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$*$-Subvarieties of the Variety Generated by $\big(M_2(\mathbb{K}), t\big)$

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Abstract. Let $\mathbb{K}$ be a field of characteristic zero, and $* = t$ the transpose involution for the matrix algebra $M_2(\mathbb{K})$. Let $\mathfrak{U}$ be a proper subvariety of the variety of algebras with involution generated by $\big(M_2(\mathbb{K}), *\big)$. We define two sequences of algebras with involution $\mathcal{R}_p$, $\mathcal{S}_q$, where $p, q \in \mathbb{N}$. Then we show that $T_*(\mathfrak{U})$ and $T_*(\mathcal{R}_p \oplus \mathcal{S}_q)$ are $*$-asymptotically equivalent for suitable p, q.

1 Introduction

Let $\mathbb{K}$ be a field of characteristic zero and let A be an associative $\mathbb{K}$-algebra with involution $*$ of first kind, *i.e.*, such that $(\alpha r)^* = \alpha r^*$ for all $\alpha \in \mathbb{K}$, $r \in A$. If $X = \{x_1, x_2, \dots\}$ is a countable set of unknowns, we denote by $\mathbb{K}\langle X, *\rangle = \mathbb{K}\langle x_1, x_1^*, x_2, x_2^*, \dots\rangle$ the free associative algebra with involution generated by X over $\mathbb{K}$. Recall that an element $f(x_1, x_1^*, \dots, x_m, x_m^*)$ of $\mathbb{K}\langle X, *\rangle$ is a $*$-polynomial identity for A if $f(a_1, a_1^*, \dots, a_m, a_m^*) = 0$ for all substitutions $a_1, \dots, a_m \in A$. Moreover $T_*(A)$, the set of all $*$-polynomial identities of A, is a T_*-ideal of $\mathbb{K}\langle X, *\rangle$, *i.e.*, an ideal invariant under all endomorphisms of $\mathbb{K}\langle X, *\rangle$ commuting with $*$, and $\mathbb{K}\langle X, *\rangle / T_*(A)$ is the relatively free $*$-algebra of countable rank in the $*$-variety $\mathcal{V}(A, *)$ generated by A.

In case $A = M_k(\mathbb{K})$, $k \geq 2$, is the algebra of $k \times k$ matrices over $\mathbb{K}$ two involutions play a very important role in the study of $*$-polynomial identities: the transpose involution, denoted $* = t$, and the canonical symplectic involution, denoted $* = s$ and defined only in case $k = 2l$ is even. In fact, it is well known (see [9, Theorem 3.1.62]) that, if $\mathbb{K}$ is an infinite field and $*$ is an involution in $M_k(\mathbb{K})$, then either $T_*\big(M_k(\mathbb{K}), *\big) = T_*\big(M_k(\mathbb{K}), t\big)$ or $T_*\big(M_k(\mathbb{K}), *\big) = T_*\big(M_k(\mathbb{K}), s\big)$.

A complete study of $T_*\big(M_2(\mathbb{K}), t\big)$ and of $T_*\big(M_2(\mathbb{K}), s\big)$, in characteristic zero, has been made by Drensky and Giambruno in [3] and by Procesi in [8] respectively.

The purpose of this paper is to determine a description of the $*$-subvarieties of the $*$-variety $\mathcal{V}\big(M_2(\mathbb{K}), t\big)$ or, equivalently, of the T_*-ideals properly containing $T_*\big(M_2(\mathbb{K})\big)$ by using the method due to Drensky [2] in the case of ordinary T-ideals containing $T\big(M_2(\mathbb{K})\big)$.

First, we shall introduce the notion of $*$-asymptotic equivalence for T_*-ideals of $\mathbb{K}\langle X, *\rangle$. Then we shall construct two sequences of $*$-algebras, (R_p) and (S_q), with $T_*(R_p) \cap T_*(S_q) \supset T_*\big(M_2(\mathbb{K})\big)$ and establish that if U is any T_*-ideal of $\mathbb{K}\langle X, *\rangle$ such

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that $U \supset T_*\big(M_2(\mathbb{K})\big)$, then U and $T_*(R_p) \cap T_*(S_q)$ are $*$-asymptotically equivalent for suitable integers p, q.

2 Preliminaries

Throughout the paper we shall denote by $\mathbb{K}$ a field of characteristic zero. It is useful to regard the free associative algebra with involution $\mathbb{K}\langle X, *\rangle$ as generated by symmetric and skew-symmetric variables, *i.e.*, if we set $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$, for $i = 1, 2, \dots$, then $\mathbb{K}\langle X, *\rangle = \mathbb{K}\langle Y, Z\rangle = \mathbb{K}\langle y_1, z_1, y_2, z_2, \dots\rangle$. Moreover, any $\mathbb{K}$-algebra A with involution $*$ is a direct sum $A = A^+ \oplus A^-$, where $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ are the spaces of symmetric and skew-symmetric elements of A respectively. Therefore $f(y_1, \dots, y_r, z_1, \dots, z_s) \in \mathbb{K}\langle Y, Z\rangle$ is a $*$-polynomial identity for A if $f(a_1, \dots, a_r, b_1, \dots, b_s) = 0$ for all $a_i \in A^+$, $b_j \in A^-$, $i = 1, \dots, r$, $j = 1, \dots, s$.

Let us denote by $F_m(*) = \mathbb{K}\langle y_1, \dots, y_m, z_1, \dots, z_m\rangle$ the free associative algebra with involution generated by the symmetric variables $y_1, \dots, y_m$ and the skew symmetric variables $z_1, \dots, z_m$. Let $U = \mathrm{Span}_{\mathbb{K}}\{y_1, \dots, y_m\}$ and $V = \mathrm{Span}_{\mathbb{K}}\{z_1, \dots, z_m\}$. The group $\mathrm{GL}(U) \times \mathrm{GL}(V) \cong \mathrm{GL}_m \times \mathrm{GL}_m$ acts on the left on the vector space $U \oplus V$ and we can extend this action diagonally to get an action on $F_m(*)$. For every T_*-ideal $T_*(A)$, $F_m(*) \cap T_*(A)$ is invariant under the above action of $\mathrm{GL}_m \times \mathrm{GL}_m$. Hence we can view $F_m(A, *) = F_m(*)/\big(F_m(*) \cap T_*(A)\big)$ as a $\mathrm{GL}_m \times \mathrm{GL}_m$-module. Its homogeneous component of degree n, $F_m^{(n)}(A, *)$, is a $\mathrm{GL}_m \times \mathrm{GL}_m$-submodule of $F_m(A, *)$.

Now we describe briefly the representation theory of $\mathrm{GL}_m \times \mathrm{GL}_m$ acting on $F_m^{(n)}(*)$. The isomorphism classes of the irreducible modules are described by pairs of partitions (λ, μ), where λ and μ are partitions of $r, n - r$ respectively in not more than m parts, for all $r = 0, 1, \dots, n$. We write $W_{\lambda,\mu}$ for a representative of the corresponding isomorphism class of $\mathrm{GL}_m \times \mathrm{GL}_m$-modules. More precisely, let $(\mathbb{K}\langle y_1, \dots, y_m\rangle)^{(r)}$ be the homogeneous component of degree r of $\mathbb{K}\langle y_1, \dots, y_m\rangle$. Let W_λ be an irreducible $\mathrm{GL}(U)$-submodule of $(\mathbb{K}\langle y_1, \dots, y_m\rangle)^{(r)}$ corresponding to λ. It is well known that there exists an isomorphic copy of W_λ in $(\mathbb{K}\langle y_1, \dots, y_m\rangle)^{(r)}$, generated by the following product of standard polynomials

$$w_\lambda(y_1, \dots, y_p) := \prod_{j=1}^{\lambda_1} s_{p_j}(y_1, \dots, y_{p_j}),$$

where p_j is the height of the j-th column of the Young diagram D_λ associated to λ, $p := p_1$ and the s_{p_j} are the standard polynomials of degree p_j. Similarly we may define a $\mathrm{GL}(V)$-submodule W_μ of $(\mathbb{K}\langle z_1, \dots, z_m\rangle)^{(n-r)}$ with a generator w_μ. Now the irreducible $\mathrm{GL}_m \times \mathrm{GL}_m$-module $W_{\lambda,\mu}$ is isomorphic to the tensor product $W_\lambda \otimes_{\mathbb{K}} W_\mu$. A generator of an $W_{\lambda,\mu}$ is $w_\lambda(y_1, \dots, y_p)w_\mu(z_1, \dots, z_q)$, where q is the height of the first column of D_μ. Any isomorphic copy of $W_{\lambda,\mu}$ in $F_m^{(n)}(*)$ is generated by a non-zero element

$$w_{\lambda,\mu} := w_\lambda(y_1, \dots, y_p)w_\mu(z_1, \dots, z_q) \sum_{\sigma \in S_n} \alpha_\sigma \sigma \quad (\alpha \in \mathbb{K})$$

where the symmetric group S_n acts by place permutation of the variables (*right action* of S_n) (see for instance [3, p. 720],). The polynomial $w_{\lambda,\mu}$ is the so-called *highest weight vector (h.w.v.)* of the module.

We refer to [4] for a complete treatment of the representation theory of $\mathrm{GL}_m \times \mathrm{GL}_m$ acting on $F_m(*)$.

As in [3] we bring the notion of Y-proper $*$-polynomial up. First, we define (higher) commutators by

$$[v_1, v_2] = v_1 v_2 - v_2 v_1$$

$$[v_1, \ldots, v_{n-1}, v_n] = \big[[v_1, \ldots, v_{n-1}], v_n\big], \quad (n > 2).$$

We say that a $*$-polynomial $f(y_1, \ldots, y_m, z_1, \ldots, z_m) \in \mathbb{K}\langle Y, Z\rangle$ is Y-proper if the y's occur in commutators only.

By the Poincaré-Birkhoff-Witt theorem $F_m(*)$ has a basis

$$\{y_1^{s_1} \cdots y_m^{s_m} z_1^{t_1} \cdots z_m^{t_m} u_1^{r_1} \cdots u_n^{r_n} \mid s_h, t_i, r_j \geq 0\},$$

where $u_1, u_2, \ldots$ are higher commutators.

We denote by $B_m(*)$ the vector subspace of $F_m(*)$ spanned by all products

$$\{z_1^{r_1} \ldots z_m^{r_m} u_1^{s_1} \ldots u_n^{s_n} \mid r_i, s_j \geq 0\}.$$

Hence the Y-proper $*$-polynomials are the elements of $B_m(*)$. $B_m^{(n)}(*)$ denotes its homogeneous component of degree n.

An alternative definition of Y-proper polynomials is the following: f is Y-proper if all formal partial derivatives $\partial f/\partial y_i$, defined by $\partial y_j/\partial y_i := \delta_{i,j}$ (Kronecker delta), are zero for all $i = 1, \ldots, m$.

By Lemma 2.1. in [3], we have that all $*$-polynomial identities of an algebra A with involution $*$ follow from the Y-proper ones. This means that the set $T_*(A) \cap B_m(*)$ generates the whole $T_*(A) \cap F_m(*)$ as a T_*-ideal. Let us denote by $B_m(A, *) := B_m(*)/\big(T_*(A) \cap B_m(*)\big)$.

The relation between the Y-proper and all the $*$-polynomial identities of a PI-algebra A with involution $*$ are stated by the following:

Theorem 2.1 **([3, Theorem 2.3 (iv), (v)])** *The following* $\mathrm{GL}_m \times \mathrm{GL}_m$*-module isomorphism holds:*

$$F_m(A, *) \cong \mathbb{K}[y_1, \ldots, y_m] \otimes B_m(A, *)$$

where the algebra $\mathbb{K}[y_1, \ldots, y_m]$ *is under the canonical* $\mathrm{GL}(U)$*-action and the trivial* $\mathrm{GL}(V)$*-action. In particular, if*

$$F_m(A, *) \cong \sum a_{\lambda,\mu} W_\lambda \otimes W_\mu$$

$$B_m(A, *) \cong \sum b_{\nu,\mu} W_\nu \otimes W_\mu$$

then $a_{\lambda,\mu} = \sum b_{\nu,\mu}$ *where for fixed* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *and* μ *the summation runs over all partitions* $\nu = (\nu_1, \ldots, \nu_m)$ *such that*

$$\lambda_1 \geq \nu_1 \geq \cdots \geq \lambda_m \geq \nu_m.$$

As in [6] for T-ideals we give the following

Definition 2.2 Let $B^{(n)}(*)$ be the space of all Y-proper polynomials of degree n in $\mathbb{K}\langle Y, Z\rangle$. The T_*-ideals of $\mathbb{K}\langle Y, Z\rangle$, U_1 and U_2, are $*$-asymptotically equivalent if there exists $\nu_0 \in \mathbb{N}$ such that for all $n \geq \nu_0$

$$U_1 \cap B^{(n)}(*) = U_2 \cap B^{(n)}(*)$$

and we write

$$U_1 \approx_* U_2.$$

In this paper we shall investigate the case $(A, *) = \big(M_2(\mathbb{K}), *\big)$, where $*$ is the transpose involution. We shall give an asymptotic description of the T_*-ideals properly containing $T_*\big(M_2(\mathbb{K}), *\big)$.

3 Partial Linearization and the Koshlukov's Criterion

Definition 3.1 Let $f = f(x_1, \ldots, x_n)$ be a multi-homogeneous polynomial. By the symbol

$$f(x_1|x_2|\ldots|x_{i-1}|x_i, u|x_{i+1}|\ldots|x_{n-1}|x_n)$$

we denote the homogeneous component of the polynomial

$$f(x_1, x_2, \ldots, x_{i-1}, x_i + u, x_{i+1}, \ldots, x_{n-1}, x_n)$$

of degree 1 with respect to the variable u, and we shall refer to it as to the *partial linearization* of f with respect to x_i. Analogously,

$$f(x_1|x_2|\ldots|x_{i-1}|x_i, u_1, u_2, \ldots, u_k|x_{i+1}|\ldots|x_{n-1}|x_n)$$

is the homogeneous component of first degree with respect to each of the variables u_j of

$$f(x_1, x_2, \ldots, x_{i-1}, x_i + u_1 + u_2 + \cdots + u_k, x_{i+1}, \ldots, x_{n-1}, x_n),$$

and finally

$$f(x_1|\ldots|x_i, u_1, u_2|\ldots|x_j, v_1, v_2|\ldots|x_n)$$

will denote the homogeneous component of first degree with respect to u_1, u_2, v_1, v_2 of the polynomial

$$f(x_1, \ldots, x_i + u_1 + u_2, \ldots, x_j + v_1 + v_2, \ldots, x_n).$$

Examples 3.2 Let $f = f(x_1, x_2, x_3, x_4) = x_2x_1^2x_2x_3x_4x_2$. Then

$$f(x_1|x_2, u|x_3|x_4) = ux_1^2x_2x_3x_4x_2 + x_2x_1^2ux_3x_4x_2 + x_2x_1^2x_2x_3x_4u$$

$$\begin{aligned} f(x_1|x_2, u, v|x_3|x_4) = {} & ux_1^2vx_3x_4x_2 + ux_1^2x_2x_3x_4v + vx_1^2ux_3x_4x_2 \\ & + x_2x_1^2ux_3x_4v + vx_1^2x_2x_3x_4u + x_2x_1^2vx_3x_4u. \end{aligned}$$

Very often we shall consider a given polynomial $f = f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ and write $f(x_1| \dots |x_{i-1}|x_i, g_1, g_2, \dots |x_{i+1}| \dots |x_n)$, where $g_1, g_2, \dots$ are polynomials, to mean the polynomial obtained from the partial linearization $f(x_1| \dots |x_{i-1}|x_i, u_1, u_2, \dots |x_{i+1}| \dots |x_n)$ by substituting g_1 to the linear variable u_1, g_2 to the linear variable u_2 and so on.

The following criterion due to Koshlukov [7] is an effective tool to establish if a given polynomial f of multidegree $\lambda = (\lambda_1, \dots, \lambda_m)$ is a h.w.v. for a GL_m-module $W \cong W_\lambda$:

Proposition 3.3 **(Koshlukov's Criterion, [7, Lemma 1.3.1])** *A multi-homogeneous polynomial $f = f(x_1, \dots, x_m)$ of degree $\lambda = (\lambda_1, \dots, \lambda_m)$ is the h.w.v. for a GL_m-module $W \cong W_\lambda$ if and only if $f \neq 0$ and*

$$f(x_1| \dots |x_i, x_j| \dots |x_m) = 0 \quad \forall i = 1, \dots, m, \ j < i.$$

4 The Algebras $\big(M_2(\mathbb{K}), *\big)$, $\mathcal{R}_p$, $\mathcal{S}_q$

Let $A = M_2(\mathbb{K})$. We recall that just two kinds of involution do define different T_*-ideals: the transpose and the symplectic involution. The transpose involution acts on $M_2(\mathbb{K})$ by

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^* = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}.$$

In this case, we know the following decomposition:

Theorem 4.1 **([3, Theorem 3.4, (i)])**

$$B_m\big(M_2(\mathbb{K}), *\big) \cong \bigoplus W_{\lambda,\mu} \cong \bigoplus W_\lambda \otimes W_\mu$$

where the summation is over all partitions $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1)$ and $\lambda_2 \neq 0$ when $\lambda_1 \neq 0$ and $\mu_1 = 0$.

We define two sequences of finite dimensional algebras with involution which will be essential to our description of the T_*-ideals properly containing $T_*\big(M_2(\mathbb{K})\big)$.

Let $k \geq 1$ and $\mathcal{C}_k = \mathbb{K}[t]/(t^k)$ be the polynomial algebra modulo the ideal generated by t^k.

We define the following algebras with involution

$$\mathcal{R}_p = \mathcal{C}_p\mathbf{e} + t\mathcal{C}_p\mathbf{a} + t\mathcal{C}_p\mathbf{b} + \mathcal{C}_p\mathbf{c}$$
$$\mathcal{S}_q = \mathcal{C}_q\mathbf{e} + \mathcal{C}_q\mathbf{a} + t\mathcal{C}_q\mathbf{b} + t\mathcal{C}_q\mathbf{c}$$

where $\mathbf{e}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the matrices

$$\mathbf{e} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{a} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{c} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since the algebras $M_2(\mathcal{C}_p)$ and $M_2(\mathcal{C}_q)$ have the same $*$-polynomial identities as $M_2(\mathbb{K})$, and $\mathcal{R}_p$ and $\mathcal{S}_q$ are subalgebras of $M_2(\mathcal{C}_p)$ and $M_2(\mathcal{C}_q)$, respectively, we obtain that the $\mathrm{GL}_m \times \mathrm{GL}_m$-modules $B_m(\mathcal{R}_p, *)$ and $B_m(\mathcal{S}_q, *)$ are homomorphic images of $B_m\big(M_2(\mathbb{K}), *\big)$. By Theorem 4.1, the latter $\mathrm{GL}_m \times \mathrm{GL}_m$-module decomposes into irreducible submodules associated to pairs of partitions of kind $\big((\lambda_1, \lambda_2), (k)\big)$. So $B_m(\mathcal{R}_p, *)$ and $B_m(\mathcal{S}_q, *)$ do the same. Now we are going to check which among these modules occur in these decompositions, actually. By Theorem 4.1, it will suffice to work in $B_2(*)$ and consider Y-proper polynomials in which just one z occurs, *i.e.*, Y-proper polynomials in y_1, y_2, z_1. The following lemma will prove itself useful in direct computations:

Lemma 4.2 *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be the matrices defined at the beginning of this section.*

(1) *The following relations hold:*

$$\mathbf{a}^2 = \mathbf{b}^2 = -\mathbf{c}^2 = \mathbf{e}$$
$$\mathbf{ab} = \mathbf{c} = -\mathbf{ba} \quad \mathbf{ac} = \mathbf{b} = -\mathbf{ca} \quad \mathbf{cb} = \mathbf{a} = -\mathbf{bc}.$$

(2) *The previous relations yield*

$$[\mathbf{a}, \mathbf{b}] = 2\mathbf{c} \quad [\mathbf{a}, \mathbf{c}] = 2\mathbf{b} \quad [\mathbf{c}, \mathbf{b}] = 2\mathbf{a}.$$

(3) *For higher commutators, the following relations hold*

$$[\mathbf{c}, \underbrace{\mathbf{a}, \ldots, \mathbf{a}}_{p}] = 2^p\mathbf{c}\mathbf{a}^p \quad [\mathbf{b}, \underbrace{\mathbf{a}, \ldots, \mathbf{a}}_{p}] = 2^p\mathbf{b}\mathbf{a}^p.$$

Proof The statements (1) and (2) are straightforward. A simple induction on $p \geq 1$ yields (3). If $p = 1$, then $[\mathbf{b}, \mathbf{a}] = 2\mathbf{ba} = -2\mathbf{c}$ and $[\mathbf{c}, \mathbf{a}] = 2\mathbf{ca} = -2\mathbf{b}$. Assume $p \geq 1$ and consider

$$[\mathbf{c}, \underbrace{\mathbf{a}, \ldots, \mathbf{a}}_{p+1}] = \big[[\mathbf{c}, \underbrace{\mathbf{a}, \ldots, \mathbf{a}}_{p}], \mathbf{a}\big] = [2^p\mathbf{c}\mathbf{a}^p, \mathbf{a}] = 2^p[\mathbf{c}, \mathbf{a}]\mathbf{a}^p$$

since $\mathbf{a}^p$ and $\mathbf{a}$ commute. Similar arguments prove the other equality. ∎

Now we are going to compute the h.w.v. of the irreducible components of $B_m\big(M_2(\mathbb{K}), *\big)$.

Lemma 4.3 *Let W be the irreducible component of $B_2\big(M_2(\mathbb{K}), *\big)$ associated to the pair $\big((\lambda_1, \lambda_2), (k)\big)$. Then*

$$w = [z, \underbrace{y_1, \ldots, y_1}_{\lambda_1-\lambda_2}][y_2, y_1]^{\lambda_2} z^{k-1}, \quad \textit{if } k > 0,$$

$$w = \big[[y_2, y_1], \underbrace{y_1, \ldots, y_1}_{\lambda_1-\lambda_2}\big][y_2, y_1]^{\lambda_2-1}, \quad \textit{if } k = 0$$

is its highest weight vector.

Proof The polynomials w are not zero, the symmetric variables y_1, y_2 occur in commutators only (hence they are Y-proper) and replacing in w one y_2 by y_1 the polynomials $w(y_1|y_2, y_1|z)$ vanish identically (Koshlukov's criterion). The only thing to check is that they are not polynomial identities for $\big(M_2(\mathbb{K}), *\big)$. A straightforward calculation (using Lemma 4.2) shows that $w(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$. ■

Some of the polynomials listed in Lemma 4.3 may be a $*$-PI for the algebras $\mathcal{R}_p, \mathcal{S}_q$, and now we shall see which ones. The proofs of the next two lemmas are very similar. We give the proof of the last one only.

Lemma 4.4 *Let $k = 0$ and, from Lemma 4.3, consider the highest weight vector w for the irreducible component of $B_2\big(M_2(\mathbb{K}), *\big)$ associated to the pair $\big((\lambda_1, \lambda_2), (0)\big)$. Then*

1. *w is a $*$-polynomial identity for $\mathcal{R}_p$ if and only if $\lambda_1 + \lambda_2 \geq p$;*
2. *w is a $*$-polynomial identity for $\mathcal{S}_q$ if and only if $\lambda_2 \geq q$.*

Similarly, in the general case:

Lemma 4.5 *Let $k > 0$ and, from Lemma 4.3, consider the highest weight vector w for the irreducible component of $B_2\big(M_2(\mathbb{K}), *\big)$ associated to the pair $\big((\lambda_1, \lambda_2), (k)\big)$. Then*

1. *w is a $*$-polynomial identity for $\mathcal{R}_p$ if and only if $\lambda_1 + \lambda_2 \geq p$;*
2. *w is a $*$-polynomial identity for $\mathcal{S}_q$ if and only if $\lambda_2 + k \geq q$.*

Proof The symmetric part of $\mathcal{R}_p$ is spanned over $\mathcal{C}_p$ by the elements $\mathbf{e}, t\mathbf{a}, t\mathbf{b}$, and the skew-symmetric part by $\mathbf{c}$. Since in the polynomial w the variables y_1, y_2 occur in commutators only, we may assume that the generic substitution for w is of kind

$$\varphi\colon\, y_i \longmapsto \alpha_i t\mathbf{a} + \beta_i t\mathbf{b}, \quad z \longmapsto \mathbf{c}.$$

Let $n := \lambda_1 + \lambda_2$. Then

$$w\big(\varphi(y_1), \varphi(y_2), \varphi(z)\big) = t^n w(\alpha_1 \mathbf{a} + \beta_1 \mathbf{b}, \alpha_2 \mathbf{a} + \beta_2 \mathbf{b}, \mathbf{c}).$$

Hence if $n \geq p$ then $\varphi(w) = 0$ and $w \in T_*(\mathcal{R}_p)$. Conversely if $n < p$ the substitution $w(t\mathbf{a}, t\mathbf{b}, \mathbf{c}) = t^n w(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is not zero.

So w is a $*$-PI for the algebra $\mathcal{R}_p$ if and only if $\lambda_1 + \lambda_2 \geq p$. Similar arguments hold for the algebra $\mathcal{S}_q$. ■

5 Consequences of a Highest Weight Vector

If $f \in \mathbb{K}\langle Y, Z\rangle$ we denote by $(f)_{T_*}$ the T_*-ideal of $\mathbb{K}\langle Y, Z\rangle$ generated by f. Recall that a $*$-polynomial g is a *consequence* of f if $g \in (f)_{T_*}$.

Definition 5.1 Let $W, \bar{W}$ be irreducible components of $B_2\big(M_2(\mathbb{K}), *\big)$, and $w, \bar{w}$ their corresponding highest weight vectors. If $\bar{w}$ is a consequence of w, then we say that $\bar{W}$ is a *higher consequence* of W, and that the polynomial $\bar{w}$ is a higher consequence of w for $\bar{W}$.

Since $B_2\big(M_2(\mathbb{K}), *\big)$ decomposes into irreducible components with multiplicities 1 by Theorem 4.1, we shall identify the components $W_{(\lambda_1,\lambda_2),(k)} \cong W_{(\lambda_1,\lambda_2)} \otimes W_{(k)}$ with the corresponding diagrams $(\lambda_1, \lambda_2) \otimes (k)$.

We are interested in the consequences of degree $n + 1$ of the highest weight vector of degree n for a fixed irreducible submodule $\begin{array}{|l|}\hline p+q \\ \hline p \\ \hline \end{array} \otimes \boxed{k}$ of $B_2\big(M_2(\mathbb{K}), *\big)$.

We shall prove that, for any diagram $(a, b) \otimes (c)$ which can be obtained from $(p + q, p) \otimes (k)$ by one the following operations

1. glue a new box to one row
2. delete a box from a row and glue a new box to each other row,

the highest weight vector of the corresponding irreducible component of $B_2\big(M_2(\mathbb{K}), *\big)$ is a higher consequence of the highest weight vector of $(p+q, p) \otimes (k)$.

As in Lemma 4.3, the highest weight vector w of $(p + q, p) \otimes (k)$ is of type

$$w = [z, \underbrace{y_1, \ldots, y_1}_{q}][y_2, y_1]^p z^{k-1}, \quad \text{if } k > 0,$$

$$w = \big[[y_2, y_1], \underbrace{y_1, \ldots, y_1}_{q}\big][y_2, y_1]^{p-1}, \quad \text{if } k = 0.$$

We may rephrase Proposition 3.3 to get an effective tool in order to verify that a given polynomial is the h.w.v. for the irreducible submodule $W_{(\lambda_1,\lambda_2),(k)}$ of $B_2\big(M_2(\mathbb{K}), *\big)$.

Remark 5.2 Let $f = f(y_1, y_2, z_1)$ be a polynomial in $F_2\big(M_2(\mathbb{K}), *\big)$ of multidegree $(\lambda_1, \lambda_2, k)$. Then f is the h.w.v. for the irreducible submodule $(\lambda_1, \lambda_2) \otimes (k)$ if:

(1) $\partial f/\partial y_i = 0$ for $i = 1, 2$. This is to check that f is Y-proper.
(2) $f(y_1|y_2, y_1|z_1) = 0$. This is Koshlukov's criterion.
(3) $f \notin T_*\big(M_2(\mathbb{K})\big)$.

Since the formal derivative $\partial f/\partial y_i$ equals the partial linearization of y_i by 1 in f, if we write

$$\underset{y_i \mapsto u}{f} := f(\dots|y_i, u|\dots)$$

we have to check

$$\underset{y_1\mapsto 1}{f} = \underset{y_2\mapsto 1}{f} = \underset{y_2\mapsto y_1}{f} = 0, \quad f(a_1, a_2, b_1) \neq 0$$

for $a_1, a_2 \in A^+$, $b_1 \in A^-$.

We start dealing with the simplest case.

Lemma 5.3 *The polynomial*

$$\bar{w} = wz$$

is a higher consequence of w for $(p+q, p) \otimes (k+1)$.

Proof The polynomial $\bar{w}$ is a consequence of w, is Y-proper, has the right multidegree and satisfies the Koshlukov's criterion as in Remark 5.2. Finally it is not a $*$-polynomial identity for $M_2(\mathbb{K})$ since $\bar{w}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = w(\mathbf{a}, \mathbf{b}, \mathbf{c})\mathbf{c}$ and $w(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$ because w is the highest weight vector for the irreducible component of $B_2\big(M_2(\mathbb{K}), *\big)$ associated to $(p+q, p) \otimes (k)$. ■

The remaining cases are investigated in the following five lemmas. These lemmas can be proved by straightforward calculations as for Lemma 5.3. We shall omit the proofs of Lemmas 5.5 and 5.8 because they are through and through similar to those investigated.

More precisely, we shall prove that any polynomial $\bar{w}$ (listed in the following lemmas) is a consequence of w. Each $\bar{w}$ is a linear combination of suitable partial linearizations of w. We remark that it is essential that in the partial linearizations we replace (skew-)symmetric elements by elements of the free algebra which are of the same type. This ensures that $\bar{w}$ is in the T_*-ideal generated by w. Next, we follow the steps listed in Remark 5.2 and prove that $\bar{w}$ is the highest weight vector for the corresponding irreducible submodule of $B_2\big(M_2(\mathbb{K}), *\big)$.

In the following we shall adopt the standard notation of the so-called *Jordan product*, $u \circ v := uv + vu$. Also, we shall write $c := [y_2, y_1]$ for shortness.

Lemma 5.4 *The polynomial*

$$\bar{w} := w(y_1, y_1^2|y_2|z) - 2(p+q)wy_1 \quad \textit{if } p+q \neq 0$$

or

$$\bar{w} := w(y_1|y_2|z, z \circ y_1) - 2kwy_1 \quad \text{if } p + q = 0$$

is a higher consequence of w *for* $(p + q + 1, p) \otimes (k)$.

Proof Assume at first that $p + q \neq 0$ and $k > 0$. Then, by Lemma 4.3,

$$w = [z, \underbrace{y_1, \ldots, y_1}_{q}][y_2, y_1]^p z^{k-1}$$

and write

$$w_1 := \underset{y_1 \mapsto y_1^2}{w} = \sum_{i=1}^{q} [z, y_1, \ldots, \underset{\substack{\uparrow \\ i}}{y_1^2}, \ldots, y_1][y_2, y_1]^p z^{k-1}$$

$$+ [z, \underbrace{y_1, \ldots, y_1}_{q}] \sum_{i+j=p-1} [y_2, y_1]^i [y_2, y_1^2][y_2, y_1]^j z^{k-1}$$

Here we shall denote by $\sum_i^q [z, \ldots, u, \ldots]$ the summation

$$[z, \underset{\substack{\uparrow \\ 1}}{u}, y_1, \ldots, y_1] + [z, y_1, \underset{\substack{\uparrow \\ 2}}{u}, y_1, \ldots, y_1] + \cdots + [z, y_1, \ldots, y_1, \underset{\substack{\uparrow \\ q}}{u}].$$

We shall write $[z, \ldots]$ as a shortcut for $[z, \underbrace{y_1, \ldots, y_1}_{q}]$ if there is no ambiguity. Finally, we shall shorten $\sum_{i+j=\alpha}$, $\sum_{i+j+h=\alpha}$ in $\sum_{i,j}^{\alpha}$, $\sum_{i,j,h}^{\alpha}$ respectively.

In order to simplify the notation, we shall use these shortcuts through the rest of this section. Actually, we shall add some more shortcuts in the following lemmas. We shall emphasize them as soon as they occur.

Since in the partial linearization w_1 we are replacing the symmetric element y_1 by the symmetric element y_1^2, w_1 is a consequence of w. Hence $\bar{w}$ is a consequence of w, as well.

It is worth noticing that

$$[y_2, y_1^2] = y_1 \circ [y_2, y_1]$$

$$[z, \ldots, y_1^2, \ldots] = y_1 \circ [z, \ldots]$$

as direct calculations show. Hence

$$w_1 = q(y_1 \circ [z, \ldots])c^p z^{k-1} + [z, \ldots] \sum_{i,j}^{p-1} c^i (y_1 \circ c) c^j z^{k-1}.$$

Since in $\bar{w}$ the variable y_2 occurs in $[y_2, y_1]$ only, $\bar{w}$ is y_2-proper and $\underset{y_2 \mapsto y_1}{\bar{w}} = 0$. Moreover, since w is Y-proper, $\underset{y_1 \mapsto 1}{w} = 0$. Hence

$$\underset{y_1 \mapsto 1}{\bar{w}} = 2q[z, \dots]c^p z^{k-1} + 2p[z, \dots]c^p z^{k-1} - 2(p+q)w = 0$$

and $\bar{w}$ is y_1-proper, as well. We have to prove that $\bar{w}$ is not a polynomial identity in $\big(M_2(\mathbb{K}), *\big)$. Since $\mathbf{a}^2 = \mathbf{e}$ and y_1^2 occurs in w_1 in commutators only, deduce that $w_1(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$ hence $\bar{w}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -2(p+q)w(\mathbf{a}, \mathbf{b}, \mathbf{c})\mathbf{a} \neq 0$.

Now assume that $k = 0$. Then, by Lemma 4.3,

$$w = [c, \dots]c^{p-1}$$

and

$$\begin{aligned}
\bar{w} &= \big[[y_2, y_1^2], \dots\big] + \sum_{i}^{q} [c, \dots, y_1^2, \dots]c^{p-1} \\
&\quad + [c, \dots] \sum_{i,j}^{p-2} c^i [y_2, y_1^2] c^j - 2(p+q)wy_1 \\
&= [y_1 \circ c, \dots] + q(y_1 \circ [c, \dots])c^{p-1} + [c, \dots] \sum_{i,j}^{p-2} c^i (y_1 \circ c) c^j - 2(p+q)wy_1 \\
&= (q+1)(y_1 \circ [c, \dots])c^{p-1} + [c, \dots] \sum_{i,j}^{p-2} c^i (y_1 \circ c) c^j - 2(p+q)wy_1.
\end{aligned}$$

Then

$$\underset{y_1 \mapsto 1}{\bar{w}} = 2(q+1)w + 2(p-1)w - 2(p+q)w = 0.$$

As before, $\underset{y_2 \mapsto 1}{\bar{w}} = \underset{y_2 \mapsto y_1}{\bar{w}} = 0$ because y_2 occurs in $[y_2, y_1]$ only, and again $\bar{w}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -2(p+q)w(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$. In both cases ($k = 0$ and $k \neq 0$) the polynomial $\bar{w}$ is a h.w.v. for $(p+q+1, p) \otimes (k)$. Hence $\bar{w}$ is a higher consequence of w for the module $(p+q+1) \otimes (k)$.

Now assume $p + q = 0$. Hence $w = z^k$ ($k > 0$), by Lemma 4.3, and

$$\bar{w} = \sum_{i,j}^{k-1} z^i (y_1 \circ z) z^j - 2kz^k y_1.$$

Since in the partial linearization $\underset{z \mapsto z \circ y_1}{w}$ we are replacing the skew-symmetric element z by the skew-symmetric element $z \circ y_1$, $\underset{z \mapsto z \circ y_1}{w}$ is a consequence of w. Hence $\bar{w}$ is a consequence of w.

Clearly

$$\bar{w}_{y_1 \mapsto 1} = 2kz^k - 2kz^k = 0,$$

so $\bar{w}$ is y_1-proper (and Y-proper). Since $\mathbf{c}, \mathbf{a}$ anti-commute, $\mathbf{c} \circ \mathbf{a} = 0$. Hence

$$\bar{w}(\mathbf{a}, \mathbf{c}) = -2k\mathbf{c}^k\mathbf{a} \neq 0.$$

So $\bar{w}$ is a higher consequence of w. ■

The proof of the following lemma is very similar, and we shall omit it.

Lemma 5.5 *Let $q \geq 1$.*

(1) *If $k > 0$, then*

$$\bar{w} = w(y_1, y_2|y_2|z, z \circ y_1) - 2kw(y_1, y_2|y_2|z)y_1 - qw(y_1|y_2|z, z \circ y_2) + 2kqwy_2$$

is a higher consequence of w for $(p+q, p+1) \otimes (k)$.

(2) *If $k = 0$, then*

$$\begin{aligned}\bar{w} = {} & 2pqwy_2 - 2pw(y_1, y_2|y_2|z)y_1 + w(y_1|y_2, y_2^2|z) \\ & + w(y_1, y_2|y_2, y_2 \circ y_1|z) + qw(y_1, y_2^2|y_2, y_1|z)\end{aligned}$$

is a higher consequence of w for $(p+q, p+1) \otimes (0)$.

So far, we proved that the components $(a, b) \otimes (c)$ obtained by gluing an extra box to one row of $(p+q, p) \otimes (k)$ are higher consequences of the given irreducible component of $B_2\big(M_2(\mathbb{K}), *\big)$. Now we start with deleting one box from a row and gluing a new box in each of the remaining rows.

Lemma 5.6 *Let $k > 0$. Then*

$$\bar{w} = w(y_1|y_2|z, [y_2, y_1])$$

is a higher consequence of w for $(p+q+1, p+1) \otimes (k-1)$.

Proof Since $k > 0$, by Lemma 4.3

$$w = [z, \dots]c^p z^{k-1}.$$

Then

$$\bar{w} = [c, \dots]c^p z^{k-1} + [z, \dots]c^p \sum_{i,j}^{k-2} z^i c z^j.$$

It is a consequence of w because in the partial linearization we are replacing the skew-symmetric variable z by the skew-symmetric element $[y_2, y_1]$ of the free algebra. The polynomial $\bar{w}$ is clearly Y-proper, has the right degree and $\underset{y_2 \mapsto y_1}{\bar{w}} = 0$. Thus we have to check that $\bar{w} \notin T_*\big(M_2(\mathbb{K})\big)$. It is the case, since

$$\begin{aligned}\bar{w}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= -2[\mathbf{c}, \mathbf{a}, \dots, \mathbf{a}](-1)^p 2^p \mathbf{c}^p \mathbf{c}^{k-1} - 2[\mathbf{c}, \mathbf{a}, \dots, \mathbf{a}](-1)^p 2^p \mathbf{c}^p \sum_{i,j}^{k-2} \mathbf{c}^i \mathbf{c} \mathbf{c}^j \\ &= (-2)^{p+1} k [\mathbf{c}, \mathbf{a}, \dots, \mathbf{a}] \mathbf{c}^{p+k-1} \neq 0.\end{aligned}$$

Hence $\bar{w}$ is a higher consequence of w. ∎

The proof of the following lemma involves much more calculation:

Lemma 5.7 *Let $q \geq 2$.*

(1) *If $k > 0$ then*

$$\begin{aligned}\bar{w} &= (q^2 - 1) w(y_1, [z, y_2] | y_2 | z) - (q+1) w(y_1, y_2, [z, y_1] | y_2 | z) \\ &\quad + (q-1) w(y_1, y_2 | y_2, [z, y_2] | z) - w(y_1, y_2, y_2 | y_2, [z, y_1] | z)\end{aligned}$$

is a higher consequence of w for $(p+q-1, p+1) \otimes (k+1)$.

(2) *If $k = 0$ then*

$$\begin{aligned}\bar{w} &= (p-q-1)(q-1) w(y_1, [z, y_2] | y_2 | z) + (q-1) w(y_1, y_2, [z, y_2] | y_2, y_1 | z) \\ &\quad + w(y_1, y_2, y_2 | y_2, [z, y_1] | z) + (q+1) w(y_1, y_2, [z, y_1] | y_2 | z)\end{aligned}$$

is a higher consequence of w for $(p+q-1, p+1) \otimes (1)$.

Proof First assume that $k > 0$. Thus $w = [z, \dots] c^p z^{k-1}$ by Lemma 4.3. Split $\bar{w}$ into the linearizations w_1, w_2, w_3, w_4 of w which are the summands of $\bar{w}$. Since $[z, y_1], [z, y_2]$ are symmetric elements, these summands are consequences of w. Then, we follow the steps listed in Remark 5.2 for each of them in order to prove that $\bar{w}$ is an highest weight vector in $B_2\big(M_2(\mathbb{K}), *\big)$. Thus we shall compute, for $i = 1, 2, 3, 4$, the partial linearizations $\underset{y_1 \mapsto 1}{w_i}$, $\underset{y_2 \mapsto 1}{w_i}$, $\underset{y_2 \mapsto y_1}{w_i}$, and the values $w_i(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in $\big(M_2(\mathbb{K}), *\big)$.

$$\begin{aligned}w_1 &= w(y_1, [z, y_2] | y_2 | z) \\ &= \sum_i^q [z, \dots, [z, y_2], \dots] c^p z^{k-1} - [z, \dots] \sum_{i,j}^{p-1} c^i [z, y_2, y_2] c^j z^{k-1}\end{aligned}$$

$$\underset{y_1 \mapsto 1}{w_1} = \underset{y_2 \mapsto 1}{w_1} = 0$$

$$\underset{y_2\mapsto y_1}{w_1} = \sum_i \big[z, \dots, [z, y_1], \dots\big]\, c^p z^{k-1} - 2[z, \dots] \sum_{i,j}^{p-1} c^i [z, y_2, y_1] c^j z^{k-1}$$

$$- [z, \dots] \sum_{i,j}^{p-1} c^i [c, z] c^j z^{k-1}$$

$$w_1(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 2^{p+q+1}(-1)^{p+q}(p+q)\mathbf{a}^q \mathbf{c}^{p+k}$$

(note that in computing $\underset{y_2\mapsto y_1}{w_1}$ we use $[z, y_1, y_2] = [z, y_2, y_1] + [y_2, y_1, z]$ by the Jacobi law).

Here and for the rest of this section we adopt the notation

$$\sum_{i\neq j} [z, \dots, f, \dots, g, \dots]$$

to mean the summation

$$[z, f, g, \dots] + [z, g, f, \dots] + [z, f, \cdot, g, \dots]$$
$$+ [z, g, \cdot, f, \dots] + \dots + [z, \dots, f, g] + [z, \dots, g, f].$$

Now let us consider the second summand.

$$w_2 = w(y_1, y_2, [z, y_1] | y_2 | z) = \sum_{i\neq j} \big[z, \dots, y_2, \dots, [z, y_1], \dots\big]\, c^p z^{k-1}$$

$$- \sum_i [z, \dots, y_2, \dots] \sum_{i,j}^{p-1} c^i [z, y_2, y_1] c^j z^{k-1}$$

$$- \sum_i [z, \dots, y_2, \dots] \sum_{i,j}^{p-1} c^i [c, z] c^j z^{k-1}$$

$$\underset{y_1\mapsto 1}{w_2} = \underset{y_2\mapsto 1}{w_2} = 0$$

$$\underset{y_2\mapsto y_1}{w_2} = (q-1) \sum_i \big[z, \dots, [z, y_1], \dots\big]\, c^p z^{k-1} - q[z, \dots] \sum_{i,j}^{p-1} c^i [z, y_2, y_1] c^j z^{k-1}$$

$$- \sum_i [z, \dots, y_2, \dots] \sum_{i,j} c^i [z, y_1, y_1] c^j z^{k-1} - q[z, \dots] \sum_{i,j}^{p-1} c^i [c, z] c^j z^{k-1}.$$

In order to compute $w_2(\mathbf{a}, \mathbf{b}, \mathbf{c})$, note that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$ and so is any 3-commutator with these three elements, for all their place permutations. Furthermore, note that

$$\sum_{i\neq j} [\mathbf{c}, \dots, \mathbf{b}, \dots, \mathbf{b}, \dots] = 2 \sum_{\substack{i<q \\ i \text{ odd}}} [\mathbf{c}, \dots, \underset{\substack{\uparrow \\ i}}{\mathbf{b}}, \mathbf{b}, \dots] = 2 \sum_{\substack{i<q \\ i \text{ odd}}} [\mathbf{c}, \dots, \underset{\substack{\uparrow \\ i}}{\mathbf{a}}, \mathbf{a}, \dots].$$

We shall denote by $d(1, q-1)$ the number of odd integers between 1 and $q-1$. With this notation,

$$\sum_{i\neq j}[\mathbf{c},\dots,\mathbf{b},\dots,\mathbf{b}\dots] = 2^{q+1}d(1,q-1)\mathbf{c}\mathbf{a}^q.$$

Hence we get

$$\begin{aligned} w_2(\mathbf{a},\mathbf{b},\mathbf{c}) &= -2\sum_{i\neq j}[\mathbf{c},\dots,\mathbf{b},\dots,\mathbf{b},\dots](-1)^p 2^p \mathbf{c}^{p+k-1} \\ &= -2(-1)^{p+q}2^{p+q+1}d(1,q-1)\mathbf{a}^q\mathbf{c}^{p+k}. \end{aligned}$$

Let us proceed with the next polynomial.

$$\begin{aligned} w_3 &= w(y_1, y_2|y_2,[z,y_2]|z) \\ &= \sum_i [z,\dots,y_2,\dots]\sum_{i,j}^{p-1} c^i[z,y_2,y_1]c^j z^{k-1} + [z,\dots]\sum_{i,j}^{p-1} c^i[z,y_2,y_2]c^j z^{k-1} \end{aligned}$$

$$\underset{y_1\mapsto 1}{w_3} = \underset{y_2\mapsto 1}{w_3} = 0$$

$$\begin{aligned} \underset{y_2\mapsto y_1}{w_3} &= (q+2)[z,\dots]\sum_{i,j}^{p-1} c^i[z,y_2,y_1]c^j z^{k-1} + [z,\dots]\sum_{i,j}^{p-1} c^i[c,z]c^j z^{k-1} \\ &\quad + \sum_i [z,\dots,y_2,\dots]\sum_{i,j}^{p-1} c^i[z,y_1,y_1]c^j z^{k-1} \end{aligned}$$

$$w_3(\mathbf{a},\mathbf{b},\mathbf{c}) = (-1)^{p+q-1}2^{p+q+1}p\mathbf{a}^q\mathbf{c}^{p+k}.$$

Similarly,

$$\begin{aligned} w_4 &= w(y_1,y_2,y_2|y_2,[z,y_1]|z) \\ &= \sum_{i\neq j}[z,\dots,y_2,\dots,y_2\dots]\sum_{i,j}^{p-1} c^i[z,y_1,y_1]c^j z^{k-1} \\ &\quad + 2\sum_i [z,\dots,y_2,\dots]\sum_{i,j}^{p-1} c^i[z,y_2,y_1]c^j z^{k-1} \\ &\quad + 2\sum_i [z,\dots,y_2,\dots]\sum_{i,j}^{p-1} c^i[c,z]c^j z^{k-1} \end{aligned}$$

$$\underset{y_1\mapsto 1}{w_4} = \underset{y_2\mapsto 1}{w_4} = 0$$

$$\underset{y_2\mapsto y_1}{w_4} = 2q\sum_i [z,\dots,y_2,\dots]\sum_{i,j}^{p-1} c^i[z,y_1,y_1]c^j z^{k-1}$$

$$+ 2q[z,\dots]\sum_{i,j}^{p-1} c^i[z,y_2,y_1]c^j z^{k-1} + 2q[z,\dots]\sum_{i,j}^{p-1} c^i[c,z]c^j z^{k-1}$$

$$w_4(\mathbf{a},\mathbf{b},\mathbf{c}) = 2(-1)^{p+q-1}2^{p+q+1}pd(1,q-1)\mathbf{a}^q\mathbf{c}^{p+k}.$$

Since $\bar{w} = (q^2-1)w_1 - (q+1)w_2 + (q-1)w_3 - w_4$, by the previous calculations we got $\underset{y_1\mapsto 1}{\bar{w}} = \underset{y_2\mapsto 1}{\bar{w}} = \underset{y_2\mapsto y_1}{\bar{w}} = 0$. Moreover,

$$\bar{w}(\mathbf{a},\mathbf{b},\mathbf{c}) = (-1)^{p+q}2^{p+q+1}\Big((q^2-1)(p+q) + 2(q+1)d(1,q-1) - p(q-1) + 2pd(1,q-1)\Big)\mathbf{a}^q\mathbf{c}^{p+k}$$

which is not zero since $q \geq 2$ implies $q^2-1 > q-1$ and $p+q > p$, hence $(q^2-1)(p+q) > p(q-1)$.

Now assume that $k=0$. In this case, $w = [c,\dots]c^{p-1}$ by Lemma 4.3. Similarly to what was done in the previous case, we split the calculations for each summand of $\bar{w}$.

$$w_1 = w(y_1,[z,y_2]|y_2|z) = -\big[[z,y_2,y_2],\dots\big]c^{p-1}$$

$$+\sum_i \big[c,\dots,[z,y_2],\dots\big]c^{p-1} - [c,\dots]\sum_{i,j}^{p-2} c^i[z,y_2,y_2]c^j$$

$$\underset{y_1\mapsto 1}{w_1} = \underset{y_2\mapsto 1}{w_1} = 0$$

$$\underset{y_2\mapsto y_1}{w_1} = -2\big[[z,y_2,y_1],\dots\big]c^{p-1} - \big[[c,z],\dots\big]c^{p-1} + \sum_i\big[c,\dots,[z,y_1],\dots\big]c^{p-1}$$

$$-2[c,\dots]\sum_{i,j}^{p-2} c^i[z,y_2,y_1]c^j - [c,\dots]\sum_{i,j}^{p-2} c^i[c,z]c^j$$

$$w_1(\mathbf{a},\mathbf{b},\mathbf{c}) = (-1)^{p+q}2^{p+q+1}(p+q)\mathbf{a}^q\mathbf{c}^p.$$

Then, consider w_2, w_3, w_4

$$w_2 = w(y_1,y_2,[z,y_2]|y_2,y_1|z) = -p\sum_i\big[c,\dots,[z,y_2],\dots\big]c^{p-1}$$

$$+(p-1)[c,\dots]\sum_{i,j}^{p-2} c^i[z,y_2,y_2]c^j - \sum_i\big[[z,y_2,y_1],\dots,y_2,\dots\big]c^{p-1}$$

$$-\sum_i[c,\dots,y_2,\dots]\sum_{i,j}^{p-2} c^i[z,y_2,y_1]c^j + (p-1)\big[[z,y_2,y_2],\dots\big]c^{p-1}$$

$$\underset{y_1\mapsto 1}{w_2} = \underset{y_2\mapsto 1}{w_2} = 0$$

$$\underset{y_2 \mapsto y_1}{w_2} = -p \sum_i \big[c, \dots, [z, y_1], \dots \big]\, c^{p-1}$$

$$+ (2p - q - 2)\Big([c, \dots] \sum_{i,j}^{p-2} c^i [z, y_2, y_1] c^j + \big[[z, y_2, y_1], \dots \big]\, c^{p-1} \Big)$$

$$+ (p-1)\Big([c, \dots] \sum_{i,j}^{p-2} c^i [c, z] c^j + \big[[c, z], \dots \big]\, c^{p-1} \Big)$$

$$- \sum_i [c, \dots, y_2, \dots] \sum_{i,j}^{p-2} c^i [z, y_1, y_1] c^j$$

$$- \sum_i \big[[z, y_1, y_1], \dots, y_2, \dots \big]\, c^{p-1}$$

$$w_2(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -(-1)^{p+q} 2^{p+q+1} p(p + q - 1) \mathbf{a}^q \mathbf{c}^p$$

$$w_3 = w(y_1, y_2, y_2 | y_2, [z, y_1] | z)$$

$$= 2\Big(\sum_i [c, \dots, y_2, \dots] \sum_{i,j}^{p-2} c^i [z, y_2, y_1] c^j + \sum_i \big[[z, y_2, y_1], \dots, y_2, \dots \big]\, c^{p-1} \Big)$$

$$+ 2\Big(\sum_i \big[[c, z], \dots, y_2, \dots \big]\, c^{p-1} + \sum_i [c, \dots, y_2, \dots] \sum_{i,j}^{p-2} c^i [c, z] c^j \Big)$$

$$+ \sum_{i \neq j} \big[[z, y_1, y_1] \dots, y_2, \dots, y_2, \dots \big]\, c^{p-1}$$

$$+ \sum_{i \neq j} [c, \dots, y_2, \dots, y_2, \dots] \sum_{i,j}^{p-2} c^i [z, y_1, y_1] c^j$$

$$\underset{y_1 \mapsto 1}{w_3} = \underset{y_2 \mapsto 1}{w_3} = 0$$

$$\underset{y_2 \mapsto y_1}{w_3} = 2q\Big(\sum_i \big[[z, y_1, y_1] \dots, y_2, \dots \big]\, c^{p-1}$$

$$+ \sum_i [c, \dots, y_2, \dots] \sum_{i,j}^{p-2} c^i [z, y_1, y_1] c^j \Big)$$

$$+ 2q\Big(\big[[z, y_2, y_1], \dots \big]\, c^{p-1} + [c, \dots] \sum_{i,j}^{p-2} c^i [z, y_2, y_1] c^j \Big)$$

$$+ 2q\Big([c, \dots] \sum_{i,j}^{p-2} c^i [c, z] c^j + \big[[c, z], \dots \big]\, c^{p-1} \Big)$$

$$w_3(\mathbf{a},\mathbf{b},\mathbf{c}) = -2(-1)^{p+q}2^{p+q+1}pd(1,q-1)\mathbf{a}^q\mathbf{c}^p$$

$$w_4 = w(y_1,y_2,[z,y_1]|y_2|z) = \sum_{i\neq j}\big[c,\dots,y_2,\dots,[z,y_1]\dots\big]\,c^{p-1}$$

$$-\sum_i\big[[z,y_2,y_1],\dots,y_2,\dots\big]\,c^{p-1} - \sum_i[c,\dots,y_2,\dots]\sum_{i,j}^{p-2}c^i[z,y_2,y_1]c^j$$

$$-\sum_i\big[[c,z],\dots,y_2,\dots\big]\,c^{p-1} - \sum_i[c,\dots,y_2,\dots]\sum_{i,j}^{p-2}c^i[c,z]c^j$$

$$\underset{y_1\mapsto 1}{w_4} = \underset{y_2\mapsto 1}{w_4} = 0$$

$$\underset{y_2\mapsto y_1}{w_4} = (q-1)\sum_i\big[c,\dots,[z,y_1],\dots\big]\,c^{p-1} - \sum_i\big[[z,y_1,y_1]\dots,y_2,\dots\big]\,c^{p-1}$$

$$-\sum_i[c,\dots,y_2,\dots]\sum_{i,j}^{p-2}c^i[z,y_1,y_1]c^j - q\big[[z,y_2,y_1]\dots\big]\,c^{p-1}$$

$$-q[c,\dots]\sum_{i,j}^{p-2}c^i[z,y_2,y_1]c^j - q\big[[c,z],\dots\big]\,c^{p-1} - q[c,\dots]\sum_{i,j}^{p-2}c^i[c,z]c^j$$

$$w_4(\mathbf{a},\mathbf{b},\mathbf{c}) = -2(-1)^{p+q}2^{p+q+1}d(1,q-1)\mathbf{a}^q\mathbf{c}^p.$$

Hence $\underset{y_1\mapsto 1}{\bar{w}} = \underset{y_2\mapsto 1}{\bar{w}} = \underset{y_2\mapsto y_1}{\bar{w}} = 0$. Moreover, $\bar{w}(\mathbf{a},\mathbf{b},\mathbf{c}) = (-1)^{p+q}2^{p+q+1}\alpha\mathbf{a}^q\mathbf{c}^p$ where

$$\alpha = -\big(q(q-1)+2d(1,q-1)\big)\,(p+q+1)\neq 0.$$

■

The proof of the next lemma is very similar to the previous one, and we shall omit it.

Lemma 5.8 *Let $p>0$. Then*

$$\bar{w} = w(y_1|y_2,[z,y_1]|z)$$

is a higher consequence of w for $(p+q+1,p-1)\otimes(k+1)$.

We summarize the results in Lemmas 5.3–5.8 in the following corollary:

Corollary 5.9 *Let $W,\bar{W}$ be the irreducible components of $B_2\big(M_2(\mathbb{K}),*\big)$ corresponding to the diagrams $(p+q,p)\otimes(k)$, $(a,b)\otimes(c)$ respectively. If $(a,b)\otimes(c)$ is obtained by $(p+q,p)\otimes(k)$ as the result of one of the following operations:*

1. *glue a new box to one row,*

2. *delete a box from a row and glue a new box to each other row,*

then $\bar{W}$ *is a higher consequence of* W.

Now we look for consequences of bigger degrees.

Lemma 5.10 *Let* W *be the irreducible submodule of* $B_2\big(M_2(\mathbb{K}),*\big)$ *corresponding to* $\begin{array}{|l|}\hline \mu_1 \quad\quad \\ \hline \mu_2 \\ \hline \end{array} \otimes \boxed{k\quad}$. *If* $\bar{W}$ *is the irreducible submodule of* $B_2\big(M_2(\mathbb{K}),*\big)$ *corresponding to the diagram* $(a,b)\otimes(c)$ *satisfying*

$$\begin{cases} a+b+c \geq 2(\mu_1+\mu_2+k) \\ a+b \geq \mu_1+\mu_2 \\ b+c \geq \mu_2+k \end{cases}$$

then $\bar{W}$ *is a higher consequence of* W.

Proof By comparing the length of the rows of the two diagrams we distinguish eight cases. It is clear by our assumptions that the cases $\begin{cases} a<\mu_1 \\ b<\mu_2 \\ c\geq k \end{cases}$, $\begin{cases} a\geq\mu_1 \\ b<\mu_2 \\ c<k \end{cases}$ and $\begin{cases} a<\mu_1 \\ b<\mu_2 \\ c<k \end{cases}$ are not possible. Moreover the case $\begin{cases} a<\mu_1 \\ b\geq\mu_2 \\ c<k \end{cases}$ is not possible as well. Indeed it holds: $c+a<\mu_1+k$, therefore $\mu_1+k+b>a+b+c\geq 2(\mu_1+\mu_2+k)$ and $b>\mu_1+2\mu_2+k$. Hence $\mu_1>a\geq b>\mu_1+2\mu_2+k$, and $2\mu_2+k<0$ which is impossible. Then we study the following four cases:

$$1.\ \begin{cases} a\geq\mu_1 \\ b\geq\mu_2 \\ c\geq k \end{cases},\quad 2.\ \begin{cases} a\geq\mu_1 \\ b<\mu_2 \\ c\geq k \end{cases},\quad 3.\ \begin{cases} a<\mu_1 \\ b\geq\mu_2 \\ c\geq k \end{cases} \text{ and } 4.\ \begin{cases} a\geq\mu_1 \\ b\geq\mu_2 \\ c<k \end{cases}.$$

(1) If $\begin{cases} a\geq\mu_1 \\ b\geq\mu_2 \\ c\geq k \end{cases}$, then we can apply Lemma 5.3 $(c-k)$ times to obtain a higher consequence of w for the module corresponding to $(\mu_1,\mu_2)\otimes(c)$. Hence, by applying successively Lemma 5.4 and Lemma 5.5, we obtain a higher consequence $\bar{w}$ of w for $(a,b)\otimes(c)$.

(2) If $\begin{cases} a\geq\mu_1 \\ b<\mu_2 \\ c\geq k \end{cases}$, then we have to remove (μ_2-b) boxes. Hence we apply Lemma 5.8 (μ_2-b) times. We obtain a higher consequence of w for $(\mu_1+\mu_2-b,b)\otimes(\mu_2+k-b)$. We can apply Lemma 5.3 and Lemma 5.4 suitable times because $c\geq\mu_2+k-b$ and $a\geq\mu_1+\mu_2-b$. In this way we obtain a higher consequence $\bar{w}$ of w for $(a,b)\otimes(c)$.

(3) If $\begin{cases} a < \mu_1 \\ b \geq \mu_2 \\ c \geq k \end{cases}$, then we can use Lemma 5.7 to obtain a higher consequence of w for $(a, \mu_1 + \mu_2 - a) \otimes (\mu_1 + k - a)$ (notice that $\mu_1 + \mu_2 - a \leq b \leq a$). It is trivial to prove that $\mu_1 + \mu_2 - a \leq b$. Moreover, our assumptions on a, b, c force $\mu_1 + k \leq a + c$. Indeed, suppose on the contrary $\mu_1 + k > a + c$. It follows $\mu_1 + k - c + b > a + b$, therefore $\mu_1 + k + b > a + b + c \geq 2\mu_1 + 2\mu_2 + 2k$. Hence $b > \mu_1 + 2\mu_2 + k$ and it follows $\mu_1 + k - c > a \geq b > \mu_1 + 2\mu_2 + k$, so $-c > 2\mu_2$ which is impossible. Therefore $\mu_1 + k - a \leq c$ and we can use Lemma 5.3 and Lemma 5.5 to glue the remaining boxes and obtain a higher consequence $\bar{w}$ of w for $(a, b) \otimes (c)$.

(4) Finally, if $\begin{cases} a \geq \mu_1 \\ b \geq \mu_2 \\ c < k \end{cases}$, then we move away $(k - c)$ boxes applying $(k - c)$ times Lemma 5.6, and we get the higher consequence for $(\mu_1 + k - c, \mu_2 + k - c) \otimes (c)$. We note that $\mu_2 + k - c \leq b$ holds; on the other hand it is $\mu_1 + k \leq a + c$, that is $\mu_1 + k - c \leq a$. Hence we may apply Lemmas 5.4 and 5.5 and get the higher consequence of w for $(a, b) \otimes (c)$. ■

6 A Description of the Proper Subvarieties

For convenience of the reader,we recall Definition 2.2: Let $B^{(n)}(*)$ be the space of all Y-proper polynomials of degree n in $\mathbb{K}\langle Y, Z\rangle$. The T_*-ideals of $\mathbb{K}\langle Y, Z\rangle$, U_1 and U_2, are $*$-asymptotically equivalent if there exists $\nu_0 \in \mathbb{N}$ such that for all $n \geq \nu_0$

$$U_1 \cap B^{(n)}(*) = U_2 \cap B^{(n)}(*)$$

and we write

$$U_1 \approx_* U_2.$$

Here is the main result of our investigation:

Theorem 6.1 *Let $* = t$ the transpose involution. If U is the T_*-ideal of $\mathbb{K}\langle Y, Z\rangle$ of a proper subvariety of the variety of algebras with involution generated by $\big(M_2(\mathbb{K}), *\big)$, then*

$$U \approx_* T_*(\mathcal{R}_p) \cap T_*(\mathcal{S}_q)$$

for suitable p and q.

Proof Let U be a T_*-ideal properly containing $T_*\big(M_2(\mathbb{K})\big)$. By Theorem 4.1 it is enough to consider Y-proper polynomials in $B_2(*)$. We may consider the submodule $\bar{U} := \big(U \cap B_2(*)\big) / \Big(T_*\big(M_2(\mathbb{K})\big) \cap B_2(*)\Big)$ of the $\mathrm{GL}_2 \times \mathrm{GL}_2$-module $B_2\big(M_2(\mathbb{K}), *\big)$. It describes all Y-proper identities in $U \cap B_2(*)$ which do not hold for $\big(M_2(\mathbb{K}), *\big)$.

Since $U \supsetneqq T_*\big(M_2(\mathbb{K})\big)$ and U is generated by its Y-proper polynomials, we obtain that $\bar{U}$ is not zero. For simplicity of notation, we denote the irreducible $\mathrm{GL}_2 \times \mathrm{GL}_2$-module $W_{(\alpha,\beta)} \otimes W_{(\gamma)}$ by $(\alpha, \beta) \otimes (\gamma)$. If the unique irreducible submodule W of $B_2\big(M_2(\mathbb{K}), *\big)$ associated to $(a, b) \otimes (c)$ occurs in the decomposition of $\bar{U}$, then all its higher consequences occur in the decomposition of $\bar{U}$, as well.

Now write:

$$p_1 := \min .\{\alpha + \beta \mid (\alpha, \beta) \otimes (\gamma) \text{ occurs in } \bar{U}\},$$
$$q_1 := \min .\{\beta + \gamma \mid (\alpha, \beta) \otimes (\gamma) \text{ occurs in } \bar{U} \text{ and } \alpha + \beta = p_1\}.$$
$$\text{Choose } (a_1, b_1) \otimes (c_1) \text{ in } \bar{U} \text{ such that } a_1 + b_1 = p_1 \text{ and } b_1 + c_1 = q_1.$$

Let

$$q_2 := \min .\{\beta + \gamma \mid (\alpha, \beta) \otimes (\gamma) \text{ occurs in } \bar{U}\},$$
$$p_2 := \min .\{\alpha + \beta \mid (\alpha, \beta) \otimes (\gamma) \text{ occurs in } \bar{U} \text{ and } \beta + \gamma = q_2\}.$$
$$\text{Choose } (a_2, b_2) \otimes (c_2) \text{ in } \bar{U} \text{ such that } b_2 + c_2 = q_2 \text{ and } a_2 + b_2 = p_2.$$

Note that $p_1 \leq p_2$ and $q_1 \geq q_2$. Now set $p = p_1$, $q = q_2$ and $V := T_*(\mathcal{R}_p) \cap T_*(\mathcal{S}_q)$. We want to show that $U \approx_* V$. By Theorem 4.1 it is enough to show that the irreducible Y-proper submodules occurring in $U \cap B_2^{(n)}(*)$ and $V \cap B_2^{(n)}(*)$ are the same from a suitable positive integer n on. Since U and V both contain $T_*\big(M_2(\mathbb{K})\big)$ we may work modulo $T_*\big(M_2(\mathbb{K})\big)$.

Note that if $(\alpha, \beta) \otimes (\gamma) \in \bar{U} = \big(U \cap B_2(*)\big) / \Big(T_*\big(M_2(\mathbb{K})\big) \cap B_2(*)\Big)$ (with abuse of notation), then

i. $\alpha + \beta \geq p_1$ and, if $\alpha + \beta = p_1$, then $\beta + \gamma \geq q_1 \geq q_2$;
ii. $\beta + \gamma \geq q_2$ and, if $\beta + \gamma = q_2$, then $\alpha + \beta \geq p_2 \geq p_1$.

Certainly, it is already true that if $(\alpha, \beta) \otimes (\gamma) \in \big(U \cap B_2(*)\big) / \Big(T_*\big(M_2(\mathbb{K})\big) \cap B_2(*)\Big)$, then

$$(\alpha, \beta) \otimes (\gamma) \in \big(V \cap B_2(*)\big) / \Big(T_*\big(M_2(\mathbb{K})\big) \cap B_2(*)\Big),$$

as a consequence of the choice made for p and q in view of Lemmas 4.4 and 4.5. To complete the proof, set

$$n_0 := \max\{2(q_1 + a_1), 2(p_2 + c_2), p_2 + q_1\}.$$

If $(\alpha, \beta) \otimes (\gamma) \in (V \cap B_2) / \Big(T_*\big(M_2(\mathbb{K})\big) \cap B_2\Big)$ is of degree at least n_0, *i.e.*, $\gamma + \alpha + \beta \geq n_0$, then by Lemmas 4.4 and 4.5 it satisfies $\begin{cases} \alpha + \beta \geq p_1 \\ \beta + \gamma \geq q_2 \end{cases}$. Hence, if $\alpha + \beta \geq p_2$, then $(\alpha, \beta) \otimes (\gamma)$ is a higher consequence of $(a_2, b_2) \otimes (c_2)$ by Lemma 5.10; so it is in

$\left(U \cap B_2(*)\right) / \Big(T_*\left(M_2(\mathbb{K})\right) \cap B_2(*)\Big)$. Suppose this is not the case, so $\alpha + \beta < p_2$. If $\beta + \gamma \geq q_1$, then Lemma 5.10 applies, so $(\alpha, \beta) \otimes (\gamma)$ is a higher consequence of $(a_1, b_1) \otimes (c_1)$. Therefore we suppose that this is not the case, that is $\beta + \gamma < q_1$. But this yields a contradiction, since we get

$$\begin{cases} \alpha + \beta < p_2 \\ \beta + \gamma < q_1 \end{cases} \quad \text{yields} \quad \gamma + \alpha + \beta \geq n_0 \geq p_2 + q_1 > \gamma + \alpha + 2\beta.$$

This ends the proof. ■

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