$\frac{dp(t_1)}{dx_0} = \int_{t_1}^{t_2} \Phi_2^T(s,t_1)L_{xx}(x(s))\frac{dx(s)}{dx_0} + \frac{d\Phi_2^T(s,t_1)}{dt_1}L_x^T(x_s)\frac{dt_1}{dx_0}ds - L_x^T(x_1)\Phi_2(t_1,t_1)\frac{dt_1}{dx_0} \tag{13}$$

To get $\frac{dx(s)}{dx_0}$ notice that $x(t_1) = \varphi_1(t_1, t_0, x_0)$, hence $x(s) = \varphi_2(s, t_1, \varphi_1(t_1, t_0, x_0))$ for $s \in [t_1, t_2]$, thus,

$$\frac{dx(s)}{dx_0} = \frac{\partial x(s)}{\partial t_1}\frac{dt_1}{dx_0} + \frac{\partial x(s)}{\partial x_1}\frac{\partial x_1}{\partial t_1}\frac{dt_1}{dx_0} + \frac{\partial x(s)}{\partial x_1}\frac{\partial x_1}{\partial x_0} \tag{14}$$

Now, $\partial x(s)/\partial t_1 = -f_2(x(s))$ [1], $\partial x(s)/\partial x_1 = \Phi_2(s,t_1)$, $\partial x_1/\partial x_0 = \Phi_1(t_1,t_0)$, $\partial x_1/\partial t_1 = f_1(x_1)$, $\Phi_2(t_1,t_1) = I$,

$$\frac{d}{dt_1}\Phi_2(s,t_1) = -\Phi_2(s,t_1)\frac{\partial f_2(x_1)}{\partial x} \tag{15}$$

[1] For time invariant dynamics, $[\varphi(s,t+h,x) - \varphi(s,t,x)]/h = [\varphi(s-h,t,x) - \varphi(s,t,x)]/h = -f(x(s)) + o(h)$.

(to be transposed). It results:

$$\frac{d}{dx_0}p(t_1) = I_1 - I_2 + I_3 - I_4 - K \tag{16}$$

where

$$\begin{aligned}
I_1 &= \int_{t_1}^{t_2} \Phi_2^T(s,t_1)L_{xx}(x(s))\Phi_2(s,t_1)f_1(x_1)\frac{dt_1}{dx_0} \\
I_2 &= \int_{t_1}^{t_2} \Phi_2^T(s,t_1)L_{xx}(x(s))f_2(x(s))\frac{dt_1}{dx_0} \\
I_3 &= \int_{t_1}^{t_2} \Phi_2^T(s,t_1)L_{xx}(x(s))\Phi_2(s,t_1)\Phi_1(t_1,t_0)ds \\
I_4 &= \int_{t_1}^{t_2} f_{2x}^T(x_1)\Phi_2^T(s,t_1)L_x^T(x(s))\frac{dt_1}{dx_0}ds \\
K &= L_x^T(x_1)\frac{dt_1}{dx_0}
\end{aligned} \tag{17}$$

To handle these, integrate by parts I_2 (letting $\frac{dt_1}{dx_0}$), taking into account that

$$\int L_{xx}(x(s))f_2(x(s))ds = L_x^T(x(s))$$

we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \Phi_2^T(s,t_1)L_{xx}(x(s))f_2(x(s))ds = \\
&-I_4 + \Phi_2^T(s,t_1)L_x^T(x(s))\Big|_{t_1}^{t_2} = \\
&-I_4 + \Phi_2^T(t_2,t_1)L_x^T(x_2) - K
\end{aligned} \tag{18}$$

This leads to the cancellation of I_4 and K in (16).

To complete, let's compute $dR(x_1)/dx_0$. Again, notice that $x_1 = x(t_1) = x[t_1(x_0), x_0]$, hence,

$$\begin{aligned}
\frac{d}{dx_0}R(x_1) &= \frac{\partial R}{\partial x}(x_1)\left[\frac{\partial x_1}{\partial t_1}\frac{dt_1}{dx_0} + \frac{\partial x_1}{\partial x_0}\right] = \\
&\frac{\partial R}{\partial x}(x_1)\left[f_1(x_1)\frac{dt_1}{dx_0} + \Phi_1(t_1,t_0)\right]
\end{aligned} \tag{19}$$

Multiplying this by $p^T(t_1)$, (16) by R^T from the left and summing up we finally obtain:

$$\begin{aligned}
&\frac{dc(t_1)}{dx_0} = \\
&\left[R^T(Qf_1 - \Phi_2^T(t_2,t_1)L_x^T(x_2)) + p^T(t_1)R_x f_1\right]\frac{dt_1}{dx_0} \\
&+ \left[R^T Q - p^T(t_1)R_x\right]\Phi_1(t_1,t_0)
\end{aligned} \tag{20}$$

where $f_1 := f_1(x_1)$, and

$$Q := \int_{t_1}^{t_2} \Phi_2^T(s,t_1)L_{xx}(x(s))\Phi_2(s,t_1)ds \tag{21}$$

which is a kind of quadratic form co-costate. Notice that the term multiplying $\frac{dt_1}{dx_0}$ above, is a scalar. So, if we know that t_1^* is a local optimum for an evolution starting from x_0, then, assuming to start from $\tilde{x}_0 = x_0 + \delta x_0$, we simply must switch at $t_1^* + \delta t_1^* + o(\delta x_0)$. According to (20),

$$\delta t_1^* = \frac{-[R^T Q - p^T(t_1)R_x]\Phi_1(t_1,t_0)\ \delta x_0}{R^T(Qf_1 - \Phi_2^T(t_2,t_1)L_x^T(x_2)) + p^T(t_1)R_x f_1} \tag{22}$$

3. TOWARD THE CONSTRUCTION OF THE OPTIMAL SWITCHING SURFACES

To put in use Eq. (22) assume that *one* optimal switching time has been derived for a certain "sample" evolution of the system, e.g. one starting in $\hat{x}_0$. Then the optimal switching surfaces are defined by the optimal switching *states* yielded by the variation on the optimal switching times when initial conditions different than $\hat{x}_0$ are considered. However, it must be paid attention to the fact that the formula derived above works for a fixed final time: indeed for the case of evolution ending at a terminal manifold the following result holds,

Theorem 1. Consider a nominal and a perturbed execution of the set χ_g, $x(\cdot)$ and $y(\cdot)$, respectively, the first starting at x_0 and the latter starting from a point y_0 which lies on the nominal trajectory; i.e., assume that it exists an interval δt_0 such that $y_0 = \varphi_1(t_0 + \delta t_0, t_0, x_0)$. Then, the optimal switching time

$$t_1^*(y_0) = t_1^*(x_0) - \delta t_0 \tag{23}$$

for all $\delta t_0 < t_1^* - t_0$

Proof Denote by $a \to b$ a trajectory from point a to b, and let $x(t_1^*) = x_1^*$ and $x(t_2^*) = x_2^*$ where t_2^* is the terminal time if the switching time from mode 1 to mode 2 is t_1^*. If (23) did not hold then assume

$$t_1^*(y_0) = t_1^*(x_0) - \delta t_0 + \epsilon \tag{24}$$

for some ϵ (assume with no loss of generality $\epsilon > 0$). Let the nominal trajectory that switches at $t_1^*(x_0) + \epsilon$ terminate at x_2^ϵ. Denote $A = x(t_0) \to x(t_0 + \delta t_0)$, $B = x(t_0 + \delta t_0) \to x_1^*$, $C = x_1^* \to x_2^*$, $D = x(t_1^*) \to x(t_1^* + \epsilon)$ and $E = x(t_1^* + \epsilon) \to x_2^\epsilon$. According to (23) the optimal nominal trajectory is $A \cup B \cup C$ paying for this the cost $J(A) + J(B) + J(C)$. On the other hand by (24) the perturbed trajectory $B \cup C$ incurs in a greater cost than $B \cup D \cup E$, which implies that $J(C) > J(D) + J(E)$. This in turn means that if the nominal trajectory switches at $t_1^*(x_0) + \epsilon$ then it pays less than if the switch take place at t_1^* $\square$

Notice that Theorem 1 easily extends to negative δt_0, i.e., if y_0 is chosen such that the evolution starting from y_0 will reach x_0 we must *add* the time needed to reach x_0 from y_0 to the optimal (nominal) switching time.

In case of fixed terminal time the optimal switching state may vary because the perturbed trajectory described in Theorem 1 above, switching at $t_1^* - \delta t_0$, reaches the point $x(t_2)$ (of the nominal trajectory) at time instant $t_2 - \delta t_0$, thence "visits" additional states from $t_2 - \delta t_0$ to t_2 (in other words $\tilde{x}(\cdot)_{(t_2 - \delta t_0, t_2]}$ is a set of states not visited by $x(\cdot)$). Such remnants of the perturbed trajectory add further costs, so that two different trajectories, even if the starting point of one of them lies in the trajectory of the other, cannot really be properly compared, in terms of optimal switching states.

This can be actually seen: take an i.c. $y_0 = \varphi_1(t_0 + \delta t_0, t_0, x_0)$ very close to x_0, so that $\delta x_0 = f_1(x_0)\delta t_0 + o(\delta t_0)$. Multiplying (22) by such δx_0, we have that its numerator (plus higher order terms) is:

$$\begin{aligned} -[R^T Q + p^T(t_1) R_x]\Phi_1(t_1, t_0)\delta x_0 = \\ -[R^T Q f_1 - p^T(t_1) R_x f_1]\delta t_0 \end{aligned} \tag{25}$$

where $\Phi_1(t_1, t_0) f_1(x_0) = f_1(x1)$ is due to the fact that vector fields obey their variational dynamics [2] . Hence in this case

$$\delta t_1^* = \frac{-[R^T Q f_1 - p^T(t_1) R_x f_1]\ \delta t_0}{R^T(Q f_1 - \Phi_2^T(t_2, t_1) L_x^T(x_2)) + p^T(t_1) R_x f_1} \tag{26}$$

In this case condition (23) is equivalent to $\delta t_1^* = -\delta t_0$, so that to be verified, denominator and numerator should have had the same terms, opposed in sign. Here, the only term making the difference, preventing (23) to hold (as expected) is $-R^T \Phi_2^T(t_2, t_1) L_x^T(x_2)$.

According to Theorem 1, and to the above "verification", the procedure to build optimal switching surfaces should be better pursued considering evolution ending at terminal manifolds, since variations in the switching times define soundly optimal switching *states* as well. Theorem 1 also gives an hint about the set of initial conditions that should be considered to set such procedure. Indeed, it seems reasonable account only for that set of initial conditions which are transversal to the flow defined by the vector field of the initial dynamics (here f_1) which contains $\hat{x}_0$. Such set of initial condition is a surface itself and can be described by $s(x) = 0$ where s is a $\mathbb{R}$-valued function such that $s(\hat{x}_0) = 0$ and such that $s_x(x)$ is collinear with $f_1(x)$, so that s would be a kind of *potential* of the vector field. This choice is justified by Theorem 1, since the components of the variation δx_0 on some x_0 which are tangent to the flow yield no difference on the optimal switching *state*, hence giving no contribution to the construction of an optimal switching surface which is optimal for the executions determined by any possible initial condition.

4. CONCLUSION AND FUTURE WORKS

This paper presents the first steps to determine optimal switching surfaces for hybrid systems

[2] Indeed, the variational system $\dot{z}(t) = \frac{\partial f(x(t))}{\partial x} z(t)$ has the solution $z(t) = f(x(t))$, which can be seen from the chain rule $\dot{f} = f_x f$

with autonomous modes. Future work will be devoted to the derivation of an analogue formula of (22) for executions ending at a terminal manifold. This in order to pursue the program outlined above, about the investigation of the impact of transverse variations in the initial condition on the switching states.

REFERENCES

Arkin, R.C. (1998). *Behavior Based Robotics.* The MIT Press. Cambridge, MA.

Boccadoro, M., Y. Wardi, M. Egerstedt and E. Verriest (2005). Optimal control of switching surfaces in hybrid dynamical systems. *JDEDS* **15**(4), 433 – 448.

Broucke, M. and A. Arapostathis (2002). Continuous selections of trajectories of hybrid systems. *Systems and Control Letters* **47**, 149–157.

Egerstedt, M., Y. Wardi and F. Delmotte (2003). Optimal control of switching times in switched dynamical systems. In: *42nd IEEE Conference on Decision and Control (CDC '03).* Maui, Hawaii, USA.

Giua, A., C. Seatzu and C. Van Der Mee (2001). Optimal control of switched autonomous linear systems. In: 40^{th} *IEEE Conf. on Decision and Control (CDC 2001).* Orlando, FL, USA. pp. 2472–2477.

Khmelnitsky, E. and M. Caramanis (1998). One-machine n-part-type optimal setup scheduling: analytical characterization of switching surfaces. *IEEE Trans. on Automatic Control* **43**(11), 1584–1588.