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Reinforced random processes in continuous time

Pietro Muliere[a], Piercesare Secchi[b,*], Stephen G. Walker[c,1]

[a] *Istituto di Metodi Quantitativi, Università "L. Bocconi", Via Sarfatti 25, I-20136 Milano, Italy*

[b] *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20123 Milano, Italy*

[c] *Department of Mathematical Sciences, University of Bath, BA2 7AY, UK*

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Abstract

We introduce a stochastic process based on nonhomogeneous Poisson processes and urn processes which can be reinforced to produce a mixture of semi-Markov processes. By working with the notion of exchangeable blocks within the process, we present a Bayesian nonparametric framework for handling data which arises in the form of a semi-Markov process. That is, if units provide information as a semi-Markov process and units are regarded as being exchangeable then we show how to construct the sequence of predictive distributions without explicit reference to the de Finetti measure, or prior.

Keywords: Reinforced urn processes; Nonhomogeneous Poisson processes; Mixture of semi-Markov processes; Bayesian nonparametrics

1. Introduction

The idea of reinforced random walks goes back to Coppersmith and Diaconis (1986) and Pemantle (1988).

> The term random process with reinforcement is intended to delimit a class of discrete time processes of which the Pólya urn process is prototypical (Pemantle, 1988).

* Corresponding author. Fax: +39-02-23994568.
E-mail address: secchi@mate.polimi.it (P. Secchi).
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The aim of this paper is to introduce a class of *continuous time* reinforced random processes through an extensive use of Pólya urns. The reinforcement is done in such a way that the notion of *exchangeability* plays a prominent role. That is, the reinforcement can be understood as a form of Bayesian learning. This was the key to the paper of Muliere et al. (2000) who introduced discrete time reinforced urn processes, relevant to Bayesian nonparametric inference. The present paper can be thought of as a generalization of the Muliere et al. paper to continuous time. The basic building blocks of the continuous time reinforced process are a nonhomogeneous Poisson process and the notion of a continuum of Pólya urns along the time axis.

The practical relevance of our paper is that we can undertake Bayesian nonparametric inference, i.e. prediction, without explicit knowledge of the prior. That is, we can construct exchangeable information (in the form of a semi-Markov process) without necessarily being able to compute the de Finetti measure (i.e. the prior). However, via straightforward updating rules, we are able to provide an explicit form for predictive information. If individuals provide information in the form of a continuous time semi-Markov process and individuals are regarded as exchangeable then this paper provides a framework for inference. To our knowledge, little work has been done on constructing mixtures of semi-Markov processes in continuous time. Bühlmann (1963) presents a characterisation of mixtures of Lévy processes and Freedman (1963) a characterisation of mixtures of Markov chains in continuous time. See also Freedman (1996). Mixtures of semi-Markov processes have been characterized by Epifani et al. (2001) through partial exchangeability of the array of successor states and holding times.

In Section 2 we present background material by introducing a reinforced renewal process, which forms the building block for our reinforced continuous time process. Section 3 also provides some background material on product integrals. Section 4 introduces our processes and establishes the property of being a mixture of semi-Markov processes. Finally, in Section 5 we present a practical use for the process.

2. A reinforced renewal process

The aim of this section is to construct a continuous time point process useful for situations classically modeled through renewal processes but with the additional advantage of incorporating learning by past observations through reinforcing. In particular via a nonhomogeneous Poisson process we will describe a sequence of exchangeable random times whose de Finetti measure is that of the beta-Stacy process of Walker and Muliere (1997).

Let F be a distribution function on $[0,\infty)$ and assume that $\{V_n\}$ is an infinite sequence of independent random variables with values in $[0,\infty)$ and identical distribution equal to F. The point process

$$N(t) = \sup\left\{n \geqslant 0 : \sum_{i=1}^{n} V_i \leqslant t\right\}$$

defined for all $t \geqslant 0$, is called an *ordinary renewal process* and F is called the distribution function of the interarrival times of the process. When F is the exponential distribution, $N = \{N(t), t \geqslant 0\}$ is the classical Poisson process. Now consider a component subject to sequential failures and take F to be the distribution of the random times between failures which are for now assumed to be independent and identically distributed. If F is exponential, the hazard rate is constant over time; in many applications one needs to model times between failures in such a way that the probability of having a failure in the time interval $(t, t + \mathrm{d}t)$ given that there has been no failure before time t changes with t. A natural model for these situations, which generalizes the Poisson process, is given by a nonhomogeneous Poisson process which we now define. Let α be a positive measure on the Borel sets of $[0,\infty)$ such that $\alpha(\{0\}) = 0$; indicate with $0 \leqslant a_1 < a_2 < \cdots < a_n < \cdots \in [0,\infty)$ the points where α concentrates a positive mass and let α_c be the continuous part of α. Thus, for all $t > 0$,

$$\alpha_c(0,t] = \alpha(0,t] - \sum_{a_i \leqslant t} \alpha(\{a_i\}).$$

Let $\beta : [0,\infty) \to (0,\infty)$ be a positive and measurable function. With $F_{\alpha,\beta}$ we indicate a nondecreasing, right continuous function defined, for all $t > 0$, by

$$F_{\alpha,\beta}(t) = 1 - \left(\prod_{a_i \leqslant t} \left[1 - \frac{\alpha(\{a_i\})}{\alpha(\{a_i\}) + \beta(a_i)} \right] \right) \exp\left(- \int_0^t \frac{\mathrm{d}\alpha_c(v)}{\beta(v)} \right)$$

while $F_{\alpha,\beta}(t) = 0$ for $t \leqslant 0$. When

$$\left(\prod_{a_i < \infty} \left[1 - \frac{\alpha(\{a_i\})}{\alpha(\{a_i\}) + \beta(a_i)} \right] \right) \exp\left(- \int_0^\infty \frac{\mathrm{d}\alpha_c(v)}{\beta(v)} \right) = 0, \tag{1}$$

$F_{\alpha,\beta}$ is a proper distribution function. Let α and β satisfy condition (1); an ordinary renewal process with distribution $F_{\alpha,\beta}$ for the interarrival times is called a *nonhomogeneous Poisson process with parameters* (α, β).

A nonhomogeneous Poisson process is allowed to have a hazard rate changing over time; moreover, it might have a countable set of discontinuities. Note that when $\alpha(0,t] = \lambda t$, for $\lambda > 0$ and $t > 0$, and $\beta(t) = 1$ for all $t \geqslant 0$, we obtain an ordinary Poisson process with parameter λ.

For an ordinary renewal process, the conditional distribution of the $(n+1)$th interarrival time V_{n+1}, given the previous interarrival times $V_1, \ldots, V_n$, is equal to the distribution F for all $n \geqslant 1$. Therefore, we have no means to incorporate learning from past observations into the model. The aim of this section is to construct a model which is suited for applied situations where one would otherwise naturally consider a nonhomogeneous Poisson process with parameters (α, β), but which incorporates, through reinforcement of the parameters (α, β), information produced along time by past observations.

We now give the constructive definition of a reinforced renewal process. Given α and β satisfying (1), let $\tau_1 \in [0,\infty)$ have distribution $F_{\alpha,\beta}$ and, for $n \geqslant 1$, define recursively the conditional distribution of τ_{n+1}, given $\tau_1,\dots,\tau_n$, to be equal to F_{α_n,β_n} where, for all $t>0$,

$$\alpha_n(0,t]=\alpha(0,t]+\sum_{i=1}^{n} I[\tau_i\leqslant t] \quad \text{and} \quad \beta_n(t)=\beta(t)+\sum_{i=1}^{n} I[\tau_i>t]. \tag{2}$$

The countably infinite sequence of times $\{\tau_n\}$, or equivalently the point process

$$N(t)=\sup\left\{n\geqslant 0:\sum_{i=1}^{n}\tau_i\leqslant t\right\},\qquad t\geqslant 0, \tag{3}$$

will be called a *reinforced renewal process* with parameters (α,β). Observe that if α and β satisfy condition (1), then with probability one α_n and β_n satisfy the same condition for all $n\geqslant 1$; hence the process $\{\tau_n\}$ is well defined.

In order to understand how learning from past observations is incorporated into the model through reinforcement, let us recall the definition of a Pólya urn. Let U be an urn with initial composition $C=(c_0,\dots,c_k)$: that is, U contains $c_0\geqslant 0$ balls of color 0, $c_1\geqslant 0$ balls of color $1,\dots,c_k\geqslant 0$ balls of color k. We assume that $\sum_{j=0}^{k} c_j>0$, but we do not require the quantities $c_0,\dots,c_k$ to be integers. The urn U is called a Pólya urn if its composition changes, when the urn is sampled, according to the following rule: every ball sampled from the urn is replaced into it along with another of the same color. This obviously *reinforces* the probability that a ball of same color as the one currently sampled will be sampled in the future. As is well known, the infinite sequence of colors produced by a Pólya urn with initial composition $C=(c_0,\dots,c_k)$ is exchangeable with de Finetti measure equal to a Dirichlet distribution with parameters $(c_0,\dots,c_k)$. Going back to the process defined in (3), imagine that to each infinitesimal time interval $(t,t+\mathrm{d}t)$ is associated a Pólya urn with initial composition of $\mathrm{d}\alpha(t)$ balls of color 0 and $\beta(t)$ balls of color 1. The first renewal time τ_1 is generated by sequentially sampling the urns associated to each infinitesimal time interval starting from $t=0$. If the ball sampled from the urn associated to $(t,t+\mathrm{d}t)$ is of color 1 we move to the next infinitesimal time interval and we sample the associated urn; if the color of the sampled ball is 0, we set $\tau_1=t$ and we proceed to the generation of τ_2 by sequentially sampling the urns associated to each infinitesimal time interval starting again from $t=0$, and so on. This idea becomes clearer, and more natural, if one were to represent the distribution $F_{\alpha,\beta}$ of the initial observation τ_1 by means of a product integral: we will expand on this point in the next section.

Lemma 1. $P[\tau_n<\infty]=1$, *for all* $n\geqslant 1$.

Proof. $P[\tau_1<\infty]=1$ since α and β satisfy (1). Furthermore, for all $n\geqslant 1$, given $\tau_1,\dots,\tau_n$, α_n and β_n satisfy (1) with probability one; thus

$$P[\tau_{n+1}<\infty\mid\tau_1,\dots,\tau_n]=1$$

with probability one, and therefore $P[\tau_{n+1}<\infty]=1$. $\square$

The beta-Stacy law for random probability distributions has been introduced and studied by Walker and Muliere (1997) and is widely used in Bayesian nonparametric survival studies. The next result shows that it is intimately linked with the reinforced renewal process just defined.

Theorem 2. *The sequence* $\{\tau_n\}$ *is exchangeable and its de Finetti measure is a beta-Stacy process with parameters* (α, β).

Proof. Let $\{Y_n\}$ be an infinite exchangeable sequence of random variables with values in $[0, \infty)$ and de Finetti measure equal to a beta-Stacy process with parameters (α, β). Then, for all $t \in [0, \infty)$,

$$P[Y_1 \leqslant t] = F_{\alpha,\beta}(t)$$

and

$$P[Y_{n+1} \leqslant t | Y_1, \ldots, Y_n] = F_{\alpha_n,\beta_n}(t)$$

for $n \geqslant 1$, with probability one where α_n and β_n are constructed as in (2) with the Y's replacing the τ's. For details, see Walker and Muliere (1997). Since all the predictive distributions of the sequences $\{Y_n\}$ and $\{\tau_n\}$ are the same, for all $n \geqslant 1$ and $t_1, \ldots, t_n \in [0, \infty)$,

$$P[Y_1 \leqslant t_1, \ldots, Y_n \leqslant t_n] = P[\tau_1 \leqslant t_1, \ldots, \tau_n \leqslant t_n].$$

Therefore, by applying de Finetti's Representation Theorem, the sequence $\{\tau_n\}$ is exchangeable and its unique de Finetti measure is a beta-Stacy process with parameters (α, β). □

Remark 3. We point out that in the case of the beta-Stacy process, if we let α be a finite, positive measure and, for all $t > 0$, we constrain $\beta(t) = \alpha(t, \infty)$, then we have a Dirichlet process with parameter α. Consequently, the process $\{\tau_n\}$ can be thought as being generated by the generalized Pólya urn scheme of Blackwell and MacQueen (1973) with parameter α.

We conclude the section by proving a lemma which shows that the point process $\{\tau_n\}$ is nonexplosive.

Lemma 4. $P[\sum_{n=1}^{\infty} \tau_n < \infty] = 0$.

Proof. Let G be a beta-Stacy process with parameters (α, β) and assume that, given G, the random variables $\tau_1, \tau_2, \ldots$ are independent and identically distributed with distribution G. Since $\alpha(\{0\}) = 0$, $P[G(0) = 0] = 1$; therefore

$$P\left[\int_0^{\infty} t \, \mathrm{d}G(t) > 0\right] = 1.$$

Hence, the law of large numbers implies that

$$P\left[\sum_{n=1}^{\infty} \tau_n = \infty \,|\, G\right] = 1$$

on a set of probability one. Thus $P[\sum_{n=1}^{\infty} \tau_n < \infty] = 0$. □

The process $\{\tau_n\}$ is of interest for Bayesian nonparametric inference in applied situations where the reference model is a renewal process. In Section 4, we will use reinforced renewal processes as stepping stones for the construction of more general continuous time reinforced processes with values in a finite state space. Heuristically, we associate with each state a different clock whose ringing marks the time when the process moves to a new state; in fact, the sequence of successive ringings of a clock will be modeled by a reinforced renewal process. Movements between states are controlled by Pólya urns. There are a number of ways of exploiting this idea for constructing reinforced continuous time processes on a finite state space with special properties; such as a mixture of semi-Markov processes. Differences may arise as a result of when we decide to start the learning process i.e. re-set the clock. For example, in Section 4 we consider re-setting clocks to zero and updating their parameters every time the process enters a new state. The distribution of the random time that will be spent by the process in a particular state on the second visit is updated as a consequence of the time spent in it during the first visit, and so on.

3. The product integral representation

Although not necessary, it is useful and evocative to represent reinforced renewal processes by means of product integrals (for a survey on product integrals see Gill and Johansen, 1990).

Let us begin with the representation of the distribution function $F_{\alpha,\beta}$ defining a nonhomogeneous Poisson process with parameters (α,β). For all $t>0$, set

$$A(t)=\int_0^t \frac{\mathrm{d}F_{\alpha,\beta}(v)}{F_{\alpha,\beta}(v-)}=\int_0^t \frac{\mathrm{d}\alpha_c(v)}{\beta(v)}+\sum_{a_i\leqslant t}\frac{\alpha(\{a_i\})}{\beta(a_i)+\alpha(\{a_i\})}$$

to be the cumulative hazard rate of $F_{\alpha,\beta}$. Then, for $t>0$, the hazard rate of $F_{\alpha,\beta}$ becomes

$$\mathrm{d}A(t)=\frac{\mathrm{d}\alpha(t)}{\beta(t)+\alpha(\{t\})}$$

and we can write

$$F_{\alpha,\beta}(t)=1-\mathbf{\Pi}_{v\leqslant t}\{1-\mathrm{d}A(v)\}=1-\mathbf{\Pi}_{v\leqslant t}\left\{1-\frac{\mathrm{d}\alpha(v)}{\beta(v)+\alpha(\{v\})}\right\}, \tag{4}$$

where $\mathbf{\Pi}$ is the symbol used for the product integral. The expression

$$\mathbf{\Pi}_{v\leqslant t}\left\{1-\frac{\mathrm{d}\alpha(v)}{\beta(v)+\alpha(\{v\})}\right\}$$

appearing in (4), is interpreted as the product over many small time intervals $(v,v+\mathrm{d}v)$ of the probability

$$1-\frac{\mathrm{d}\alpha(v)}{\beta(v)+\alpha(\{v\})}; \tag{5}$$

this in turn, being equal to 1 minus the hazard rate of $F_{\alpha,\beta}$ computed in v, can be interpreted as the probability of an observation from $F_{\alpha,\beta}$ greater than or equal to $v + \mathrm{d}v$ given that it is greater than or equal to v. It seems then natural to associate to each time interval $(v, v + \mathrm{d}v)$ an urn with $\mathrm{d}\alpha(v)$ balls of color 0 and $\beta(v)$ balls of color 1 and to imagine that an observation from $F_{\alpha,\beta}$ is produced by sequentially sampling the urns associated to each time interval $(v, v + \mathrm{d}v)$ starting from time 0 until a ball of color 0 is produced. The need to take into account information provided by past observations moves us to qualify these urns as Pólya urns; i.e. to reinforce their composition according to the samples they produced in the past. The definition of a reinforced renewal process $\{\tau_n\}$ follows consequently: if $\tau_1 \in [0, \infty)$ has distribution $F_{\alpha,\beta}$, for $n = 1, 2, \ldots$ and $t > 0$ set

$$P[\tau_{n+1} > t \mid \tau_1, \ldots, \tau_n] = \mathbf{\Pi}_{v \leqslant t} \left\{ 1 - \frac{\mathrm{d}\alpha_n(v)}{\beta_n(v) + \alpha_n(\{v\})} \right\}$$

with α_n and β_n defined in (2).

4. A continuous time reinforced urn process

The aim of this section is to construct a reinforced, continuous time process $\{X_t, t \geqslant 0\}$ on a finite state space: the process will be shown to be a special mixture of semi-Markov processes. Following Epifani et al. (2001), we will proceed by first defining for each state, the sequence of holding times in the state for the process; these will be independent reinforced renewal processes constructed as in the previous section. Then, for every element of the state space, conditionally on its sequence of holding times, we will define a sequence of successor states for the process; its law will be generated by means of Pólya urns.

Let $\mathscr{L} = \{0, \ldots, k\}$ be a finite set of states equipped with the sigma-field of all its subsets; $\mathscr{L}$ will be the state space for the process $\{X_t, t \geqslant 0\}$. For each $i \in \mathscr{L}$, let $\{\tau_n^i\}$ be a reinforced renewal process generated, as in the previous section by a positive measure α^i defined on the Borel sets of $[0, \infty)$ such that $\alpha^i(\{0\}) = 0$, and a measurable function $\beta^i : [0, \infty) \to (0, \infty)$; that is, $\{\tau_n^i\}$ is an exchangeable sequence of random variables with values in $[0, \infty)$, equipped with the Borel sigma-field, and de Finetti measure equal to a beta-Stacy process with parameters α^i and β^i. The sequence $\{\tau_n^i\}$ will be that of the successive holding times in state i for the process $\{X_t, t \geqslant 0\}$ whenever it visits state i. We assume that the sequences $\{\tau_n^0\}, \{\tau_n^1\}, \ldots, \{\tau_n^k\}$ are independent. For each $i \in \mathscr{L}$ and $t \geqslant 0$, let $U^i(t)$ be a Pólya urn with initial composition $C^i(t) = (c_0^i(t), c_1^i(t), \ldots, c_k^i(t))$. Set $\phi^i : [0, \infty) \to [0, \infty)$. Conditionally on the sequence of holding times $\{\tau_n^i\}$, the collection of Pólya urns $\{U^i(t), t \geqslant 0\}$ and the function ϕ^i generate the law of the sequence $\{s_n^i\}$ of states visited after state i. We call this sequence the sequence of successor states; it is recursively constructed as follows. Given τ_1^i, let s_1^i be the color produced by Pólya urn $U^i(\phi^i(\tau_1^i))$. For $n \geqslant 1$, given $\tau_1^i, s_1^i, \ldots, \tau_n^i, s_n^i, \tau_{n+1}^i$, let s_{n+1} be the color produced by Pólya urn $U^i(\phi^i(\tau_{n+1}^i))$.

Lemma 5. *For every $i \in \mathscr{L}$, the sequence $\{(\tau_n^i, s_n^i)\}$ is exchangeable. Moreover, the sequences $\{(\tau_n^0, s_n^0)\}, \{(\tau_n^1, s_n^1)\}, \ldots, \{(\tau_n^k, s_n^k)\}$ are independent.*

Proof. Fix $i \in \mathscr{L}$ and let $B_1, \ldots, B_n$ be Borel subsets of $[0, \infty)$ and $l_1, \ldots, l_n \in \mathscr{L}$. We now compute

$$P[(\tau_1^i, s_1^i) \in B_1 \times \{l_1\}, (\tau_2^i, s_2^i) \in B_2 \times \{l_2\}, \ldots, (\tau_n^i, s_n^i) \in B_n \times \{l_n\}]$$

$$= \int_{B_1 \times \cdots \times B_n} P[s_1^i = l_1, \ldots, s_n^i = l_n \mid \tau_1^i = t_1, \ldots, \tau_n^i = t_n] \, \mathrm{d}P_{\tau_1^i, \ldots, \tau_n^i}(t_1, \ldots, t_n)$$

where $P_{\tau_1^i, \ldots, \tau_n^i}$ indicates the probability distribution induced on $[0, \infty)^n$ by the random vector $(\tau_1^i, \ldots, \tau_n^i)$. For all $(t_1, \ldots, t_n) \in [0, \infty)^n$, let $d_1, \ldots, d_{n'}$ be the distinct values among $\phi_i(t_1), \ldots, \phi_i(t_n)$ with respective multiplicities $m_1, \ldots, m_{n'}$. Moreover, for $l \in \mathscr{L}$, define $r(d_j, l)$ to be the number of times the Pólya urn $U^i(d_j)$ produced the color l along the sequence $((t_1, l_1), \ldots, (t_n, l_n))$. Then, for $(t_1, \ldots, t_n)$ in a subset of $[0, \infty)^n$ of $P_{\tau_1, \ldots, \tau_n}$ probability one

$$P[s_1^i = l_1, s_2^i = l_2, \ldots, s_n^i = l_n \mid \tau_1^i = t_1, \tau_2^i = t_2, \ldots, \tau_n^i = t_n]$$

$$= \prod_{d_j} \left[\frac{\prod_{l \in \mathscr{L}} \prod_{q=0}^{r(d_j, l)-1} (c_l^i(d_j) + q)}{\prod_{q=0}^{m_j - 1} (q + \sum_{l \in \mathscr{L}} c_l^i(d_j))} \right]$$

with the convention that $\prod_0^{-1}(\cdot) = 1$. Therefore, for $(t_1, \ldots, t_n)$ in a subset of $[0, \infty)^n$ of $P_{\tau_1, \ldots, \tau_n}$ probability one and for any permutation $\pi = (\pi(1), \ldots, \pi(n))$ of $(1, \ldots, n)$,

$$P[s_1^i = l_1, s_2^i = l_2, \ldots, s_n^i = l_n \mid \tau_1^i = t_1, \tau_2^i = t_2, \ldots, \tau_n^i = t_n]$$

$$= P[s_1^i = l_{\pi(1)}, s_2^i = l_{\pi(2)}, \ldots, s_n^i = l_{\pi(n)} \mid \tau_1^i = t_{\pi(1)}, \tau_2^i = t_{\pi(2)}, \ldots, \tau_n^i = t_{\pi(n)}].$$

Hence

$$P[(\tau_1^i, s_1^i) \in B_1 \times \{l_1\}, (\tau_2^i, s_2^i) \in B_2 \times \{l_2\}, \ldots, (\tau_n^i, s_n^i) \in B_n \times \{l_n\}]$$

$$= \int_{B_1 \times \cdots \times B_n} P[s_1^i = l_{\pi(1)}, \ldots, s_n^i = l_{\pi(n)} \mid \tau_1^i = t_{\pi(1)}, \ldots, \tau_n^i = t_{\pi(n)}]$$

$$\mathrm{d}P_{\tau_1^i, \ldots, \tau_n^i}(t_1, \ldots, t_n)$$

$$= \int_{B_{\pi(1)} \times \cdots \times B_{\pi(n)}} P[s_1^i = l_{\pi(1)}, \ldots, s_n^i = l_{\pi(n)} \mid \tau_1^i = t_{\pi(1)}, \ldots, \tau_n^i = t_{\pi(n)}]$$

$$\mathrm{d}P_{\tau_1^i, \ldots, \tau_n^i}(t_{\pi(1)}, \ldots, t_{\pi(n)})$$

$$= P[(\tau_1^i, s_1^i) \in B_{\pi(1)} \times \{l_{\pi(1)}\}, (\tau_2^i, s_2^i) \in B_{\pi(2)} \times \{l_{\pi(2)}\}, \ldots,$$

$$(\tau_n^i, s_n^i) \in B_{\pi(n)} \times \{l_{\pi(n)}\}],$$

where the second equality holds because the sequence $\{\tau_n^i\}$ is exchangeable. This is enough to prove that $\{(\tau_n^i, s_n^i)\}$ is exchangeable. Independence of the sequences

$\{(\tau_n^0, s_n^0)\}, \{(\tau_n^1, s_n^1)\}, \ldots, \{(\tau_n^k, s_n^k)\}$ follows from the assumed independence of the sequences $\{\tau_n^0\}, \ldots, \{\tau_n^k\}$. $\square$

We are now ready for a synthetic definition of the process $\{X_t, t \geqslant 0\}$. Let $X_0 = L_0 \in \mathscr{L}$ denote the initial state of the process and set $T_0 = 0$. For $n \geqslant 1$ and given $L_0, \ldots, L_{n-1} \in \mathscr{L}$ and $T_0, \ldots, T_{n-1} \in [0, \infty)$, indicate with $\natural(L_0, \ldots, L_{n-1})$ the number of times state L_{n-1} appears in the string $L_0, \ldots, L_{n-1}$ and set

$$L_n = s^{L_{n-1}}_{\natural(L_0,\ldots,L_{n-1})} \quad \text{and} \quad T_n = T_{n-1} + \tau^{L_{n-1}}_{\natural(L_0,\ldots,L_{n-1})}.$$

For $t > 0$ define

$$N(t) = \sup\{n \geqslant 0 : T_n \leqslant t\};$$

then let

$$X_t = L_{N(t)}.$$

Therefore, the process $\{X_t, t \geqslant 0\}$ starts in state L_0 where it stays for a time $\tau_1^{L_0}$; at that time the process moves to state $L_1 = s_1^{L_0}$ where it stays for a time $\tau^{L_1}_{\natural(L_0,L_1)}$. Then the process moves to state $L_2 = s^{L_1}_{\natural(L_0,L_1)}$ where it stays for a time $\tau^{L_2}_{\natural(L_0,L_1,L_2)}$, and so on. We will refer to the process $\{L_n\}$ as the *embedded chain* for the process $\{X_t, t \geqslant 0\}$, while $\{v_n\}$ with $v_n = T_n - T_{n-1}$ for $n = 1, 2, \ldots$ will be called the sequence of elapsed times for the process $\{X_t, t \geqslant 0\}$.

By definition the trajectories of $\{X_t, t \geqslant 0\}$ are right continuous. The next lemma proves that the process is also regular; that is, the probability that the process makes an infinite number of transitions in a finite time is zero.

Lemma 6. $P[N(t) = \infty] = 0$ *for all* $t < \infty$.

Proof. For $t > 0$, consider the event $\{N(t) = \infty\}$ which is true if the process $\{X_t, t \geqslant 0\}$ makes an infinite number of transitions before time t; since $\mathscr{L}$ is finite, this is possible only if there is a state where the process $\{X_t, t \geqslant 0\}$ sojourns an infinite number of times before time t. Therefore

$$\{N(t) = \infty\} \subseteq \bigcup_{i=0}^{k} \left\{ \sum_{n=1}^{\infty} \tau_n^i \leqslant t \right\} \subseteq \bigcup_{i=0}^{k} \left\{ \sum_{n=1}^{\infty} \tau_n^i < \infty \right\}.$$

However, for $i \in \mathscr{L}$, condition $\alpha^i(\{0\}) = 0$ implies that $P\left[\sum_{n=0}^{\infty} \tau_n^i < \infty\right] = 0$ as follows from Lemma 4. Therefore $P[N(t) = \infty] = 0$. $\square$

For a continuous time process $\{Y_t, t \geqslant 0\}$ with values in $\mathscr{L}$, let L_0' denote the initial state of the process and, for $n \geqslant 1$, let L_n' denote the state of the process immediately after the nth transition has occurred. Moreover, indicate with v_n' the elapsed time between the $(n-1)$th and the nth transition. Then $\{Y_t, t \geqslant 0\}$ is said to be a *semi-Markov process* on $\mathscr{L}$ if the embedded chain $\{L_n'\}$ is a Markov chain on $\mathscr{L}$ and for all $n \geqslant 0$, given $L_0' = l_0, \ldots, L_{n+1}' = l_{n+1}$, the elapsed times $v_1', \ldots, v_n'$ are conditionally independent with distributions $G_{l_0,l_1}, G_{l_1,l_2}, \ldots, G_{l_n,l_{n+1}}$, respectively. (cf. Ross, 1970 and references therein.)

Theorem 7. *The process $\{X_t, t \geqslant 0\}$ is a mixture of semi-Markov processes.*

Proof. Consider the double array $\mathscr{A}$ with entries $a_{i,n} = (\tau_n^i, s_n^i)$, for $i \in \mathscr{L}$ and $n = 1, 2, \ldots$. Because of Lemma 5, for every $i \in \mathscr{L}$ the sequence $\{(\tau_n^i, s_n^i)\}$ is exchangeable and the sequences $\{(\tau_n^0, s_n^0)\}, \{(\tau_n^1, s_n^1)\}, \ldots, \{(\tau_n^k, s_n^k)\}$ are independent; these facts easily imply that the array $\mathscr{A}$ is partially exchangeable (de Finetti, 1938). That is, for all $n_0, \ldots, n_k \geqslant 1$ and for all permutations $\pi_0, \pi_1, \ldots, \pi_k$ of $(1, \ldots, n_0), (1, \ldots, n_1), \ldots, (1, \ldots, n_k)$, respectively,

$$\begin{aligned} &P[(\tau_1^0, s_1^0) \in B_1^0 \times \{l_1^0\}, \ldots, (\tau_{n_0}^0, s_{n_0}^0) \in B_{n_0}^0 \times \{l_{n_0}^0\}, \\ &\quad \ldots, (\tau_1^k, s_1^k) \in B_1^k \times \{l_1^k\}, \ldots, (\tau_{n_k}^k, s_{n_k}^k) \in B_{n_k}^k \times \{l_{n_k}^k\}] \\ &\qquad = P[(\tau_1^0, s_1^0) \in B_{\pi_0(1)}^0 \times \{l_{\pi_0(1)}^0\}, \ldots, (\tau_{n_0}^0, s_{n_0}^0) \in B_{\pi_0(n_0)}^0 \times \{l_{\pi_0(n_0)}^0\}, \\ &\qquad\quad \ldots, (\tau_1^k, s_1^k) \in B_{\pi_k(1)}^k \times \{l_{\pi_k(1)}^k\}, \ldots, (\tau_{n_k}^k, s_{n_k}^k) \in B_{\pi_k(n_k)}^k \times \{l_{\pi_k(n_k)}^k\}] \end{aligned}$$

for $(B_1^0, \ldots, B_{n_0}^0, \ldots, B_1^k, \ldots, B_{n_k}^k)$ Borel subsets of $[0, \infty)$ and

$$(l_1^0, \ldots, l_{n_0}^0, \ldots, l_1^k, \ldots, l_{n_k}^k)$$

states in $\mathscr{L}$. Therefore, de Finetti's Representation Theorem for partially exchangeable arrays (see Regazzini, 1991) implies that there exist $Q_0, \ldots, Q_k$ random and independent probability distributions on $[0, \infty) \times \mathscr{L}$ equipped with the product sigma-field such that, conditionally on $Q_0, \ldots, Q_k$, for every $i \in \mathscr{L}$ the random elements of the sequence $\{(\tau_n^i, s_n^i)\}$ are independent and identically distributed with probability distribution Q_i. Hence, conditionally on $Q_0, \ldots, Q_k$, the embedded chain $\{L_n\}$ is a homogeneous Markov chain with transition probabilities

$$P_{ij} = Q_i(\{j\} \times [0, \infty))$$

for $i, j \in \mathscr{L}$; furthermore, for $n \geqslant 1$ and given $L_0 = l_1, \ldots, L_{n+1} = l_{n+1}$, the elapsed times $v_1, \ldots, v_n$ are conditionally independent and their conditional probability distributions on $[0, \infty)$ are, respectively,

$$G_{l_0, l_1}(\cdot) = \frac{Q_{l_0}(\{l_1\} \times \cdot)}{Q_{l_0}(\{l_1\} \times [0, \infty))}, \ldots, G_{l_n, l_{n+1}}(\cdot) = \frac{Q_{l_n}(\{l_{n+1}\} \times \cdot)}{Q_{l_n}(\{l_{n+1}\} \times [0, \infty))}.$$

This proves that $\{X_t, t \geqslant 0\}$ is a mixture of semi-Markov processes. □

Let now $l_0 \in \mathscr{L}$ be a recurrent state for $\{X_t, t \geqslant 0\}$ and fix l_0 to be the initial state of the process; that is assume that $L_0 = l_0$ and

$$P[L_n = l_0 \text{ for infinitely many } n] = 1.$$

Set $\xi_0 = T_0 = 0$ and, for $n \geqslant 1$, define $\xi_n = \inf\{n > \xi_{n-1} : L_n = l_0\}$: the random quantities ξ_n's mark the indexes of successive transitions into state l_0 for the process $\{X_t, t \geqslant 0\}$. For $n \geqslant 1$, call

$$\mathscr{B}_n = ((L_{\xi_{n-1}}, v_{\xi_{n-1}}), \ldots, (L_{\xi_n - 1}, v_{\xi_n - 1}))$$

an l_0-*block* for the process $\{X_t, t \geqslant 0\}$. Since l_0 is recurrent for $\{X_t, t \geqslant 0\}$, l_0-blocks are almost surely well defined finite sequences of elements of $\mathscr{L} \times [0,\infty)$; moreover the sequence $\{\mathscr{B}_n\}$ is infinite. Let $S = \mathscr{L} \times [0,\infty)$ be endowed with the product sigma-field and consider the space S^* of all finite sequences of S : equip S^* with the sigma-field generated by sets of the type

$$(\{l_0\} \times B_0) \times (\{l_1\} \times B_1) \times \cdots (\{l_n\} \times B_n)$$

for $n = 0, 1, 2, \ldots$ natural numbers, $l_0, \ldots, l_n \in \mathscr{L}$ and $B_0, \ldots, B_n$ Borel subsets of $[0,\infty)$. The l_0-blocks $\mathscr{B}_1, \mathscr{B}_2, \ldots$ of $\{X_t, t \geqslant 0\}$ are random elements of S^*. Observe that when $\{Y_t, t \geqslant 0\}$ is a continuous time, semi-Markov process on $\mathscr{L}$ with initial state l_0 and l_0 is recurrent, the l_0-blocks for $\{Y_t, t \geqslant 0\}$ are independent and identically distributed random elements of S^*. Thus Theorem 7 has the following immediate corollary.

Corollary 8. *Let $l_0 \in \mathscr{L}$ be the initial state of $\{X_t, t \geqslant 0\}$ and assume that l_0 is recurrent. Then the sequence $\{\mathscr{B}_n\}$ of l_0-blocks for $\{X_t, t \geqslant 0\}$ is exchangeable.*

In fact, by appealing to the particular constructive definition of the process $\{X_t, t \geqslant 0\}$, one could directly prove the exchangeability of its l_0-blocks without making use of Theorem 7; needless to say, the proof becomes cumbersome and less attractive. Blocks exchangeability makes the process $\{X_t, t \geqslant 0\}$ an interesting model for applications in Bayesian nonparametric statistics, when histories of exchangeable individuals are sequentially observed, each from an initial event, represented by a transition of $\{X_t, t \geqslant 0\}$ into state l_0, until a certain terminal event. Each block of $\{X_t, t \geqslant 0\}$ then represents the history of an individual. Given the simple rules for updating the parameters controlling the law of the process $\{X_t, t \geqslant 0\}$, predictive distributions for blocks are easily computed or simulated, without having to characterize their prior distribution: this will be illustrated in the next section. For the time being, let us remark that if ψ is a function which maps measurably S^* into another space, then the sequence $\{\psi(\mathscr{B}_n)\}$ is also exchangeable. For instance, for every $\mathscr{B}_n = ((L_{\xi_{n-1}}, v_{\xi_{n-1}}), \ldots, (L_{\xi_n - 1}, v_{\xi_n - 1}))$, we may define

$$\psi(\mathscr{B}_n) = v_{\xi_{n-1}} + v_{\xi_{n-1}+1} + \cdots + v_{\xi_n - 1} = T_{\xi_n} - T_{\xi_{n-1}}.$$

Then $\psi(\mathscr{B}_n)$ measures the time elapsed between the nth and the $(n+1)$th transition to the initial state l_0 by the process $\{X_t, t \geqslant 0\}$; or what we would call the total survival time of individual n if blocks represented histories of individuals.

5. Examples and concluding remarks

5.1. Reinforced urn processes

From Theorem 7, it follows that the embedded chain $\{L_n\}$ for the process $\{X_t, t \geqslant 0\}$ defined in Section 3 is a mixture of Markov chains with values in $\mathscr{L}$. By setting to be constants the functions ϕ_i's appearing in the construction of the sequences of successor states for $\{X_t, t \geqslant 0\}$, we get that $\{L_n\}$ is a reinforced urn process as defined in Muliere et al. (2000). In fact, for all $i \in \mathscr{L}$, let us take $\phi^i(t) = \gamma^i \in [0, \infty)$ for all $t \in [0, \infty)$. Therefore, when the process is in i, transitions to the next state are always generated by the same Pólya urn $U^i(\gamma^i)$ disregarding the amount of time spent in state i. Then, conditionally on a random transition matrix M with independent rows and such that its ith row has Dirichlet distribution with the same parameters as the initial composition of urn $U^i(\gamma^i)$, $\{L_n\}$ is a Markov chain on $\mathscr{L}$ (with transition matrix M).

5.2. A two state example

Let $\mathscr{L} = \{0, 1\}$. Successor states for the process $\{X_t, t \geqslant 0\}$ are taken to be non-random: state 1 always follows state 0 and state 0 always follows state 1. That is, $s_n^0 = 1 - s_n^1 = 1$ for $n = 1, 2, \ldots$. The process starts in state $0 = l_0$. Given two positive measures α^0 and α^1 on $[0, \infty)$ such that $\alpha^0(\{0\}) = \alpha^1(\{0\}) = 0$ and measurable functions $\beta^0, \beta^1 : [0, \infty) \to (0, \infty)$, the sequences of holding times in state 0 and in state 1 for $\{X_t, t \geqslant 0\}$ are defined to be reinforced renewal processes with parameters (α^0, β^0) and (α^1, β^1), respectively. The process $\{X_t, t \geqslant 0\}$ is well defined and states 0 and 1 are both recurrent if (α^0, β^0) and (α^1, β^1) satisfy condition (1). For instance, this happens if α^0 and α^1 are continuous and

$$\int_0^\infty \frac{d\alpha^0(v)}{\beta^0(v)} = \int_0^\infty \frac{d\alpha^1(v)}{\beta^1(v)} = \infty.$$

The sequence $\{\mathscr{B}_n\}$ of 0-blocks for the process $\{X_t, t \geqslant 0\}$ is characterized by the transition times from 0 to 1 and then from 1 to 0, say $\{(\tau_n^0, \tau_n^1)\}$ for $n = 1, 2, \ldots$. Conditionally on the blocks $\mathscr{B}_1, \ldots, \mathscr{B}_M$ the revised α's and β's are:

$$\alpha_M^0(0, t] = \alpha^0(0, t] + \sum_{m=1}^M I[\tau_m^0 \leqslant t],$$

$$\alpha_M^1(0, t] = \alpha^1(0, t] + \sum_{m=1}^M I[\tau_m^1 \leqslant t],$$

$$\beta_M^0(t) = \beta^0(t) + \sum_{m=1}^M I[\tau_m^0 > t],$$

$$\beta_M^1(t) = \beta^1(t) + \sum_{m=1}^M I[\tau_m^1 > t]$$

for $t > 0$. We may now easily compute the predictive distribution of the $M+1$th block since:

$$P(\tau_{M+1}^0 > t \mid \mathscr{B}_1, \ldots, \mathscr{B}_M) = \mathbf{\Pi}_{v \leqslant t} \left\{ 1 - \frac{d\alpha_M^0(v)}{\alpha_M^0(\{v\}) + \beta_M^0(v)} \right\}$$

and

$$P(\tau^1_{M+1} > t \mid \tau^0_{M+1}, \mathscr{B}_0, \ldots, \mathscr{B}_M) = \mathbf{\Pi}_{v \leqslant t} \left\{ 1 - \frac{\mathrm{d}\alpha^1_M(v)}{\alpha^1_M(\{v\}) + \beta^1_M(v)} \right\}$$

for $t > 0$. If blocks represented histories of individuals, from the expected values of the predictive distributions of τ^0_{M+1} and τ^1_{M+1} we would get a Bayesian nonparametric estimator for the history of the $(M+1)$st individual.

This example is highly reminiscent of an analogous one treated in Section 5 of Muliere et al. (2000) where a reinforced urn process on two levels was introduced for modeling survival of patients subject to a two-state disease. Here the two levels are represented by the two states of $\mathscr{L}$. Besides discreteness and continuity of time, the main difference between the two examples is that in the former the two clocks measuring the time spent by a patient at each disease's level are reset to zero when, and only when, state 0 is reached, i.e. a new patient is considered, whereas in the example above both clocks are reset to zero whenever a new state, either 0 or 1, is entered by the process $\{X_t, t \geqslant 0\}$. It is however intuitive how to modify the expressions above in order to obtain continuous time versions for the predictive distribution of the $M+1$th block in the situation considered in the 2000 paper.

A possibility is to reset clocks to zero and update their parameters only after a specific event has occurred; e.g. a specified state, called the l_0-state, has been visited. Within the time elapsing between two consecutive visits to the l_0 state, a particular state may be visited any number of times but the updates of its clock only start to work once the l_0 state has been visited. So, for our example here, taking $l_0 = 0$ and given blocks $\mathscr{B}_1, \ldots, \mathscr{B}_M$, parameters α's and β's should be updated as above except that now

$$\beta^1_M(t) = \beta^1(t) + \sum_{m=1}^{M} I[\tau^0_m \leqslant t < \tau^1_m],$$

while

$$P(\tau^0_{M+1} > t \mid \mathscr{B}_1, \ldots, \mathscr{B}_M) = \mathbf{\Pi}_{v \leqslant t} \left\{ 1 - \frac{\mathrm{d}\alpha^0_M(v)}{\alpha^0_M(\{v\}) + \beta^0_M(v)} \right\}$$

and

$$P(\tau^1_{M+1} > t \mid \tau^0_{M+1}, \mathscr{B}_1, \ldots, \mathscr{B}_M) = \mathbf{\Pi}_{\tau^0_{M+1} < v \leqslant t} \left\{ 1 - \frac{\mathrm{d}\alpha^1_M(v)}{\alpha^1_M(\{v\}) + \beta^1_M(v)} \right\}$$

for $t > 0$. This shows the versatility of reinforcement as a way of incorporating information obtained by past observations into predictive distributions. In fact, for more states and a random successor state, formulae and predictives are more complex but follow the same principle.

Without reinforcement the processes described in this paper would be very familiar objects; essentially renewal processes or semi-Markov processes. Reinforcement is the key. This allows us to learn in a Bayesian way about the mechanism driving the process as it materialises. We do not rely on the traditional notion of a prior and posterior, indeed the prior seems to be intractable. Nevertheless, we have shown how to derive explicit forms for the predictive via reinforcement which provides us with the necessary exchangeability.

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