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A determinantal approach to Appell polynomials

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ABSTRACT

A new definition by means of a determinantal form for Appell (1880) [1] polynomials is given. General properties, some of them new, are proved by using elementary linear algebra tools. Finally classic and non-classic examples are considered and the coefficients, calculated by an ad hoc Mathematica code, for particular sequences of Appell polynomials are given.

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1. Introduction

In 1880 [1] Appell introduced and widely studied sequences of n-degree polynomials

$$A_n(x), \quad n = 0, 1, \ldots \tag{1}$$

satisfying the recursive relations

$$\frac{\mathrm{d}A_n(x)}{\mathrm{d}x} = nA_{n-1}(x), \quad n = 1, 2, \ldots. \tag{2}$$

In particular, Appell noticed the one-to-one correspondence of the set of such sequences $\{A_n(x)\}_n$ and the set of numerical sequences $\{\alpha_n\}_n$, $\alpha_0 \neq 0$ given by the explicit representation

$$A_n(x) = \alpha_n + \binom{n}{1}\alpha_{n-1}x + \binom{n}{2}\alpha_{n-2}x^2 + \cdots + \alpha_0 x^n, \quad n = 0, 1, \ldots. \tag{3}$$

Eq. (3), in particular, shows explicitly that for each $n \geq 1$ $A_n(x)$ is completely determined by $A_{n-1}(x)$ and by the choice of the constant of integration α_n. Furthermore Appell provided an alternative general method to determine such sequences of polynomials that satisfy (2). In fact, given the power series:

$$a(h) = \alpha_0 + \frac{h}{1!}\alpha_1 + \frac{h^2}{2!}\alpha_2 + \cdots + \frac{h^n}{n!}\alpha_n + \cdots, \quad \alpha_0 \neq 0 \tag{4}$$

with α_i $i = 0, 1, \ldots$ real coefficients, a sequence of polynomials satisfying (2) is determined by the power series expansion of the product $a(h)\mathrm{e}^{hx}$, i.e.:

$$a(h)\mathrm{e}^{hx} = A_0(x) + \frac{h}{1!}A_1(x) + \frac{h^2}{2!}A_2(x) + \cdots + \frac{h^n}{n!}A_n(x) + \cdots. \tag{5}$$

The function $a(h)$ is called the 'generating function' of the sequence of polynomials $A_n(x)$.

Well known examples of sequences of polynomials verifying (2) or, equivalently (3) and (5), now called Appell Sequences, are:

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(1) the sequences of growing powers of variable x

$$1, x, x^2, \ldots, x^n, \ldots,$$

as already stressed in [1];
(2) the Bernoulli sequence $B_n(x)$ [2,3];
(3) the Euler sequences $E_n(x)$ [4,3];
(4) the Hermite normalized sequences $H_n(x)$ [5];
(5) the Laguerre sequences $L_n(x)$ [5].

Moreover, further generalizations of above polynomials have been considered [5–7].

Sequences of Appell polynomials have been well studied because of their remarkable applications in Mathematical and Numerical Analysis, as well as in Number theory, as both classic literature [1,8,9,5,10] and more recent literature [11–15,6,7,16] testify.

In a recent work [17], a new approach to Bernoulli polynomials was given, based on a determinantal definition. The authors, through basic tools of linear algebra, have recovered the fundamental properties of Bernoulli polynomials; moreover the equivalence, with a triangular theorem, of all previous approaches is given.

The aim of this work is to propose a similar approach for more general Appell polynomials, for the following motivations:

(i) the algebraic approach provides a unifying theory for all classes of polynomials considered in (1)–(5) and their very natural generalization;
(ii) it is possible to compute the coefficients or the value in a chosen point, for particular sequences of Appell polynomials, through an efficient and stable Gaussian algorithm;
(iii) this theory is simpler that the classical analytic approaches based, for example, on the method of generating functions;
(iv) the proposed algebraic approach allows the solution of the following remarkable general linear interpolation problem, which is in an advanced phase of study and will appear later:

Let $\mathscr{P}_n$ be the space of univariate polynomials of degree $\leq n$ and L a linear functional defined on $C^n[a,b]$ such that $L(1) \neq 0$. Let $\omega_0, \omega_1, \ldots, \omega_n \in \mathbb{R}$. Then, there exists a unique polynomial $P_n(x) \in \mathscr{P}_n$ such that

$$L\left(\frac{\frac{\mathrm{d}^i}{\mathrm{d}x^i} P_n(x)}{i!}\right) = \omega_i \quad i = 0, 1, \ldots, n.$$

The solution can be expressed, using the determinantal form, by a basis of Appell polynomials. Relevant examples are:

(a) $L(f) = \int_0^1 w(x) f(x) \mathrm{d}x$, where $w(x)$ is a general weight function. In this case the basis is realized in [2] or generalized Bernoulli polynomials [7].
(b) $L(f) = \frac{w_1 f(1) + w_2 f(0)}{w_1 + w_2}$, $w_1, w_2 > 0$. In this case the basis is realized in [4] or generalized Euler polynomials (4.4).

The work is organized as follows:

In Section 2 we introduce the new approach and we establish its equivalence with previous characterizations through a circular theorem. In Section 3 we give general properties by employing basic tools of linear algebra. In Section 4, we consider classic examples, in particular Bernoulli, Euler, Hermite and Laguerre polynomials and their possible generalizations not studied in the literature so far. Finally, in Section 5, we provide explicitly the same classes of Appell polynomials, by using an ad hoc Mathematica code based on the new definition.

2. A determinantal definition

Let us consider $P_n(x),\ n = 0, 1, \ldots$ the sequence of polynomials of degree n defined by

$$\begin{cases} P_0(x) = \dfrac{1}{\beta_0} \\ P_n(x) = \dfrac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \ldots \end{cases} \tag{6}$$

where $\beta_0, \beta_1, \ldots, \beta_n \in \mathbb{R},\ \beta_0 \neq 0$.

Then we have

Theorem 1. *The following relation holds*

$$P_n'(x) = nP_{n-1}(x) \quad n = 1, 2, \ldots. \tag{7}$$

Proof. Using the properties of linearity we can differentiate the determinant (6), expand the resulting determinant with respect to the first column and recognize the factor $P_{n-1}(x)$ after multiplication of the i-th row by $i-1,\ i = 2, \ldots, n$ and j-th column by $\frac{1}{j},\ j = 1, \ldots, n$. □

Theorem 2. *The polynomials $P_n(x)$, defined in (6), can be written in the form:*

$$P_n(x) = \alpha_n + \binom{n}{1}\alpha_{n-1}x + \binom{n}{2}\alpha_{n-2}x^2 + \cdots + \alpha_0 x^n, \quad n = 0, 1, \ldots \tag{8}$$

where

$$\alpha_0 = \frac{1}{\beta_0}, \tag{9}$$

$$\alpha_i = \frac{(-1)^i}{(\beta_0)^{i+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{i-1} & \beta_i \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} \\ 0 & \beta_0 & \cdots & \cdots & \binom{i-1}{2}\beta_{i-3} & \binom{i}{2}\beta_{i-2} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \binom{i}{i-1}\beta_1 \end{vmatrix}, \quad i = 1, 2, \ldots, n. \tag{10}$$

Proof. The desired result follows by expanding the determinant $P_n(x)$ with respect to the first row. □

Corollary 3. *For the polynomials $P_n(x)$ we have*

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j} P_{n-j}(0)x^j, \quad n = 0, 1, \ldots. \tag{11}$$

Proof. Taking into account

$$P_i(0) = \alpha_i, \quad i = 0, 1, \ldots, n, \tag{12}$$

relation (11) is a consequence of (8). □

Remark 4 (*Computation*). For computation we can observe that α_n is a n-order determinant of an upper Hessenberg matrix and it is known that the algorithm of Gaussian elimination without pivoting for computing the determinant of an upper Hessenberg matrix is stable [18, p. 27]. With the same algorithm, from (6), we can calculate the value of an Appell polynomial in a fixed point, without the explicit calculation of the coefficients.

Theorem 5. *For the coefficients α_i in (8) the following relations hold*

$$\alpha_0 = \frac{1}{\beta_0}, \tag{13}$$

$$\alpha_i = -\frac{1}{\beta_0}\sum_{k=0}^{i-1} \binom{i}{k} \beta_{i-k}\alpha_k, \quad i = 1, 2, \ldots, n. \tag{14}$$

Proof. Set $\overline{\alpha}_i = (-1)^i (\beta_0)^{i+1} \alpha_i$ for $i = 1, 2, \ldots, n$. From (10) α_i is a determinant of an upper Hessenberg matrix of order i and for that [17] we have

$$\overline{\alpha}_i = \sum_{k=0}^{i-1} (-1)^{i-k-1} h_{k+1,i} q_k(i) \overline{\alpha}_k, \tag{15}$$

where

$$h_{l,m} = \begin{cases} \beta_m & \text{for } l = 1, \\ \binom{m}{l-1} \beta_{m-l+1} & \text{for } 1 < l \leq m+1, \qquad l, m = 1, 2, \ldots, i, \\ 0 & \text{for } l > m+1, \end{cases} \tag{16}$$

and

$$q_k(i) = \prod_{j=k+2}^{i} h_{j,j-1} = (\beta_0)^{i-k-1}, \quad k = 0, 1, \ldots, i-2, \tag{17}$$

$$q_{i-1}(i) = 1. \tag{18}$$

By virtue of the previous setting, (15) implies

$$\begin{aligned} \overline{\alpha}_i &= \sum_{k=0}^{i-2} (-1)^{i-k-1} \binom{i}{k} \beta_{i-k} (\beta_0)^{i-k-1} \overline{\alpha}_k + \binom{i}{i-1} \beta_1 \overline{\alpha}_{i-1} \\ &= (-1)^i (\beta_0)^{i+1} \left(-\frac{1}{\beta_0} \sum_{k=0}^{i-1} \binom{i}{k} \beta_{i-k} \frac{1}{(-1)^k (\beta_0)^{k+1}} \overline{\alpha}_k \right) \\ &= (-1)^i (\beta_0)^{i+1} \left(-\frac{1}{\beta_0} \sum_{k=0}^{i-1} \binom{i}{k} \beta_{i-k} \alpha_k \right) \end{aligned}$$

and the proof is concluded. □

Theorem 6. *Let $P_n(x)$ be the sequence of Appell polynomials with generating function $a(h)$ as in (4) and (5). If $\beta_0, \beta_1, \ldots, \beta_n$, with $\beta_0 \neq 0$, are the coefficients of Taylor series expansion of function $\frac{1}{a(h)}$ we have*

$$P_0(x) = \frac{1}{\beta_0} \tag{19}$$

$$P_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \ldots. \tag{20}$$

Proof. Let $P_n(x)$ be the sequence of Appell polynomials with generating function $a(h)$ i.e.

$$a(h) = \alpha_0 + \frac{h}{1!}\alpha_1 + \frac{h^2}{2!}\alpha_2 + \cdots + \frac{h^n}{n!}\alpha_n + \cdots \tag{21}$$

and

$$a(h)e^{hx} = \sum_{n=0}^{\infty} P_n(x) \frac{h^n}{n!}. \tag{22}$$

Let $b(h)$ be such that $a(h)b(h) = 1$. We can write $b(h)$ as its Taylor series expansion (in h) at the origin, that is

$$b(h) = \beta_0 + \frac{h}{1!}\beta_1 + \frac{h^2}{2!}\beta_2 + \cdots + \frac{h^n}{n!}\beta_n + \cdots. \tag{23}$$

Then, according to the Cauchy-product rules, we find

$$a(h)b(h) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\alpha_k\beta_{n-k}\frac{h^n}{n!}$$

by which

$$\sum_{k=0}^{n}\binom{n}{k}\alpha_k\beta_{n-k} = \begin{cases}1 & \text{for } n=0,\\ 0 & \text{for } n>0.\end{cases}$$

Hence

$$\begin{cases}\beta_0 = \dfrac{1}{\alpha_0},\\ \beta_n = -\dfrac{1}{\alpha_0}\left(\displaystyle\sum_{k=1}^{n}\binom{n}{k}\alpha_k\beta_{n-k}\right), \quad n=1,2,\ldots.\end{cases} \tag{24}$$

Let us multiply both sides of Eq. (22) by $\frac{1}{a(h)}$ and, in the same equation, replace functions e^{hx} and $\frac{1}{a(h)}$ by their Taylor series expansion at the origin; then (22) becomes

$$\sum_{n=0}^{\infty}\frac{x^n h^n}{n!} = \sum_{n=0}^{\infty}P_n(x)\frac{h^n}{n!}\sum_{n=0}^{\infty}\frac{h^n}{n!}\beta_n. \tag{25}$$

By multiplying the series on the left hand side of (25) according to the Cauchy-product rules, the previous equality leads to the following system of infinite equations in the unknown $P_n(x),\ n=0,1,\ldots$

$$\begin{cases}P_0(x)\beta_0 = 1,\\ P_0(x)\beta_1 + P_1(x)\beta_0 = x,\\ P_0(x)\beta_2 + \dbinom{2}{1}P_1(x)\beta_1 + P_2(x)\beta_0 = x^2,\\ \vdots\\ P_0(x)\beta_n + \dbinom{n}{1}P_1(x)\beta_{n-1} + \cdots + P_n(x)\beta_0 = x^n,\\ \vdots\end{cases} \tag{26}$$

The special form of the previous system (lower triangular) allows us to work out the unknown $P_n(x)$ operating with the first $n+1$ equations, only by applying the Cramer rule:

$$P_n(x) = \frac{\begin{vmatrix}\beta_0 & 0 & 0 & \cdots & 0 & 1\\ \beta_1 & \beta_0 & 0 & \cdots & 0 & x\\ \beta_2 & \dbinom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & x^2\\ \vdots & & & \ddots & & \vdots\\ \beta_{n-1} & \dbinom{n-1}{1}\beta_{n-2} & \cdots & \cdots & \beta_0 & x^{i-1}\\ \beta_n & \dbinom{n}{1}\beta_{n-1} & \cdots & \cdots & \dbinom{n}{n-1}\beta_1 & x^i\end{vmatrix}}{\begin{vmatrix}\beta_0 & 0 & 0 & \cdots & 0 & 0\\ \beta_1 & \beta_0 & 0 & \cdots & 0 & 0\\ \beta_2 & \dbinom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & 0\\ \vdots & & & \ddots & & \vdots\\ \beta_{n-1} & \dbinom{n-1}{1}\beta_{n-2} & \cdots & \cdots & \beta_0 & 0\\ \beta_n & \dbinom{n}{1}\beta_{n-1} & \cdots & \cdots & \dbinom{n}{n-1}\beta_1 & \beta_0\end{vmatrix}}$$

$$= \frac{1}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 1 \\ \beta_1 & \beta_0 & 0 & \cdots & 0 & x \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & x^2 \\ \vdots & & & \ddots & & \vdots \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \cdots & \cdots & \beta_0 & x^{i-1} \\ \beta_n & \binom{n}{1}\beta_{n-1} & \cdots & \cdots & \binom{n}{n-1}\beta_1 & x^i \end{vmatrix}.$$

By transposition of the previous, we have

$$P_n(x) = \frac{1}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \\ 1 & x & x^2 & \cdots & x^{n-1} & x^n \end{vmatrix}, \quad n = 1, 2, \ldots \tag{27}$$

that is exactly (20) after n circular row exchanges: more precisely, the i-th row moves to the $(i+1)$-th position for $i = 1, \ldots, n-1$, the n-th row goes to the first position. □

Theorems 1, 2 and 6 concur to assert the validity of the following

Theorem 7 (*Circular*). *For Appell polynomials we have*

$$\begin{array}{ccc} (2 \text{ and } 3) & \longrightarrow & (4 \text{ and } 5) \\ \nwarrow & & \swarrow \\ & (6) & \end{array} \tag{28}$$

Proof.

(2 and 3) ⇒ (4 and 5) Follows from Appell's proof [1].
(4 and 5) ⇒ (6) Follows from Theorem 6.
(6) ⇒ (2 and 3) Follows from Theorems 1 and 2. □

Therefore we can give, now, the following

Definition 8. The Appell polynomial of degree n, denoted by $A_n(x)$, is defined by

$$A_0(x) = \frac{1}{\beta_0} \tag{29}$$

$$A_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \ldots \tag{30}$$

where $\beta_0, \beta_1, \ldots, \beta_n \in \mathbb{R}$, $\beta_0 \neq 0$.

3. General properties of Appell polynomials

By elementary tools of linear algebra we can prove general properties of Appell polynomials, some of them known, others not known.

Theorem 9 (*Recurrence*). *For Appell sequence $A_n(x)$ we have*

$$A_n(x) = \frac{1}{\beta_0}\left(x^n - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}A_k(x)\right), \quad n = 1, 2, \dots. \tag{31}$$

Proof. The claimed thesis follows by observing that $A_n(x)$ is a determinant of an upper Hessenberg matrix of order $n+1$ [17] as for Theorem 5. □

Corollary 10. *If $A_n(x)$ is an Appell polynomial then*

$$x^n = \sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}A_k(x), \quad n = 0, 1, \dots. \tag{32}$$

Proof. The result follows from (31). □

Let us consider two sequences of Appell polynomials

$$A_n(x), \qquad B_n(x) \quad n = 0, 1, \dots$$

and indicate with $(AB)_n(x)$ the polynomial that is obtained replacing in $A_n(x)$ powers $x^0, x^1, \dots, x^n$, respectively, with the polynomials $B_0(x), B_1(x), \dots, B_n(x)$. The following theorem can be proven.

Theorem 11. *The sequences*

(i) $\lambda A_n(x) + \mu B_n(x),\ \lambda, \mu \in \mathbb{R}$,
(ii) $(AB)_n(x)$

are sequences of Appell polynomials again.

Proof. (i) follow from the property of linearity of determinant.
(ii) by definition we have

$$(AB)_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} B_0(x) & B_1(x) & B_2(x) & \cdots & \cdots & B_{n-1}(x) & B_n(x) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}.$$

Expanding the determinant $(AB)_n(x)$ with respect to the first row we obtain

$$\begin{aligned}(AB)_n(x) &= \frac{(-1)^n}{(\beta_0)^{n+1}}\sum_{j=0}^{n}(-1)^j(\beta_0)^j\binom{n}{j}\overline{\alpha}_{n-j}B_j(x) \\ &= \sum_{j=0}^{n}\frac{(-1)^{n-j}}{(\beta_0)^{n-j+1}}\binom{n}{j}\overline{\alpha}_{n-j}B_j(x),\end{aligned} \tag{33}$$

where

$$\overline{\alpha}_0 = 1,$$

$$\overline{\alpha}_i = \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{i-1} & \beta_i \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} \\ 0 & \beta_0 & \cdots & \cdots & \binom{i-1}{2}\beta_{i-3} & \binom{i}{2}\beta_{i-2} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \binom{i}{i-1}\beta_1 \end{vmatrix}, \quad i = 1, 2, \dots, n.$$

We observe that

$$A_i(0) = \frac{(-1)^i}{(\beta_0)^{i+1}}\overline{\alpha}_i, \quad i = 1, 2, \dots, n$$

and hence (33) becomes

$$(AB)_n(x) = \sum_{j=0}^{n} \binom{n}{j} A_{n-j}(0) B_j(x). \tag{34}$$

Differentiating both hand sides of (34) and since $B_j(x)$ is a sequence of Appell polynomials, we deduce

$$\begin{aligned} ((AB)_n(x))' &= \sum_{j=0}^{n} \binom{n}{j} A_{n-j}(0) B_j'(x) \\ &= \sum_{j=1}^{n} j \binom{n}{j} A_{n-j}(0) B_{j-1}(x) \\ &= n \sum_{j=1}^{n} \binom{n-1}{j-1} A_{n-j}(0) B_{j-1}(x) \\ &= n \sum_{j=0}^{n-1} \binom{n-1}{j} A_{n-1-j}(0) B_j(x) \\ &= n (AB)_{n-1}(x). \quad \square \end{aligned}$$

Theorem 12 ([10, p. 27]). *For Appell polynomials $A_n(x)$ we have*

$$A_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) y^{n-i}, \quad n = 0, 1, \dots. \tag{35}$$

Proof. Starting with the definition in (30) and using the identity

$$(x+y)^i = \sum_{k=0}^{i} \binom{i}{k} y^k x^{i-k}, \tag{36}$$

we infer

$$A_n(x+y) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & (x+y)^1 & \cdots & (x+y)^{n-1} & (x+y)^n \\ \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \beta_0 & \beta_1 \binom{n}{n-1} \end{vmatrix}$$

$$= \sum_{i=0}^{n} \frac{(-1)^n y^i}{(\beta_0)^{n+1}} \begin{vmatrix} 0 & 0 & \cdots & 0 & \binom{i}{i} & \cdots & \binom{n-1}{i} x^{n-1-i} & \binom{n}{i} x^{n-i} \\ \beta_0 & \beta_1 & \cdots & \beta_{i-1} & \beta_i & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \beta_{i-2}\binom{i-1}{1} & \beta_{i-1}\binom{i}{1} & \cdots & \beta_{n-2}\binom{n-1}{1} & \beta_{n-1}\binom{n}{1} \\ \vdots & \ddots & \ddots & & & & & \vdots \\ \vdots & & \ddots & \beta_0 & \beta_1\binom{i}{i-1} & \cdots & & \beta_{n-i+1}\binom{n}{i-1} \\ \vdots & & & 0 & \beta_0 & \cdots & & \beta_{n-i}\binom{n}{i} \\ \vdots & & & & 0 & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \beta_0 & \beta_1\binom{n}{n-1} \end{vmatrix}$$

$$= \sum_{i=0}^{n} y^i \frac{(-1)^{n-i}}{(\beta_0)^{n-i+1}} \begin{vmatrix} \binom{i}{i} & \binom{i+1}{i} x^1 & \binom{i+2}{i} x^2 & \cdots & \binom{n-1}{i} x^{n-i-1} & \binom{n}{i} x^{n-i} \\ \beta_0 & \beta_1\binom{i+1}{i} & \beta_2\binom{i+2}{i} & \cdots & \beta_{n-i-1}\binom{n-1}{i} & \beta_{n-i}\binom{n}{i} \\ 0 & \beta_0 & \beta_1\binom{i+2}{i+1} & \cdots & \beta_{n-i-2}\binom{n-1}{i+1} & \beta_{n-i-1}\binom{n}{i+1} \\ \vdots & & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \beta_1\binom{n}{n-1} \end{vmatrix}.$$

We divide, now, each j-th column, $j = 2, \ldots, n-i+1$, for $\binom{i+j-1}{i}$ and multiply each h-th row, $h = 3, \ldots, n-i+1$, for $\binom{i+h-2}{i}$. Thus we finally obtain

$$A_n(x+y) = \sum_{i=0}^{n} \frac{\binom{i+1}{i}\cdots\binom{n}{i}}{\binom{i+1}{i}\cdots\binom{n-1}{i}} y^i \frac{(-1)^{n-i}}{(\beta_0)^{n-i+1}} \begin{vmatrix} 1 & x^1 & x^2 & \cdots & x^{n-i-1} & x^{n-i} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-i-1} & \beta_{n-i} \\ 0 & \beta_0 & \beta_1\binom{2}{1} & \cdots & \beta_{n-i-2}\binom{n-i-1}{1} & \beta_{n-i-1}\binom{n-i}{1} \\ \vdots & & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \beta_1\binom{n-i}{n-i-1} \end{vmatrix}$$

$$= \sum_{i=0}^{n} \binom{n}{i} A_{n-i}(x) y^i = \sum_{i=0}^{n} \binom{n}{i} A_i(x) y^{n-i}. \quad \square$$

Corollary 13 (*Forward Difference*). *For Appell polynomials $A_n(x)$ we have*

$$\Delta A_n(x) \equiv A_n(x+1) - A_n(x) = \sum_{i=0}^{n-1} \binom{n}{i} A_i(x), \quad n = 0, 1, \ldots. \tag{37}$$

Proof. The desired result follows from (35) with $y = 1$. $\square$

Corollary 14 (*Multiplication Theorem*). *For Appell polynomials $A_n(x)$ we have*

$$A_n(mx) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) (m-1)^{n-i} x^{n-i}, \qquad \begin{array}{l} n = 0, 1, \ldots, \\ m = 1, 2, \ldots. \end{array} \tag{38}$$

Proof. The desired result follows from (35) with $y = x(m-1)$. $\square$

Theorem 15 (*Symmetry*). *For Appell polynomials $A_n(x)$ the following relation holds*

$$\left(A_n\left(h-x\right)=(-1)^n A_n(x)\right) \Leftrightarrow \left(A_n(h)=(-1)^n A_n(0)\right), \qquad \begin{matrix} h\in\mathbb{R} \\ n=0,1,\ldots. \end{matrix} \tag{39}$$

Proof. ($\Rightarrow$) Follows from the hypothesis with $x=0$
($\Leftarrow$) Using (35) we find

$$\begin{aligned} A_n\left(h-x\right) &= \sum_{i=0}^{n}\binom{n}{i}A_i\left(h\right)\left(-x\right)^{n-i} \\ &= (-1)^n\sum_{i=0}^{n}\binom{n}{i}A_i\left(h\right)(-1)^i x^{n-i} \\ &= (-1)^n\sum_{i=0}^{n}\binom{n}{i}A_{n-i}(h)\,(-1)^{\,n-i}x^i. \end{aligned}$$

Therefore, using the assumptions and (11), we have

$$\begin{aligned} A_n\left(h-x\right) &= (-1)^n\sum_{i=0}^{n}\binom{n}{i}A_{n-i}(0)x^i \\ &= (-1)^n A_n(x). \quad \square \end{aligned}$$

Lemma 16. *For the numbers α_{2n+1} and β_{2n+1} we have*

$$(\alpha_{2n+1}=0) \Longleftrightarrow (\beta_{2n+1}=0)\,, \quad n=0,1,\ldots. \tag{40}$$

Proof. As in (24), we know that

$$\begin{cases} \beta_0=\dfrac{1}{\alpha_0}, \\ \beta_n=-\dfrac{1}{\alpha_0}\left(\displaystyle\sum_{k=1}^{n}\binom{n}{k}\alpha_k\beta_{n-k}\right), \quad n=1,2,\ldots. \end{cases}$$

Hence

$$\begin{cases} \beta_1=-\dfrac{1}{\alpha_0}\alpha_1\beta_0, \\ \beta_{2n+1}=-\dfrac{1}{\alpha_0}\dbinom{2n+1}{1}\alpha_1\beta_{2n}-\dfrac{1}{\alpha_0}\left(\displaystyle\sum_{k=1}^{n}\left[\binom{2n+1}{2k}\alpha_{2k}\beta_{2(n-k)+1}+\binom{2n+1}{2k+1}\alpha_{2k+1}\beta_{2(n-k)}\right]\right), \\ \quad n=1,2,\ldots \end{cases}$$

and

$$\begin{aligned} \alpha_{2n+1} &= 0, \quad n=0,1,\ldots \\ &\Rightarrow \begin{cases} \beta_1=0 \\ \beta_{2n+1}=-\dfrac{1}{\alpha_0}\displaystyle\sum_{k=1}^{n}\binom{2n+1}{2k}\alpha_{2k}\beta_{2(n-k)+1}, \quad n=1,2,\ldots \end{cases} \\ &\Rightarrow \beta_{2n+1}=0, \quad n=0,1,\ldots. \end{aligned}$$

In the same way, again from (24), we have

$$\begin{cases} \alpha_0=\dfrac{1}{\beta_0} \\ \alpha_n=-\dfrac{1}{\beta_0}\left(\displaystyle\sum_{k=0}^{n-1}\binom{n}{k}\alpha_k\beta_{n-k}\right), \quad n=1,2,\ldots. \end{cases}$$

As a consequence

$$\begin{cases} \alpha_1=-\dfrac{1}{\beta_0}\alpha_0\beta_1, \\ \alpha_{2n+1}=-\dfrac{1}{\beta_0}\left(\displaystyle\sum_{k=0}^{n-1}\left[\binom{2n+1}{2k}\alpha_{2k}\beta_{2(n-k)+1}+\binom{2n+1}{2k+1}\alpha_{2k+1}\beta_{2(n-k)}\right]\right)-\dfrac{1}{\beta_0}\dbinom{2n+1}{2n}\alpha_{2n}\beta_1, \\ \quad n=1,2,\ldots \end{cases}$$

and

$$\begin{aligned} \beta_{2n+1} &= 0, \quad n = 0, 1, \ldots \\ &\Rightarrow \begin{cases} \alpha_1 = 0, \\ \alpha_{2n+1} = -\dfrac{1}{\beta_0} \displaystyle\sum_{k=0}^{n-1} \binom{2n+1}{2k+1} \alpha_{2k+1}\beta_{2(n-k)}, \quad n = 1, 2, \ldots \end{cases} \\ &\Rightarrow \alpha_{2n+1} = 0, \quad n = 0, 1, \ldots . \quad \square \end{aligned}$$

Theorem 17. *For Appell polynomials $A_n(x)$ the following relation holds*

$$\left(A_n(-x) = (-1)^n A_n(x)\right) \Longleftrightarrow \left(\beta_{2n+1} = 0\right), \quad n = 0, 1, \ldots . \tag{41}$$

Proof. By Theorem 15 with $h = 0$ and Lemma 16, we find

$$(A_n(-x) = (-1)^n A_n(x)) \Longleftrightarrow (A_n(0) = (-1)^n A_n(0)) \Longleftrightarrow (A_{2n+1}(0) = 0) \Longleftrightarrow (\alpha_{2n+1} = 0) \Longleftrightarrow (\beta_{2n+1} = 0). \quad \square$$

Theorem 18. *For each $n \geq 1$ it is true that*

$$\int_0^x A_n(x)\mathrm{d}x = \frac{1}{n+1}\left[A_{n+1}(x) - A_{n+1}(0)\right] \tag{42}$$

and

$$\int_0^1 A_n(x)\mathrm{d}x = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}A_i(0). \tag{43}$$

Proof. Equality (42) follows from (7). Moreover, for $x = 1$ we find

$$\int_0^1 A_n(x)\mathrm{d}x = \frac{1}{n+1}\left[A_{n+1}(1) - A_{n+1}(0)\right] \tag{44}$$

and, using (35) with $x = 0$ and $y = 1$, we obtain

$$A_{n+1}(1) = \sum_{i=0}^{n+1}\binom{n+1}{i}A_i(0), \tag{45}$$

so, by (45), relation (44) becomes

$$\begin{aligned} \int_0^1 A_n(x)\mathrm{d}x &= \frac{1}{n+1}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}A_i(0) - A_{n+1}(0)\right] \\ &= \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}A_i(0). \quad \square \end{aligned}$$

4. Examples

In this section we present some examples.

4.1. Bernoulli polynomials

Placing

$$\beta_0 = 1, \tag{46}$$

$$\beta_i = \frac{1}{i+1}, \quad i = 1, \ldots, n, \tag{47}$$

in (29) and (30), the resulting Appell polynomial is known as Bernoulli polynomial [2]. The determinantal form of this polynomial has been considered in [17] and the fundamental properties have also been obtained through elementary algebraic tools.

Moreover the following identity can, now, be derived.

Theorem 19. *For Bernoulli polynomials $B_n(x)$ we have*

$$m^{n-1}\sum_{i=0}^{m-1} B_n\left(x+\frac{i}{m}\right)=\sum_{i=0}^{n}\binom{n}{i}B_i(x)\,(m-1)^{n-i}\,x^{n-i},\qquad \begin{matrix} n=0,1,\ldots, \\ m=1,2,\ldots. \end{matrix} \tag{48}$$

Proof. It is known [19] that

$$B_n\,(mx)=m^{n-1}\sum_{i=0}^{m-1}B_n\left(x+\frac{i}{m}\right),\qquad \begin{matrix} n=0,1,\ldots, \\ m=1,2,\ldots \end{matrix} \tag{49}$$

and hence from (38) and (49) the proof is concluded. □

4.1.1. Generalized Bernoulli polynomials

By direct inspection of (46) and (47) we deduce

$$\beta_i=\int_0^1 x^i \mathrm{d}x,\quad i=0,1,\ldots,n. \tag{50}$$

Analogously, we can consider the weighted coefficients

$$\beta_i^w=\int_0^1 w(x)x^i \mathrm{d}x,\quad i=0,1,\ldots,n, \tag{51}$$

where $w(x)$ is a general weight function.

In particular by taking the classical Jacobi weight, $w(x)=(1-x)^{\alpha}\,x^{\beta},\ \alpha,\beta>-1$, we obtain

$$\beta_i^w=\frac{\Gamma\,(\alpha+1)\,\Gamma\,(\beta+i+1)}{\Gamma\,(\alpha+\beta+i+2)},\quad i=0,1,\ldots,n. \tag{52}$$

The relative Appell polynomials, called now Bernoulli–Jacobi, are not considered in the literature to our knowledge, except for the case $\alpha=\beta=0$, for which we find again the Bernoulli polynomials. For the case $\alpha=\beta=-1/2$ it is useful to normalize by setting

$$\beta_i^w=\frac{1}{\pi}\frac{\Gamma\,(\alpha+1)\,\Gamma\,(\beta+i+1)}{\Gamma\,(\alpha+\beta+i+2)},\quad i=0,1,\ldots,n. \tag{53}$$

4.2. Hermite normalized polynomials

Assuming

$$\beta_0=1, \tag{54}$$

$$\beta_i^w=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}x^i\mathrm{d}x=\begin{cases}0 & \text{for } i \text{ odd}\\ \dfrac{(i-1)(i-3)\cdot\cdots\cdot 3\cdot 1}{2^{\frac{i}{2}}} & \text{for } i \text{ even}\end{cases},\quad i=1,\ldots,n, \tag{55}$$

in (29) and (30), the related Appell polynomials coincide with the well-known Hermite normalized polynomials [5].

It is known [9] that Hermite normalized polynomials are the only ones which are, at the same time, orthogonal and Appell polynomials.

The Hessenberg determinantal form does not seem to be known in literature.

4.2.1. Generalized Hermite polynomials

Assuming

$$\beta_i^w=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-|x|^{\alpha}}x^i\mathrm{d}x=\begin{cases}0 & \text{for } i \text{ odd}\quad i=0,1,\ldots,n,\\ \dfrac{2}{\alpha\sqrt{\pi}}\Gamma\left(\dfrac{i+1}{\alpha}\right) & \text{for } i \text{ even}\quad \alpha>0,\end{cases} \tag{56}$$

in (29) and (30), we obtain a wider class of Appell polynomials.

4.2.2. Generalized Laguerre polynomials

Placing

$$\beta_i = \int_0^{+\infty} e^{-sx} x^i dx = \frac{1}{s}\Gamma\left(\frac{i+1}{s}\right), \quad s > 0,\ i = 1, \ldots, n, \tag{57}$$

in (29) and (30), we obtain a new class of Appell polynomials, called now Appell–Laguerre, that does not seem to be known in literature, except for the case $s = 1$ [5].

4.3. Euler polynomials

Placing

$$\beta_0 = 1, \tag{58}$$

$$\beta_i = \frac{1}{2}, \quad i = 1, \ldots, n, \tag{59}$$

in (29) and (30), the resulting Appell polynomials are known as Euler polynomials [4]. The determinantal form seems new. In fact we have

$$E_0(x) = 1, \tag{60}$$

$$E_n(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1} & \cdots & \cdots & \frac{1}{2}\binom{n-1}{1} & \frac{1}{2}\binom{n}{1} \\ 0 & 0 & 1 & \cdots & \cdots & \frac{1}{2}\binom{n-1}{2} & \frac{1}{2}\binom{n}{2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \frac{1}{2}\binom{n}{n-1} \end{vmatrix}, \quad n = 1, 2, \ldots. \tag{61}$$

Concerning Euler polynomials, all the properties proved in general for Appell polynomials hold. In particular we have the following result.

Theorem 20. *For Euler polynomials $E_n(x)$ we have*

$$E_n(x) = x^n - \frac{1}{2^n}\sum_{k=0}^{n-1}\binom{n}{k} E_k(x), \quad n = 1, 2, \ldots. \tag{62}$$

Proof. The claimed thesis follows from (31). □

Theorem 21. *For Euler polynomials $E_n(x)$ we have*

$$\sum_{i=0}^{n}\binom{n}{i} E_i(x)\,(m-1)^{n-i}\,x^{n-i} = \begin{cases} m^n \sum_{i=0}^{m-1}(-1)^i E_n\left(x + \frac{i}{m}\right), & n = 0, 1, \ldots,\ m = 1, 3, \ldots, \\ -\frac{2}{n+1} m^n \sum_{i=0}^{m-1}(-1)^i B_{n+1}\left(x + \frac{i}{m}\right), & n = 0, 1, \ldots,\ m = 2, 4, \ldots. \end{cases} \tag{63}$$

Proof. In literature [19] it is known that

$$E_n\,(mx) = \begin{cases} m^n \sum_{i=0}^{m-1}(-1)^i E_n\left(x + \frac{i}{m}\right), & n = 0, 1, \ldots,\ m = 1, 3, \ldots, \\ -\frac{2}{n+1} m^n \sum_{i=0}^{m-1}(-1)^i B_{n+1}\left(x + \frac{i}{m}\right), & n = 0, 1, \ldots,\ m = 2, 4, \ldots \end{cases} \tag{64}$$

and therefore, from (38) and (64), the desired result follows. □

4.4. Generalized Euler polynomials

From (59) we can write

$$\beta_i = Mx^i, \quad i = 1, \ldots, n, \tag{65}$$

where $Mf = \frac{f(1)+f(0)}{2}$. In a similar way, we can consider the weighted coefficients

$$\beta_i^w = M^w x^i, \quad i = 1, \ldots, n, \tag{66}$$

where $M^w f = \frac{w_1 f(1)+w_2 f(0)}{w_1+w_2},\ w_1, w_2 > 0$, i.e.:

$$\beta_i^w = \frac{w_1}{w_1+w_2}, \quad i = 1, \ldots, n. \tag{67}$$

5. Numerical examples

By the choice of the coefficients β_i in definition (30) we can compute, using a Mathematica code, the relative Appell polynomial

$$A_n(x) = c_0 + c_1 x + \cdots + c_n x^n. \tag{68}$$

5.1. Bernoulli–Jacobi/Tchebycheff polynomials

Placing in (53) $\alpha = \beta = -\frac{1}{2}$ we have

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$n=0$	1								
$n=1$	$-\frac{1}{2}$	1							
$n=2$	$\frac{1}{8}$	-1	1						
$n=3$	$\frac{1}{16}$	$\frac{3}{8}$	$-\frac{3}{2}$	1					
$n=4$	$-\frac{7}{128}$	$\frac{1}{4}$	$\frac{3}{4}$	-2	1				
$n=5$	$-\frac{13}{256}$	$-\frac{35}{128}$	$\frac{5}{8}$	$\frac{5}{4}$	$-\frac{5}{2}$	1			
$n=6$	$\frac{71}{1024}$	$-\frac{39}{128}$	$-\frac{105}{128}$	$\frac{5}{4}$	$\frac{15}{8}$	-3	1		
$n=7$	$\frac{187}{2048}$	$\frac{497}{1024}$	$-\frac{273}{256}$	$-\frac{245}{128}$	$\frac{35}{16}$	$\frac{21}{8}$	$-\frac{7}{2}$	1	
$n=8$	$-\frac{5479}{32768}$	$\frac{187}{256}$	$\frac{497}{256}$	$-\frac{91}{32}$	$-\frac{245}{64}$	$\frac{7}{2}$	$\frac{7}{2}$	-4	1

5.2. Generalized Laguerre polynomials

Setting in (57) $s = 1$ we have the normalized Appell–Laguerre polynomials [5]

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
$n=0$	1										
$n=1$	−1	1									
$n=2$	0	−2	1								
$n=3$	0	0	−3	1							
$n=4$	0	0	0	−4	1						
$n=5$	0	0	0	0	−5	1					
$n=6$	0	0	0	0	0	−6	1				
$n=7$	0	0	0	0	0	0	−7	1			
$n=8$	0	0	0	0	0	0	0	−8	1		
$n=9$	0	0	0	0	0	0	0	0	−9	1	
$n=10$	0	0	0	0	0	0	0	0	0	−10	1

5.3. Generalized Euler polynomials

Placing in (67) $w_1 = \frac{1}{2},\ w_2 = \frac{1}{3}$ we find

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$n=0$	1								
$n=1$	$-\frac{3}{5}$	1							
$n=2$	$\frac{3}{25}$	$-\frac{6}{5}$	1						
$n=3$	$\frac{33}{125}$	$\frac{9}{25}$	$-\frac{9}{5}$	1					
$n=4$	$-\frac{141}{625}$	$\frac{132}{125}$	$\frac{18}{25}$	$-\frac{12}{5}$	1				
$n=5$	$-\frac{267}{625}$	$-\frac{141}{125}$	$\frac{66}{25}$	$\frac{6}{5}$	-3	1			
$n=6$	$\frac{2751}{3125}$	$-\frac{1602}{625}$	$-\frac{423}{125}$	$\frac{132}{25}$	$\frac{9}{5}$	$-\frac{18}{5}$	1		
$n=7$	$\frac{20109}{15625}$	$\frac{19257}{3125}$	$-\frac{5607}{625}$	$-\frac{987}{125}$	$\frac{231}{25}$	$\frac{63}{25}$	$-\frac{21}{5}$	1	
$n=8$	$-\frac{448761}{78125}$	$\frac{160872}{15625}$	$\frac{77028}{3125}$	$-\frac{14952}{625}$	$-\frac{1974}{125}$	$\frac{1848}{125}$	$\frac{84}{25}$	$-\frac{24}{5}$	1

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