Journal of Mathematical Analysis and Applications **261,** 271–281 (2001)
doi:10.1006/jmaa.2001.7503, available online at http://www.idealibrary.com on IDEAL®

On Property (M) and Its Generalizations

Hong-Kun Xu[1]

Department of Mathematics, University of Durban-Westville, Private Bag X54001, Durban 4000, South Africa
E-mail: hkxu@pixie.udw.ac.za

and

G. Marino and P. Pietramala

Dipartimento di Matematica, Universitá della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy
E-mail: gmarino@unical.it, pietramala@unical.it

Submitted by Richard Aron

Received November 8, 1999

Properties strict (M) and uniform (M) are introduced. It is shown that if X has property (M) and is uniformly convex in every direction, then X has both strict (M) and uniform (M). It is also shown that if X^* is separable, then strict (M) implies uniform (M) and property (M) implies weak uniform normal structure. Relations with other geometrical properties of Banach spaces are also discussed.

Key Words: properties (M); strict (M); uniform (M); weakly continuous duality map; weak uniform normal structure; (locally) uniform Opial property; GGLD.

1. INTRODUCTION

In order to classify those separable Banach spaces X such that the ideal $\mathscr{K}(X)$ of linear compact operators is an M-ideal in the algebra $\mathscr{L}(X)$ of all linear bounded operators, Kalton [K] introduced the notion of property (M). This requires that if u, v satisfy $\|u\| = \|v\|$ and if (x_n) is a weakly null sequence then

$$\limsup_{n\to\infty}\|u + x_n\| = \limsup_{n\to\infty}\|v + x_n\|.$$

[1] Supported in part by NRF (South Africa). To whom correspondence should be addressed.

0022-247X/01 $35.00

It is not hard to see that Hilbert spaces and l_p $(1 \leq p < \infty)$ have property (M). It is also known that c_0 has property (M) and hence property (M) does not imply weak normal structure. (Recall that a Banach space has weak normal structure [BM] if every weakly compact convex subset C consisting of more than one point has a nondiametral point; i.e., there exists an $x \in C$ such that $\sup\{\|x - y\|: y \in C\} < \operatorname{diam}(C)$.) It is, however, shown in [GS] that property (M) implies the weak fixed point property for nonexpansive mappings. Namely, if X is a Banach space having property (M), if C is a weakly compact convex nonempty subset of X, and if T: $C \to C$ is nonexpansive (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$), then T has a fixed point. It is not hard to see ([K, Lemma 2.1]) that X has property (M) if and only if, for each weakly null sequence (x_n), the type $\psi_{(x_n)}(u) := \limsup_n \|u + x_n\|$ is nondecreasing in $\|u\|$. However, if X is a Hilbert space, we have $\psi_{(x_n)}(u) = (\limsup_n \|x_n\|^2 + \|u\|^2)^{1/2}$ and if X is l_p $(1 < p < \infty)$ we have $\psi_{(x_n)}(u) = (\limsup_n \|x_n\|^p + \|u\|^p)^{1/p}$. In both cases, we see that each type $\psi_{(x_n)}(u)$ is indeed strictly increasing in $\|u\|$. This motivates us to introduce the notion of property strict (M) (Definition 2.2). In order to measure property (M) we also introduce the concept of uniform (M) (Definition 2.3).

The purpose of this paper is to discuss the relations among properties (M), strict (M), uniform (M), and other geometrical properties of Banach spaces. We show, among others, that if a Banach space X has property (M) and is uniformly convex in every direction, then X has both strict (M) and uniform (M). We also show that if X has property (M) and X^* is separable, then X has weak uniform normal structure. Moreover, if X has a weakly continuous duality map, then we obtain a formula for the M-modulus κ_X of X.

Notation. $\rightharpoonup$ for weak convergence; $\to$ for strong convergence.

2. STRICT (M) AND UNIFORM (M)

We begin with Kalton's property (M) [K].

DEFINITION 2.1. A Banach space X has *property* (M) if whenever $u, v \in X$ are such that $\|u\| = \|v\|$ and (x_n) is a weakly null sequence in X we have

$$\limsup_{n \to \infty} \|u + x_n\| = \limsup_{n \to \infty} \|v + x_n\|.$$

Property (M) is one of the essential ingredients in Kalton's characterization ([K, Theorem 2.4(5)]) of those separable Banach spaces X for which the ideal $\mathscr{K}(X)$ of compact linear operators is an M-ideal of the algebra $\mathscr{L}(X)$ of all bounded linear operators.

It is easily seen ([K, Lemma 2.1]) that a Banach space X has property (M) if and only if, for each weakly null sequence (x_n) in X, the type

$$\psi_{(x_n)}(u) := \limsup_n \|u + x_n\|, \; u \in X$$

is nondecreasing in $\|u\|$; that is, if $u, v \in X$ satisfy $\|u\| \le \|v\|$, then $\psi_{(x_n)}(u) \le \psi_{(x_n)}(v)$. However, if X is a Hilbert space, we have

$$\psi_{(x_n)}(u) = \left(\limsup_n \|x_n\|^2 + \|u\|^2 \right)^{1/2}$$

and if X is l_p $(1 < p < \infty)$, we have

$$\psi_{(x_n)}(u) = \left(\limsup_n \|x_n\|^p + \|u\|^p \right)^{1/p}.$$

We actually have, in either case, that every type $\psi_{(x_n)}$ is a strictly increasing function of $\|u\|$. So it is natural to introduce the following notion.

DEFINITION 2.2. We say that a Banach space X has *property strict* (M) if, for each weakly null sequence (x_n) in X, the type $\psi_{(x_n)}$ is strictly increasing in $\|u\|$; that is, if $u, v \in X$ have $\|u\| < \|v\|$, then $\psi_{(x_n)}(u) < \psi_{(x_n)}(v)$.

Thus, both Hilbert spaces and l_p $(1 < p < \infty)$ have strict (M), while c_0 has (M) but fails to have strict (M). Indeed, if (x_n) is a weakly null sequence in c_0, we have $\psi_{(x_n)}(u) = \max\{\limsup_n \|x_n\|, \|u\|\}$.

Recall that a Banach space X has the *locally uniform Opial property* [LTX] if, for each weakly null sequence (x_n) in X such that $\|x_n\| \to 1$ and $c > 0$, there exists a constant $r > 0$ such that

$$\psi_{(x_n)}(u) \ge 1 + r \quad \forall u \in X, \; \|u\| \ge c.$$

THEOREM 2.1. *Let X be a Banach space which has property strict (M). Then X has the locally uniform Opial property.*

Proof. Assume (x_n) is a weakly null sequence in X satisfying $\|x_n\| \to 1$ and $c > 0$. Set $r = \psi_{(x_n)}(u_0) - 1$, where $u_0 \in X$ is any element with norm c. Since $\psi_{(x_n)}$ is strictly increasing in $\|u\|$, we have $r > \psi_{(x_n)}(0) - 1 = 0$. Hence, for $u \in X$ such that $\|u\| \ge c$, we have $\psi_{(x_n)}(u) \ge \psi_{(x_n)}(u_0) = 1 + r$. ∎

We next introduce the notion of M-modulus in order to measure property (M).

DEFINITION 2.3. The M-modulus of a Banach space X is defined by

$$\kappa_X(u) := \inf\{\psi_{(x_n)}(u) : x_n \rightharpoonup 0, \|x_n\| \to 1\}.$$

If X has property (M) and κ_X is strictly increasing in $\|u\|$ (i.e., $\kappa_X(u) < \kappa_X(v)$ if $\|u\| < \|v\|$), then X is said to have *property uniform* (M).

Since $\kappa_X(u) = (1 + \|u\|^2)^{1/2}$ if X is a Hilbert space and $\kappa_{l_p}(u) = (1 + \|u\|^p)^{1/p}$ for $1 < p < \infty$, Hilbert spaces and l_p $(1 < p < \infty)$ have uniform (M). A more general case is presented below.

Recall that a function φ: $[0, \infty) \to [0, \infty)$ is a *gauge* if φ is continuous, strictly increasing, $\varphi(0) = 0$, and $\lim_{t\to\infty}\varphi(t) = \infty$. To a gauge φ, we can associate a duality map J_φ; $X \to X^*$ by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between X and X^*. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau.$$

Then $J_\varphi(x)$ is the subdifferential $\partial\Phi(\|x\|)$ of the convex function $\Phi(\|\cdot\|)$ at x. Following Browder [B], we say that X has a *weakly continuous duality map* if there exists a gauge φ such that the duality map J_φ is single-valued and (sequentially) continuous from X, with the weak topology, to X^*, with the weak* topology. It is known that l_p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi = t^{p-1}$.

THEOREM 2.2. *If a Banach space X has a weakly continuous duality map with gauge φ, then X has both properties strict (M) and uniform (M), and*

$$\kappa_X(u) = \Phi^{-1}(\Phi(1) + \Phi(\|u\|)), \ u \in X.$$

Proof. Since $J_\varphi(x)$ is the Gateaux derivative of the convex function $\Phi(\|x\|)$, it follows that

$$\Phi(\|x + u\|) = \Phi(\|x\|) + \int_0^1 \langle h, J_\varphi(x + tu) \rangle\, dt, \ x, u \in X. \tag{1}$$

Let now (x_n) be a weakly null sequence in X such that $\|x_n\| \to 1$. Since $J_\varphi(x_n + tu) \overset{*}{\rightharpoonup} J_\varphi(tu)$, we get by (1)

$$\begin{aligned}\limsup_n \Phi(\|x_n + u\|) &= \limsup_n \Phi(\|x_n\|) + \int_0^1 \langle u, J_\varphi(tu)\rangle\, dt \\ &= \Phi(1) + \|u\|\int_0^1 \varphi(t\|u\|)\, dt \\ &= \Phi(1) + \Phi(\|u\|).\end{aligned}$$

Namely, for any $x \rightharpoonup 0$ with $\|x_n\| \to 1$, we have

$$\psi_{(x_n)}(u) = \Phi^{-1}(\Phi(1) + \Phi(\|u\|)),\ u \in X.$$

This finishes the proof. ∎

Recall that a Banach space X has the *uniform Opial property* ([P], [LTX]) if the Opial modulus of X, $r_X(c) > 0$ for all $c > 0$, where

$$r_X(c) = \inf\left\{\limsup_n \|x_n + x\| - 1:\ x_n \rightharpoonup 0,\ \limsup_n \|x_n\| \geq 1,\ \|x\| \geq c\right\}.$$

It is not hard to see that uniform (M) implies the uniform Opial property. Indeed we have $r_X(c) = \kappa_X(u_0) - 1$ for any $u_0 \in X$ such that $\|u_0\| = c$. However, the uniform Opial property does not in general imply uniform (M). The space $l_{p,q}$ $(1 < p, q < \infty)$ is the space l_p renormed by

$$|x|_{p,q} := \left(\|x^+\|_p^q + \|x^-\|_p^q\right)^{1/q},$$

where $x^+(x^-)$ is the positive (negative) part of x and $\|\cdot\|_p$ is the usual l_p norm. It is shown [X] that for all $1 < p$, $q < \infty$, $l_{p,q}$ has the uniform Opial property. But consider the type $\psi_{(e_n)}$, (e_n) being the standard unit basis of l_p. We obtain $\psi_{(e_n)}(e_1) = 2^{1/p}$ and $\psi_{(e_n)}(-e_1) = 2^{1/q}$. So $l_{p,q}$ fails to have property (M) if $p \neq q$.

Recall that a Banach space X is uniformly convex in every direction (UCED) if, for each $z \in X$ such that $\|z\| = 1$ and $\varepsilon > 0$, we have

$$\delta_x(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\|:\ \|x\| \leq 1,\ \|y\| \leq 1,\ x - y = tz,\ |t| \geq \varepsilon\right\} > 0.$$

THEOREM 2.3. *Suppose that a Banach space X has property (M) and is also UCED. Then X has both strict (M) and uniform (M).*

Proof. Given $u, v \in X$ such that $\|u\| < \|v\|$ and a weakly null sequence (x_n) in X. We may assume by property (M), with no loss of generality, that

$u = \lambda v$, where $\lambda = \|u\|/\|v\| \in [0,1)$. Put $r := \psi_{(x_n)}(v) > 0$. It follows that

$$\begin{aligned}\psi_{(x_n)}\left(\frac{\lambda+1}{2}v\right) &= \psi_{(x_n)}\left(\frac{u+v}{2}\right)\\ &= \limsup_n \left\|\frac{(x_n+u)+(x_n+v)}{2}\right\|\\ &\le r\left[1-\delta_z\left(\frac{\|v\|-\|u\|}{r}\right)\right] < r,\end{aligned}$$

where $z = v/\|v\|$. Since $\frac{\lambda+1}{2}\|v\| = \frac{1}{2}(\|u\| + \|v\|) > \|u\|$, we get by property (M),

$$\psi_{(x_n)}\left(\frac{\lambda+1}{2}v\right) \ge \psi_{(x_n)}(u).$$

We conclude from the last two inequalities that $\psi_{(x_n)}(u) < \psi_{(x_n)}(v)$ and X has strict (M). If $\|x_n\| \to 1$, then $r \le 1 + \|v\|$ and we also conclude that

$$\psi_{(x_n)}(u) \le \psi_{(x_n)}(v)\left[1-\delta_z\left(\frac{\|v\|-\|u\|}{1+\|v\|}\right)\right].$$

Hence

$$\kappa_X(u) \le \kappa_X(v)\left[1-\delta_z\left(\frac{\|v\|-\|u\|}{1+\|v\|}\right)\right] < \kappa_X(v)$$

and X has uniform (M). ∎

THEOREM 2.4. *If X is a Banach space such that X^* is separable, then for each $u \in X$ there exists a weakly null sequence (z_m), $\|z_m\| \to 1$, for which $\psi_{(z_m)}(u) = \kappa_X(u)$. In particular, in this case, strict (M) implies uniform (M).*

Proof. Let $(f_n) \subset X^*$ be a countable family which is dense in X^*. For each $m \ge 1$, we can find a weakly null sequence (x_n^m) satisfying $x_n^m \rightharpoonup 0$, $\|x_n^m\| \to 1$, and

$$\kappa_X(u) \le \psi_{(x_n^m)}(u) < \kappa_X(u) + \frac{1}{m}.$$

Pick n_m such that

$$\|x_{n_m}^m + u\| < \kappa_X(u) + \frac{1}{m},$$
$$\left|f_j\left(x_{n_m}^m\right)\right| < \frac{1}{m}(1 \le j \le m) \text{ and } 1 - \frac{1}{m} < \|x_{n_m}^m\| < 1 + \frac{1}{m}. \tag{2}$$

Set $z_m = x_{n_m}^m$, $m \ge 1$. It follows that $z_m \rightharpoonup 0$ and $\|z_m\| \to 1$. Hence, by (2), $\psi_{(z_m)}(u) = \kappa_X(u)$.

Next assume that X has strict (M) and u, v satisfy $\|u\| < \|v\|$. Let (z_n) be chosen such that $\psi_{(z_n)}(v) = \kappa_X(v)$. We then have, by strict (M), $\kappa_X(u) \le \psi_{(z_n)}(u) < \psi_{(z_n)}(v) = \kappa_X(v)$. ∎

Since c_0 has property (M), property (M) does not imply weak normal structure. Garcia and Sims [GS] showed that if X is a Banach space with property (M) and if X satisfies the property

$(*)$ There exists a point $x_0 \in X$, $\|x_0\| = 1$, such that whenever $y_n \rightharpoonup x_0$ and $\|y_n\| \to 1$, the separation index $\gamma(y_n) := \sup \inf\{\|y_{n_k} - y_{n_m}\|: k \neq m\} < 1$, where the sup is taken over all subsequences (y_{n_k}) of (y_n),

then X has weak normal structure. Using similar arguments we shall show that more can be obtained.

Recall that a Banach space X has the *Generalized Gossez–Lami Dozo property* $(GGLD)$ [Ji] if for any weakly null sequence (x_n) in X such that $\|x_n\| \to 1$, we have $D(x_n) > 1$, where

$$D(x_n) = \limsup_n \limsup_n \|x_n - x_m\|.$$

It is known that GGLD implies weak normal structure.

THEOREM 2.5. *If X is a Banach space with properties (M) and $(*)$, then X has GGLD.*

Proof. Suppose X fails to have GGLD. Then we have a weakly null sequence (x_n), $\|x_n\| \to 1$, such that $D(x_n) = 1$. Property (M) then implies that $\psi_{(x_n)}(u) = 1$ for $u \in X$ with $\|u\| \le 1$. Consider now the sequence $y_n := x_0 - x_n$. We have $y_n \rightharpoonup x_0$ and $\limsup_n \|y_n\| = \psi_{(x_n)}(x_0) = 1$ for $\|x_0\| = 1$. It follows from property $(*)$ that $\gamma(x_n) = \gamma(y_n) < 1$. This yields a subsequence $(x_{n'})$ of (x_n) satisfying

$$\|x_{n'} - x_{m'}\| < 1 - \varepsilon \text{ for all } n', m' \ge 1 \text{ and for some } \varepsilon > 0.$$

Hence $D(x_{n'}) \le 1 - \varepsilon$. This is impossible since $D(x_{n'}) \ge \lim \|x_{n'}\| = 1$. ∎

COROLLARY 2.1. *If X has property (M) and satisfies any of the following, then X has GGLD.*

(i) *X has the Kadec–Klee property (i.e., the relative weak and strong topologies agree on the unit sphere of X).*

(ii) *X is reflexive.*

(iii) *X has the Radom–Nikodym property.*

(iv) *X has the point of continuity property: for every weakly closed bounded subset A of X, the identity map (A, weak) to (A, norm) has at least one point of continuity. (See [EW] for details.)*

(v) *The unit sphere of X contains at least one point at which the relative weak and norm topologies agree.*

THEOREM 2.6. *Let X be a separable Banach space with property strict (M). Then we have*

(a) *For any infinite-dimensional closed subspace Y of X, there exists p, $1 \leq p < \infty$, such that for every $\varepsilon > 0$, Y contains a subspace E such that the Banach–Mazur distance $d(E, l_p) < 1 + \varepsilon$.*

(b) *If X contains no copy of l_1, then there exist $1 < p < \infty$ and a weakly null sequence (x_n) such that for every $u \in X$ and every α,*

$$\lim_n \|u + \alpha x_n\| = \left(\|u\|^p + |\alpha|^p\right)^{1/p}.$$

(c) *If X contains no copy of l_1 and is also stable [KM]; that is, for any pair of bounded sequences $(u_m), (v_n)$ in X,*

$$\lim_n \lim_m \|u_m + v_n\| = \lim_n \lim_m \|u_m + v_n\|,$$

then there exists p, $1 < p < \infty$ such that for every weakly null sequence (x_n) in X and every $x \in X$,

$$\limsup_n \|u + x_n\|^p = \|u\|^p + \limsup_n \|x_n\|^p.$$

Hence X has uniform (M) and $\kappa_X(u) = (1 + \|u\|^p)^{1/p}$.

Proof. (a) and (b) follow from the fact that strict (M) rules out c_0 in [K, Propositions 3.8 and 3.9], while (c) follows from the proof of Theorem 3.10 in [K]. ∎

Garcia–Sims [GS] observed that if X has property (M) and $c_0 \not\hookrightarrow X$, then X has weak normal structure. The following is an improvement.

THEOREM 2.7. *If X is a Banach space with property (M), then*

(i) *X has GGLD if and only if $c_0 \not\hookrightarrow X$.*

(ii) *if $c_0 \not\hookrightarrow X$ and if X^* is separable, then X has weak uniform normal structure. Namely, Bynum's constant of X* ([By]), $WCS(X) > 1$, *where*

$$WCS(X) = \inf\{D(x_n): x_n \rightharpoonup 0, \|x_n\| \to 1\}.$$

Proof. (i) Dowling [D, Theorem 8] showed GGLD implies $c_0 \not\hookrightarrow X$. Assume now X fails to have GGLD. Then we have a sequence (x_n) in X with $x_n \rightharpoonup 0$ and $\|x_n\| \to 1$, but $D(x_n) = 1$. It then follows by property (M) that

$$\psi_{(x_n)}(u) = 1 \quad \forall \|u\| \leq 1.$$

The result now follows by the argument found in Remark 2 of [GS].

(ii) Assume $WCS(X) = 1$. Then for each integer $k \geq 1$, we have a weakly null sequence (x_n^k), $\|x_n^k\| \to 1$, such that

$$\limsup_m \limsup_n \|x_n^k - x_m^k\| < 1 + \frac{1}{k}. \tag{3}$$

We may assume with no loss of generality that $\|x_n^k\| = 1$ (otherwise replace (x_n^k) by $(x_n^k/\|x_n^k\|)$.)

Repeat the argument in the proof of Theorem 2.4 to get a diagonal sequence $z_m := x_{n_m}^m$ $(m \geq 1)$ such that $z_m \rightharpoonup 0$, $\|z_m\| \to 1$. We next show that we may choose (z_m) so that $D(z_m) = 1$. Indeed by (3) and property (M) we see that $\psi_{(x_n^k)}(u) < 1 + 1/k$ for all k, where $\|u\| = 1$. In particular, $\|x_n^k + u\| < 1/k$ for large enough n. Hence by ensuring that n_k is chosen large enough we have $\|z_k + u\| < 1 + 1/k$. This implies that $\psi_{(z_m)}(u) \leq 1$ and thus $D(z_m) = 1$. Now we can repeat the proof of (i) above to get the contradiction $c_0 \hookrightarrow X$. ∎

Recall that a Banach space X satisfies Opial's property [O] if, whenever $x_n \rightharpoonup 0$ and $x \neq 0$, we have

$$\limsup_n \|x_n\| < \limsup_n \|x_n - x\|.$$

Similar to [D, Theorem 8] we have the following result.

THEOREM 2.8. *If X has both property (M) and Opial's property, then X contains no isomorphic copy of c_0.*

Proof. Assume X contains an isomorphic copy of c_0. Then by the James distortion theorem for c_0 [Ja], there are a null sequence (ε_n) of positive real numbers and a sequence (x_n) in X so that

$$(1-\varepsilon_n)\sup_{k\geq n}|t_k| \leq \left\| \sum_{k=n}^{\infty} t_k x_k \right\| \leq (1+\varepsilon_n)\sup_{k\geq n}|t_k|, \tag{4}$$

for all $(t_k)\in c_0$ and for all $n\geq 1$. By passing to a subsequence if necessary, we may assume without loss of generality that (ε_n) is decreasing. Since (x_n) is equivalent to the standard unit basis of c_0, (x_n) is weakly null. Clearly (4) implies $\|x_n\|\to 1$ and $1-\varepsilon_k \leq \|x_n - x_k\| \leq 1+\varepsilon_k$ for $n>k$ which in turn implies

$$\lim_{k\to\infty}\psi_{(x_n)}(x_k) = 1. \tag{5}$$

So by the fact that $\|x_k\|\to 1$ and the continuity of $\psi_{(x_n)}$, it follows from (5) and property (M) that $\psi_{(x_n)}(x) = 1$ for all $x\in X$ such that $\|x\|\leq 1$. This clearly contradicts the Opial property of X. ∎

We conclude this article with the following remark.

Remark. Recall that a Banach space X has property (M^*) ([K]) if whenever $u^*, v^* \in X^*$ with $\|u^*\|\leq\|v^*\|$ and whenever (x_n^*) is a weak* null sequence in X^* then

$$\limsup_n \|u^* + x_n^*\| \leq \limsup_n \|v^* + x_n^*\|.$$

Kalton [K] showed that if X has property (M^*), then X property (M). In an obvious way one can define strict (M^*) and uniform (M^*) for X. One then would conjecture that strict (M^*) (uniform (M^*)) would imply strict (M) (uniform (M)). The answer is, however, negative. For example, consider $(c_0)^* = l^1$. If a sequence (x_n) is weak* null in l^1, then we have (see [L])

$$\limsup_n \|x + x_n\| = \limsup_n \|x_n\| + \|x\|, \quad x\in l^1.$$

Hence c_0 has both strict (M^*) and uniform (M^*), but it has neither strict (M), nor uniform (M).

ACKNOWLEDGMENT

This work was carried out while the first author was visiting the University of Calabria (Cosenza). He thanks the mathematics department for its hospitality.

REFERENCES

[B] F. E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Rational Mech. Anal.* **24** (1967), 82–90.

[BM] M. S. Brodskii and D. P. Mil'man, On the center of a convex set, *Dolk. Akad. Nauk SSSR (N.S.)* **59** (1948), 837–840.

[By] W. L. Bynum, Normal structure coefficients for Banach spaces, *Pacific J. Math.* **86** (1980), 427–436.

[D] P. N. Dowling, Asymptotically isometric copies of c_0 and renorming of Banach spaces, *J. Math. Anal. Appl.* **228** (1998), 265–271.

[EW] G. Edgar and R. F. Wheeler, Topological properties of Banach spaces, *Pacific J. Math.* **115** (1984), 317–350.

[GS] J. Garcia-Falset and B. Sims, Property (M) and the weak fixed point property, *Proc. Amer. Math. Soc.* **125** (1997), 2891–2896.

[Ja] R. C. James, Uniformly non-square Banach spaces, *Ann. Math.* **80** (1964), 542–555.

[Ji] A. Jiménéz-Melado, Stability of weak normal structure in James' quasi relexive space, *Bull. Austral. Math. Soc.* **46** (1992), 367–372.

[K] N. J. Kalton, *M*-ideals of compact operators, *Illinois J. Math.* **37** (1993), 147–169.

[KM] J. L. Krivine and B. Maurey, Espaces de Banach stables, *Israel J. Math.* **39** (1981), 273–295.

[L] T. C. Lim, Asymptotic centers and nonexpansive mappings in conjugate spaces, *Pacific J. Math.* **90** (1980), 135–143.

[LTX] P. K. Lin, K. K. Tan, and H. K. Xu, Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings, *Nonlinear Anal.* **24** (1995), 929–946.

[O] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591–597.

[P] S. Prus, Banach spaces with the uniform Opial property, *Nonlinear Anal.* **18** (1992), 697–704.

[X] H. K. Xu, Geometrical coefficients of Banach spaces and nonlinear mappings, *in* "Recent Advances on Metric Fixed Point Theory" (T. Dominguez-Benavides, Ed.), pp. 161–178, Universidad de Seville Press, Seville, 1996.