An analogue circuit to study the forced and quadratically damped Duffing oscillator

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Abstract: In this communication we study the forced and quadratically damped Duffing oscillator system experimentally through an analogue circuit. For this purpose we design a circuit and observe its dynamical behaviour under various sets of control parameters and also compare the experimental results with the numerical simulation results.

Keywords: nonlinear damping; Duffing oscillator; chaos; periodic attractor.

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1   Introduction

The study of effect of nonlinearity on the behaviour of natural and man-made devices has been an active area of research during last few decades. Particularly the study of chaos (its existence and analysis) in complex dynamical systems has been of interest for scientists, engineers and physicists (Sparrow, 1982; Rossler, 1976). There has been recent interest to study the nonlinear phenomena and chaos in various physical systems through modelling them via electronic circuits (Chua and Lin, 1990; Carroll and Pecora, 1991). For simulating the real complex nonlinear systems, electronic circuits have attracted special attention of researchers in the field due to the fact that electronic circuits provide a very useful medium for both modelling and experimental studies of nonlinear systems. And moreover the real complex physical system can be simulated in real-time through an electronic circuit costing very less (see Murali et al., 1994; Kurt et al., 2000; Kasap and Kurt, 1998; Kiers et al., 2004). Many of these circuits could be modelled using second-order nonlinear differential equations. A simple circuit consisting of a resistor, inductor and diode showed the period doubling route to chaos (Linsay, 1981). Number of circuits such as Matsumoto's, Chua's, Bronson's and mixed mode have been explored (see Murali et al., 1994; Matsumoto et al., 1987; Bitton et al., 1997) and found to exhibit very rich nonlinear behaviours including periodic, quasiperiodic, quasi-periodic and folded-torus, chaotic etc. (see Kiers et al., 2004; Ikezi et al., 1983; Su et al., 1989; Murali et al., 1995). Matsumoto et al. (1987) reported a torus breakdown in the three segment piece-wise sub-circuit. A dissipative non-autonomous chaotic circuit was proposed by

Murali et al. (1994) using a complex simulation, where the Chua's diode was considered as the source for nonlinearity. Some physical mechanisms have been reported in a simple nonlinear circuit (Bitton et al., 1997) based on the proposal of Matsumoto (1984). Vincent et al. (2014) suggested that the chaotic behaviour could be conveniently tamed in practical experiment. Various nonlinear electronic circuits have been designed and analysed to understand the dynamics of many complex physical systems of diverse nature under real-time conditions like: nonlinear physical pendulum, forced Duffing oscillator, forced van der pol oscillator, forced Duffing-van der pol oscillator, parametrically driven Duffing oscillator etc. (Murali and Lakshmanan, 1996).

Recently, Sharma et al. (2012) have analysed the dynamical behaviour of forced Duffing oscillator under the presence of nonlinear damping term analytically as well as computationally. They have particularly observed the effect of nonlinear damping on the global dynamical behaviour of forced Duffing oscillator through Melnikov analysis, characterisation of parameter space through Lyapunov spectrum calculation, fractalness of basin boundaries and fractal dimensions etc. In this communication we intend to study the same dynamical system i.e., the forced Duffing oscillator system under the presence of nonlinear damping experimentally through an analogue circuit. For this purpose we have designed a circuit and observed its dynamical behaviour under various sets of control parameters and also compared the experimental results with the numerical simulation results. In the next paragraph we briefly describe the forced and quadratically damped Duffing oscillator system.

Duffing oscillator describes the motion of a particle in double-well potential and it is represented by a ubiquitous nonlinear differential equation (Sharma et al., 2012). It has a number of applications not only in physical and engineering problems but also in biological problems. The generalised form of nonlinear equation of Duffing oscillator is:

$$\ddot{x} + \alpha\dot{x} - \omega_o^2 x + \beta x^3 = f\cos\omega t \qquad (1)$$

where first term represents the force on a unit mass particle, second term is velocity term in which α is damping coefficient, whose value is greater than zero. Third term represents the restoring force in which ω_o is natural frequency. Fourth term is nonlinear term in which nonlinear parameter β is known as stiffness constant. The right hand side term represents the external force applied to the system which is cosine in nature and f and ω are the external forcing amplitude and external frequency respectively. In our present work we consider a nonlinear damping term i.e., proportional to the pth power of velocity. Thus we consider the following generalised form of model equation:

$$\ddot{x} + \alpha\dot{x}\,|\,\dot{x}\,|^{p-1} - \omega_o^2 x + \beta x^3 = f\cos\omega t \qquad (2)$$

Here modulus of velocity is taken to consider the sign of velocity. For all calculations we have considered $\omega_o^2 = 1$, $\beta = 1$ and $\omega = 1$. We have studied the dynamics of the system for different value of damping coefficient (α) and external forcing amplitude (F). For our study, we have designed an electronic circuit for the model equation (2) and simulated it through the PSpice circuit simulator and implemented it in terms of a real-time circuit for the laboratory simulations. In the next section we give the description of the electronic circuit designed for our study. In Section 3, we discuss our circuit simulation results and compare them with the PSpice simulations as well as numerical simulation results and Section 4 concludes the paper.

2   Description of the circuit

We have constructed an analogue circuit equivalent to equation (2) using conventional operational amplifiers and four quadrant multipliers, which is shown in Figure 1.

The circuit is constructed using conventional operational amplifiers and four quadrant multipliers. Here A1 and A2 represent two integrators; A3, A4 and A5 are inverters; A6 is an absolute rectifier; M1, M2, M3 and M4 are four quadrant multipliers; and A7 is a summing amplifier. The operational amplifiers used in A1, A2, A3, A4, A5, A6 and A7 are op-amp μA741C and the four multipliers M1, M2, M3 and M4 are analogue devices multiplier AD633 with the inbuilt gain of 0.1. Analogue devices multiplier AD633 is the low cost chip with 2% error at full scale.

Figure 1   Analogue circuit for the forced and quadratically damped Duffing oscillator, equivalent to equation (2) (see online version for colours)
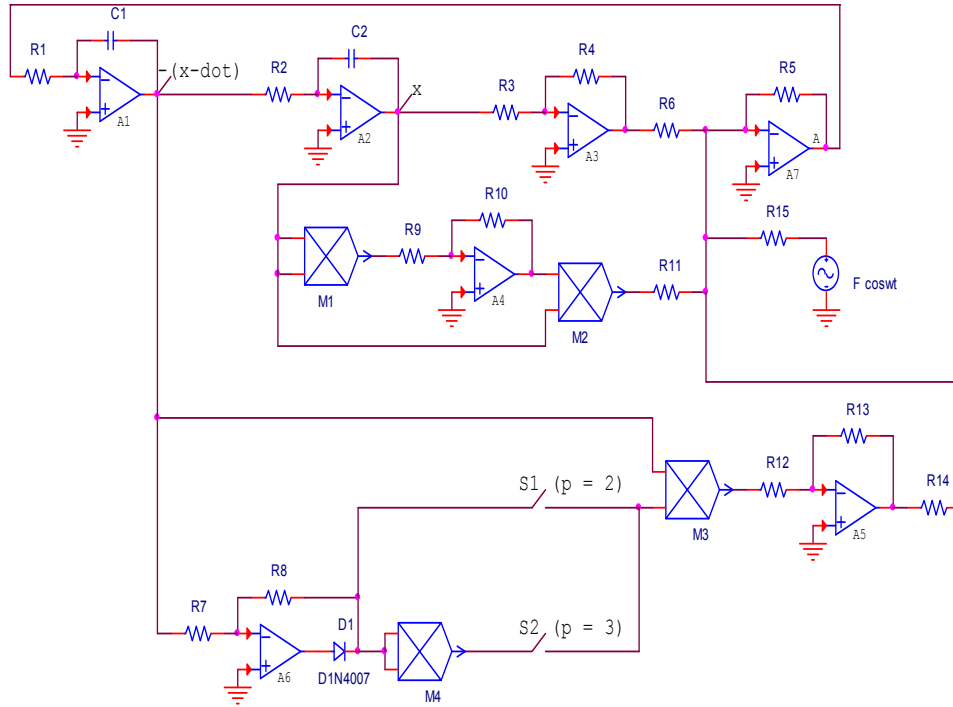
At junction A i.e., at the output of summing amplifier, we obtain the following equation:

$$\ddot{x} = -\alpha \dot{x} \, | \, \dot{x} \, |^{p-1} + \omega_0^2 x - \beta x^3 - F \cos \omega t$$

where $p = 2$ (switch) S1 closed) or 3 (switch S2 closed), $| \, \omega_0^2 \, | = \dfrac{R5}{R6}$, $\alpha = \dfrac{R5}{R14}$, and

$\beta = \dfrac{(0.1)R5}{R11}$.

The values of various resistors and capacitors used in the circuit are as given below:

$R1 = 100k\Omega$, $R2 = 100k\Omega$, $R3 = 1k\Omega$, $R4 = 1k\Omega$, $R5 = 10k\Omega$, $R6 = 10k\Omega$, $R7 = 1k\Omega$,
$R8 = 1k\Omega$, $R9 = 1k\Omega$, $R10 = 10k\Omega$, $R11 = 1k\Omega$, $R12 = 1k\Omega$, $R13 = 10k\Omega$ (for $p = 2$)
or $100k\Omega$ (for $p = 3$), $R14 = 20k\Omega$, $R15 = 10k\Omega$, and $C1 = C2 = 0.01\,\mu F$.

Here $\omega_0^2 = 1$, $\alpha = 0.5$, $\beta = 1$ and $\omega = 1$. $\omega = 1$ is obtained at external frequency f $\approx$160 Hz since here $\omega = 2\pi * f * (R1 * C1)$ or $2\pi * f* (R2 * C2)$.

A snapshot of the real circuit implemented on the breadboard with the simulation result on digital CRO for the selected set of parameters ($\omega_0^2 = 1$, $\alpha = 0.5$, $\beta = 1$, $\omega = 1$, $p = 2$ and forced amplitude F = 0.845 V) has been depicted in Figure 2.
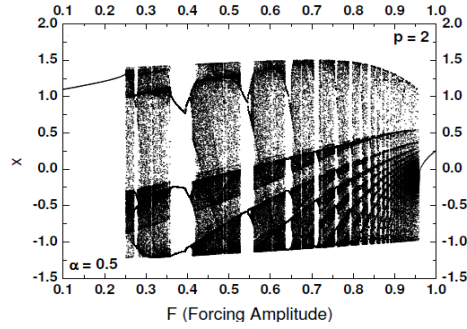
Figure 2   A snapshot of the real circuit (see online version for colours)
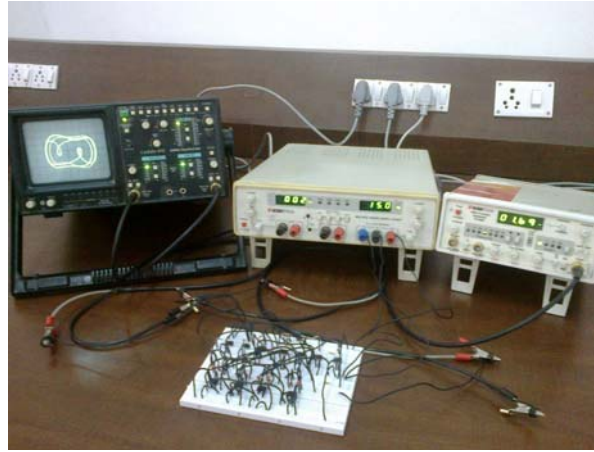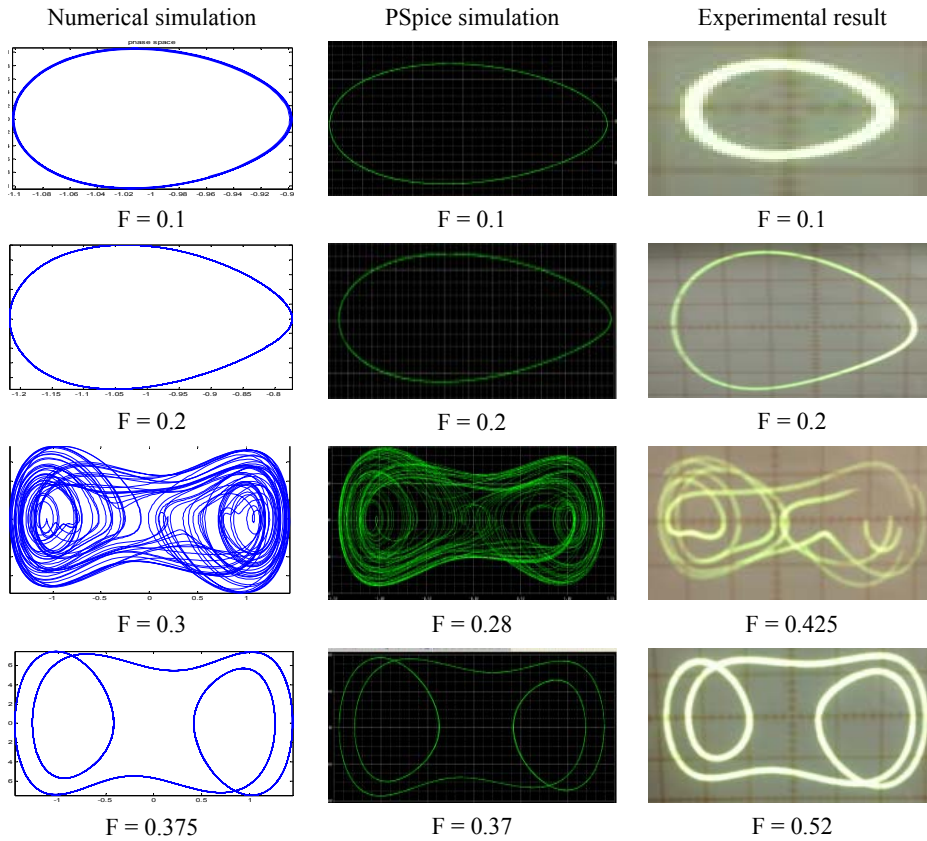

3   Results and discussion

In this section we present the results of experimental study and their comparison with the numerical results. In Figure 3, we show the result for the fixed values of parameters $\omega_0^2 = 1$, $\beta = 1$, $\omega = 1$, $\alpha = 0.5$, $p = 2$ (drag like damping) and various values of forcing amplitudes. In Figure 3(a), we have depicted a numerically computed bifurcation diagram showing the complete sequence of dynamical behaviour for the selected set of parameters against the various values of forcing amplitude. However in Figure 3(b), the first column depicts the value of the forcing amplitude and the corresponding numerical simulation results produced by integrating the second order differential equation (2) through MATLAB, the second and third columns respectively show thePSpice simulator results and experimental results along with the corresponding values of forcing amplitudes.

Figure 3   (a) Bifurcation diagram for the nonlinearly damped forced Duffing oscillator for
$\omega_0^2 = 1$, $\beta = 1$, $\omega = 1$, $\alpha = 0.5$, $p = 2$ (drag like damping) (b) Phase plots for the
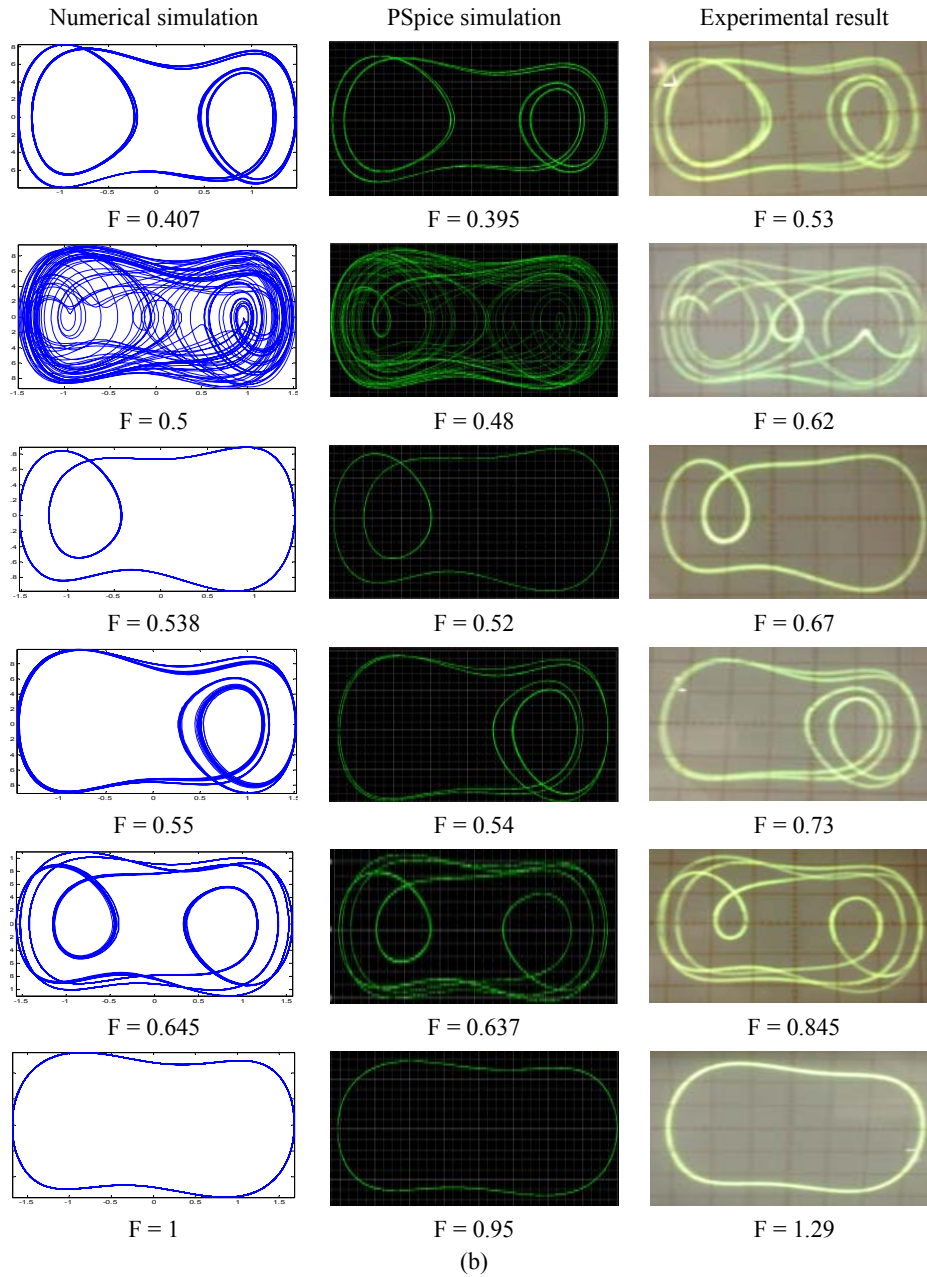nonlinearly damped forced Duffing oscillator (see online version for colours)

(a)

Numerical simulation	PSpice simulation	Experimental result

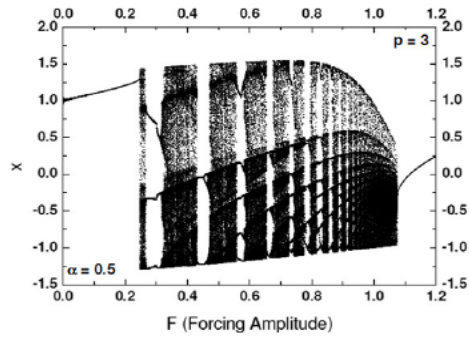| F = 0.1	F = 0.1	F = 0.1
F = 0.2	F = 0.2	F = 0.2
F = 0.3	F = 0.28	F = 0.425
F = 0.375	F = 0.37	F = 0.52

(b)

Notes: First, second and third columns in Figure 3(b) respectively show the
computationally obtained phase plots, PSpice simulation results and the third the
experimental results obtained from the circuit shown in Figure 1.

Figure 3 (a) Bifurcation diagram for the nonlinearly damped forced Duffing oscillator for $\omega_0^2 = 1$, $\beta = 1$, $\omega = 1$, $\alpha = 0.5$, $p = 2$ (drag like damping) (b) Phase plots for the nonlinearly damped forced Duffing oscillator (continued) (see online version for colours)

Numerical simulation	PSpice simulation	Experimental result

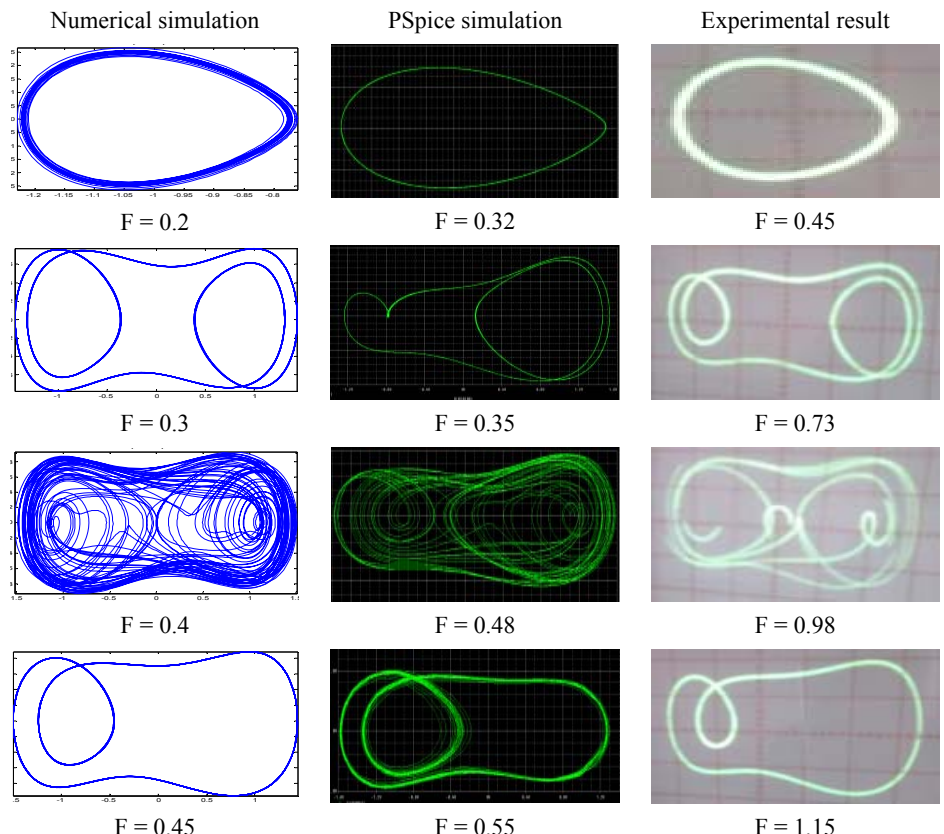| F = 0.407	F = 0.395	F = 0.53
F = 0.5	F = 0.48	F = 0.62
F = 0.538	F = 0.52	F = 0.67
F = 0.55	F = 0.54	F = 0.73
F = 0.645	F = 0.637	F = 0.845
F = 1	F = 0.95	F = 1.29

(b)

Notes: First, second and third columns in Figure 3(b) respectively show the computationally obtained phase plots, PSpice simulation results and the third the experimental results obtained from the circuit shown in Figure 1.

Figure 4   (a) Bifurcation diagram for the nonlinearly damped forced Duffing oscillator for $\omega_0^2 = 1$, $\beta = 1$, $\omega = 1$, $\alpha = 0.5$, $p = 3$ (nonlinear third-power damping) (b) Phase plots for the nonlinearly damped forced Duffing oscillator (see online version for colours)

(a)

Numerical simulation	PSpice simulation	Experimental result

F = 0.2	F = 0.32	F = 0.45
F = 0.3	F = 0.35	F = 0.73
F = 0.4	F = 0.48	F = 0.98
F = 0.45	F = 0.55	F = 1.15

(b)

Notes: First, second and third columns in Figure 4(b) respectively show the computationally obtained phase plots, PSpice simulation results and the third the experimental results obtained from the circuit shown in Figure 1.

We observe from Figure 3 that the designed circuit is able to produce the correct sequence of the dynamical behaviour as obtained with the numerical simulation however the corresponding values of the forcing amplitudes are shifted to lower side for the PSpice simulations and to the upper side for the experimental simulations. This difference in the values of forcing amplitude can be explained in the following way: in PSpice, the difference in the forcing amplitude is due to the input and output offset voltages of analogue devices multiplier IC AD633 that creates 2% error at full scale however in the experimental circuit, the difference in forcing amplitude is due to AD633 and $\pm 10 - 15$ % tolerances of electronic components used.

Another set of results for the fixed values of parameters $\omega_0^2 = 1$, $\beta = 1$, $\omega = 1$, $\alpha = 0.5$, $p = 3$ (nonlinear third-power damping) and various values of forcing amplitudes have been depicted in Figure 4. Particularly in Figure 4(a), we have depicted a numerically computed bifurcation diagram showing the complete sequence of dynamical behaviour for the selected set of parameters against the various values of forcing amplitude. However in Figure 4(b), the first column depicts the value of the forcing amplitude and the corresponding numerical simulation results produced by integrating the second order differential equation (2) through MATLAB, the second and third columns respectively show the PSpice simulator results and experimental results along with the corresponding values of forcing amplitudes. We observe that the difference in the forcing amplitude values for the numerical computation, Pspice simulation and experimental results is large as compared to the case $p = 2$ (Figure 3) is due to use of one additional analogue multiplier IC AD633.

4   Conclusions

We have designed and implemented an analogue circuit equivalent to a forced and quadratically damped Duffing oscillator to study the effect of nonlinear damping on the dynamical behaviour of the forced oscillator. We have studied the designed circuit extensively for various combinations of system parameters and observed that the designed circuit is able to produce the correct sequence of the dynamical behaviour as obtained with the numerical simulation. We have also compared the experimental results, PSpice simulations and numerical results and found good degree of agreement. The designed circuit may be useful to understand the real time behaviour of complex physical systems modelled by forced and quadratically damped Duffing differential equation.

Acknowledgements

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