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Pollution control quasi-equilibrium problems with joint implementation of environmental projects

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ABSTRACT

In this paper, we show how one of the Kyoto Protocol mechanisms, the so-called joint implementation in environmental projects, can be transformed into and studied as an infinite-dimensional quasi-variational inequality. Specifically, we examine the situation in which different countries attempt to fulfill Kyoto commitments by investing in emission reduction or emission removal projects in countries where the abatement costs are lower. We derive the equilibrium conditions and prove their characterization in terms of an infinite-dimensional quasi-variational inequality problem. Finally, we discuss the existence of solutions.

1. The model

The 1997 Kyoto Protocol prescribes that some industrialized countries, labeled as "Annex I Parties", must reduce their greenhouse gas emissions to at least 5% below the 1990 levels for the 2008–2012 period (see [1]). Under the Treaty, countries must meet their targets primarily through national measures; however, some other market-based mechanisms are offered. In this paper, we focus on the so-called joint implementation (JI), a mechanism that allows Parties, with emission reduction or limitation commitments, to collect rewards in the form of emission reduction units (ERUs) from an emission reduction or emission removal project in another Annex I Party, where the abatement costs are lower.

In this paper, we present a new approach in the study of the JI mechanism, based on variational inequality theory. It is by now recognized how variational inequality theory can be fruitfully applied to describe a large variety of equilibrium problems, such as the traffic equilibrium problem, the spatial price equilibrium problem, the financial equilibrium problem, the Walras equilibrium problem, migration networks, and electric power supply chain networks (see for instance [2–6]). Our aim is to show how the JI mechanism can be transformed into and studied as a quasi-variational inequality. We emphasize that a quasi-variational inequality framework is applied to such a model for the first time. It is out of the scope of the present paper to discuss theoretical and computational advances on quasi-variational inequalities; however, we address the interested reader to the books [7,8] and, amongst others, to the paper [9].

In addition, we explicitly take into account the evolution of the system with respect to time, and hence assume that all data are time dependent, and study the evolution of the system during the time horizon $[0, \bar{t}]$, with $\bar{t} > 0$. The importance of studying evolutionary problems, in order to represent models that are able to involve equilibria adjustment processes or retarded equilibria (see for instance [10–14]), is indeed well known.

Following [15–17], we now present the features of the model. Let N be the number of countries involved in the program of control pollution. Let $e^i(t)$ denote the gross emissions resulting from the industrial production of country i at time $t \in [0, \bar{t}]$, which are proportional to the industrial output of the country. Let the revenue R_i of country i be defined as the function

$$R_i(t, e^i(t)) : [0, \bar{t}] \times \mathbb{R}_+ \to \mathbb{R}_+.$$

Emissions can be reduced by investing in local or foreign projects. Let $I^i_j(t)$ be the amount of environmental investments held by country i in country j at time $t \in [0, \bar{t}]$. Further, we group the total emissions and the investments held by

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country i in the column vectors $e(t) = (e^1(t), \dots, e^N(t))^T$ and $I^i(t) = (I_1^i(t), \dots, I_N^i(t))^T$, respectively. The benefits of investments lie in the acquisition of emission reduction units (ERUs), which are assumed to be proportional to the investment, namely have the form $\gamma_{ij}(t)I_j^i(t)$. Here, $\gamma_{ij}(t)$ is a positive technological efficiency parameter depending on both the investor i and the host country j, because in general there is a dependence on both the investor's technologies and laws, and the situation in the host country. In addition, we assume that $\gamma_{ji}(t)$ and $\frac{1}{\gamma_{ij}(t)}$, for $j = 1, \dots, N$, belong to $L^\infty([0, \bar{t}])$. We denote by $I^{-i}(t)$ the $N(N-1)$-vector of all the investments held by all the countries except i. Analogously, $e^{-i}(t)$ represents the $(N-1)$-vector of the gross emissions of countries different from i. We further group the total amount of investments into the N^2-vector $I(t) = (I^i(t), I^{-i}(t))$, and denote by $e(t) = (e^i(t), e^{-i}(t))$ the total gross emissions. In order to shorten the notation, sometimes we will omit the dependence on t. The net emission in country i, namely, the difference between the gross emissions and the reduction resulting from local and foreign investments in the same country, is denoted by $\eta_i(t, e^i, I_i^1, \dots, I_i^N) = e^i(t) - \gamma_{ii}(t)I_i^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t)I_i^j(t) \geq 0$ a.e. in $[0, \bar{t}]$. Moreover, the accounted-for emissions of country i, given by its own emissions minus the ERUs gained by investing in environmental projects, must be kept below a prescribed level $E_i(t) > 0$ a.e. in $[0, \bar{t}]$; namely, the following environmental constraint holds: $\varepsilon_i(t, e^i, I^i) = e^i(t) - \gamma_{ii}(t)I_i^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ij}(t)I_j^i(t) \leq E_i(t)$ a.e. in $[0, \bar{t}]$. Let the local investment cost be given by

$$C_i(t, I^i) : [0, \bar{t}] \times \mathbb{R}_+^N \to \mathbb{R}_+.$$

We also assume that pollution in one country can affect also other countries; hence, the damage from pollution in one country depends on the net emissions of all countries according to the function $D_i(t, e, I) : [0, \bar{t}] \times \mathbb{R}_+^{N+N^2} \to \mathbb{R}_+$, where

$$D_i(t, e, I) = D_i\left(t, \eta_1(t, e^1, I_1^1, \dots, I_1^N), \dots, \eta_N(t, e^N, I_N^1, \dots, I_N^N)\right).$$

We choose as our functional settings the Hilbert spaces $L^2([0, \bar{t}], \mathbb{R}^{1+N})$ of square-integrable functions defined in the closed interval $[0, \bar{t}]$, endowed with the scalar product $\langle \cdot, \cdot \rangle_{L^2} = \int_0^{\bar{t}} \langle \cdot, \cdot \rangle dt$ and the usual associated norm $\| \cdot \|_{L^2}$.

We assume that, for all i, $R_i(t, e^i)$, $C_i(t, I^i)$, and $D_i(t, e, I)$ are measurable in t and continuous with respect to the other arguments. Moreover, we assume that $\frac{\partial R_i(t,e^i)}{\partial e^i}$, $\frac{\partial D_i(t,e,I)}{\partial e^i}$, $\frac{\partial C_i(t,I^i)}{\partial I_j^i}$, and $\frac{\partial D_i(t,e,I)}{\partial I_j^i}$, $j = 1, \dots, N$, exist and are measurable in t and continuous with respect to the other arguments. In addition, we require the following growth conditions in order to ensure that the above functions are square-integrable.

$$\exists \delta_1^i, \delta_2^i \in L^2([0, \bar{t}]) : \left| \frac{\partial R_i(t, e^i)}{\partial e^i} \right| \leq \delta_1^i(t) + |e^i|, \qquad \left| \frac{\partial C_i(t, I^i)}{\partial I_j^i} \right| \leq \delta_2^i(t) + |I^i|, \tag{1}$$

$$\exists \delta_3^i, \delta_4^i \in L^2([0, \bar{t}]) : \left| \frac{\partial D_i(t, e, I)}{\partial e^i} \right| \leq \delta_3^i(t) + |e|, \qquad \left| \frac{\partial D_i(t, e, I)}{\partial I_j^i} \right| \leq \delta_4^i(t) + |I|. \tag{2}$$

The goal of country i, given the other players' strategies (e^{*-i}, I^{*-i}), consists in choosing the strategy (e^i, I^i) that maximizes the revenue and minimizes the investments in emission reduction as well as the damage from pollution. Therefore, each country has to solve the following optimization problem:

$$\max_{(e^i, I^i) \in K_i(e^{*-i}, I^{*-i})} \int_0^{\bar{t}} W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))dt, \tag{3}$$

where

$$W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t)) = R_i(t, e^i(t)) - C_i(t, I^i(t)) - D_i(t, e(t), I(t))$$

and

$$K_i(e^{*-i}, I^{*-i}) = \left\{ (e^i, I^i) \in L^2([0, \bar{t}], \mathbb{R}^{1+N}) : e^i(t) \geq 0, I_j^i(t) \geq 0, j = 1, \dots, N, \right.$$

$$e^i(t) - \gamma_{ii}(t)I_i^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ij}(t)I_j^i(t) \leq E_i(t), \tag{4}$$

$$e^i(t) - \gamma_{ii}(t)I_i^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t)I_i^{*j}(t) \geq 0, \tag{5}$$

$$\left. e^{*k}(t) - \gamma_{ik}(t)I_k^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{jk}(t)I_k^{*j}(t) \geq 0, k = 1, \dots, N, \quad k \neq i, \text{ a.e. in } [0, \bar{t}] \right\}. \tag{6}$$

Let us set $K(e^*, I^*) = \prod_{i=1}^{N} K_i(e^{*-i}, I^{*-i})$.

Definition 1. A vector of emissions and investments $(e^*(t), I^*(t)) \in K(e^*, I^*)$ is an equilibrium of the evolutionary environmental pollution control problem if and only if, for each $i = 1, \ldots, N$, and a.e. in $[0, \bar{t}]$, it satisfies the conditions

$$-\frac{\partial R_i(t, e^{*i})}{\partial e^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t) \geq 0, \tag{7}$$

$$\frac{\partial C_i(t, I^{*i})}{\partial I_i^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_i^i} - \overline{\nu_i}(t)\gamma_{ii}(t) + \overline{\tau_i}(t)\gamma_{ii}(t) \geq 0, \tag{8}$$

$$\frac{\partial C_i(t, I^{*i})}{\partial I_j^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_j^i} - \overline{\nu_i}(t)\gamma_{ij}(t) \geq 0, \quad j \neq i, \tag{9}$$

$$\left(-\frac{\partial R_i(t, e^{*i})}{\partial e^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t)\right) e^{*i}(t) = 0, \tag{10}$$

$$\left(\frac{\partial C_i(t, I^{*i})}{\partial I_i^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_i^i} - \overline{\nu_i}(t)\gamma_{ii}(t) + \overline{\tau_i}(t)\gamma_{ii}(t)\right) I_i^{*i}(t) = 0, \tag{11}$$

$$\left(\frac{\partial C_i(t, I^{*i})}{\partial I_j^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_j^i} - \overline{\nu_i}(t)\gamma_{ij}(t)\right) I_j^{*i}(t) = 0, \quad j \neq i. \tag{12}$$

simultaneously, where $\overline{\nu_i}(t), \overline{\tau_i}(t) \in L^2([0, \bar{t}])_+$ are the Lagrange multipliers attached to the environmental constraint and the non-negativity constraint of net emissions, respectively.

2. Quasi-variational inequality formulation

In this section, we provide the quasi-variational inequality formulation of our problem.

Theorem 1. *A vector of emissions and investments (e^*, I^*) is an equilibrium of the evolutionary environmental pollution control problem if and only if it satisfies the following quasi-variational inequality:*

$$\begin{aligned}\sum_{i=1}^{N} \int_0^{\bar{t}} \Bigg\{ & \left(-\frac{\partial R_i(t, e^{*i})}{\partial e^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t)\right)(e^i(t) - e^{*i}(t)) \\ & + \left(\frac{\partial C_i(t, I^{*i})}{\partial I_i^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_i^i} - \overline{\nu_i}(t)\gamma_{ii}(t) + \overline{\tau_i}(t)\gamma_{ii}(t)\right)(I_i^i(t) - I_i^{*i}(t)) \\ & + \sum_{j=1}^{N} \left(\frac{\partial C_i(t, I^{*i})}{\partial I_j^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_j^i} - \overline{\nu_i}(t)\gamma_{ij}(t)\right)(I_j^i(t) - I_j^{*i}(t)) \Bigg\} dt \geq 0, \quad \forall (e, I) \in K(e^*, I^*). \end{aligned} \tag{13}$$

Proof. Let $(e^*(t), I^*(t)) \in K(e^*, I^*)$ be an equilibrium solution. Therefore, a.e. in $[0, \bar{t}]$, and for all $i = 1, \ldots, N$, we may write

$$\left(-\frac{\partial R_i(t, e^{*i})}{\partial e^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t)\right)(e^i(t) - e^{*i}(t)) \geq 0,$$

$$\left(\frac{\partial C_i(t, I^{*i})}{\partial I_i^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_i^i} - \overline{\nu_i}(t)\gamma_{ii}(t) + \overline{\tau_i}(t)\gamma_{ii}(t)\right)(I_i^i(t) - I_i^{*i}(t)) \geq 0,$$

$$\sum_{j=1}^{N} \left(\frac{\partial C_i(t, I^{*i})}{\partial I_j^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_j^i} - \overline{\nu_i}(t)\gamma_{ij}(t)\right)(I_j^i(t) - I_j^{*i}(t)) \geq 0.$$

Summing up the above inequalities and integrating, we immediately find (13).

We now establish that, if $(e^*(t), I^*(t)) \in K(e^*, I^*)$ satisfies quasi-variational inequality (13), it also verifies equilibrium conditions (7)–(12). By contradiction, we assume that the equilibrium conditions are not verified. Without loss of generality, we can suppose that conditions (7) and (10) are not satisfied. Hence, there exist an index $s \in \{1, \ldots, N\}$ and a set $E \subset [0, \bar{t}]$ with positive measure, such that

$$-\frac{\partial R_s(t, e^{*s})}{\partial e^s} + \frac{\partial D_s(t, e^*, I^*)}{\partial e^s} + \overline{\nu_s}(t) - \overline{\tau_s}(t) < 0, \quad \text{a.e. on } E. \tag{14}$$

Let us choose $I_j^i(t) = I_j^{*i}(t)$, for $i, j = 1, \ldots, N$, a.e. in $[0, \bar{t}]$, $e^i(t) = e^{*i}(t)$, for $i \neq s$ in $[0, \bar{t}]$, and

$$e^s(t) \begin{cases} = e^{*s}(t) & \text{if } t \in [0, \bar{t}] \setminus E, \\ > e^{*s}(t) & \text{if } t \in E. \end{cases}$$

By virtue of the choices of $e(t)$ and $I(t)$, and due to condition (14), quasi-variational inequality (13) reduces to

$$\int_E \left(-\frac{\partial R_s(t, e^{*s})}{\partial e^s} + \frac{\partial D_s(t, e^*, I^*)}{\partial e^s} + \overline{\nu_s}(t) - \overline{\tau_s}(t) \right) (e^s(t) - e^{*s}(t))dt < 0,$$

which is an absurd assertion. □

Remark 1. It is worth noting that the existence of Lagrange multipliers $\overline{\nu_i}(t), \overline{\tau_i}(t)$ can be ensured by exploiting recent results in [18–25] that use a generalized constraint qualification assumption, referred as Assumption S, based on the notion of quasi-relative interior.

3. An existence result

Before showing our result, we recall the following theorem adapted to our case (see [26]), which will be useful for our purposes. We also address the interested reader to [7] for general existence results.

Theorem 2. *Let Y be a topological linear locally convex Hausdorff space, and let $E \subset Y$ be a convex, compact, and nonempty subset. Let $F : E \to Y'$ be a continuous function, and let $K : E \to 2^E$ be a closed lower semi-continuous set-valued map with $K(X), X \in E$, convex, compact, and nonempty. Then, there exists $X^* \in K(X^*)$ such that*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in K(X^*).$$

In order to simplify notation, for $i = 1, \ldots, N$, we set

$$F_i(t, e^i, I^i) = \begin{pmatrix} -\frac{\partial R_i(t, e^{*i})}{\partial e^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t) \\ \frac{\partial C_i(t, I^{*i})}{\partial I_i^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_i^i} - \overline{\nu_i}(t)\gamma_{ii}(t) + \overline{\tau_i}(t)\gamma_{ii}(t) \\ \sum_{j=1}^{N} \left(\frac{\partial C_i(t, I^{*i})}{\partial I_j^i} + \frac{\partial D_i(t, e^*, I^*)}{\partial I_j^i} - \overline{\nu_i}(t)\gamma_{ij}(t) \right) \end{pmatrix}.$$

Now, we are able to prove our outcome. We note that the theorem has recourse to weak topology arguments (see, for example, [27] for definitions and properties). The advantage of such a framework is that the (weak) compactness of the set $K_i(e^{*-i}, I^{*-i})$ can be deduced without requiring any compactness on set E; see also [6,28].

Theorem 3. *Let us assume that, for all $i = 1, \ldots, N$, the function*

$$F_i : [0, \bar{t}] \times \mathbb{R}^{1+N} \to \mathbb{R}^{1+N}$$

satisfies the following conditions.

(a) *$F_i(t, e^i, I^i)$ is measurable in $t \forall e^i, I^i$, and continuous with respect to e^i, I^i a.e. in $[0, \bar{t}]$. Moreover, let us suppose that $\frac{\partial R_i(t,e^i)}{\partial e^i}$, $\frac{\partial D_i(t,e,I)}{\partial e^i}$, $\frac{\partial C_i(t,I^i)}{\partial I_j^i}$ and $\frac{\partial D_i(t,e,I)}{\partial I_j^i}$, $j = 1, \ldots, N$ are measurable in t and continuous with respect to the other arguments, and that the growth conditions (1) and (2) are verified.*
(b) *$F_i(t, e^i, I^i)$ is convex in e^i, I^i a.e. in $[0, \bar{t}]$, and upper semi-continuous with respect to the weak topology in e^i, I^i a.e. in $[0, \bar{t}]$.*
(c) *$\gamma_{ji}(t)$ and $\frac{1}{\gamma_{ij}(t)}$, for $j = 1, \ldots, N$, belong to $L^\infty([0, \bar{t}])$.*

Then, quasi-variational inequality problem (13) admits solutions.

Proof. First, we fix $i \in \{1, \ldots, N\}$ and reduce the quasi-equilibrium constraints (5) and (6) to equality constraints, by introducing the slack variables $s_j^i(t) \geq 0$, $j = 1, \ldots, N$ (we also set $s^i(t) = (s_1^i(t), \ldots, s_N^i(t))^T$). Thus, we may write

$$e^i(t) - \gamma_{ii}(t)I_i^i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{N} \gamma_{ji}(t)I_i^{*j}(t) - s_i^i(t) = 0, \tag{15}$$

$$e^{*k}(t) - \gamma_{ik}(t)I_k^i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{N} \gamma_{jk}(t)I_k^{*j}(t) - s_k^i(t) = 0, \tag{16}$$

$k = 1, \ldots, N, k \neq i,$ a.e. in $[0, \bar{t}]$.

We proceed using arguments of weak topology. Under hypothesis (a), and if $(e^{*i}, I^{*i}) \in L^2([0,\bar{t}], \mathbb{R}^{1+N})$, it results that $F_i(t, e^{*i}, I^{*i}) \in L^2([0,\bar{t}], \mathbb{R}^{1+N})$. Moreover, $F_i(t, e^{*i}, I^{*i})$ belongs to the class of Nemytskii operators (see Example 2.5.5, p. 159, and Example 4.7.3, p. 341, in [29] for definitions and main properties of Nemytskii operators); therefore, if $\{(e_n^{*i}, I_n^{*i})\}$ strongly converges to (e^{*i}, I^{*i}) in L^2, then

$$\|F_i(t, e_n^{*i}, I_n^{*i}) - F_i(t, e^{*i}, I^{*i})\|_{L^2} \to 0,$$

and the function F_i is continuous in L^2 with respect to the strong topology.

Now, in order to prove that $K_i(e^{*-i}, I^{*-i})$ is a weakly closed set-valued map, we show that it is strongly closed, i.e.,

$$\forall\{(e_n^{*i}, I_n^{*i}, s_n^{*i})\} \to (e^{*i}, I^{*i}, s^{*i}), \quad \forall\{(e_n^{i}, I_n^{i}, s_n^{i})\} \to (e^{i}, I^{i}, s^{i}),$$

with $(e_n^i, I_n^i, s_n^i) \in K_i(e_n^{*-i}, I_n^{*-i}) \forall n \in \mathbb{N}$, $(e^i, I^i, s^i) \in K_i(e^{*-i}, I^{*-i})$.

Let $\{(e_n^{*i}, I_n^{*i}, s_n^{*i})\}$ and $\{(e_n^{i}, I_n^{i}, s_n^{i})\}$ be two arbitrary sequences strongly converging to (e^{*i}, I^{*i}, s^{*i}) and (e^i, I^i, s^i), respectively. Since $(e_n^i, I_n^i, s_n^i) \in K_i(e_n^{*-i}, I_n^{*-i})$, we have, for $n \in \mathbb{N}$, and a.e. in $[0,\bar{t}]$,

(c1) $e_n^i(t), I_{j,n}^i(t), s_{j,n}^i(t) \geq 0, \quad \forall j = 1, \dots, N,$

(c2) $e_n^i(t) - \gamma_{ii}(t) I_{i,n}^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ij}(t) I_{j,n}^i(t) \leq E_i(t),$

(c3) $e_n^i(t) - \gamma_{ii}(t) I_{i,n}^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t) I_{i,n}^{*j}(t) - s_{i,n}^i(t) = 0,$

(c4) $e_n^{*k}(t) - \gamma_{ik}(t) I_{k,n}^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{jk}(t) I_{k,n}^{*j}(t) - s_{k,n}^i(t) = 0, \quad k \neq i.$

Due to the convergence of (e_n^i, I_n^i, s_n^i) in L^2, conditions (c1) and (c2) are verified. To prove (c3), we observe that the following inequality chain holds:

$$\begin{aligned} 0 &\leq \left\| e^i(t) - \gamma_{ii}(t) I_i^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t) I_i^{*j}(t) - s_i^i(t) \right\|_{L^2} \\ &\leq \Big\| e^i(t) - \gamma_{ii}(t) I_i^i(t) - s_i^i(t) - \big(e_n^i(t) - \gamma_{ii}(t) I_{i,n}^i(t) - s_{i,n}^i(t)\big) \\ &\quad + \left(e_n^i(t) - \gamma_{ii}(t) I_{i,n}^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t) I_{i,n}^{*j}(t) - s_{i,n}^i(t) \right) + \left(\sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t) I_{i,n}^{*j}(t)) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t) I_i^{*j}(t) \right) \Big\|_{L^2} \to 0. \end{aligned}$$

We now verify condition (c4).

$$\begin{aligned} 0 &\leq \left\| e^{*k}(t) - \gamma_{ik}(t) I_k^i(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{jk}(t) I_k^{*j}(t) - s_k^i(t) \right\|_{L^2} \\ &\leq \Bigg\| \big(-\gamma_{ik}(t) I_k^i(t) - s_k^i(t)\big) + \big(\gamma_{ik}(t) I_{k,n}^i(t) + s_{k,n}^i(t)\big) + \left(-\gamma_{ik}(t) I_{k,n}^i(t) - s_{k,n}^i(t) + e_n^{*k}(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{jk}(t) I_{k,n}^{*j}(t) \right) \\ &\quad \times \left(-e_n^{*k}(t) + \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{jk}(t) I_{k,n}^{*j}(t) + e^{*k}(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{jk}(t) I_k^{*j}(t) \right) \Bigg\|_{L^2} \to 0. \end{aligned}$$

Therefore, the feasibility of (e^i, I^i, s^i) is proved. In addition, it is easy to show that $K_i(e^{*-i}, I^{*-i})$ is a convex set. As a consequence, $K_i(e^{*-i}, I^{*-i})$ being convex and strongly closed, it is also weakly closed.

In order to prove that $K_i(e^{*-i}, I^{*-i})$ is a lower semi-continuous set-valued map with respect to the weak topology, we show that, for all $\{(e_n^{*i}, I_n^{*i}, s_n^{*i})\}$ weakly convergent to (e^{*i}, I^{*i}, s^{*i}), briefly $\{(e_n^{*i}, I_n^{*i}, s_n^{*i})\} \rightharpoonup (e^{*i}, I^{*i}, s^{*i})$, and $\forall (e^i, I^i, s^i) \in K_i(e^{*-i}, I^{*-i})$, there exists $\{(e_n^i, I_n^i, s_n^i)\}$ such that

$$\{(e_n^i, I_n^i, s_n^i)\} \rightharpoonup (e^i, I^i, s^i), \quad \text{with} \quad (e_n^i, I_n^i, s_n^i) \in K_i(e_n^{*-i}, I_n^{*-i}), \ \forall n \in \mathbb{N}.$$

Let us consider an arbitrary sequence $\{(e_n^{*i}, I_n^{*i}, s_n^{*i})\} \rightharpoonup (e^{*i}, I^{*i}, s^{*i})$, $(e^i, I^i, s^i) \in K_i(e^{*-i}, I^{*-i})$ and $t \in [0,\bar{t}]$. We introduce the following sets:

$$B_i = \left\{ j \in \{1, \dots, N\} : \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t) I_i^{*j}(t) - \sum_{\substack{j=1\\ j\neq i}}^N \gamma_{ji}(t) I_{i,n}^{*j}(t) \leq 0 \right\},$$

$$C_i = \left\{ j \in \{1, \ldots, N\} : 0 < \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t) < I_j^i(t) \right\},$$

$$D_i = \left\{ j \in \{1, \ldots, N\} : I_j^i(t) \leq \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t) \right\}.$$

Let us also construct the sequence $\{(e_n^i, I_n^i, s_n^i)\}$, where $I_n^i(t) = (I_{1,n}^i(t), \ldots, I_{N,n}^i(t))^T$ and $s^i(t) = (s_1^i(t), \ldots, s_N^i(t))^T$, such that $e_n^i(t) = e^i(t), s_{j,n}^i(t) = s_j^i(t),\ j = 1, \ldots, N,$

$$I_{j,n}^i(t) = \begin{cases} I_j^i(t) & \text{if } j \in B_i \cup D_i \\ I_j^i(t) + \dfrac{\sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t)}{\gamma_{ij}(t)} & \text{if } j \in C_i, \end{cases}$$

$$I_{k,n}^i(t) = \begin{cases} I_k^i(t) & \text{if } k \in B_i \cup D_i \\ I_k^i(t) + \dfrac{\sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{jk}(t) I_k^{*j}(t) - \sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{jk}(t) I_{k,n}^{*j}(t)}{\gamma_{ik}(t)} & \text{if } k \in C_i. \end{cases}$$

In particular, we set

$$I_{i,n}^i(t) = \begin{cases} I_i^i(t) & \text{if } i \in B_i \cup D_i \\ I_i^i(t) + \dfrac{\sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t)}{\gamma_{ii}(t)} & \text{if } i \in C_i. \end{cases}$$

If $j \in B_i \cup D_i$, then $e_n^i(t) = e^i(t), s_{j,n}^i(t) = s_j^i(t)$ and $I_{j,n}^i(t) = I_j^i(t), j = 1, \ldots, N$. Moreover, since $(e^i, I^i, s^i) \in K_i(e^{*-i}, I^{*-i})$, the sequence satisfies conditions (c1)–(c4).

If $j \in C_i$, we prove that conditions (c1)–(c4) are verified.

Condition (c1)

$$e_n^i(t) = e^i(t) \geq 0, \qquad s_{j,n}^i(t) = s_j^i(t) \geq 0, \qquad I_{j,n}^i(t) = I_j^i(t) \geq 0, \quad \forall j = 1, \ldots, N.$$

Condition (c2)

$$\begin{aligned} & e_n^i(t) - \gamma_{ii}(t) I_{i,n}^i(t) - \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ij}(t) I_{j,n}^i(t) \\ &= e^i(t) - \gamma_{ii}(t) I_i^i(t) - \frac{\gamma_{ii}}{\gamma_{ii}} \left(\sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t) \right) \\ &\quad - \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ij}(t) I_j^i(t) - \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ij}(t) \left(\frac{\sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum\limits_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t)}{\gamma_{ij}(t)} \right) \\ &= \underbrace{e^i(t) - \gamma_{ii}(t) I_i^i(t) - \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ij}(t) I_j^i(t)}_{\leq E_i(t)} \underbrace{- \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) + \sum_{\substack{j=1\\j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t)}_{\leq 0} \end{aligned}$$

$$- \Bigg(\underbrace{\sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t)}_{>0} \Bigg)$$
$$\leq E_i(t).$$

Condition (c3)

$$e_n^i(t) - \gamma_{ii}(t) I_{i,n}^i(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t) - s_{i,n}^i(t) = e^i(t) - \gamma_{ii}(t) I_i^i(t) - \frac{\gamma_{ii}}{\gamma_{ii}} \left(\sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t) \right) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t) - s_i^i(t)$$
$$= e^i(t) - \gamma_{ii}(t) I_i^i(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - s_i^i(t) = 0.$$

Condition (c4)

$$e_n^{*k}(t) - \gamma_{ik}(t) I_{k,n}^i(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{jk}(t) I_{k,n}^{*j}(t) - s_{k,n}^i(t) = e^{*k}(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{jk}(t) I_{k,n}^{*j}(t) - \gamma_{ik}(t) I_k^i(t) - \gamma_{ik}(t) \frac{\sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{jk}(t) I_k^{*j}(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{jk}(t) I_{k,n}^{*j}(t)}{\gamma_{ik}(t)} - s_k^i(t)$$
$$= e^{*k}(t) - \gamma_{ik}(t) I_k^i(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{jk}(t) I_k^{*j}(t) - s_k^i(t) = 0. \quad \square$$

In order to prove that $\{(e_n^i, I_n^i, s_n^i)\}$ weakly converges to (e^i, I^i, s^i), we show that

$$\forall f(t) \in L^2([0, \bar{t}]), \quad \lim_{n\to\infty} \int_0^{\bar{t}} f(t) \left((e_n^i, I_n^i, s_n^i) - (e^i, I^i, s^i) \right) dt = 0.$$

Due to the construction of the sequence, it is sufficient to prove the above relationship for $I_n^i(t)$. We have

$$\left| \int_0^{\bar{t}} f(t)(I_n^i(t) - I^i(t)) dt \right| = \left| \int_0^{\bar{t}} f(t) \left[\sum_{j\in B_i\cup D_i} (I_{i,n}^i(t) - I_i^i(t)) + \sum_{j\in C_i} (I_{j,n}^i(t) - I_j^i(t)) \right] dt \right|$$
$$= \left| \int_0^{\bar{t}} f(t) \sum_{j\in C_i} \frac{\sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_i^{*j}(t) - \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ji}(t) I_{i,n}^{*j}(t)}{\gamma_{ij}(t)} dt \right| \to 0.$$

Thus the last expression of the above equality chain converges to zero, and we obtain that $\{(e_n^i, I_n^i, s_n^i)\}$ weakly converges to (e^i, I^i, s^i). Moreover, as E is convex, closed, and bounded, it is weakly compact, and hence $K_i(e^{*-i}, I^{*-i})$ is also weakly compact for all (e^{*i}, I^{*i}). Finally, assumption (b) and the strong continuity of F_i imply that F_i is weakly continuous. Thus, by Theorem 2, the existence of at least one solution is ensured.

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