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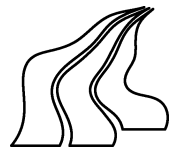# OPTIMAL ACYCLIC EDGE-COLOURING OF CUBIC GRAPHS

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Abstract

An *acyclic edge-colouring* of a graph is a proper edge-colouring such that the subgraph induced by the edges of any two colours is acyclic. The *acyclic chromatic index* of a graph G is the smallest number of colours in an acyclic edge-colouring of G. We prove that the acyclic chromatic index of a connected cubic graph G is 4, unless G is K_4 or $K_{3,3}$; the acyclic chromatic index of K_4 and $K_{3,3}$ is 5.

1 Introduction

Various types of edge-colourings of graphs have occurred in graph theory for more than a century. Among these colourings the most important are *proper* colourings – those that do not allow two adjacent edges to have the same colour. Considerable effort has been devoted to proper edge-colourings where cycles that contain only two colours are forbidden.

A proper k-edge-colouring of G such that there are no two-coloured cycles in G is called an *acyclic k-edge-colouring* of G. The concept of acyclic colourings was introduced by Grünbaum in [8].

In this paper all the considered graphs are finite and simple, i.e. without multiple edges and loops. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in a graph G. Throughout the paper, any proper k-edge-colouring uses the colours denoted by $1, 2, \ldots, k$. The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the least number of colours needed to colour the edges of G by a proper edge-colouring. Similarly we define the *acyclic chromatic index* (also called acyclic edge chromatic

number) of a graph G, denoted by $a'(G)$, to be the least k such that G has an acyclic k-edge-colouring. Obviously, $a'(G) \geq \chi'(G)$. An alternative definition of the acyclic chromatic index gives a slightly different point of view: the acyclic chromatic index of a graph G is the minimum number of matchings which suffice to cover all edges of G in such a way that the union of any two matchings does not contain a cycle.

It is known that a simple graph G with maximal degree Δ has chromatic index either Δ or $\Delta + 1$. How many new colours do we have to use if we do not want to allow two-coloured cycles? It is known that $a'(G) \leq 16\Delta$ (see [2] and [9]), but recent results (e. g. [1], [3], and [10]) suggest that this bound is far from tight. The conjecture of Fiamčík [7] and later by Alon, Sudakov, and Zaks [1] says that $a'(G) \leq \Delta + 2$ for every graph G. Burnstein's result in [4] implies that for cubic graphs $a'(G) \leq 5$, therefore the conjecture is true for $\Delta = 3$. In 1980, Fiamčík [5] published a paper claiming that K_4 is the only cubic graph requiring five colours in an acyclic edge-colouring, nevertheless both the result and the proof were incorrect. Four years later, in [6], he corrected one of the errors of the previous paper, and stated a correct result that there are two exceptional cubic graphs requiring five colours in an acyclic edge-colouring. However, the proof still contains a big gap: usage of Lemma 2 in [5] eliminates the cycle of colours 1 and 2, but it does not ensure that no two-coloured cycles of other pairs of colours are created. This may be the reason why the result of Fiamčík has fallen into obscurity.

In this paper we focus on graphs with $\Delta = 3$. In [3], Basavaraju and Chandran proved that $a'(G) \leq 4$ for all subcubic graphs (graphs with maximal degree at most 3) containing a vertex of degree at most 2.

In every proper 3-edge-colouring of a cubic graph G the edges coloured by any of the colours form a perfect matching. Hence the subgraph of G induced by the edges of any two colours form a 2-factor, which is a nonempty set of two-coloured cycles. Therefore to have an acyclic edge-colouring of a cubic graph we need at least four colours. We prove that four colours are optimal for all connected cubic graphs with the exception of K_4 and $K_{3,3}$, the two graphs mentioned by Fiamčík in [6], for which 5 colours are optimal.

Our main result is captured in the following theorem. The proof in Section 2 is algorithmic and does not rely on any probabilistic arguments. As a simple corollary we are able to determine the acyclic chromatic index of all cubic graphs.

Theorem 1.1 *Let G be a connected graph with $\Delta(G) \leq 3$ different from K_4 and $K_{3,3}$. Then $a'(G) \leq 4$.*

Corollary 1.2 *The acyclic chromatic index of a connected cubic graph G is 4 unless G is K_4 or $K_{3,3}$, for which $a'(K_4) = a'(K_{3,3}) = 5$.*

Proof. For a cubic graph G we have proved that $a'(G) \geq 4$. An acyclic 5-edge-colouring of the graphs K_4 and $K_{3,3}$ is shown in Fig. 1.

It remains to prove that the acyclic chromatic index is at least 5 for both of these two graphs. Suppose that we have an acyclic 4-edge-colouring of K_4. The graph K_4 has 6 edges, and the edges coloured by any of the colours form a matching, hence no colour is used on more than two edges. Therefore there are two colours such that both of them colour exactly two edges, and edges coloured by these colours form a 2-factor, which is a two-coloured cycle of length 4 in our case. We have derived a contradiction.

Suppose that we have an acyclic 4-edge-colouring of $K_{3,3}$. The graph $K_{3,3}$ has 9 edges. If there are two colours such that both of them colour three edges, the edges coloured by these colours form a 2-factor, which is a two-coloured cycle of length 6 in our case. Otherwise exactly one colour, say 1, colours three edges and any other colour colours two edges. Choose a cycle C of length 4 in $K_{3,3}$ such that two of its
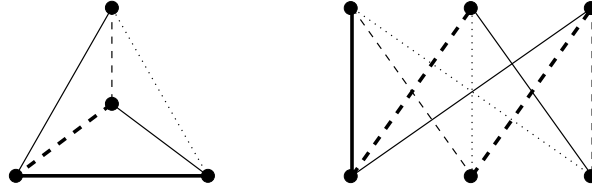
Figure 1: An acyclic 5-edge-colouring of K_4 and $K_{3,3}$

edges are coloured by the colour 1. The other edges in this cycle must be of different colours, say 2 and 3. Look at one of the edges not belonging to C and adjacent with the edge of C that is coloured by the colour 2. Its colour can be either 3 or 4. Since our colouring is proper and each colour different from 1 is used twice, the colours of all the other edges are determined. It is easy to check that in both cases there is a two-coloured cycle, hence the graph $K_{3,3}$ cannot have an acyclic 4-edge-colouring. □

2 Acyclic edge-colouring of subcubic graphs

In this section we prove Theorem 1.1. For the sake of completeness we let the proof cover the case where the minimum degree $\delta(G) \leq 2$, although this case was proved in [3], as already noted.

We proceed by induction on the number of vertices of G. If G is a 1-vertex graph, then the assertion is trivial. We therefore assume that G is a connected graph on at least two vertices with $\Delta(G) \leq 3$, $G \neq K_4$, $G \neq K_{3,3}$, and for every connected graph H of smaller order, with $\Delta(H) \leq 3$ and different from K_4 and $K_{3,3}$, $a'(H) \leq 4$. The proof splits into several cases depending on whether G contains a bridge or not, and on $\delta(G)$.

Case 1: G contains a bridge. Let e be a bridge. Let V_1 and V_2 be the vertex-sets of the two components created by removing e and let G_i be the graph induced by V_i, $i = 1, 2$. Both G_1 and G_2 have fewer vertices than G and each of them has at least one vertex of degree at most 2, therefore both these graphs are different from K_4 and $K_{3,3}$, and, in turn, have an acyclic 4-edge-colouring. For $i = 1, 2$, let φ_i be an acyclic 4-edge-colouring of G_i. We can obtain an acyclic 4-edge-colouring φ of G by letting $\varphi(e)$ be any colour and permuting the colours of φ_1 and φ_2 on G_1 and G_2 so that the edges of G adjacent to e have colours different from $\varphi(e)$.

Case 2: G is bridgeless and $\delta(G) = 2$. (When G is bridgeless $\delta(G)$ is at least 2.) Let v be a vertex of degree 2 and let v_1 and v_2 be the two neighbours of v.

First suppose that v_1 and v_2 are not adjacent. We construct a new graph G' by removing the vertex v from G together with the incident edges v_1v and v_2v and adding a new edge v_1v_2; clearly, G' is simple and connected. If $G' = K_4$ or $G' = K_{3,3}$, then G together with an acyclic 4-edge-colouring is depicted in Fig. 2. If $G' \neq K_4$ and $G' \neq K_{3,3}$, we have $a'(G') \leq 4$ by the induction hypothesis. Let φ' be an acyclic 4-edge-colouring of G'. We construct an acyclic 4-edge-colouring φ of G. We set $\varphi(e) = \varphi'(e)$ for every $e \neq vv_1$ and $e \neq vv_2$.

If the degree of one of the neighbours of v, say v_2, is 2, denote by v_3 the neighbour of v_2 such that $v \neq v_3$. We choose $\varphi(vv_1)$ to be $\varphi'(v_1v_2)$. To finish the colouring we chose $\varphi(vv_2)$ to be any colour different from $\varphi(vv_1)$ and $\varphi(v_2v_3)$. The resulting colouring is proper and clearly, from the induction hypothesis, if there is a 2-coloured cycle in φ, it has to contain the vertex v. But this is not possible because $\varphi(vv_1) \neq \varphi(v_2v_3)$.
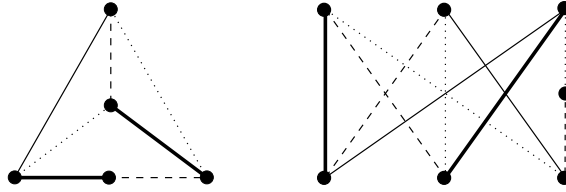
Figure 2: Graphs K_4 and $K_{3,3}$ with a subdivided edge and their acyclic 4-edge-colourings

If, on the other hand, $\deg(v_1) = 3$ and $\deg(v_2) = 3$, let e_1 and e_2 be the edges incident with v_1 but different from vv_1 and let e_3 and e_4 be the edges incident with v_2 but different from vv_2. Since φ' is a proper 4-edge-colouring of G' which includes the edge v_1v_2, we have $\{\varphi(e_1), \varphi(e_2)\} \cap \{\varphi(e_3), \varphi(e_4)\} \neq \emptyset$. If $\{\varphi(e_1), \varphi(e_2)\} = \{\varphi(e_3), \varphi(e_4)\}$, we choose for each of $\varphi(vv_1)$ and $\varphi(vv_2)$ one of the two remaining colours different from $\varphi(e_1)$ and $\varphi(e_2)$, different from each other. If $|\{\varphi(e_1), \varphi(e_2)\} \cap \{\varphi(e_3), \varphi(e_4)\}| = 1$, and, say $\varphi(e_1) = \varphi(e_3)$, we set $\varphi(vv_1) = \varphi(e_4)$ and $\varphi(vv_2) = \varphi'(v_1v_2)$. It can be easily seen that in both cases φ is an acyclic 4-edge-colouring of G.

Now suppose that v_1 and v_2 are adjacent. Since G is bridgeless, either G is a triangle, G is K_4 minus an edge or the vertices v_1 and v_2 are of degree 3 and there is no vertex different from v adjacent to both v_1 and v_2. If G is a triangle or K_4 minus an edge, then $a'(G) \leq 4$; in the latter case let u_1 be the neighbour of v_1 different from v and v_2 and let u_2 be the neighbour of v_2 different from v and v_1. As noted, $u_1 \neq u_2$. We remove from G the vertices v, v_1, and v_2 together with the incident edges and add a new vertex w of degree 2, joined by an edge to u_1 and u_2. We denote the resulting graph G'. The graph G' is simple and connected and has fewer vertices than G, and it is not K_4 or $K_{3,3}$ as it has a vertex of degree 2, therefore by the induction hypothesis $a'(G') \leq 4$. Let φ' be an acyclic 4-edge-colouring of G'. Define the colouring φ of G as follows: $\varphi(e) = \varphi'(e)$ for every edge e that is not incident with any of v, v_1, and v_2. Further set $\varphi(u_1v_1) = \varphi(vv_2) = \varphi'(u_1w)$, $\varphi(u_2v_2) = \varphi(vv_1) = \varphi'(u_2w)$ and colour the edge v_1v_2 by any of the two remaining colours different from $\varphi(vv_1)$ and $\varphi(vv_2)$ (see Fig. 3). Clearly, φ is an acyclic 4-edge-colouring of G.
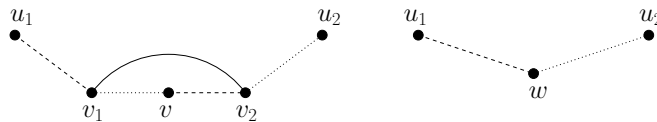

Figure 3: Creating G' from G in Case 2, v_1 and v_2 adjacent

Case 3: G is a bridgeless cubic graph. Here the proof splits into several subcases depending on the girth of G (denoted by $g(G)$ in what follows).
Case 3a: The girth of G is even and $g(G) \geq 6$.

Let C be any of the shortest cycles in G. Let the vertices of C be $u_0, v_0, u_1, v_1, \ldots, u_{k-1}, v_{k-1}$ where $k = g(G)/2 \geq 3$ (the indices will be taken mod k in what follows). Let u_i' and v_i' be the neighbours of u_i and v_i, respectively, which do not belong to C (the vertices $u_0', v_0', u_1', v_1', \ldots$ are distinct, because if a vertex outside C had two edges to C there would be a cycle of shorter length than C). Let G' be the graph constructed in the following way: we remove from G the cycle C together with all the edges adjacent to this cycle and join u_i' to v_i' for $i = 0, 1, 2, \ldots, k-1$ (see Fig. 4). The resulting graph G' is cubic and simple because $g(G) > 4$: a multiple edge in G' yields a cycle of length 4 in G. Note that G' might not be connected but it contains

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no component isomorphic to K_4 or $K_{3,3}$. Indeed, such a component would yield a cycle shorter than C in G; we prove this assertion in the next paragraphs.

Assume that G' contains a component K isomorphic to K_4 or $K_{3,3}$. Since G does not contain cycles of length 4, any such cycle in K must contain at least one of the edges added in the construction of G' (we call such edges *added* in what follows). If $K = K_4$, one can easily check that K must contain two adjacent added edges, contradicting that $u'_0, v'_0, u'_1, v'_1, \ldots$ are distinct.

If $K = K_{3,3}$, then since K does not contain two adjacent added edges, it must contain three added edges forming a 1-factor (otherwise there is a 4-cycle in G). Since no other edges in K are added, the remaining six edges of K forming a cycle of length 6 belong to G. Thus the girth of G is 6, the same is the number of vertices of C. Therefore any vertex of C is adjacent to a vertex of K. It is easy to check that in any case G contains a cycle of length 4 or 5.

Now we return to the main argument of the Case 3a. The induction hypothesis used on each component of G' yields an acyclic 4-edge-colouring φ' of G', we extend this colouring to an acyclic 4-edge-colouring φ of G.
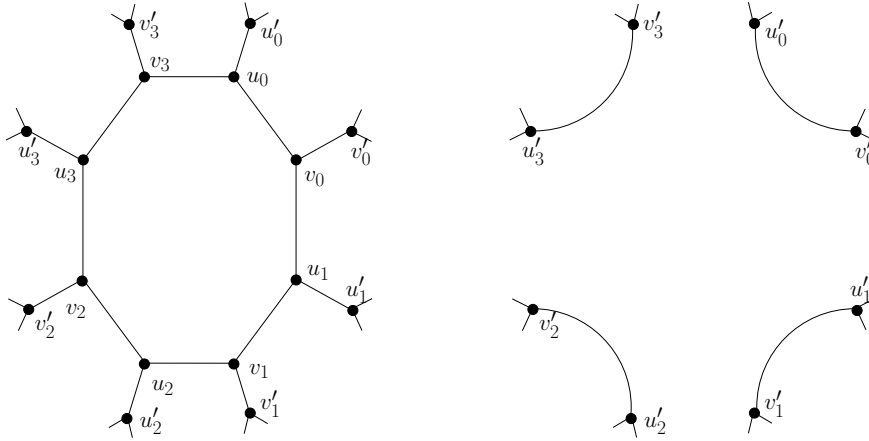

Figure 4: Creating G' from G in Case 3a

Let $\varphi(e) = \varphi'(e)$ for all edges e common for G and G' and $\varphi(u_i u'_i) = \varphi(v_i v'_i) = \varphi'(u'_i v'_i)$. It remains to define φ on the edges of C. We do this in several steps. The direction $u_0, v_0, u_1, \ldots$ along C will be referred to as *clockwise*. Let D be a cycle in G other than C. We say that D *enters* C at a vertex w and *leaves* C at a vertex w', if the path w–w' belongs to both C and D, goes along C in the clockwise direction and cannot be extended in any direction. We say that D *is of intersection* ℓ, if the path w–w' has length ℓ. The cycle D can have more than one common path with C, in such a case we choose w and w' so that their clockwise distance along C is maximal.

Now we define φ on the remaining edges of G, after this step φ will be a proper 4-edge-colouring of G such that any cycle of intersection 1 will contain at least three distinct colours. Consider the edge $v_i u_{i+1}$. If the colours $\varphi(v_i v'_i)$ and $\varphi(u_{i+1} u'_{i+1})$ are different, we choose any of the two remaining colours for $\varphi(v_i u_{i+1})$. Otherwise we set $\varphi(v_i u_{i+1})$ to be the colour different from the colours of the three edges incident with v'_i. We thus colour all k edges $v_i u_{i+1}$.

Consider the edge $u_i v_i$. It is adjacent to four edges, but two of them ($u_i u'_i$ and $v_i v'_i$) are coloured by the same colour, so there is at least one free colour to colour the edge $u_i v_i$. We choose one such free colour. We do this for $i = 0, 1, \ldots, k-1$. Now we have coloured all the edges of G.

All cycles disjoint with C are not two-coloured by the induction hypothesis. It is easy to verify that a cycle of intersection 1 entering C in v_i contains at least three

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colours. Consider a cycle of intersection 1 entering C in u_i. This cycle contains at least three colours by the induction hypothesis (even if it enters C several times).

Now we modify φ to be a proper 4-edge-colouring of G such that any cycle other than C contains at least three colours. The only problematic cycles are those of intersection at least 2. Such a cycle containing only two colours cannot enter C in u_i: the colours of u_iu_i' and v_iv_i' are the same, hence the colours $\varphi(u_iu_i')$, $\varphi(u_iv_i)$, and $\varphi(v_iu_{i+1})$ are three different colours contained in this cycle.

Let D be a two-coloured cycle which enters C in v_i. Then D cannot leave C in v_{i+1}, because the colours of v_iu_{i+1} and $v_{i+1}v_{i+1}'$ cannot be the same. Hence the colours of v_iu_{i+1} and $v_{i+1}u_{i+2}$ are the same. Now set $\varphi(u_{i+1}v_{i+1})$ to be the colour different from $\varphi(v_iv_i')$, $\varphi(v_iu_{i+1})$ and $\varphi(u_{i+1}u_{i+1}')$. Any cycle of intersection 1 containing the edge $u_{i+1}v_{i+1}$ contains at least three colours by the induction hypothesis, thus we did not introduce a new two-coloured cycle.

After these modifications the only possibly two-coloured cycle is C. Assume that it contains only two colours, say 1 and 2. The edges incident with vertices of C but not belonging to C are coloured by colours 3 and 4. If $\varphi(u_1u_1') = \varphi(u_2u_2') = 3$, we can set $\varphi(u_2v_2) = 4$ to obtain an acyclic 4-edge-colouring of G. Otherwise we may assume that the edges u_iu_i' for $i = 1, 2 \ldots, k$ are coloured alternately by colours 3 and 4 (this assumption allows us to replace colour 1 on the edges of C by colour 2 and vice versa without introducing two-coloured cycles of intersection 1). Moreover we may assume that $\varphi(u_1u_1') = \varphi(v_1v_1') = 3$, $\varphi(u_2u_2') = \varphi(v_2v_2') = 4$, $\varphi(u_1v_1) = 2$ and $\varphi(v_1u_2) = 1$. There are three cases for the colours f_1 and f_2 of the two edges incident with u_2' and not incident with u_2:
- if $\{f_1, f_2\} = \{2, 3\}$ we set $\varphi(v_1u_2) = 4$ and $\varphi(u_2u_2') = 1$,
- if $\{f_1, f_2\} = \{1, 2\}$ we set $\varphi(v_1u_2) = 4$ and $\varphi(u_2u_2') = 3$,
- if $\{f_1, f_2\} = \{1, 3\}$ we set $\varphi(v_1u_2) = 4$ and $\varphi(u_2u_2') = 2$ and colour the edges of C different from v_1u_2 by colours 1 and 2 alternately to obtain a proper 4-edge-colouring.

The resulting colouring φ is an acyclic 4-edge-colouring of G.

Case 3b: The girth of G is at least 5 and is odd.

Let C be one of the shortest cycles of G. Let $u_0, v_0, u_1, v_1, \ldots, u_{k-1}, v_{k-1}, w$ be the vertices of this cycle in the order in which they lie on the cycle. Let u_i', v_i', and w' be the neighbours of u_i, v_i, and w, respectively, which do not lie on the cycle C.

From the graph G we construct a graph G' by removing the cycle C with all the edges adjacent to this cycle and joining u_i' to v_i' for $i = 0, 1, 2, \ldots, k - 1$. As the girth of G is at least 5 and G is a shortest cycle, the resulting graph G' contains no loops and no multiple edges. Moreover G' has $\delta = 2$, $\Delta = 3$ and no component of G is isomorphic to K_4 or $K_{3,3}$. The last assertion can be proved in almost the same way as we have done it in the Case 3a, the difference occurs only for $K = K_{3,3}$, where we have three added edges forming a 1-factor: since the graph G in this case ($K_{3,3}$ without a 1-factor) has girth at most 6 and the girth is odd, it is equal to 5, hence C has only 5 vertices. This is a contradiction to the fact that K has 6 distinct vertices that have 6 pairwise distinct neighbours lying on C. The induction hypothesis applied to each component of G' gives an acyclic 4-edge-colouring φ' of G'. We extend this colouring to an acyclic 4-edge-colouring φ of G.

First, set $\varphi(e) = \varphi'(e)$ for each edge e of G that have no vertex in common with C and set $\varphi(u_iu_i') = \varphi(v_iv_i') = \varphi'(u_i'v_i')$.

Next, we describe the colouring of the three edges adjacent with w. In what follows we may assume that the colours used in φ' are $1, 2, 3, 4$ and the colours of the edges incident with w' but not incident with w are 1 and 2 and $\varphi(v_{k-1}v_{k-1}') \leq \varphi(u_0u_0')$. There are several possibilities for the colours of u_0u_0' and $v_{k-1}v_{k-1}'$, all of them with the desired colouring of the edges incident with w are in Fig. 5. Note that in any possibility there is no two-coloured cycle passing through any two of the three

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edges ww', u_0u_0' and $v_{k-1}v_{k-1}'$. The direction $u_0, v_0, u_1, \ldots$ along C is referred to as *clockwise*. Recall the definitions of the terms *enter*, *leave* and *to be of intersection* for a cycle that has an edge in common with C, they apply also in this case.
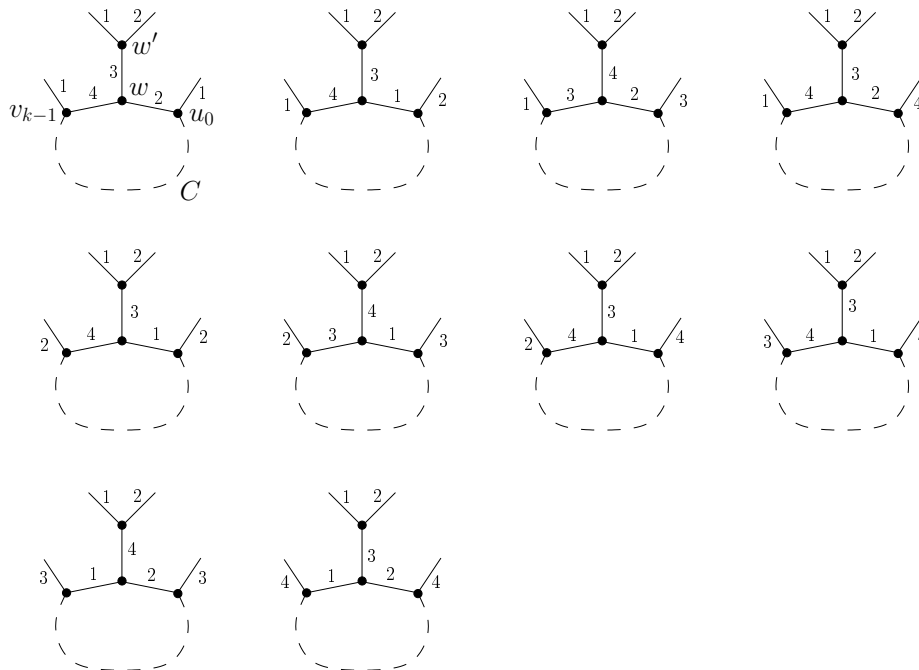

Figure 5: Colouring of the edges incident with w in Case 3b

As the next step we colour one edge of C so that there will be no two-coloured cycle that enters C at w. Any two-coloured cycle in a proper edge-colouring has its edges coloured alternately by two colours. We utilize this fact and choose the colour of one of the edges of C so that this edge together with the already coloured edge ww' will exclude the possibility of a two-coloured cycle entering C in w.

Consider one of the possibilities depicted in Fig. 5. Assume that there exists a smallest integer $j \in \{0, 1, \ldots, k-1\}$ such that v_jv_j' and $u_{j+1}u_{j+1}'$ are not of the same colour. Any two-coloured cycle entering C at w would have to use the path $wu_0v_0 \ldots v_ju_{j+1}$. There are two possible colours for the edge v_ju_{j+1}, we choose one of them so that this edge will break the possibly two-coloured cycle entering C in w. If there is no integer j with the desired property, there is no problem with a cycle entering C in w: all the edges u_iu_i' and v_iv_i' are of the same colour that is not contained in a two-coloured cycle entering C in w, hence this cycle cannot leave C.

We continue by colouring the edges v_iu_{i+1} which have not been coloured yet, we do this in the same manner as we have done in the Case 3a. Then we colour the edges u_iv_i to get a proper 4-edge-colouring φ of G. The induction hypothesis assures that any two-coloured cycle in G has to contain at least one edge from C. The cycle C itself has odd length, hence it cannot be two-coloured in a proper edge-colouring. From the construction of φ we know that no cycle of intersection 1 is two-coloured. We have dealt with cycles entering C in w in the previous paragraph. Since φ is proper, no two-coloured cycle can enter C in u_i. Put together this means that the only possibly problematic cycles is of intersection at least 2 and enter C in v_i for some i. For $i = 0, 1, 2, \ldots, k-2$ we successively apply the same recolouring as we have done for graphs of even girth to change φ so that any such cycle will contain at least three colours. Finally, almost the same recolouring applies also for a cycle entering C in v_{k-1} — which has to reach u_0, the only difference is that the path v_{k-1}–u_0 has length 2 instead of 1 and we possibly change the colour of u_0v_0. After this step φ is an acyclic 4-edge-colouring of G, hence $a'(G) \leq 4$.

Case 3c: The girth of G is 4.

Let $C_4 = v_1v_2v_3v_4$ be a 4-cycle in G. First assume that C_4 contains two non-adjacent edges such that neither of them is contained in a 4-cycle different from C_4. For $i = 1, 2, 3, 4$, let v_i' denote the neighbour of v_i not belonging to C_4. There are no triangles in G and thus no two vertices of v_1', v_2', v_3', and v_4' coincide. Moreover, since G is bridgeless, the graph $G - C_4$ has either one or two components.

If $G - C_4$ is disconnected, there are two vertices in any of the two components of $G - C_4$ joined by an edge to C_4 in G. Without loss of generality we can assume that v_1' and v_2' are in different components of $G - C_4$. We create the graph G' by removing the vertices of C_4 together with the incident edges and adding edges joining v_1' with v_2' and v_3' with v_4'. The graph G' is a cubic, connected and of smaller order than G and it contains a 2-edge cut, therefore $G' \neq K_4$ and $G' \neq K_{3,3}$. By the induction hypothesis, G' has an acyclic 4-edge-colouring φ'. We define an acyclic 4-edge-colouring φ of G by setting $\varphi(e) = \varphi'(e)$ for every edge e not incident with a vertex of C_4, $\varphi(v_1v_1') = \varphi(v_2v_2') = \varphi'(v_1'v_2')$, and $\varphi(v_3v_3') = \varphi(v_4v_4') = \varphi'(v_3'v_4')$. Further, if $\varphi'(v_1'v_2') = \varphi'(v_3'v_4')$, we set $\varphi(v_2v_3)$ to be the colour not assigned to an edge incident with v_2' in G, $\varphi(v_1v_4)$ to be the colour not assigned to an edge incident with v_1' in G and if $\varphi(v_2v_3) \neq \varphi(v_1v_4)$ we set $\varphi(v_1v_2)$ and $\varphi(v_3v_4)$ to be the colour different from $\varphi(v_1v_1')$, $\varphi(v_2v_3)$, and $\varphi(v_1v_4)$; otherwise, if $\varphi(v_2v_3) = \varphi(v_1v_4)$, we set $\varphi(v_1v_2)$ and $\varphi(v_3v_4)$ to be the two different colours, both different from $\varphi(v_1v_1')$ and $\varphi(v_2v_3)$. If, on the other hand, the colours $\varphi'(v_1'v_2')$ and $\varphi'(v_3'v_4')$ are different, we set $\varphi(v_1v_2) = \varphi(v_3v_3')$, $\varphi(v_3v_4) = \varphi(v_1v_1')$ and set $\varphi(v_2v_3)$ and $\varphi(v_1v_4)$ to be two different colours, both different from $\varphi(v_1v_1')$ and $\varphi(v_3v_3')$. In both cases it can be easily checked that the resulting colouring is acyclic.

Now suppose that $G - C_4$ is connected and the edges v_1v_2 and v_3v_4 are in no other 4-cycle of G than C_4. We form the graph G' in the same way as if $G - C_4$ was disconnected. If $G' \neq K_4$ and $G' \neq K_{3,3}$ we find an acyclic 4-edge-colouring of G analogously. If $G' = K_4$ or $G' = K_{3,3}$ the graph G together with an acyclic 4-edge-colouring is depicted in Fig. 11(a), 11(b) and 11(c).

Now we proceed to the case that C_4 does not contain a pair of non-adjacent edges such that neither of them is contained in a 4-cycle different from C_4. Then C_4 has to contain two adjacent edges, each of them contained in a 4-cycle different from C_4. Let v_1v_2 and v_2v_3 be two such edges.

We first deal with the case that G contains a 4-cycle D different from C_4 which contains both v_1v_2 and v_2v_3. Since G is simple, neither v_3v_4 nor v_4v_1 is contained in D. Therefore, there is a vertex v_5 different from v_2 and v_4, adjacent to both v_1 and v_3. Let v_2', v_4', and v_5' be the neighbour of v_2, v_4, and v_5, respectively, that is not contained in $\{v_1, v_2, \ldots, v_5\}$. Since G is bridgeless, the vertices v_2', v_4', and v_5' exist but they need not be pairwise distinct. Moreover, it could not happen $v_2' = v_4' = v_5'$, for otherwise $G = K_{3,3}$.

Assume that the vertices v_2', v_4', and v_5' are pairwise distinct. We form a new graph G' from G in the following way. We remove from G the vertices $v_1, v_2, \ldots, v_5$ and add a new vertex u linked by an edge with the vertices v_2', v_4', and v_5' (see Fig. 6). Clearly, G' is simple, cubic and connected and is distinct from K_4 since G does not contain triangles. If G' is distinct from $K_{3,3}$, then $a'(G') \leq 4$ by the induction hypothesis. Let φ' be an acyclic 4-edge-colouring of G'. An acyclic 4-edge-colouring φ of G can be constructed by setting $\varphi(e) = \varphi'(e)$ for every edge e in G which is not incident with any of $v_1, v_2, \ldots, v_5$ and $\varphi(v_1v_2) = \varphi(v_3v_5) = \varphi(v_4v_4') = \varphi'(uv_4')$, $\varphi(v_1v_5) = \varphi(v_2v_2') = \varphi(v_3v_4) = \varphi'(uv_2')$, and $\varphi(v_1v_4) = \varphi(v_2v_3) = \varphi(v_5v_5') = \varphi'(uv_5')$. If $G' = K_{3,3}$, an acyclic 4-edge-colouring of G is depicted in Fig. 11(d).

Assume that $v_2' \neq v_4' = v_5'$. Let v_6 be the neighbour of v_4' distinct from v_4 and v_5; as G is bridgeless, $v_6 \neq v_2'$. We create a new graph G' by deleting the vertices $v_1, v_2, \ldots, v_5, v_4'$ and adding a new vertex v joined to v_2' and v_6. Then G' has fewer
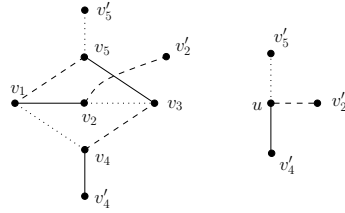
Figure 6: Creating G' from G in Case 3c if there is a 4-cycle D containing both v_1v_2 and v_2v_3 and v_2', v_4', and v_5' are pairwise distinct

vertices than G, it is connected, contains a vertex of degree 2 and has maximum degree 3. Hence $a'(G') \le 4$ by the induction hypothesis. Let φ' be an acyclic 4-edge-colouring of G'. We extend φ' to an acyclic 4-edge-colouring of G in the following way. Without loss of generality we can assume that $\varphi'(vv_2') = 1$ and $\varphi'(vv_6) = 4$.

We set $\varphi(e) = \varphi'(e)$ for every edge e that is not incident with any of $v_1, v_2, \ldots, v_5, v_4'$ and let $\varphi(v_1v_5) = \varphi(v_3v_4) = \varphi(v_2v_2') = 1$, $\varphi(v_3v_5) = \varphi(v_4v_4') = 2$, $\varphi(v_1v_4) = \varphi(v_2v_3) = \varphi(v_5v_4') = 3$, and $\varphi(v_1v_2) = \varphi(v_4'v_6) = 4$, see Fig. 7. If $v_4' \ne v_2' = v_5'$ or $v_5' \ne v_2' = v_4'$ the proof can be done in a similar way.
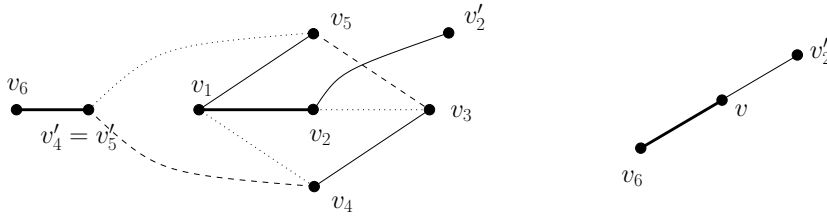

Figure 7: Creating G' from G in Case 3c if there is a 4-cycle D containing both v_1v_2 and v_2v_3 and $v_2' \ne v_4' = v_5'$

Now assume that there is no 4-cycle different from C_4 containing two adjacent edges of C_4, and C_4 contains two adjacent edges, say v_1v_2 and v_2v_3, each contained in a 4-cycle different from C_4. Let $D_1 = v_1v_2v_5v_6$ be a 4-cycle such that $\{v_3, v_4\} \cap \{v_5, v_6\} = \emptyset$ and let D_2 be a 4-cycle different from C_4 containing v_2v_3. Since G is cubic and simple, D_1 and D_2 share either one or two edges.
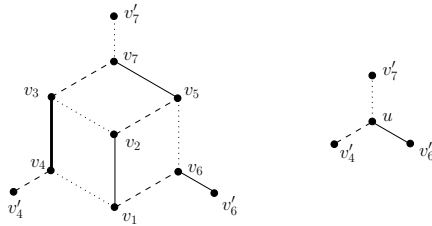

Figure 8: Creating G' from G in Case 3c if there is a 4-cycle D_1 containing v_1v_2 and a 4-cycle D_2 containing v_2v_3, D_1 and D_2 share exactly one edge, and v_4', v_6', and v_7' are pairwise distinct

If D_1 and D_2 share exactly one edge then there exists a vertex v_7 different from $v_1, v_2, \ldots, v_6$ and adjacent to both v_3 and v_5. Note that no two vertices of $\{v_4, v_6, v_7\}$ are joined by an edge, for otherwise G would contain a triangle. Let v_4', v_6', and v_7' be the neighbour of v_4, v_6, and v_7, respectively, different from all the vertices $v_1, v_2, \ldots, v_7$; the vertices v_4', v_6', and v_7' are not necessarily distinct. If v_4', v_6', and v_7' are pairwise distinct, we obtain G' from G by removing the vertices $v_1, v_2, \ldots, v_7$ together with the incident edges and adding a new vertex u joined by

an edge to v_4, v_6, and v_7. Since G is triangle-free, G' is different from K_4. If G' is also different from $K_{3,3}$, then since it is connected and cubic and has fewer vertives than G, G' has an acyclic 4-edge-colouring φ' by the induction hypothesis. We construct an acyclic 4-edge-colouring φ of G by setting $\varphi(e) = \varphi'(e)$ for every edge e which is incident with no vertex from $\{v_1, v_2, \ldots, v_7\}$ and further set $\varphi(v_1 v_2) = \varphi(v_6 v_6') = \varphi(v_5 v_7) = \varphi'(u v_6')$, $\varphi(v_1 v_4) = \varphi(v_2 v_3) = \varphi(v_5 v_6) = \varphi(v_7 v_7') = \varphi'(u v_7')$, $\varphi(v_1 v_6) = \varphi(v_2 v_5) = \varphi(v_3 v_7) = \varphi(v_4 v_4') = \varphi'(u v_4')$, and $\varphi(v_3 v_4)$ be the colour different from the colours of edges incident with u in G' (see Fig. 8). If G' is $K_{3,3}$, an acyclic 4-edge-colouring of G can be found in Fig. 11(f).

Now we deal with the case that two of the vertices v_4', v_6', and v_7' are equal and different from the third. Assume that $v_4' = v_6' \neq v_7'$, the proof for other pairs can be done analogously (in fact the situations are completely symmetric). Let u be the neighbour of v_4' different from v_4 and v_6. Since G is bridgeless, u is different from v_7'. We form the graph G' by removing the vertices $v_1, v_2, \ldots, v_7$ and v_4' from G and adding a new vertex v joined to each of v_7' and u by an edge. The graph G' is different from K_4 and $K_{3,3}$ as it has a vertex of degree 2, and it is connected, has maximum degree 3 and has fewer vertices than G. We take an acyclic 4-edge-colouring φ' of G' and extend it to an acyclic colouring φ of G. Without loss of generality we may assume that $\varphi'(v v_7') = 1$ and $\varphi'(vu) = 4$. We set $\varphi(e) = \varphi'(e)$ for every edge not incident with any of $v_1, v_2, \ldots, v_7$ and $\varphi(v_1 v_6) = \varphi(v_2 v_5) = \varphi(v_3 v_4) = \varphi(v_7 v_7') = 1$, $\varphi(v_1 v_4) = \varphi(v_2 v_3) = \varphi(v_5 v_7) = \varphi(v_6 v_4') = 2$, $\varphi(v_1 v_2) = \varphi(v_3 v_7) = \varphi(v_4 v_4') = 3$ and $\varphi(v_5 v_6) = \varphi(v_4' u) = 4$ (see Fig. 9).
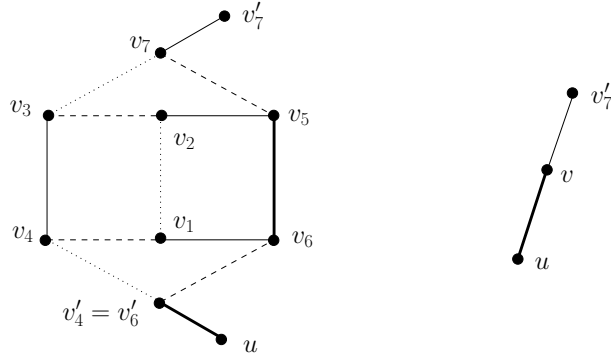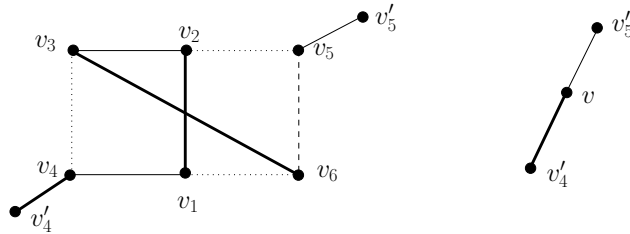

Figure 9: Creating G' from G in Case 3c if there is a 4-cycle D_1 containing $v_1 v_2$ and a 4-cycle D_2 containing $v_2 v_3$, D_1 and D_2 share exactly one edge, and $v_4' = v_6' \neq v_7'$

If all three vertices v_4', v_6', and v_7' coincide, graph G is the 3-dimensional cube Q_3. An acyclic 4-edge-colouring of Q_3 is shown in Fig. 11(g).

Figure 10: Creating G' from G in Case 3c if there is a 4-cycle D_1 containing $v_1 v_2$ and a 4-cycle D_2 containing $v_2 v_3$, D_1 and D_2 share two edges

Assume finally that D_1 and D_2 share two edges. Then, G has to contain the edge $v_3 v_6$. Let v_4' be the neighbour of v_4 different from v_1 and v_3 and let v_5' be the neighbour of v_5 different from v_2 and v_6. Since $G \neq K_{3,3}$, the vertices v_4' and v_5

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are distinct and since G is bridgeless we have $v_4' \neq v_5'$. We construct a connected graph G' with fewer vertices than G by removing the vertices $v_1, v_2, \ldots, v_6$ together with the incident edges from G and adding a new vertex v joined to v_4' and v_5'. The graph G' is different from K_4 and $K_{3,3}$, since it has a vertex of degree 2. We extend an acyclic 4-edge-colouring φ' of G' to an acyclic 4-edge-colouring φ of G according to Fig. 10.
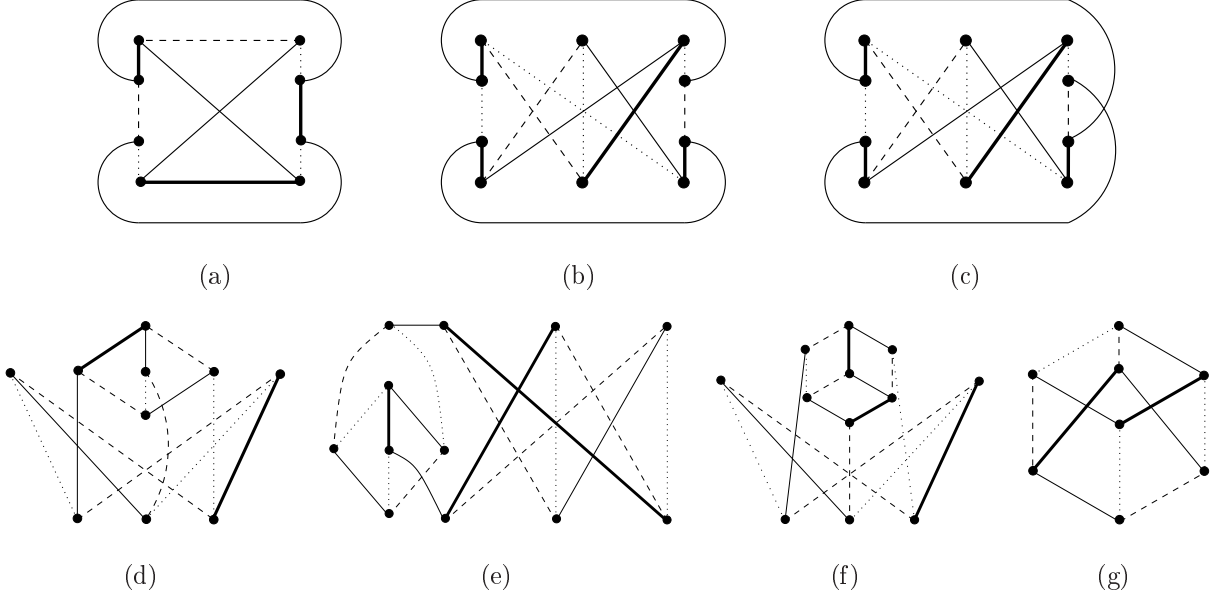

Figure 11: Acyclic 4-edge-colourings of small graphs from Case 3c

Case 3d: The girth of G is 3. First, suppose that G contains a triangle with no edge included in another triangle; denote one such triangle C_3 and let $C_3 = v_1 v_2 v_3$. Let v_i' be the neighbour of v_i not contained in C_3. Create a new graph G' by removing the vertices of C_3 together with the incident edges and introducing a new vertex u connected to v_1', v_2', and v_3'. Since no edge of C_3 was included in another triangle, G' is simple.
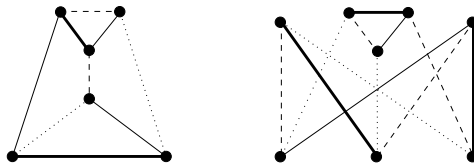

Figure 12: Acyclic 4-edge-colouring of K_4 and $K_{3,3}$ with a vertex expanded to a triangle

If $G' = K_4$ or $G' = K_{3,3}$ then G is isomorphic to one of the graphs in Fig. 12; an acyclic 4-edge-colouring of G is given in the figure. Otherwise, G' is connected and has fewer vertices than G, thus by the induction hypothesis, $a'(G') \leq 4$. Let φ' be an acyclic 4-edge-colouring of G'. We extend φ' to an acyclic 4-edge-colouring of G: we set $\varphi(e) = \varphi'(e)$ for every edge e not incident with a vertex of C_3, set $\varphi(v_i' v_i) = \varphi'(v_i' u)$, $\varphi(v_1 v_2) = \varphi'(v_3 v_3')$, $\varphi(v_1 v_3) = \varphi'(v_2 v_2')$, and $\varphi(v_2 v_3) = \varphi'(v_1 v_1')$.

Now suppose that each triangle in G has an edge in common with another triangle. Fix a triangle $C_3 = v_1 v_2 v_3$ in G. Since $G \neq K_4$, C_3 has exactly one edge, say $v_1 v_2$, included in a triangle different from C_3. Let v_4 be the common neighbour of v_1 and v_2 different from v_3. Let v_3' be the neighbour of v_3 different from v_1 and v_2 and let v_4' be the neighbour of v_4 different from v_1 and v_2. Since G does not contain a bridge, we have $v_3 \neq v_4$. Construct a new graph G' by removing the vertices v_1,

v_2, v_3, and v_4 from G and introducing a new vertex u of degree 2 adjacent to v_3' and v_4'. Let φ' be an acyclic 4-edge-colouring of G'; G' admits such a colouring by the induction hypothesis. We construct a colouring φ of G as follows: we set $\varphi(e) = \varphi'(e)$ for every edge e which is not incident with either of v_i for $i = 1, 2, 3, 4$ and $\varphi(v_3'v_3) = \varphi(v_2v_4) = \varphi'(v_3'u)$, $\varphi(v_4'v_4) = \varphi(v_1v_3) = \varphi'(v_4'u)$. Further, let c_1 and c_2 be the two colours different from $\varphi'(uv_3')$ and $\varphi'(uv_4')$. To finish the definition of the colouring set $\varphi(v_2v_3) = \varphi(v_1v_4) = c_1$, and, finally $\varphi(v_1v_2) = c_2$ (see Fig. 13). Clearly φ is acyclic. This concludes the proof.
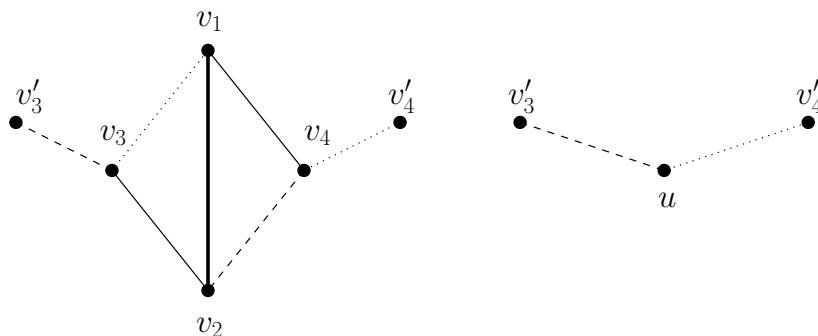

Figure 13: Creating G' from G in Case 3a if every triangle shares an edge with another triangle

Acknowledgements

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