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On the total positivity of restricted Stirling numbers☆

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ABSTRACT

This note shows that the matrix whose (n, k) entry is the number of set partitions of $\{1, \dots, n\}$ into k blocks with size at most m is never totally positive for $m \geq 3$; thus answering a question posed in [H. Han, S. Seo, Combinatorial proofs of inverse relations and log-concavity for Bessel numbers, European J. Combin. 29 (2008) 1544–1554].

1. Introduction

Let $S_{\leq m}(n, k)$ be the number of set partitions of $\{1, \dots, n\}$ into k blocks with size $\leq m$. In [3], Han and Seo study these numbers only when $m = 2$ and formulate the following problem: decide when the matrix $S_{\leq m} := \left(S_{\leq m}(n, k)\right)_{k,n\in\mathbb{N}}$ is totally positive i.e. all its minors are nonnegative (for information about totally positive matrices, see e.g. [4]). The authors show in [3] that the answer is positive when $m = 2$, by using techniques introduced in [1]. Moreover, if m goes to infinity, we get that the limit matrix is the matrix whose entries are the Stirling numbers of the second kind, which is again totally positive (see [1]).

In this note we show that the previous two cases (and the trivial case $m = 1$) are the only ones such that the matrix $S_{\leq m}$ is totally positive.

2. Counterexamples

We denote the matrix of the Stirling numbers of the second kind by $S := (S(n, k))_{k,n\in\mathbb{N}}$, i.e.

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 3 & 7 & \cdots \\ 0 & 0 & 0 & 1 & 6 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Recall (see e.g. [2], Section 5.1, Theorem A) that the Stirling numbers of the second kind satisfy the following relation for all $n, k \geq 1$

$$S(n,k) = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^n = \sum_{r=1}^{k}(-1)^{k-r}\frac{r^{n-1}}{(r-1)!(k-r)!}. \tag{1}$$

This formula will be useful since if the maximum size m of each block is at least $n-k+1$ then $S_{\leq m}(n,k) = S(n,k)$.

Proposition 1. *For all $m \geq 3$ the matrix $S_{\leq m}$ is not totally positive.*

Proof. In the following, for all $m \geq 3$ we denote by D_m the following determinant

$$\begin{vmatrix} S_{\leq m}(m-1,1) & S_{\leq m}(m,1) & S_{\leq m}(m+1,1) & S_{\leq m}(m+2,1) \\ S_{\leq m}(m-1,2) & S_{\leq m}(m,2) & S_{\leq m}(m+1,2) & S_{\leq m}(m+2,2) \\ S_{\leq m}(m-1,3) & S_{\leq m}(m,3) & S_{\leq m}(m+1,3) & S_{\leq m}(m+2,3) \\ S_{\leq m}(m-1,4) & S_{\leq m}(m,4) & S_{\leq m}(m+1,4) & S_{\leq m}(m+2,4) \end{vmatrix}. \tag{2}$$

Our goal is to show that D_m is negative when $m \geq 5$ thus proving our claim, except for the cases $m = 3$, $m = 4$. By the previous discussion, we can rewrite D_m as follows

$$\begin{vmatrix} S(m-1,1) & S(m,1) & 0 & 0 \\ S(m-1,2) & S(m,2) & S(m+1,2) & S(m+2,2)-a \\ S(m-1,3) & S(m,3) & S(m+1,3) & S(m+2,3) \\ S(m-1,4) & S(m,4) & S(m+1,4) & S(m+2,4) \end{vmatrix} \tag{3}$$

where a is the number of partitions of $m+2$ into 2 blocks, one of them with size at least $m+1$. It is easy to compute a, since it counts the number of partitions of $m+2$ into 2 blocks whose sizes are exactly $m+1$ and 1. Therefore, $a = m+2$.

We now use (1) for each entry in (3) and we obtain that

$$12 \cdot D_m = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1+2^{m-2} & -1+2^{m-1} & -1+2^m & -1+2^{m+1}-(m+2) \\ 1-2^{m-1}+3^{m-2} & 1-2^m+3^{m-1} & 1-2^{m+1}+3^m & 1-2^{m+2}+3^{m+1} \\ -1+3\cdot 2^{m-2}-3^{m-1}+4^{m-2} & -1+3\cdot 2^{m-1}-3^m+4^{m-1} & -1+3\cdot 2^m-3^{m+1}+4^m & -1+3\cdot 2^{m+1}-3^{m+2}+4^{m+1} \end{vmatrix}.$$

We now subtract the first column from the second and we get (after a trivial Laplace expansion about the first row) that $12D_m$ is equal to

$$\begin{vmatrix} 2^{m-2} & 2^m-1 & 2^{m+1}-m-3 \\ 2\cdot 3^{m-2}-2^{m-1} & 3^m-2^{m+1}+1 & 3^{m+1}-2^{m+2}+1 \\ 3\cdot 4^{m-2}-2\cdot 3^{m-1}+3\cdot 2^{m-2} & 4^m-3^{m+1}+3\cdot 2^m-1 & 4^{m+1}-3^{m+2}+3\cdot 2^{m+1}-1 \end{vmatrix}.$$

By adding two times the first row to the second and three times the first and second rows to the third we obtain

$$\begin{vmatrix} 2^{m-2} & 2^m-1 & 2^{m+1}-m-3 \\ 2\cdot 3^{m-2} & 3^m-1 & 3^{m+1}-2m-5 \\ 3\cdot 4^{m-2} & 4^m-1 & 4^{m+1}-3m-7 \end{vmatrix}.$$

By multilinearity of the determinant, $12D_m$ is equal to

$$\begin{vmatrix} 2^{m-2} & 2^m-1 & 2^{m+1}-3 \\ 2\cdot 3^{m-2} & 3^m-1 & 3^{m+1}-5 \\ 3\cdot 4^{m-2} & 4^m-1 & 4^{m+1}-7 \end{vmatrix} - m\begin{vmatrix} 2^{m-2} & 2^m-1 & 1 \\ 2\cdot 3^{m-2} & 3^m-1 & 2 \\ 3\cdot 4^{m-2} & 4^m-1 & 3 \end{vmatrix}. \tag{4}$$

We will show that the first determinant in (4) is negative and the second is positive when $m \geq 5$. After manipulations, the first determinant is

$$d1_m = -\frac{1}{144}24^m + \frac{2}{9}12^m - \frac{5}{16}8^m + \frac{1}{9}6^m + \frac{3}{8}4^m - \frac{8}{9}3^m + \frac{1}{2}2^m, \tag{5}$$

and the second determinant is

$$d2_m = \frac{5}{144}12^m - \frac{1}{8}8^m + \frac{1}{12}6^m - \frac{3}{16}4^m + \frac{4}{9}3^m - \frac{1}{4}2^m. \tag{6}$$

Obviously,

$$d1_m < -\frac{1}{144}24^m + \left(\frac{2}{9} + \frac{1}{9} + \frac{3}{8} + \frac{1}{2}\right)12^m,$$
$$d2_m > \frac{5}{144}12^m - \left(\frac{1}{8} + \frac{3}{16} + \frac{1}{4}\right)8^m$$

and then

$$d1_m < 0 \quad \text{if } m \geq 8,$$
$$d2_m > 0 \quad \text{if } m \geq 7.$$

It follows that $D_m < 0$ for all $m \geq 8$. It is possible to compute the exact value of D_m when $m \leq 7$ and we find that D_7, D_6, D_5 are negative, $D_4 = 0$ and $D_3 = 10$. When $m = 3, 4$, we consider the submatrix of $S_{\leq m}$ whose entries are all the elements $S_{\leq m}(n, k)$ with $m - 1 \leq n \leq m + 3$ and $1 \leq k \leq 5$. In both cases the determinant is negative. □

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