



D'Alembert formula on finite one-dimensional networks

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Abstract

We find a d'Alembert type formula for the solution of the Cauchy problem for the wave equation on finite weighted networks. We also discuss the periodicity in time of the solution in terms of the spectrum of the discrete graph associated with the network and finally we present two significant examples to illustrate and clarify the general analysis.

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1. Introduction

Consider a graph Γ with its set of vertices V and its set of edges E . If every edge $e \in E$ is seen as a homeomorphic copy of the interval $[0, 1]$, and obvious identifications between endpoints of different edges are made in order to account for the vertices of Γ , the graph Γ becomes a topological space (in fact a one-dimensional CW complex) that is called a one-dimensional network according to several authors who have developed the analysis on these structures and studied PDEs on them (see, e.g., [1–3,5,7,10,12–16,19]). The name is reminiscent of modeling and applications in electrical engineering, but a

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famous example comes from biology, namely the Rall–Rinzel model for the neuron (see [17,18]). Applications to hydraulic networks (sewers for instance) are also in order.

Even though a network is not a manifold (because of the vertices), the homeomorphisms between the edges and the interval $[0, 1]$ provide a differentiable structure on most of Γ . Suitable conditions on the functions at the vertices (the Kirchhoff conditions in our case) will play the role of the differentiability, a notion that in the vertices has no obvious natural meaning. A fairly large amount of the standard existence and uniqueness theorems for PDEs were proven in this context (see, for instance, the book [2] and references therein).

Spectral theory for 1-dimensional finite (i.e., with a finite number of vertices and edges) networks and its connection with spectral theory for finite discrete graphs were developed by Nicaise [12,13] and partially extended to infinite networks by Cattaneo [4].

The aim of this paper is to exploit the relationship between the spectrum of the Laplacian on the network Γ and the spectrum of the discrete Laplacian on the vertices of Γ in order to solve the Cauchy problem for the wave equation on finite connected weighted networks. We look for an extension to these structures of the classical d'Alembert formula for the wave equation on the real line

$$u(t, x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds,$$

where f and g are the prescribed initial value data $u(0, x)$ and $(\partial u / \partial t)(0, x)$, respectively.

The previous formula also solves the Cauchy problem for the wave equation on the interval $[0, 1]$ with homogeneous Neumann condition at 0 and 1 if we extend f and g as even functions on $[-1, 1]$ and then further extend them to \mathbf{R} as 2-periodic functions.

On a general network, the formula cannot be so simple. The even and 2-periodic extensions described above do not help anymore. We need a different setting and different conventions. D'Alembert formula for a network must have the form

$$u(t, x) = \int_{\Gamma} \mathbf{K}(x, y, t) f(y) dy + \int_{\Gamma} \mathbf{H}(x, y, t) g(y) dy, \quad (1)$$

where x and y are points of Γ , f , g and u are functions on Γ and \mathbf{K} and \mathbf{H} are “distributions” on $\Gamma \times \Gamma$ or, to be more precise, elements in the dual of the space of the continuous functions on $\Gamma \times \Gamma$ (thus avoiding the problem of developing a full theory of distributions on a topological space lacking the differentiable structure of manifold). Notice then that a function u on Γ can be identified with a collection $(u_e)_{e \in E}$ of functions on $[0, 1]$, one for each edge of Γ . \mathbf{K} and \mathbf{H} are thus matrices whose $e_i e_j$ entry should account for the influence (in general varying with the time t) that the perturbation on the edge e_j has on the edge e_i .

Clearly, from this point of view, the formulae that we expect are rather complicated, involving a large number of terms as soon as the graph has enough edges. Ways to reduce the complexity are however available. For instance, it is possible to compute the solution at the vertices only. If it is needed on a particular edge e too, it would be enough to solve an initial value problem on the $[0, 1]$ interval with prescribed values at the end points.

Other efficient solving techniques for the Cauchy problem are available. For example, it is possible to solve the problem on each edge with free Dirichlet boundary conditions

and then glue the solutions at the vertices. This process leads to a linear algebra problem that can easily be dealt with by a computer (see, e.g., [8,9]). Nevertheless, our formula has its fine points. First of all, it is a global formula and therefore it makes easier to see the connections between properties of the network and properties of the solution of the wave equation. For instance, we can derive information about the periodicity in time of the solution from the spectrum of the network. Some sort of diagnostic on the network seems also possible. By this we refer to the problem of determining structural elements of the network (for example, the values of some of the weights $c(e)$, the degrees of some vertices, the total number of vertices or edges, or information about the spectrum) from the knowledge of the solution of the Cauchy problem restricted to a single edge or vertex. We are not however pursuing this analysis in the present paper. Similarly, we only mention that interesting development can probably be derived from the solution of the heat equation that we obtain as a consequence of our work on the wave equation.

In the next two sections we introduce our notation and collect the basic facts about networks, functions and Laplacian on them. In particular in Section 3 we reproduce Nicaise's description of the spectrum of Δ [13]. In Section 4 we reduce the problem to the determination of the kernel \mathbf{H} . The main tool is the Fourier expansion with respect to an orthonormal basis of eigenfunctions of Δ on Γ . The geometric structural properties of Γ play their important role through the matrices \mathbf{A} , $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, \mathbf{C}_α , \mathbf{D}_α , \mathbf{F}_α , \mathbf{G}_α . In Section 5 we solve the Cauchy problem (5) with $g \equiv 0$ by differentiating with respect to the time the function $\int_\Gamma \mathbf{H}(t, x, y) f(y) dy$ (Theorem 3). In Section 6 we illustrate the general theory by describing two examples, the most remarkable of which is the complete network with $n + 1$ vertices.

2. Notation and preliminaries

Let $\Gamma = (V, E)$ be a finite, connected graph with no self-loops. $V = \{v_1, \dots, v_{|V|}\}$ and $E = \{e_1, \dots, e_{|E|}\}$ are the set of the vertices and the set of the edges, respectively. For every vertex v in V , we denote by d_v the degree of v , and by E_v the set of the edges branching out from v (note that E_v has d_v elements). We say that two vertices v and v' are neighbours and write $v \sim v'$ if there exists an edge e in E such that $e = (v, v')$. A circuit of length n is a connected subgraph of Γ with n distinct vertices, each of degree 2. We identify every edge e of Γ with the real interval $[0, 1]$. In this way we associate Γ with a one-dimensional CW complex (see, e.g., [11]). Note that Γ is a metric space in a natural way.

For every vertex v in V and for every edge e in E_v we set

$$\mathbf{i}(v, e) = \begin{cases} 0 & \text{if } v \text{ is identified with } 0, \\ 1 & \text{if } v \text{ is identified with } 1. \end{cases} \quad (2)$$

We assign a positive weight $c(e)$ to every edge e of Γ . For every vertex v of Γ , $c(v)$ denotes the sum of the weights of all the edges branching out from v and $2c$ is their sum, i.e.,

$$c(v) = \sum_{e \in E_v} c(e) \quad \text{and} \quad c = \sum_{e \in E} c(e) = \frac{1}{2} \sum_{v \in V} c(v). \quad (3)$$

We will refer to Γ with the above structure as the one-dimensional weighted network with the same name.

Let $l^2(V, c)$ be the space of all the complex valued functions on V with inner product

$$(U, W)_{l^2(V, c)} = \sum_{v \in V} c(v) U(v) \overline{W(v)}.$$

For all v, v' distinct in V , we call transition probability from v to v' the following quantity

$$p(v, v') = \begin{cases} \sum_{e \in E_v \cap E_{v'}} \frac{c(e)}{c(v)} & \text{if } E_v \cap E_{v'} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

and we set

$$\hat{c}(v, v') = \begin{cases} \sum_{e \in E_v \cap E_{v'}} \frac{c(e)}{\sqrt{c(v)c(v')}} & \text{if } E_v \cap E_{v'} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For $v = v'$, we set $p(v, v') = \hat{c}(v, v') = 0$.

Proposition 1. *The matrices $P = (p(v, v'))_{v, v' \in V}$ and $C = (\hat{c}(v, v'))_{v, v' \in V}$ have the same eigenvalues and if U (respectively, \hat{U}) is an eigenvector of P (respectively, C) corresponding to λ then*

$$\hat{U}(v) = U(v) \sqrt{c(v)}.$$

We omit the easy proof.

Since C is symmetric all the eigenvalues of P are real. Notice that the operator $I - P$ is the standard discrete Laplacian on (Γ, c) (see, e.g., [6]).

In this paper we consistently denote by x and y points varying on Γ . Each x determines uniquely the edge e to which it belongs, unless x is a vertex. If x is not a vertex, by x_e we denote the number in the interval $[0, 1]$ corresponding to $x \in e$ under our identification of the edge e with $[0, 1]$. If x is a vertex, x_e is not well defined in general since it can well be the initial point for some of the edges in E_x and the terminal point for some other of those edges. However, in our formulae any vertex always appears as the initial or the terminal point of a specific edge e . In this case x_e has the obvious meaning.

We identify any function u on Γ with a family of functions $(u_e)_{e \in E}$ each defined on a single edge e of Γ and therefore, by our identification, on $[0, 1]$, in such a way that $u(x) = u_e(x_e)$. We use the same notation u_e to denote both the function on the edge e and the function on the real interval $[0, 1]$ identified with e .

The function $u = (u_e)_{e \in E}$ is continuous on Γ if and only if u_e is continuous on $[0, 1]$ for every e in E , and $u_e(\mathbf{i}(v, e)) = u_{e'}(\mathbf{i}(v, e'))$ for all the edges e, e' in E_v and for all v in V . So we can associate with every continuous function u on Γ a function U well defined on V by

$$U(v) = u_e(\mathbf{i}(v, e)),$$

where e is any one of the edges in E_v .

Integration on Γ is performed edge by edge. Namely

$$\int_{\Gamma} u(x) dx = \sum_{e \in E} c(e) \int_0^1 u_e(x_e) dx_e,$$

where dx_e denotes the Lebesgue measure on the interval $[0, 1]$.

We define the space $L^2(\Gamma, c)$ as the space of all the functions $u = (u_e)_{e \in E}$ on Γ such that $u_e \in L^2(0, 1)$ for every e in E .

Analogously, for every integer $m > 0$, we define the Sobolev space $H^m(\Gamma, c)$ as the space of all the functions $u = (u_e)_{e \in E}$ on Γ such that u is continuous on Γ , $u_e \in H^m(0, 1)$ for every e in E .

The above spaces are Hilbert spaces with inner products

$$(u, w)_{L^2(\Gamma, c)} = \sum_{e \in E} c(e) (u_e, w_e)_{L^2(0, 1)},$$

$$(u, w)_{H^m(\Gamma, c)} = \sum_{e \in E} c(e) (u_e, w_e)_{H^m(0, 1)}.$$

Notice that $u \in H^m(\Gamma, c)$ is a continuous function on Γ for every $m \geq 1$ but continuity at the vertices for the derivative $u' = (u'_e)_{e \in E}$ is not assured.

Consider the sesquilinear continuous form φ on $H^1(\Gamma, c)$ defined by

$$\varphi(u, w) = (u', w')_{L^2(\Gamma, c)}$$

and let Δ be its associated Laplacian.

It is easy to verify that Δ is a linear, unbounded, self-adjoint, dissipative operator on $L^2(\Gamma, c)$.

Its domain is the subset of $H^2(\Gamma, c)$ of the functions satisfying the following Kirchhoff type condition

$$\sum_{e \in E_v} c(e) \frac{\partial u_e}{\partial n_e}(\mathbf{i}(v, e)) = 0 \quad \text{at every } v \text{ in } V,$$

where $(\partial u_e / \partial n_e)(\mathbf{i}(v, e))$ denotes the normal exterior derivative of the function u_e at the endpoint $\mathbf{i}(v, e)$ of the interval $[0, 1]$, i.e.,

$$\frac{\partial u_e}{\partial n_e}(\mathbf{i}(v, e)) = \begin{cases} -u'_e(0_+) & \text{if } \mathbf{i}(v, e) = 0, \\ u'_e(1_-) & \text{if } \mathbf{i}(v, e) = 1 \end{cases}$$

(see, e.g., [13]).

Notice that $(\Delta u)_e = u''_e$ for every e in E and for every u in $D(\Delta)$. Moreover, if $u, v \in D(\Delta)$, the Kirchhoff condition implies that $\int_{\Gamma} v \Delta u = \int_{\Gamma} u \Delta v$.

The Cauchy problem for the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u, & t > 0, \\ u(0, \cdot) = f(\cdot), \\ \partial_t u(0, \cdot) = g(\cdot) \end{cases} \tag{5}$$

is an example of second-order evolution problem. It can be transformed into a first-order system by defining the vector $U(t, x) = (u(t, x), (\partial u / \partial t)(t, x))$. When the initial data f

and g belong to $D(\Delta)$ and $H^1(\Gamma, c)$, respectively, the theory of semigroups and the Hille–Yosida–Phillips theorem can be used to prove existence, uniqueness and regularity for the solution. In particular the vector U turns out to be a continuous function of t with values in $D(\Delta) \times H^1(\Gamma, c)$. We collect all these results in the following theorem, referring to the book [2] for the proof.

Theorem 1. *The Cauchy problem (5) with data f and g in $D(\Delta)$ and H^1 , respectively, has a unique solution $u(t, x)$ such that the vector $U = (u, \partial u / \partial t)$, as a function of t , belongs to*

$$C^1([0, \infty), H^1(\Gamma, c) \times L^2(\Gamma, c)) \cap C^0([0, \infty), D(\Delta) \times H^1(\Gamma, c));$$

moreover, $u \in H^1([0, T] \times \Gamma)$ for arbitrary $T > 0$ and $u(t, \cdot) \in H^2(\Gamma, c)$ for all $t \geq 0$.

Notice that $u(\cdot, x)$ and $(\partial u / \partial t)(\cdot, x)$ are continuous functions for all x in Γ .

We conclude this section by observing that the finite propagation property for waves holds on networks. Indeed it holds on each single edge and if f and g are concentrated on an edge e , the solution, on any other edge e' is 0 until one of the vertices of e' is influenced by the perturbation originated on e .

3. The spectrum of the Laplacian Δ

Since Δ is a self-adjoint non-positive operator on the Hilbert space $L^2(\Gamma, c)$, there exists an orthonormal basis of $L^2(\Gamma, c)$ composed of eigenfunctions of Δ . For finite networks such a basis has been described by Nicaise (see [13]). In this section we recall the main facts about the spectral decomposition of Δ omitting most of the proofs.

- (1) For any network $0 \in \sigma(\Delta)$ with multiplicity 1 and the constant $w_0 \equiv c^{-1/2}$ (where c is defined in (3)) as eigenfunction.
- (2) The numbers $-k^2\pi^2$, $k \in \mathbf{N}$, belong to $\sigma(\Delta)$ with multiplicity

$$m_k = \begin{cases} |E| - |V| + 2 & \text{if } \Gamma \text{ has no odd circuits,} \\ |E| - |V| + 1 + (-1)^k & \text{if } \Gamma \text{ has at least one odd circuit.} \end{cases}$$

To obtain a basis for the eigenspace of $(-k^2\pi^2)$ we consider the $|V| \times |E|$ matrix $\mathbf{A}^{(k)}$ whose elements are

$$A_{v,e}^{(k)} = \begin{cases} (-1)^{\mathbf{i}(v,e)(k+1)} c(e) & \text{if } e \in E_v, \\ 0 & \text{otherwise,} \end{cases}$$

and we choose an orthonormal basis $\{\mathbf{b}_j^{(k)}\}$ of $\text{Ker}(\mathbf{A}^{(k)})$ with respect to the norm

$$\sum_{e \in E} c(e) |\alpha_e|^2 = \|(\alpha_e)_{e \in E}\|.$$

Our notation can be misleading here. Notice that we are actually dealing with two matrices. One for k even and one for k odd. When k is even the dimension of $\text{Ker}(\mathbf{A}^{(k)})$

is given by $m = |E| - |V| + 1$, while if k is odd then it is $m - 1$ if Γ has at least an odd circuit and m if Γ has no odd circuits.

A basis for the eigenspace of $(-k^2\pi^2)$ is then constructed in the following way.

If k is odd and Γ has at least one odd circuit we choose the functions

$$w_{k,j,e}(x_e) = \sqrt{2} \mathbf{b}_{j,e}^{(k)} \sin k\pi x_e \quad \text{for } j = 1, \dots, m - 1 \tag{6}$$

as our orthonormal basis.

If k is odd and Γ has no odd circuits the set of the functions given in (6) must be completed by

$$w_{k,m+1,e}(x_e) = \sqrt{\frac{2}{c}} \mathbf{a}_e \cos k\pi x_e, \tag{7}$$

where \mathbf{a}_e is equal to 1 or -1 according to the condition

$$(-1)^{\mathbf{i}(v,e)} a_e = (-1)^{\mathbf{i}(v,e')} a_{e'} \quad \text{for all } e, e' \text{ in } E_v \text{ and all } v \text{ in } V$$

($\mathbf{i}(v, e)$ defined in (2)).

Finally if k is even we must complete the set (6) with the function

$$w_{k,m+1,e}(x_e) = \sqrt{\frac{2}{c}} \cos k\pi x_e. \tag{7'}$$

(3) The remaining part of $\sigma(\Delta)$ is the set

$$\{-\lambda < 0: \cos \sqrt{\lambda} \in (\sigma(P) \cap (-1, 1))\}. \tag{8}$$

The multiplicity m_λ of the eigenvalue $(-\lambda)$ is equal to the multiplicity of the eigenvalue $\cos \sqrt{\lambda}$ of P . We write each λ as $\lambda = ((2k - 1)\pi \pm \alpha)^2$, where $k \in \mathbf{N}$ and α belongs to the set

$$\mathcal{N} = \{0 < \alpha < \pi: (-\cos \alpha) \in \sigma(P)\}. \tag{9}$$

The orthonormal basis for the corresponding eigenspace is described by the following.

Proposition 2. Let $\{Z_{\alpha,j}(v)\}_{1 \leq j \leq m_\alpha}$ be an orthonormal basis in $l^2(V, c)$ of the eigenspace of the eigenvalue $(-\cos \alpha)$ of P . For every edge $e = (v, v')$ define

$$z_{\alpha,j,e}(0) = Z_{\alpha,j}(v)(1 - \mathbf{i}(v, e)) + Z_{\alpha,j}(v')(1 - \mathbf{i}(v', e)), \tag{10}$$

$$z_{\alpha,j,e}(1) = Z_{\alpha,j}(v)\mathbf{i}(v, e) + Z_{\alpha,j}(v')\mathbf{i}(v', e), \tag{11}$$

i.e., $z_{\alpha,j,e}(0)$ and $z_{\alpha,j,e}(1)$ are either $Z_{\alpha,j}(v)$ or $Z_{\alpha,j}(v')$ depending on which of the vertices is identified with 0. Then the functions

$$z_{\alpha,k,j,e}^\pm(x_e) = \frac{\mp \sqrt{2}}{\sin \alpha} \left(z_{\alpha,j,e}(0) \sin((2k - 1)\pi \pm \alpha)(1 - x_e) + z_{\alpha,j,e}(1) \sin((2k - 1)\pi \pm \alpha)x_e \right) \tag{12}$$

are an orthonormal basis in $L^2(\Gamma, c)$ of the eigenspace for the eigenvalue $-((2k - 1)\pi \pm \alpha)^2$ (by \pm we mean that we actually have two functions, in the first we select consistently $+$ and in the second $-$).

Proof. It is easy to prove that the functions $\{z_{\alpha,k,j}^{\pm}\}$ are eigenfunctions of the eigenvalue $-((2k-1)\pi \pm \alpha)^2$ of Δ . To prove that they are orthonormal (we sketch it since it is not in [13]) it is enough to show that

$$(z_{\alpha,k,j}^{\pm}, z_{\alpha,k,i}^{\pm})_{L^2(\Gamma,c)} = (Z_{\alpha,j}, Z_{\alpha,i})_{l^2(V,c)}. \quad (13)$$

We know that all the eigenvectors of the matrix P are real so we can assume that the functions $Z_{\alpha,j}$ are real. Equation (13) follows from straightforward calculations using

$$\sum_{e \in E} c(e) (z_{\alpha,j,e}(0)z_{\alpha,i,e}(0) + z_{\alpha,j,e}(1)z_{\alpha,i,e}(1)) = \sum_{v \in V} c(v) Z_{\alpha,j}(v) Z_{\alpha,i}(v)$$

and

$$\begin{aligned} & \sum_{e \in E} c(e) (z_{\alpha,j,e}(0)z_{\alpha,i,e}(1) + z_{\alpha,j,e}(1)z_{\alpha,i,e}(0)) \\ &= \sum_{v \in V} c(v) Z_{\alpha,j}(v) (P(Z_{\alpha,i})(v)) = -\cos \alpha \sum_{v \in V} c(v) Z_{\alpha,j}(v) Z_{\alpha,i}(v). \quad \square \end{aligned}$$

Let \mathbf{a} , $z_{\alpha,j}(0)$, $z_{\alpha,j}(1)$ be the $|E|$ -vectors with entries \mathbf{a}_e , $z_{\alpha,j,e}(0)$, $z_{\alpha,j,e}(1)$, respectively. Recall that in formula (6) the $\mathbf{b}_{j,e}^{(k)}$ were the entries of the vectors $\mathbf{b}_j^{(k)}$.

4. The solution of the Cauchy problem with $f = 0$

Our aim is to determine as explicitly as possible the solution of the Cauchy problem (5) where f belongs to the domain of the Laplacian and g belongs to $H^1(\Gamma, c)$.

From the point of view of spectral theory this amounts to find the two operators $\cos t(\sqrt{-\Delta})$ and $\sin t(\sqrt{-\Delta})/\sqrt{-\Delta}$, so that

$$u(t, x) = \cos t(\sqrt{-\Delta})(f)(x) + \frac{\sin t(\sqrt{-\Delta})}{\sqrt{-\Delta}}(g)(x). \quad (14)$$

Since

$$\cos t(\sqrt{-\Delta})(f)(x) = \frac{\partial}{\partial t} \frac{\sin t(\sqrt{-\Delta})}{\sqrt{-\Delta}}(f)(x)$$

we begin by determining the solution of (5) with $f = 0$.

The eigenfunctions of Δ that we have described in the previous section are an orthonormal basis of $L^2(\Gamma, c)$. Thus we have

Lemma 1. *The solution $u(t, x)$ of the Cauchy problem (5) with $f = 0$ has the following expansion relative to our orthonormal basis of $L^2(\Gamma, c)$:*

$$u(t, x) = a_0(g)t w_0(x) + \sum_{k \geq 1} \sum_{j=1}^{m_k} \frac{a_{k,j}(g)}{k\pi} \sin k\pi t w_{k,j}(x)$$

$$\begin{aligned}
 & + \sum_{\alpha \in \mathcal{N}} \sum_{k \geq 1} \sum_{j=1}^{m_\alpha} \left[\frac{b_{\alpha,k,j}(g)}{(2k-1)\pi + \alpha} \sin((2k-1)\pi + \alpha) t z_{\alpha,k,j}^+(x) \right. \\
 & \left. + \frac{c_{\alpha,k,j}(g)}{(2k-1)\pi - \alpha} \sin((2k-1)\pi - \alpha) t z_{\alpha,k,j}^-(x) \right],
 \end{aligned}$$

where

$$a_{k,j}(h) := \int_{\Gamma} h w_{k,j}, \quad b_{\alpha,k,j}(h) := \int_{\Gamma} h z_{\alpha,k,j}^+$$

and

$$c_{\alpha,k,j}(h) := \int_{\Gamma} h z_{\alpha,k,j}^-.$$

As a consequence of Lemma 1 we have that the operator $\sin(t\sqrt{-\Delta})/\sqrt{-\Delta}$ is an integral operator $g(x) \rightarrow \int_{\Gamma} \mathbf{H}(t, x, y)g(y) dy$ whose kernel is

$$\begin{aligned}
 \mathbf{H}_{e,e'}(t, x_e, y_{e'}) & = t w_{0,e}(x_e) w_{0,e'}(y_{e'}) \\
 & + \sum_{k \geq 1} \frac{\sin k\pi t}{k\pi} \sum_{j=1}^{m_k} w_{k,j,e}(x_e) w_{k,j,e'}(y_{e'}) \\
 & + \sum_{\alpha \in \mathcal{N}} \sum_{k \geq 1} \sum_{j=1}^{m_\alpha} \left[z_{\alpha,k,j,e}^+(x_e) z_{\alpha,k,j,e'}^+(y_{e'}) \frac{\sin((2k-1)\pi + \alpha)t}{(2k-1)\pi + \alpha} \right. \\
 & \left. + z_{\alpha,k,j,e}^-(x_e) z_{\alpha,k,j,e'}^-(y_{e'}) \frac{\sin((2k-1)\pi - \alpha)t}{(2k-1)\pi - \alpha} \right]. \tag{15}
 \end{aligned}$$

The sums of the series in (15) (that converge both pointwise a.e. and in $L^2(\Gamma)$) can be computed explicitly and to write them we introduce the following $|E| \times |E|$ “structure” matrices for the network Γ (all the vectors are column vectors whose entries are indexed after the edges, and a^T denotes the row vector transposed of a ; notice that aa^T is a $|E| \times |E|$ matrix):

$$\begin{aligned}
 \mathbf{A} & = \begin{cases} c^{-1} \mathbf{a} \mathbf{a}^T & \text{if } k \text{ odd and } \Gamma \text{ has no odd circuits,} \\ \mathbf{0} & \text{if } k \text{ odd and } \Gamma \text{ has an odd circuit,} \end{cases} \\
 \mathbf{B}^{(1)} & = \sum_{j=1}^{m_{2k-1}} \mathbf{b}_j^{(2k-1)} \mathbf{b}_j^{(2k-1)T}, \\
 \mathbf{B}^{(2)} & = \sum_{j=1}^{m_{2k}} \mathbf{b}_j^{(2k)} \mathbf{b}_j^{(2k)T}, \\
 \mathbf{C}_\alpha & = \sum_{j=1}^{m_\alpha} \left(-z_{\alpha,j}(0) z_{\alpha,j}^T(0) \cos 2\alpha - z_{\alpha,j}(1) z_{\alpha,j}^T(1) \right. \\
 & \quad \left. - (z_{\alpha,j}(0) z_{\alpha,j}^T(1) + z_{\alpha,j}(1) z_{\alpha,j}^T(0)) \cos \alpha \right),
 \end{aligned}$$

$$\begin{aligned} \mathbf{D}_\alpha &= \sum_{j=1}^{m_\alpha} \left(z_{\alpha,j}(0) z_{\alpha,j}^T(0) + z_{\alpha,j}(1) z_{\alpha,j}^T(1) \right. \\ &\quad \left. + (z_{\alpha,j}(0) z_{\alpha,j}^T(1) + z_{\alpha,j}(1) z_{\alpha,j}^T(0)) \cos \alpha \right), \\ \mathbf{F}_\alpha &= \sum_{j=1}^{m_\alpha} \left(2z_{\alpha,j}(0) z_{\alpha,j}^T(0) \cos \alpha + z_{\alpha,j}(0) z_{\alpha,j}^T(1) + z_{\alpha,j}(1) z_{\alpha,j}^T(0) \right), \\ \mathbf{G}_\alpha &= \sum_{j=1}^{m_\alpha} \left(-z_{\alpha,j}(0) z_{\alpha,j}^T(1) + z_{\alpha,j}(1) z_{\alpha,j}^T(0) \right), \end{aligned}$$

where c , \mathbf{a} , $\mathbf{b}^{(k)}$, $z_{\alpha,j}(0)$ and $z_{\alpha,j}(1)$ were defined in Section 3.

Then, by using the orthonormality (in $L^2(\Gamma, c)$) described in the previous section, we have

$$\begin{aligned} w_{0,e}(x_e) w_{0,e'}(y_{e'}) &= c^{-1}, \\ \sum_{j=1}^{m_{2k-1}} w_{2k-1,j,e}(x_e) w_{2k-1,j,e'}(y_{e'}) &= 2\mathbf{A}_{e,e'} \cos(2k-1)\pi x_e \cos(2k-1)\pi y_{e'} \\ &\quad + 2\mathbf{B}_{e,e'}^{(1)} \sin(2k-1)\pi x_e \sin(2k-1)\pi y_{e'}, \\ \sum_{j=1}^{m_{2k}} w_{2k,j,e}(x_e) w_{2k,j,e'}(y_{e'}) &= 2c^{-1} \cos 2k\pi x_e \cos 2k\pi y_{e'} + 2\mathbf{B}_{e,e'}^{(2)} \sin 2k\pi x_e \sin 2k\pi y_{e'}, \\ \sum_{j=1}^{m_\alpha} z_{\alpha,k,j,e}^\pm(x_e) z_{\alpha,k,j,e'}^\pm(y_{e'}) &= \mathbf{C}_{\alpha,e,e'} \frac{\cos((2k-1)\pi \pm \alpha)(x_e + y_{e'})}{\sin^2 \alpha} \\ &\quad + \mathbf{D}_{\alpha,e,e'} \frac{\cos((2k-1)\pi \pm \alpha)(x_e - y_{e'})}{\sin^2 \alpha} \\ &\quad + \mathbf{F}_{\alpha,e,e'} \frac{\sin((2k-1)\pi \pm \alpha)(x_e + y_{e'})}{\sin((2k-1)\pi \pm \alpha)} \\ &\quad + \mathbf{G}_{\alpha,e,e'} \frac{\sin((2k-1)\pi \pm \alpha)(x_e - y_{e'})}{\sin((2k-1)\pi \pm \alpha)}, \end{aligned}$$

where by \pm we mean that we actually have two formulae. In the first we select consistently $+$ and in the second $-$.

From Lemma 1 we obtain

$$\begin{aligned} \mathbf{H}_{e,e'}(t, x_e, y_{e'}) &= tc^{-1} + 2 \sum_{k \geq 1} \frac{\sin 2k\pi t}{2k\pi} \left(c^{-1} \cos 2k\pi x_e \cos 2k\pi y_{e'} + \mathbf{B}_{e,e'}^{(2)} \sin 2k\pi x_e \sin 2k\pi y_{e'} \right) \\ &\quad + 2 \sum_{k \geq 1} \frac{\sin(2k-1)\pi t}{(2k-1)\pi} \left(\mathbf{A}_{e,e'} \cos(2k-1)\pi x_e \cos(2k-1)\pi y_{e'} \right. \end{aligned}$$

$$\begin{aligned}
 &+ \mathbf{B}_{e,e'}^{(1)} \sin(2k-1)\pi x_e \sin(2k-1)\pi y_{e'} \\
 &+ \sum_{\alpha \in \mathcal{N}} \sum_{k \geq 1} \frac{\sin((2k-1)\pi \pm \alpha)t}{(2k-1)\pi \pm \alpha} \\
 &\times \left(\mathbf{C}_{\alpha,e,e'} \frac{\cos((2k-1)\pi \pm \alpha)(x_e + y_{e'})}{\sin^2 \alpha} \right. \\
 &\quad + \mathbf{D}_{\alpha,e,e'} \frac{\cos((2k-1)\pi \pm \alpha)(x_e - y_{e'})}{\sin^2 \alpha} \\
 &\quad \mp \mathbf{F}_{\alpha,e,e'} \frac{\sin((2k-1)\pi \pm \alpha)(x_e + y_{e'})}{\sin \alpha} \\
 &\quad \left. \mp \mathbf{G}_{\alpha,e,e'} \frac{\sin((2k-1)\pi \pm \alpha)(x_e - y_{e'})}{\sin \alpha} \right). \tag{16}
 \end{aligned}$$

The above double sign \pm is to be intended as the sum of the two expressions obtained selecting consistently the sign $+$ first, and then the sign $-$.

It is easy to see that the series for \mathbf{H} converges in L^2 and pointwise a.e.

By using the standard trigonometry identities we replace the products of sines and cosines in (16) by suitable sums of sines and cosines whose arguments depend on the quantities $\theta = x_e \pm y_{e'} \pm t$ for all four possible choices of the signs. In the next lemma we evaluate the trigonometric series in the variables θ that appear at this point in the expression for \mathbf{H} . We introduce the following notation: if y is not an odd integer then there is a unique decomposition

$$y = 2l_y + [y]_2 \tag{17}$$

with l_y integer and $[y]_2 \in (-1, 1)$. As y approaches an odd integer from the left or from the right, we have well defined left or right limits for l_y and $[y]_2$.

Lemma 2. *Let $\alpha \in (0, \pi)$. The following equalities hold in L^2 and pointwise (provided θ is not an integer):*

- (a) $\sum_{k \geq 1} \frac{\sin 2k\pi\theta}{2k\pi} = \frac{1}{4}(\operatorname{sgn}([\theta]_2) - 2[\theta]_2),$
- (b) $\sum_{k \geq 1} \frac{\sin(2k-1)\pi\theta}{(2k-1)\pi} = \frac{1}{4}\operatorname{sgn}([\theta]_2),$
- (c) $\sum_{k \geq 1} \left(\frac{\sin((2k-1)\pi + \alpha)\theta}{(2k-1)\pi + \alpha} + \frac{\sin((2k-1)\pi - \alpha)\theta}{(2k-1)\pi - \alpha} \right)$
 $= \frac{\operatorname{sgn}[\theta]_2 \cos(2l_\theta + \frac{1}{2}\operatorname{sgn}[\theta]_2)\alpha}{2 \cos \alpha/2},$
- (d) $\sum_{k \geq 1} \left(\frac{-\cos((2k-1)\pi + \alpha)\theta}{(2k-1)\pi + \alpha} + \frac{\cos((2k-1)\pi - \alpha)\theta}{(2k-1)\pi - \alpha} \right)$
 $= \frac{\operatorname{sgn}[\theta]_2 \sin(2l_\theta + \frac{1}{2}\operatorname{sgn}[\theta]_2)\alpha}{2 \cos \alpha/2}.$

Proof. The left-hand sides of (a) and (b) are just the Fourier series of the 2-periodic functions on the right-hand sides. To prove (c) and (d) recall that for $0 < x < 2\pi$ and $\beta \notin \mathbf{Z}$ the following formulae hold:

$$\sum_{k=-\infty}^{\infty} \frac{\sin(k + \beta)x}{k + \beta} = \pi \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \frac{\cos(k + \beta)x}{k + \beta} = \pi \cot \pi \beta \quad (18)$$

(see, e.g., [20, p.71]).

If we set $\beta = -1/2 + \beta_1$ and $x = 2\pi\theta$, divide by 2π and replace $2\pi\beta_1$ with $\alpha \in (0, \pi)$, after some changes in the summation indices, we obtain (c) and (d) for $\theta \in (0, 1)$. We extend the result to $(-1, 1)$ using the fact that \sin is odd and \cos even. Finally the extension to all non-integer θ is easily achieved by writing θ as in (17). \square

To state our formula for the kernel \mathbf{H} it is convenient to introduce some more notation. Let

$$\begin{aligned} \epsilon(\theta) &= \begin{cases} 1 & \text{if } \theta = (x_e \pm y_{e'}) + t, \\ -1 & \text{if } \theta = (x_e \pm y_{e'}) - t, \end{cases} \\ \eta(\theta) &= \begin{cases} 1 & \text{if } \theta = x_e + y_{e'} \pm t, \\ -1 & \text{if } \theta = x_e - y_{e'} \pm t, \end{cases} \\ \mathbf{S}_{\alpha, e, e'}^{\theta} &= \begin{cases} \mathbf{C}_{\alpha, e, e'} & \text{if } \theta = x_e + y_{e'} \pm t, \\ \mathbf{D}_{\alpha, e, e'} & \text{if } \theta = x_e - y_{e'} \pm t, \end{cases} \\ \mathbf{T}_{\alpha, e, e'}^{\theta} &= \begin{cases} \mathbf{F}_{\alpha, e, e'} & \text{if } \theta = x_e + y_{e'} \pm t, \\ \mathbf{G}_{\alpha, e, e'} & \text{if } \theta = x_e - y_{e'} \pm t. \end{cases} \end{aligned}$$

Let us denote by $\sum_{\theta=a \pm b} f(\theta)$ the sum $f(a + b) + f(a - b)$. More generally $\sum_{\theta=a \pm b \pm c} f(\theta)$ shall mean $\sum_{\theta=a+b \pm c} f(\theta) + \sum_{\theta=a-b \pm c} f(\theta)$.

By Lemma 2 we can transform the identity (16) as follows.

Theorem 2. *The kernel \mathbf{H} has the following expression:*

$$\begin{aligned} &\mathbf{H}_{e, e'}(t, x_e, y_{e'}) \\ &= t c^{-1} + \frac{1}{8} \sum_{\theta=x_e \pm y_{e'} \pm t} \left[-2(c^{-1} - \eta(\theta) \mathbf{B}_{e, e'}^{(2)}) \epsilon(\theta) ([\theta]_2) \right. \\ &\quad \left. + (c^{-1} + \mathbf{A}_{e, e'} - \eta(\theta) \mathbf{B}_{e, e'}^{(1)} - \eta(\theta) \mathbf{B}_{e, e'}^{(2)}) \epsilon(\theta) \operatorname{sgn}([\theta]_2) \right] \\ &\quad + \sum_{\alpha \in \mathcal{N}} \frac{1}{4 \sin^2 \alpha \cos \alpha / 2} \\ &\quad \times \sum_{\theta=x_e \pm y_{e'} \pm t} \left[\mathbf{S}_{\alpha, e, e'}^{\theta} \epsilon(\theta) \operatorname{sgn}([\theta]_2) \cos \left(2l_{\theta} + \frac{1}{2} \operatorname{sgn}[\theta]_2 \right) \alpha \right. \\ &\quad \left. - \sin \alpha \mathbf{T}_{\alpha, e, e'}^{\theta} \epsilon(\theta) \operatorname{sgn}[\theta]_2 \sin \left(2l_{\theta} + \frac{1}{2} \operatorname{sgn}[\theta]_2 \right) \alpha \right]. \end{aligned}$$

5. The Cauchy problem with $g \equiv 0$

We need a variation of the notation introduced in the previous section:

$$\mathbf{S}_{\alpha,e,e'}(s) = \begin{cases} \mathbf{C}_{\alpha,e,e'} & \text{if } s > 0, \\ \mathbf{D}_{\alpha,e,e'} & \text{if } s < 0, \end{cases}$$

$$\mathbf{T}_{\alpha,e,e'}(s) = \begin{cases} \mathbf{F}_{\alpha,e,e'} & \text{if } s > 0, \\ \mathbf{G}_{\alpha,e,e'} & \text{if } s < 0, \end{cases}$$

and we set

$$k_{1,e,e'} = c^{-1} + \mathbf{A}_{e,e'} + \operatorname{sgn}[r]_2(\mathbf{B}_{e,e'}^{(1)} + \mathbf{B}_{e,e'}^{(2)}),$$

$$k_{2,e,e'} = c^{-1} - \mathbf{A}_{e,e'} + \operatorname{sgn}[r]_2(\mathbf{B}_{e,e'}^{(1)} - \mathbf{B}_{e,e'}^{(2)}).$$

Then

Theorem 3. *If $g \equiv 0$ and $f \in D(\Delta)$, then the solution of the Cauchy problem (5) is*

$$u_e(t, x_e) = \frac{1}{4} \sum_{e' \in E} c(e') \sum_{r=x_e \pm t} \left\{ a_{e,e'}(r) f_{e'}(|[r]_2|) + b_{e,e'}(r) f_{e'}(1 - |[r]_2|) \right\},$$

where

$$a_{e,e'}(r) = k_{1,e,e'} + 2 \sum_{\alpha \in \mathcal{N}} \left(\frac{\mathbf{S}_{\alpha,e,e'}(-[r]_2)}{\sin^2 \alpha} \cos 2l_r \alpha - \frac{\mathbf{T}_{\alpha,e,e'}(-[r]_2)}{\sin \alpha} \sin 2l_r \alpha \right)$$

and

$$b_{e,e'}(r) = k_{2,e,e'} + 2 \sum_{\alpha \in \mathcal{N}} \left(\frac{-\mathbf{S}_{\alpha,e,e'}([r]_2)}{\sin^2 \alpha} \cos(2l_r + \operatorname{sgn}[r]_2) \alpha + \frac{\mathbf{T}_{\alpha,e,e'}(-[r]_2)}{\sin \alpha} \sin(2l_r + \operatorname{sgn}[r]_2) \alpha \right).$$

Proof. Let $f \in D(\Delta)$. By Theorem 1 and general facts about the wave equation, the function $v(t, x) = \int_{\Gamma} \mathbf{H}(t, x, y) f(y) dy$ is C^1 in t and its t -derivative is the solution of problem (5) when $g \equiv 0$.

In order to do the lengthy computations implied by that derivative, we write $r = x_e \pm t$ and observe that by Theorem 2, v is essentially the sum of the following integrals: $\int_0^1 [r \pm y]_2 f(y) dy$, $\int_0^1 \operatorname{sgn}([r \pm y]_2) f(y) dy$, $\int_0^1 \operatorname{sgn}([r \pm y]_2) \cos(2l_{r \pm y} + \operatorname{sgn}([r \pm y]_2)) \times \alpha f(y) dy$ and finally its companion with \cos replaced by \sin .

These integrals are continuous functions of r (and hence of t). Moreover, they are derivable at every non-integer r . When r is an integer, each of the above integrals has non-equal left and right t -derivatives. These singularities cancel out in the sum since we know (Theorem 1) that the solution is continuous. To understand how this happens, notice that for integral values of $x + t$ or $x - t$ some of the $f_{e'}$ are evaluated at the vertices of the network. If the function f vanished at the vertices, then all our integrals would be derivable in t . When that is not the case, we must recall that f belongs to the domain of the Laplacian and

so it satisfies continuity conditions at the vertices of Γ . They play the major role in the cancelling out of the singularities when we sum the contributions coming from all the edges.

We can therefore perform the t -derivative on each of the above integrals and be confident that no disturbance will arise in the end from the points where it does not exist. For example, the integral

$$\int_0^1 \operatorname{sgn}([r+y]_2) \cos(2l_{r+y} + \operatorname{sgn}([r \pm y]_2/2)\alpha f(y)) dy$$

is equal to

$$\begin{cases} \int_0^{1-[r]_2} \cos(2l_r + 1/2)\alpha f(y) dy - \int_{1-[r]_2}^1 \cos(2l_r + 3/2)\alpha f(y) dy & \text{if } [r]_2 > 0, \\ -\int_0^{-[r]_2} \cos(2l_r - 1/2)\alpha f(y) dy + \int_{-[r]_2}^1 \cos(2l_r + 1/2)\alpha f(y) dy & \text{if } [r]_2 < 0, \end{cases}$$

and its t -derivative is

$$\begin{cases} -2\epsilon(r) f(1 - [r]_2) \cos(2l_r + 1)\alpha \cos \alpha/2 & \text{if } [r]_2 > 0, \\ 2\epsilon(r) f(-[r]_2) \cos 2l_r \alpha \cos \alpha/2 & \text{if } [r]_2 < 0, \end{cases}$$

performing similar calculations on the other integrals and collecting all the terms we obtain the formula for u when $x_e \pm t$ is not an integer. The formula is then true without restrictions (but with our convention about values at the vertices) because u is a priori known to be continuous. It might be worth mentioning that the derivative of $t/c \int_{\Gamma} f(y) dy$ cancels out with part of the derivatives of the first couple of the integrals above. \square

Notice that when x is a vertex (with $x_e = 0$ for definiteness), $r = \pm t$ and observing that $l_{(-r)} = -l_r$ and $[-r]_2 = -[r]_2$ when r is not an odd integer, the formula becomes

$$\begin{aligned} u_e(t, 0) = & \frac{1}{2} \sum_{e' \in E} c(e') \left\{ \left[\frac{1}{c} + \mathbf{A}_{e,e'} + \sum_{\alpha \in \mathcal{N}} \left(\frac{\mathbf{C}_{\alpha,e,e'} + \mathbf{D}_{\alpha,e,e'}}{\sin^2 \alpha} \cos 2l_t \alpha \right. \right. \right. \\ & \left. \left. + \operatorname{sgn}[t]_2 \frac{\mathbf{F}_{\alpha,e,e'} - \mathbf{G}_{\alpha,e,e'}}{\sin \alpha} \sin 2l_t \alpha \right) \right] f_{e'}(|[t]_2|) \\ & + \left[\frac{1}{c} - \mathbf{A}_{e,e'} - \sum_{\alpha \in \mathcal{N}} \left(\frac{\mathbf{C}_{\alpha,e,e'} + \mathbf{D}_{\alpha,e,e'}}{\sin^2 \alpha} \cos(2l_t + \operatorname{sgn}[t]_2)\alpha \right. \right. \\ & \left. \left. - \operatorname{sgn}[t]_2 \frac{\mathbf{F}_{\alpha,e,e'} - \mathbf{G}_{\alpha,e,e'}}{\sin \alpha} \sin(2l_t + \operatorname{sgn}[t]_2)\alpha \right) \right] f_{e'}(1 - |[t]_2|) \right\}. \quad (19) \end{aligned}$$

A similar formula gives the solution at the vertices whose coordinates are $x_e = 1$.

At the vertex x we can also compute the solution of the problem (5) with $f \equiv 0$ obtaining

$$u_e(t, 0) = \frac{t}{c} \int_{\Gamma} g(y) dy + \lambda_1(t) \int_0^{\lfloor t \rfloor_2} g_{e'}(y) dy + \lambda_2(t) \int_0^{1-\lfloor t \rfloor_2} g_{e'}(y) dy + \lambda_3(t) \int_0^1 g_{e'}(y) dy,$$

where the λ_i depend on the network through our structure matrices and on t through the quantity $2l_t$ and sign of $\lfloor t \rfloor_2$ only. We omit the rather complicated formulae.

Of course, knowing the initial datum f_e and the solution in the vertices as t varies we can recover the solution on the edge e .

From the spectral formula (14) we can easily deduce the following

Proposition 3. *If all the $\alpha \in \mathcal{N}$ are rational multiples of π , then the solution $u(t, x)$ of the Cauchy problem (5) is a time-periodic function whose period is an integer. If at least one of the $\alpha \in \mathcal{N}$ is not a rational multiple of π , then there are data f for which the solution of the Cauchy problem (5) with $g \equiv 0$ is not periodic in time.*

Proof. If $\{\phi_k\}$ is an orthonormal basis of eigenvectors of Δ and $\{-\lambda_k\}$ are the corresponding eigenvalues, formula (14) can be written as

$$u(t) = \sum_k \cos(t\sqrt{-\lambda_k})(f, \phi_k)\phi_k + \sum_k \frac{\sin(t\sqrt{-\lambda_k})}{\sqrt{-\lambda_k}}(g, \phi_k)\phi_k. \tag{20}$$

If all $\alpha \in \mathcal{N}$ are rational multiples of π then all the $\sqrt{-\lambda_k}$ are multiple of the same rational fraction of π and therefore u is time-periodic with integer period.

If $\alpha \in \mathcal{N}$ is not a rational multiple of π let ϕ one of the eigenfunctions associated to it and let $-\lambda^2$ the corresponding eigenvalue. Consider the solution u of the Cauchy problem with $g = 0$ and $f = \phi + \psi$ where ψ is one of the eigenfunctions with eigenvalue $-\mu^2$ of the form $-k^2\pi$. Then

$$u(t) = \cos t\lambda \phi + \cos t\mu \psi.$$

Since ϕ and ψ are linearly independent, $u(t + T) = u(t)$ implies that $\cos(t + T)\lambda = \cos t\lambda$ and $\cos(t + T)\mu = \cos t\mu$ which is impossible because λ/μ is irrational. \square

We close this section by observing that, since the well known formula

$$e^{-tx^2} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} \cos(sx) ds$$

has the following operatorial counterpart:

$$e^{t\Delta} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} \cos(s\sqrt{-\Delta}) ds,$$

we can use Theorem 3 to solve the Cauchy problem for the heat equation

$$\begin{cases} \frac{\partial U}{\partial t} = \Delta U, & t > 0, \\ u(0, \cdot) = f(\cdot). \end{cases} \quad (5')$$

In fact, if $u(t, x)$ solves the Cauchy problem (5) with $g \equiv 0$, then

$$U(t, x) = e^{t\Delta}(f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} u(s, x) ds$$

is the solution of problem (20). Using the formula for u given by Theorem 3, the fact that the functions f_e are defined to be zero outside the interval $[0, 1]$, the fact that the kernel in the integral is even in s , and careful changes of variables and computations, we obtain the following formula:

$$U(t, x) = \frac{1}{2} \sum_{e' \in E} \frac{c(e')}{\sqrt{4\pi t}} \left\{ \int_0^1 f_{e'}(s) \sum_{n=0}^{\infty} (A_n^+(e, e') e^{-(n+s+x_e)^2/(4t)} + A_n^-(e, e') e^{-(n-s-x_e)^2/(4t)} + B_n^+(e, e') e^{-(n+s-x_e)^2/(4t)} + B_n^-(e, e') e^{-(n-s+x_e)^2/(4t)}) ds \right\},$$

where the coefficients $A_n^{\pm}(e, e')$ and $B_n^{\pm}(e, e')$ depend only on the structure of the network through our structure matrices.

Implicit in the above formula is the heat kernel on the network Γ . Notice that the quantity $n + s + x_e$ can be interpreted as the length of a path in Γ connecting the point $x \in e$ with the point $s \in e'$ by joining the 0-vertex of e to the 0-vertex of e' with a chain of oriented edges e_1, \dots, e_n such that the initial vertex of e_1 and the terminal vertex of e_n coincide with the 0-vertices of e and e' , respectively, and the terminal vertex of e_k coincide with the initial term of e_{k+1} for $k = 1, \dots, n - 1$, travelling back and forth being allowed. Similar interpretation are valid for $n + s - x_e$, $n - s + x_e$ and $n - s - x_e$ where the connection is established between the 0-vertex and the 1-vertex or the two 1-vertices of the edges e and e' . With this in mind, it is possible to compare our formula for the heat kernel with the one obtained by Roth in [19]. In Roth's paper all the $c(e)$ are 1, but he considers edges of arbitrary finite lengths l_1, \dots, l_n . His representation for the heat kernel $h(t, x, y)$ in terms of transmission coefficients and paths joining the two points x and y is more intuitive than ours and seems to have more physical and geometrical flavour. On the other hand, our coefficients seem easier to compute than his. But the comparison between the two formulas (obviously when the lengths and the weights of the edges are all 1) can probably lead to many interesting relations of a combinatorial nature connecting path related quantities with spectrum related ones as in Roth's theorem 1 in [19] (trace of the heat kernel). We will not however pursue these investigations in this paper.

6. Two examples

6.1. The cross shaped network

Let Γ be the network consisting of four edges branching out from a common vertex with unitary weights. A similar example can be found in [2] although Dirichlet conditions replace there our Neumann conditions at the four dead ends. At the fifth vertex, the common one, in both examples the Kirchhoff condition is required.

We denote by 1, 2, 3, 4 the four edges and we orient them in such a way that the common vertex become 0 in our standard identification with the interval $[0, 1]$. We name the remaining four vertices of the graph after the edges to which they belong.

In this case the eigenvalues are 0 (with multiplicity 1), $(-k^2\pi^2)$, for any $k \in \mathbf{N}$ (with multiplicity 1) and $((2k-1)\pi \pm \pi/2)^2$ (with multiplicity 3).

The constant function $w_0 \equiv 1/2$ and the functions $w_{k,1}$ whose restrictions to the edge j are $\sqrt{2}/2 \cos k\pi x_j$, provide an orthonormal basis for the direct sum of the eigenspaces associated to the eigenvalues with the multiplicity 1.

To construct the eigenfunctions corresponding to the eigenvalues with multiplicity 3, we need an orthonormal basis (in $l^2(V)$) for the kernel of the matrix

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for instance, the vectors

$$Z_1 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad Z_2 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad Z_3 = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Then, according to the general construction, we compute

$$z_1(0) = z_2(0) = z_3(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$z_1(1) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad z_2(1) = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad z_3(1) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Finally we have that the matrices $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, \mathbf{F}_α and \mathbf{G}_α are the null matrix, \mathbf{A} is the matrix whose entries are all equal to $1/4$, $\mathbf{D}_\alpha = -\mathbf{C}_\alpha$ and

$$\mathbf{C}_\alpha = \frac{1}{4} \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

We can therefore write the components of the kernel \mathbf{H} as follows (note that in this case we have $\cos(2l_\theta + \operatorname{sgn}([\theta]_2)/2)\alpha = (-1)^{l_\theta}/\sqrt{2}$)

$$\begin{aligned} \mathbf{H}_{i,i}(t, x_i, y_i) &= \frac{t}{4} - \frac{1}{16} \sum_{\theta=x_i \pm y_i \pm t} \epsilon(\theta)[\theta]_2 \\ &\quad + \frac{1}{16} \sum_{\theta=x_i - y_i \pm t} (1 + 3(-1)^{l_\theta}) \epsilon(\theta) \operatorname{sgn}([\theta]_2) \\ &\quad + \frac{1}{16} \sum_{\theta=x_i + y_i \pm t} (1 - 3(-1)^{l_\theta}) \epsilon(\theta) \operatorname{sgn}([\theta]_2), \\ \mathbf{H}_{i,j}(t, x_i, y_j) &= \frac{t}{4} - \frac{1}{16} \sum_{\theta=x_i \pm y_i \pm t} \epsilon(\theta)[\theta]_2 \\ &\quad + \frac{1}{16} \sum_{\theta=x_i - y_i \pm t} (1 - (-1)^{l_\theta}) \epsilon(\theta) \operatorname{sgn}([\theta]_2) \\ &\quad + \frac{1}{16} \sum_{\theta=x_i + y_i \pm t} (1 + (-1)^{l_\theta}) \epsilon(\theta) \operatorname{sgn}([\theta]_2). \end{aligned}$$

To write the solution of the Cauchy problem with $g \equiv 0$ we calculate $k_{1ij} = 1/2$ and $k_{2ij} = 0$, for $i, j = 1, 2, 3, 4$. Then we find $a_{ii} = 1/2 + (3/2)(-1)^{l_r} \operatorname{sgn}([r]_2)$ and, for $j \neq i$, $a_{ij} = 1/2 - (1/2)(-1)^{l_r} \operatorname{sgn}([r]_2)$.

Finally

$$\begin{aligned} u_i(t, x_i) &= \frac{1}{4} \sum_{r=x_i \pm t} \left(\frac{1}{2} + \frac{3}{2}(-1)^{l_r} \operatorname{sgn}([r]_2) \right) f_i(|[r]_2|) \\ &\quad + \frac{1}{4} \sum_{j \neq i} \sum_{r=x_j \pm t} \left(\frac{1}{2} - \frac{1}{2}(-1)^{l_r} \operatorname{sgn}([r]_2) \right) f_j(|[r]_2|). \end{aligned}$$

Notice that the solution is periodic in t with period 4.

6.2. The complete network with $n + 1$ vertices

The complete graph with $n + 1$ vertices has $n(n + 1)/2$ edges and constant degree equal to n at every vertex. We define complete the network Γ associated to the complete graph. We concentrate on the simplest case where all the weights are equal to 1. According to the general discussion in Section 3, the spectrum of Γ consists of the eigenvalues 0, $(-k^2\pi^2)$ (with multiplicities $(n^2 - n + 2(-1)^k)/2$) and finally $-((2k - 1)\pi \pm \alpha)^2$ (with

multiplicity n) where $\alpha = \arccos 1/n$ and $-1/n$ is the only eigenvalue of the matrix P in $(-1, 1)$.

For all the complete networks the matrix \mathbf{A} vanishes identically whereas the matrices $\mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \mathbf{C}_\alpha, \mathbf{D}_\alpha, \mathbf{F}_\alpha$ and \mathbf{G}_α must be computed time by time as n varies. Things simplify dramatically if we concentrate our attention on the solution at a fixed vertex v . In this case, the symmetry of the structure allows us to write $u(t, v)$ explicitly in terms of the initial data f and n for any n (we are assuming that the weights $c_i = 1$ for all i). Of course, since all vertices in the complete network are equivalent, we actually have the solution explicitly on all the vertices (it is just a matter of permutations of the components of the initial data f) and from that we could, if needed, reconstruct the solution on any specific edge simply by solving an initial value problem on the unit interval with prescribed values at the edges.

Theorem 4. *Let Γ be a complete network with $n + 1$ vertices and all the weights equal to 1. For definiteness, assume v is the initial vertex for all the edges in E_v . Then the solution of the Cauchy problem with data $f = (f_e)_{e \in E}$ and $g \equiv 0$ in v is given by*

$$\begin{aligned}
 u(t, v) = & -\frac{2\sqrt{2} \operatorname{sgn}([t]_2)}{(n+1)\sqrt{n(n-1)}} \sin l_t \alpha \cos \frac{2l_t + \operatorname{sgn}([t]_2)}{2} \alpha \\
 & \times \sum_{e \notin E_v} \left(f_e(|[t]_2|) + f_e(1 - |[t]_2|) \right) \\
 & + \frac{1}{n+1} \sum_{e \in E_v} \left(\left(\frac{1}{n} + \cos 2l_t \alpha \right) f_e(|[t]_2|) \right. \\
 & \left. + \left(\frac{1}{n} - \cos(2l_t + \operatorname{sgn}([t]_2)) \alpha \right) f_e(1 - |[t]_2|) \right).
 \end{aligned}$$

We remark that if v were the terminal vertex for some of the edges e in E_v the only needed modification would be the interchanging of the corresponding $f_e([t]_2)$ with $f_e(1 - [t]_2)$ in the above formulae.

Proof. Once v is fixed, the edges of our complete network fall in the two disjoint classes E_v and $E \setminus E_v$, and by the symmetry of the structure the members of each of them are interchangeable as far as their influence on v is concerned. If $f \in D(\Delta)$ has support on the edge $e \notin E_v$, then by the finite propagation property of the solution we see that for $t \in (0, 1)$, $u(t, v) = 0$ and therefore formula (19) implies

$$\begin{aligned}
 0 = & \frac{1}{2} \left[\frac{2}{n(n+1)} + \frac{n^2}{n^2-1} (\mathbf{D}_{\alpha,e,e'} + \mathbf{C}_{\alpha,e,e'}) \right] f_{e'}(t) \\
 & + \frac{1}{2} \left[\frac{2}{n(n+1)} - \frac{n^2}{n^2-1} (\mathbf{D}_{\alpha,e,e'} + \mathbf{C}_{\alpha,e,e'}) \frac{1}{n} + \mathbf{F}_{\alpha,e,e'} - \mathbf{G}_{\alpha,e,e'} \right] f_{e'}(1-t).
 \end{aligned}$$

Since $f_{e'}$ is arbitrary, we have that

$$\frac{2}{n(n+1)} + \frac{n^2}{n^2-1} (\mathbf{D}_{\alpha,e,e'} + \mathbf{C}_{\alpha,e,e'}) = 0,$$

$$\frac{2}{n(n+1)} - \frac{n}{n^2-1} (\mathbf{D}_{\alpha,e,e'} + \mathbf{C}_{\alpha,e,e'}) + \mathbf{F}_{\alpha,e,e'} - \mathbf{G}_{\alpha,e,e'} = 0,$$

and thus $\mathbf{D}_{\alpha,e,e'} + \mathbf{C}_{\alpha,e,e'} = -2(n-1)/n^3$ and $\mathbf{F}_{\alpha,e,e'} - \mathbf{G}_{\alpha,e,e'} = -2/n^2$. Plugging these values of the constants in formula (19) we have the claim for f supported on an edge not belonging to E_v .

If f has support on an edge $e \in E_v$, the finite propagation argument is not enough to determine $\mathbf{D}_{\alpha,e,e} + \mathbf{C}_{\alpha,e,e}$ and $\mathbf{F}_{\alpha,e,e} - \mathbf{G}_{\alpha,e,e}$, but since v is the initial vertex of e , for $t \in (0, 1)$, v cannot be influenced by the wave travelling toward the terminal edge of e . Therefore the coefficient of $f_e(1-t)$ in (19) must vanish, and thus we get

$$\frac{2}{n(n+1)} - \frac{n}{n^2-1} (\mathbf{D}_{\alpha,e,e} + \mathbf{C}_{\alpha,e,e}) + \mathbf{F}_{\alpha,e,e} - \mathbf{G}_{\alpha,e,e} = 0.$$

By the definition of \mathbf{C}_α and \mathbf{D}_α and the choice of v as initial vertex, we have that

$$\mathbf{D}_{\alpha,e,e} + \mathbf{C}_{\alpha,e,e} = (1 - \cos 2\alpha) \sum_{j=1}^n z_{\alpha,j,e}^2(0) = \frac{2(n^2-1)}{n^2} \sum_{j=1}^n Z_{\alpha,j}^2(v).$$

Since the vectors $Z_{\alpha,j}$, $j = 1, \dots, n$, complemented by the vector whose components are all equal to $1/\sqrt{n(n+1)}$ form an orthonormal basis of $l^2(V, c)$, elementary facts about orthogonal matrices imply that $\sum_{j=1}^n Z_{\alpha,j}^2 = 1/(n+1)$ and therefore $\mathbf{D}_{\alpha,e,e} + \mathbf{C}_{\alpha,e,e} = 2(n-1)/n^2$, whence $\mathbf{F}_{\alpha,e,e} - \mathbf{G}_{\alpha,e,e} = 0$.

Having found the structural constants, the theorem follows at once. \square

Lemma 3. *If $n \geq 3$ then the angle $\alpha \in (0, \pi/2)$ defined by $\cos \alpha = 1/n$ cannot be a rational multiple of π .*

Proof. If α were a rational multiple of π , the complex number $z_\alpha = \cos \alpha + i \sin \alpha$ would be a N th root of unity for some natural number N and therefore the quadratic polynomial

$$(z - z_\alpha)(z - \bar{z}_\alpha) = z^2 - \frac{2}{n}z + 1$$

would be a factor of $Z^N - 1$. Since all monic factors with rational coefficients of a monic polynomial with integer coefficients must have integer coefficients, the above quadratic polynomial can divide $Z^N - 1$ only if $n = 1$ or 2 . \square

The above lemma and Proposition 3 imply that on a complete network with at least four vertices we have both time-periodic and non-periodic solutions. It is enough to choose as initial data $g = 0$ and f equal to an eigenfunction of one the eigenvalues of the form $-k^2\pi^2$ to obtain periodicity while the f described in the proof of Proposition 3 produce non-periodic solutions.

However, we can also show that, on a complete network, no wave originated by an initial datum f having support concentrated on a single edge can be time-periodic.

Proposition 4. *Let Γ be a complete network with $n + 1$ vertices and all the weights equal to 1. If $g \equiv 0$ and $f \not\equiv 0$ is supported on a single edge, then the solution u of the Cauchy problem (5) cannot be periodic in time.*

Proof. If f is supported on e , consider the solution $u(t, v)$ evaluated at a vertex $v \notin e$. Suppose first $f_e(s) + f_e(1 - s) \not\equiv 0$. If u were time-periodic with period T then $u(t + kT, v) = 0$ for any natural k and any $t \in (0, 1)$. By Theorem 4 this would imply $f_e([t + kT]_2) + f_e(1 - [t + kT]_2) = 0$ for all $k \neq 0$ such that $t + kT$ is not an odd integer since the factors $\sin l_t \alpha$ and $\cos((2l_t \pm 1)/2)\alpha$ cannot vanish for $l_t > 0$ (by Lemma 3 α is not a rational multiple of π). Now, T cannot be rational, otherwise by choosing k so that kT is an even integer, we would get $f_e(t) + f_e(1 - t) = 0$ for all $t \in (0, 1)$ against our assumption. If T were not rational, then $[kT]_2$ would be dense in $(0, 1)$ as k varies in \mathbf{N} . As a result, the continuous function $f_e(s) + f_e(1 - s)$ would be 0 on a dense subset of $(0, 1)$ and thus everywhere against our assumptions.

In the remaining case, i.e., $f(s) = -f(1 - s)$ for all $s \in [0, 1]$, we evaluate the solution at the initial vertex v of e . Theorem 4 and our assumption lead to

$$u(t, 0) = \frac{2}{n+1} \cos \frac{\alpha}{2} \cos \frac{4l_t + 1}{2} \alpha f_e([t]_2)$$

when $[t]_2 \in (0, 1)$. Reasoning as above we obtain that the solution is periodic only when $f \equiv 0$. \square

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