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EXECUTION SPACES FOR SIMPLE HIGHER DIMENSIONAL AUTOMATA

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ABSTRACT. Higher Dimensional Automata (HDA) are highly expressive models for concurrency in Computer Science, cf van Glabbeek [26]. For a topologist, they are attractive since they can be modeled as cubical complexes - with an inbuilt restriction for directions of allowable (d-)paths. In Raussen [25], we developed a new method describing, for a certain subclass of HDA, the homotopy type of the space of execution paths (d-paths) as a finite simplicial complex.

Several restrictions that were made to ease the presentation in that latter paper will be removed in this article in order to make the results applicable in greater generality. Furthermore, we take a close look at semaphore models with semaphores all of arity one. It turns out that execution spaces for these are always homotopy discrete with components representing sets of "compatible" permutations. Finally, we describe a model for the complement of the execution space seen as a subspace of a product of spheres – with the aim to make the calculation of topological invariants easier and faster.

1. INTRODUCTION

1.1. **Background.** A particular model for concurrent computation in Computer Science, called *Higher Dimensional Automata* (HDA), was introduced by Pratt [20] back in 1991. Mathematically, HDA can be described as (labelled) pre-cubical sets (with n-dimensional cubes instead of simplices as building blocks; cf Brown and Higgins [4, 3]) with a preferred set of *directed paths* (respecting the natural partial orders) in any of the cubes of the model.

Compared to other well-studied concurrency models like labelled transition systems, event sturctures, Petri nets etc. (for a survey on those cf Winskel and Nielsen [27]), it has been shown by R.J. van Glabbeek [26] that Higher Dimensional Automata have the highest expressivity; on the other hand, they are certainly less studied and less often applied so far.

All concurrency models deal with sets of states and with associated sets of execution paths (with some further structure). The interest is mainly in the structure of the spaces of *execution paths*; typically, it is difficult to extract valuable information about the path space from the state space model. We use topological models for both state space and the execution (=path) space consisting of the directed paths in state space. It is particularly important to know whether the path space is *path-connected*; and, if not, to get an overview over its path components: Executions in the same path component yield

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the *same result* (decision) in a concurrent computation; different components *may* lead to different results. From a topological perspective, the ultimate aim is to determine the *homotopy type* of these path spaces.

Higher Dimensional Automata are prototypes of *directed* topological spaces, cf Grandis [13, 14]; a directed topological space consists of an (ordinary) topological space X together with a subspace $\vec{P}(X) \subseteq X^I = [I; X]$ of "directed" d-paths satisfying several natural requirements: $\vec{P}(X)$ is

- closed under concatenation
- closed under weakly increasing reparametrizations (order-preserving self maps of the unit interval I)
- contains the constant paths.

General topological properties of spaces of d-paths and of *traces* (=d-paths up to monotone reparametrizations; cf Fahrenberg and Raussen [7, 23]) in pre-cubical complexes were investigated in Raussen [24]. But so far, apart from low-dimensional examples with convincing drawings, there have been very few explicit examples of actual computations of spaces of such traces (for an attempt in dimension two, cf Raussen [21]); let alone a general method to perform such computations.

The paper Raussen [25] describes an algorithmic method to determine the homotopy types of trace spaces for Higher Dimensional Automata (and thus in particular to calculate and describe their components) through explicitly constructed *finite simplicial complexes*, but only under several restrictions for the HDA under consideration:

(1) We had to stick to *semaphore* – or PV – models as described by Dijkstra [5] – an important but restricted class of HDA. Loosely speaking, a PV-model space is a hypercube I^n – with I the unit interval $[0, 1]$ – from which a number of n-dimensional hyperrectangles has been removed; cf Section 2.1 for details.
(2) In order to make matters mathematically "clean", we restricted attention to models in which the forbidden hypercubes do not intersect the boundary of I^n. For most natural models, this will *not* be the case.
(3) Once again, in order to get a mathematically easy description, we described only path spaces from the bottom vertex $\mathbf{0}$ to the top vertex $\mathbf{1}$ in I^n. It is important also to collect information about "intermediate" path spaces between arbitrary points in the model – for example for investigations of its fundamental category (Grandis [13], Goubault etal [8, 12]) but also for inductive reasoning and calculations. In this case, it is typically necessary to consider obstruction hyperrectangles intersecting the boundary (of a smaller hypercube).

In this paper, we will elaborate how to get rid of the last two restrictions; we will still only deal with PV-models. The general idea how a path space associated to the model can be represented in simplicial terms is the same as in Raussen [25]. We will explain it in Section 2; for the mathematical proof that the simplicial model is in fact homotopy equivalent to the space of directed paths (executions), we refer to that paper.

As already explained in Raussen [25, Section 5.1], it is not difficult to include cases in which individual processes are allowed to branch, merge and loop. Each individual program is then modelled by a digraph; the state space is a product of digraphs from which a number of "hyperrectangles" has been removed. For some first ideas on how to achieve simplicial models for execution spaces on general HDA (not necessarily semaphores), we refer to Raussen [25, Section 5.2].

1.2. **Structure and overview of results.** In Section 2, we introduce model spaces for semaphore models in concurrency and review and modify methods from Raussen [25] that allow to determine combinatorial/topological models of the associated spaces of directed paths (in these spaces). The key idea is to decompose such a model space into pieces that are *contractible*, i.e., homotopy equivalent to a point; even more important, the spaces of *d-paths within every such subspace* are *either contractible or empty*.

Fix a model space X and a pair $(\mathbf{c}, \mathbf{d})$ of start and end point within X. We wish to derive a finite description of the space $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ of directed paths joining $\mathbf{c}$ to $\mathbf{d}$, or rather of the homotopy equivalent trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$; a trace is an equivalence class of directed paths up to weakly increasing reparametrizations, cf Fahrenberg and Raussen [7, 23].

We associate to X and to $(\mathbf{c}, \mathbf{d})$ a *poset category* $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$. That category is naturally isomorphic to a subcategory of a *product* of a number of poset categories consisting of non-empty subsets of the positive integers $[1 : n]$ less than or equal to n. A topological realization of this subcategory can thus be modelled on *products of simplices* and gives rise to an explicit *prodsimplicial complex*, cf Kozlov [18], called $\mathbf{T}(X)(\mathbf{c}, \mathbf{d})$. Using standard methods from algebraic topology explained in Raussen [25], we show that the space of directed paths $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ in X from $\mathbf{c}$ to $\mathbf{d}$ or equivalently, the trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$, is homotopy equivalent to that finite complex $\mathbf{T}(X)(\mathbf{c}, \mathbf{d})$. The latter in turn has the nerve $\Delta(\mathcal{C}(X)(\mathbf{c}, \mathbf{d}))$ of the poset category as a barycentric subdivision.

A similar technique works also for spaces of directed paths starting at a given point and ending on the *upper boundary of a hypercube*. This is interesting both for inductive reasoning but also for the investigation of the decision power of distributed concurrent processes of which some may die (compare Herlihy and Rajsbaum [16] and more recent papers in distributed computing).

For calculations, it is essential to determine the category $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ explicitly. It can be described as a subcategory of the order category of *binary* $l \times n$-matrices $M_{l,n} \cong (\mathbf{Z}/2)^{ln}$ (with l the number of obstructions and n the dimension of the model space) with the componentwise partial order. In Section 3, To achieve this, we describe how to achieve this: One needs to decide, for every of the contractible subspaces mentioned above, whether there exists a directed path *within* that subspace from $\mathbf{c}$ to $\mathbf{d}$. It turns out that it is enough to find out whether there exist *deadlock* points (the only d-path with a deadlock as source is trivial) in certain *associated* models and to apply a combinatorial search algorithm for deadlocks described in Fajstrup, Goubault and Raussen [9]. The outcome of a systematic search for deadlocks (in all associated models) is a set $D(X)(\mathbf{c}, \mathbf{d})$ of *minimal non-faces* – all of the same dimension $n-1$ – describing the prodsimplicial complex

$\mathbf{T}(X)(\mathbf{c},\mathbf{d})$ within the prodsimplicial complex $(\Delta^{n-1})^l$. The *maximal faces* of $\mathbf{T}(X)(\mathbf{c},\mathbf{d})$ can then be determined via minimal transversals in an associated hypergraph, as already described in [25, Section 4.2].

In Section 4, we indicate how the topology of execution spaces $\vec{T}(X)(\mathbf{c},\mathbf{d})$ *changes* under variation of end points; and in particular, when it does not change! This is important for inductive reasoning and for obtaining a complete overview over the trace category; cf Raussen [22]. The trace category determines the fundamental category $\vec{\pi}_1(X)$ of the model space by taking the connected components of the morphism spaces; this is the information needed to classify the possible outcome of partial executions.

Section 5 is devoted to an application of the results to a specific case: these are semaphore models in which all semaphores are of *arity one*, ie they allow only *one* process to proceed at any given time. In that case, the space of executions is shown to be *homotopy discrete*; all homotopy information is contained in the fundamental category - with *finite* sets of morphisms. These morphism sets can be described as subsets of *compatible* permutations within $(\Sigma_n)^k$ where k is the number of semaphores involved.

The final Section 6 deals with a computational issue from a theoretical perspective: The prodsimplicial complex $\mathbf{T}(X)(-,-)$ modelling execution spaces embeds naturally in a product $(\partial\Delta^{n-2})^l \cong (S^{n-2})^l$ of l spheres. It seems to be algorithmically easier and quicker to determine the complement $\mathbf{U}(X)(-,-) := (\partial\Delta^{n-2})^l \setminus \mathbf{T}(X)(-,-)$ of the trace complex by giving it a prodsimplicial structure. Poincaré-Alexander-Lefschetz duality can then be applied to infer information from the complement to trace space itself.

1.3. **Implementation issues.** Some first steps towards implementation, in particular the determination of the category $\mathcal{C}(X)(\mathbf{c},\mathbf{d})$, have been taken and various methods have been compared by a research student (cf A. Lang [19]) during a research internship at CEA LIST/LMeASI in France.

The outcome should now be combined with fast algorithms for the calculation of the homology of big chain complexes as eg in Kaczynski, Mrozek and Slusarek [17]; this is an ongoing project. Moreover, we suggest a systematic investigation of how to use and implement these new methods to improve applications of geometric semantics to the static analysis of concurrent programs, cf Goubault and Haucourt [11].

2. Prodsimplicial models for execution spaces

2.1. **A model space and contractible subspaces.**

2.1.1. *Geometric semaphore models.* To start with, we analyse spaces of directed paths in a simple model space that can be described as follows: A (linear) schedule for each of a number of n individual processors $P_j, 1 \le j \le n$, is modelled on the directed interval $\vec{I}_j = [0,1]$. On subintervals $I^i_j =]a^i_j, b^i_j[\subseteq I_j, 1 \le i \le l$, there is potential conflict with the schedules of the other processors. Let $\mathbf{a}^i = (a^i_1,\dots,a^i_n), \mathbf{b}^i = (b^i_1,\dots,b^i_n) \in I^n \setminus \partial I^n$ and let $R^i = \{\mathbf{x} \in I^n | a^i_j < x_j < b^i_j, 1 \le j \le n\}$ denote the "homothetic" hyperrectangle (with faces parallel to the coordinate planes) with bottom corner $\mathbf{a}^i$ and top corner $\mathbf{b}^i$.

The state space for concurrent executions of these n linear processes is the space $X = \vec{I}^n \setminus F \subset \vec{I}^n$ with the *forbidden region* $F = \bigcup_{i=1}^{l} R^i$. The forbidden region F models conflicts and cannot be entered due to guarding semaphores (Dijkstra's PV-models [5]; an interval $]a^i_j, b^i_j[$ corresponds to a call $PcVc$ to a semaphore). See Figure 1 for an example of a forbidden region. The space X inherits a partial order $\le$ from the componentwise partial order $\le$ on $\vec{I}^n$.

We study compound schedules (execution paths) in such a state space X: A *d-path* in X is a continuous path $p\colon \vec{I} \to X$ that is continuous and *order-preserving*: each coordinate $\pi_j \circ p\colon \vec{I} \to X \subset \vec{I}^n \to \vec{I}, 1 \le j \le n$, is *weakly increasing*. The set $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ consists of all d-paths in X starting at $\mathbf{c} \in X$ and ending at $\mathbf{d} \in X$; in particular, these d-paths avoid the "forbidden region" $F \subset \vec{I}^n$. Consult eg Gunawardena[15] and Fajstrup, Goubault and Raussen [9] for detailed descriptions.

As a topological space, $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ is given the subspace topology inherited from the space $P(X)(\mathbf{c}, \mathbf{d}) = [(I, 0, 1); (X, \mathbf{c}, \mathbf{d})]$ of *all* paths in X from $\mathbf{c}$ to $\mathbf{d}$ in the compact-open topology (= uniform convergence topology).

Reparametrization equivalent d-paths [7] in X have the same directed image (= *trace*) in X. Dividing out the action of the monoid of (weakly-increasing) reparametrizations of the parameter interval $\vec{I}$, we arrive at trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$ (cf Fahrenberg and Raussen) [7, 23] which is shown in Raussen [24] to be homotopy equivalent to path space $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ for a far wider class of directed spaces X; in the latter paper, it is also shown that trace spaces enjoy nice properties: They are metrizable, locally compact, locally contractible, and they have the homotopy type of a CW-complex.

It is not difficult to generalise these models to incorporate concurrent executions of non-linear processes that are allowed to branch, to merge and to loop, still governed by semaphores; cf Raussen [25, Section 5.2].

2.1.2. *Contractible subspaces.* We will now describe certain subspaces of X and then prove that associated spaces of d-paths *within* these subspaces are *either empty or contractible*. We need some notation:

- The set of elements "below" $\mathbf{d} \in X$ is denoted
  $\downarrow\mathbf{d} := \{\mathbf{x} \in X | \ \mathbf{x} \le \mathbf{d}\} = \{\mathbf{x} \in I^n | \ \mathbf{x} \le \mathbf{d}, \mathbf{x} \notin F\}$.
  Remark that it is not always possible to reach $\mathbf{d}$ from every $\mathbf{x} \in\downarrow\mathbf{d}$ by a d-path.
  Likewise $\uparrow\mathbf{c} = \{\mathbf{x} \in X | \ \mathbf{c} \le \mathbf{x}\}$ denotes the set of elements above $\mathbf{c}$.
- The upper boundary $\{\mathbf{x} \in\downarrow\mathbf{d} | \ \exists 1 \le i \le n\colon\ x_i = d_i\}$ of the hyperrectangle with corners in $\mathbf{0}$ and $\mathbf{d}$ within X will be denoted $\partial_+\downarrow\mathbf{d}$.
- $\mathbf{a}^i = (a^i_1, \dots, a^i_n), \mathbf{b}^i = (b^i_1, \dots, b^i_n)$.

Definition 2.1. (1) For $1 \leq i \leq l, 1 \leq j_i \leq n$, let

$$X_{j_1,\cdots,j_l} := \{\mathbf{x} \in X \mid \forall i\colon\ x_{j_i} \leq a^i_{j_i} \text{ or } \exists k\colon\ x_k \geq b^i_k\} \tag{2.1}$$

$$= \{\mathbf{x} \in X \mid \forall i\colon\ (\forall k\ x_k < b^i_k \Rightarrow x_{j_i} \leq a^i_{j_i})\}. \tag{2.2}$$

(2) For non-empty subsets $J_i \subseteq [1:n], 1 \leq i \leq l$, let

$$\begin{aligned} X_{J_1,\cdots,J_l} &:= \{\mathbf{x} \in X \mid \forall i\colon\ x^i_{j_i} \leq a^i_{j_i}, j_i \in J_i, \text{ or } \exists k\colon\ x_k \geq b^i_k\} \\ &= \{\mathbf{x} \in X \mid \forall i\colon\ (\forall k\ x_k < b^i_k \Rightarrow x_{j_i} \leq a^i_{j_i} \text{ for all } j \in I_j)\} \\ &= \bigcap_{j_i \in J_i} X_{j_1,\cdots,j_l}. \end{aligned}$$

For illustrations in 2D and 3D, we refer to Raussen [25, Figure 1 and 2].

Proposition 2.2. *For X as above and $(\mathbf{c}, \mathbf{d}) \in X \times X$ we have:*

(1) $\vec{T}(X)(\mathbf{c}, \mathbf{d}) = \bigcup_{[1:n]^l} \vec{T}(X_{j_1,\cdots,j_l})(\mathbf{c}, \mathbf{d})$*; and*
$\vec{T}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d}) = \bigcup_{[1:n]^l} \vec{T}(X_{j_1,\cdots,j_l})(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$.

(2) *Let $\emptyset \neq J_i \subseteq [1:n], 1 \leq i \leq l, \mathbf{c}, \mathbf{d} \in X_{J_1,\dots J_l}$. Then the trace spaces $\vec{T}(X_{J_1,\dots J_l})(\mathbf{c}, \mathbf{d})$, resp. $\vec{T}(X_{J_1,\dots J_l})(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$ are either empty or contractible.*

Proof. For trace spaces with fixed end points as in (1), this was proved in Raussen [25, Lemma 2.10, Proposition 2.8(2)]. For trace spaces in which the end point may vary on the upper boundary of a hyperrectangle as in (2), the proofs given in [25] can be modified easily. The only new input to be used is the fact that the upper boundary of a hyperrectangle is closed under the least upper boundary operation $\vee$. □

2.2. Index categories, matrix representations and homotopy equivalences.

2.2.1. *A matrix representation of a power poset.* The index multisets $(J_1, \cdots, J_l)$ with $J_i \subseteq [1:n]$ considered in the previous Section 2.1.2 may be viewed as elements of $(\mathcal{P}([1:n]))^l \cong \mathcal{P}([1:l] \times [1:n])$. Elements of the latter power set can be encoded by their characteristic functions which can be viewed as *binary $l \times n$-matrices*:

Let $M_{l,n} = M_{l,n}(\mathbf{Z}/2)$ denote the set of all binary $l \times n$-matrices – with 2^{ln} elements. The total order on $\mathbf{Z}/2$ given by $a \leq b$ unless ($a = 1$ and $b = 0$) extends to a componentwise given *partial order* $\leq$ on $M_{l,n}$. With this partial order defining the morphisms, $M_{l,n}$ will be viewed as a *poset category*.

There is a natural order-preserving bijection between the subsets of $[1:l] \times [1:n]$ (elements of the power set $\mathcal{P}([1:l] \times [1:n])$ with partial order given by inclusion) and elements in $M_{l,n}$ given by

$$J = (J_1, \dots, J_l) \mapsto M^J = (m^J_{ij}),\ m^J_{ij} = 1 \Leftrightarrow j \in J_i \tag{2.3}$$

with inverse $M = (M_{ij}) \mapsto J^M, j \in J^M_i \Leftrightarrow m_{ij} = 1$. Under this bijection, the relevant multisets $J = (J_1, \dots, J_l)$ with $J_i \neq \emptyset, 1 \leq i \leq l$, correspond to matrices in the subset

$M^R_{l,n} \subset M_{l,n}$ consisting of the $(2^n-1)^l$ matrices such that *no row vector is a zero vector*. We view $M^R_{l,n}$ as the *full subposet category* within $M_{l,n}$.

To ease notation, we will in the following write X_M instead of X_{J^M}. The relevant index category to consider here is the full subposet category $\mathcal{C}(X)(\mathbf{c},\mathbf{d}) \subset M^R_{l,n} \subset M_{l,n}$ consisting of all binary matrices M such that

$$\vec{T}(X_M)(\mathbf{c},\mathbf{d}) \text{ is non-empty.} \tag{2.4}$$

Likewise, $\mathcal{C}(X)(\mathbf{c},\partial_+\!\downarrow\!\mathbf{d}) \subset M^R_{l,n} \subset M_{l,n}$ consists of all binary matrices M such that

$$\vec{T}(X_M)(\mathbf{c},\partial_+\!\downarrow\!\mathbf{d}) \text{ is non-empty.} \tag{2.5}$$

Departing from the index category $\mathcal{C}(X)(\mathbf{c},\mathbf{d})$ we construct a *prodsimplicial complex* (in the terminology of Kozlov [18]) $\mathbf{T}(X)(\mathbf{c},\mathbf{d})$ as follows: To every matrix $M \in \mathcal{C}(X)(\mathbf{c},\mathbf{d})$ we associate the product of simplices $\Delta(M) = \prod_{i=1}^{l} \Delta_i(M) \subset (\Delta^{n-1})^l \subset \mathbf{R}^{nl}$ with

$$\Delta_i(M) := \{(t_1,\dots,t_n) \mid 0 \le t_j \le 1, \sum_{j=1}^{n} t_j = 1, m_{ij} = 0 \Rightarrow t_j = 0\} \subset \Delta^{n-1}.$$

Remark that $M \le N \Rightarrow \Delta(M) \subset \Delta(N)$.
Using the category $\mathcal{C}(X)(\mathbf{c},\mathbf{d})$ as a pasting scheme we define the colimits

$$\mathbf{T}(X)(\mathbf{c},\mathbf{d}) := \bigcup_{M \in \mathcal{C}(X)(\mathbf{c},\mathbf{d})} \Delta(M), \tag{2.6}$$

and in a completely analogous way

$$\mathbf{T}(X)(\mathbf{c},\partial_+\!\downarrow\!\mathbf{d}) := \bigcup_{M \in \mathcal{C}(X)(\mathbf{c},\partial_+\downarrow\mathbf{d})} \Delta(M). \tag{2.7}$$

2.2.2. *Homotopy equivalences between trace spaces and finite prodsimplicial complexes.* For the following result, we need a technical, but natural and generic assumption about the placement of the hyperrectangles R^i making up the forbidden region F: For every $1 \le j \le n$, no upper boundary coordinate b^i_j is equal to a lower boundary coordinate a^k_j. Under this assumption we get a homotopy equivalence between the infinite dimensional trace space and a finite prodsimplicial model:

Theorem 2.3. (1) *Trace space $\vec{T}(X)(\mathbf{c},\mathbf{d})$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{c},\mathbf{d}) \subset (\partial\Delta^{n-1})^l$ and to the nerve $\Delta(\mathcal{C}(X)(\mathbf{c},\mathbf{d}))$ of the category $\mathcal{C}(X)(\mathbf{c},\mathbf{d})$.*

(2) *Trace space $\vec{T}(X)(\mathbf{c},\partial_+\downarrow\mathbf{d})$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{c},\partial_+\downarrow\mathbf{d}) \subset (\partial\Delta^{n-1})^l$ and to the nerve $\Delta(\mathcal{C}(X)(\mathbf{c},\partial_+\downarrow\mathbf{d}))$ of the category $\mathcal{C}(X)(\mathbf{c},\partial_+\!\downarrow\!\mathbf{d})$.*

Proof. The proof is completely analogous to that of Raussen [25, Theorem 3.5] – for $\mathbf{c} = \mathbf{0}$ and $\mathbf{d} = \mathbf{1}$ – comparing the colimits $\vec{T}(X)$ and $\mathbf{T}(X)$ with the respective homotopy colimits which are homotopy equivalent to the nerve of the category. □

Remark 2.4. If a matrix $M \in M_{l,n}$ contains a row $\mathbf{m}_i = \mathbf{1} = (1, \cdots, 1), 1 \leq i \leq l$, corresponding to $J_i = [1:n]$, then it is easy to see from Definition 2.1 that $\vec{T}(X_M)(\mathbf{c}, \mathbf{d})$ is empty. This is why trace space can be embedded in $(\partial\Delta^{n-1})^l \subset (\Delta^{n-1})^l$. This observation will be exploited in Section 6 in which we view $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ as a subset of the space of matrices $\tilde{M}^R_{l,n}$ for which every row vector contains at least one 0 and at least one 1.

3. State spaces with forbidden region intersecting the boundary

3.1. **Introduction.** In Raussen [25], we had assumed that all obstruction hyperrectangles R^i are contained in the *interior* of I^n and obtained a method to enumerate the index category $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$. It is the purpose of Section 3 to modify that method and to obtain descriptions of the index category $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ in the general case and, by more or less the same method, to generalize to the categories derived in Theorem 2.3 and described below. We shall use the notation for subspaces $X_M \subseteq X$ introduced in Section 2.2.1, we will describe the index categories (with morphisms given by the partial order)

- $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ with set of objects $\{M \in M^R_{l,n} | \ \vec{T}(X_M)(\mathbf{c}, \mathbf{d}) \neq \emptyset\}$ corresponding to trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$ with traces starting at $\mathbf{c}$ and ending at $\mathbf{d}$; and
- $\mathcal{C}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$ with set of objects $\{M \in M^R_{l,n} | \ \vec{T}(X_M)(\mathbf{c}, \partial_+ \downarrow \mathbf{d}) \neq \emptyset\}$ corresponding to traces ending on the *upper boundary* of the box with corners at $\mathbf{c}$ and $\mathbf{d}$ within $X \subset I^n$.

Remark 3.1. (1) PV-models for Higher Dimensional Automata have often obstructions intersecting the boundary ∂I^n; those arrise as soon as semaphores of an arity $r < n-1$ (at most r processors can proceed concurrently) are involved. This is for example the case for the cubical model describing the dining philosophers problem, cf Dijkstra [6] and Figure 1 below:

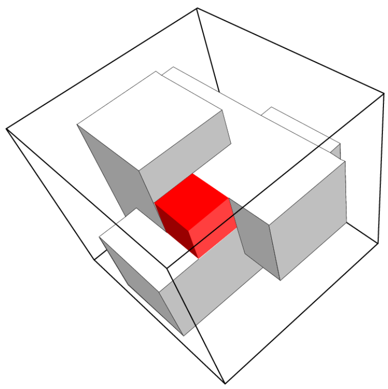

FIGURE 1. Cubical complex describing the 3-philosophers problem within the 3-cube I^3 with forbidden region F intersecting its boundary ∂I^3; the small red interior cube represents the unsafe region and is not a part of the model. Figure courtesy to A. Lang [19].

(2) The trace space $\vec{T}(X)(\mathbf{0}, \partial_+ \downarrow \mathbf{1})$ is interesting in the analysis of algorithms for wait-free protocols (cf. e.g., [16]) in which all processors with at least one exception are allowed to "die", ie cease to communicate. In this case, the accepting states correspond to the points contained in $\partial_+ \downarrow \mathbf{1}$.

As seen in Section 2.2.1, the matrix poset categories $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ and $\mathcal{C}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$ serve as pasting schemes that give rise to prodsimplicial complexes $\mathbf{T}(X)(\mathbf{c}, \mathbf{d})$ – cf (2.6); and $\mathbf{T}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$ – cf (2.7). Under the general conditions of [25, Theorem 3.5], but allowing obstruction hyperrectangles to intersect the boundary of $[\mathbf{c}, \mathbf{d}]$, we obtain using [25, Proposition 2.8] as an analogue to that theorem with essentially the same proof:

Theorem 3.2. (1) *Trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{c}, \mathbf{d})$ and to the nerve of the category $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$.*
(2) *Trace space $\vec{T}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$ and to the nerve of the category $\mathcal{C}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$.*

□

For an algorithmic determination of these index categories (as in [25, Section 4]), we need to describe several modifications of the matrix subsets $D(X)(-,-)$ (cf Section 3.2 below) and $\mathcal{C}(X)(-,-)$ with respective boundaries.

3.2. **Which trace spaces are (non-)empty?**

3.2.1. *Extended obstructions.* As in [25, Section 4.2], we define the map $\Psi : M_{l,n} \to \mathbf{Z}/2$ by $\Psi(M) = 1 \Leftrightarrow \vec{T}(X_M)(-,-) = \emptyset$ (with relevant boundaries). We wish to determine the matrices $M \in \mathcal{C}(X)(-,-) \subset M^R_{l,n} \subset M_{l,n}$ given by $\Psi(M) = 0$.

Again, one first determines "generating" matrices with $\Psi(M) = 1$ arising from a deadlock condition; it is here that several modifications become necessary as compared to [25, Section 4]. First of all, we may have fewer matrices to consider:

In both cases, hyperrectangles $R^i \subseteq [\mathbf{0}, \mathbf{1}]$ that do not intersect the box $[\mathbf{c}, \mathbf{d}]$ between $\mathbf{c}$ and $\mathbf{d}$ become irrelevant. This can be handled by reducing the number of rows in the matrices representing the index categories: We separate
$[1 : l] = [1 : l]^{in} \sqcup [1 : l]^{out}$ with $i \in [1 : l]^{in} \Leftrightarrow (1 \le j \le n \Rightarrow a^i_j < d_j, c_j < b^i_j)$ and let $l' := |[1 : l]^{in}|$.

Remark 3.3. Comparing trace spaces with varying end points, it may be necessary to take account of these irrelevant rectangles nevertheless. On the prodsimplicial side this will result in taking a product with one or several simplices Δ^{n-1}; cf. Section 4.

Lemma 3.4. *Suppose $a^i_j \le c_j$. Then $M \notin \mathcal{C}(X)(\mathbf{c}, -)$ for every matrix $M \in M^R_{l',n}$ with $m_{ij} = 1$.*

Proof. Suppose $\mathbf{x} \in X_M$ with $m_{ij} = 1$. Then $\mathbf{x} \le \mathbf{b}^i$ implies $x_j < a^i_j \le c_j$, i.e., $\mathbf{x} \notin [\mathbf{c}, \mathbf{1}]$; in particular, $\vec{T}(X_M)(\mathbf{c}, -) = \emptyset$. □

Under these circumstances, we will thus only have to investigate matrices

$$M \in M^R_{l',n}(\mathbf{c}) \text{ with the additional property: } a^i_j \le c_j \Rightarrow m_{ij} = 0. \tag{3.1}$$

As in [25, Section 4], we will deal with *extensions* $R^i_j, 0 \le i \le l', 1 \le j \le n$, of the (relevant) obstruction hyperrectangles; these are given as

$$R^i_j = \prod_{k=1}^{j-1} [0, b^i_k[\times]a^i_j, b^i_j[\times \prod_{k=j+1}^{n} [0, b^i_k[, \; 1 \le i \le l, 1 \le j \le n, \text{ and} \tag{3.2}$$

$$R^0_j = [0,1]^{j-1} \times [d_j, 1] \times [0,1]^{n-j}, \; i = 0, 1 \le j \le n. \tag{3.3}$$

For an illustration of the extensions from (3.2), we refer to Raussen [25, Figure 4]. The hyperrectangles from (3.3) are new compared to [25]; they intersect the box $[\mathbf{c}; \mathbf{d}]$ only on an upper boundary facet and may generate deadlocks on that facet.

3.2.2. *Combinatorial descriptions of index categories.* We will deal with the easier case of the index category $\mathcal{C}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$ first. The result [25, Proposition 4.3] has the following immediate modification:

Proposition 3.5. *For $M \in M \in M^R_{l',n}(\mathbf{c})$, the following are equivalent:*

1. *M is* not *an object in $\mathcal{C}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$.*
2. $\vec{T}(X_M)(\mathbf{c}, \partial_+(\downarrow \mathbf{d})) = \emptyset$.
3. *There is a map $i : [1 : n] \to [1 : l']$ such that $m_{i(j),j} = 1$ and $\bigcap_{1 \le j \le n} R^{i(j)}_j \neq \emptyset$.*
4. *There is a map $i : [1 : n] \to [1 : l']$ with $a^{i(j)}_j < b^{i(k)}_j$ for all $j, k \in [1 : n]$.*

Proof. The proof is an easy modification of the one given for [25, Proposition 4.3]: trace space is empty if every trace is bound to end in a deadlock arising from an n-tuple of extended hyperrectangles. Note that a deadlock on the boundary $\partial_+(\downarrow \mathbf{d})$ is irrelevant for paths/traces in $\vec{T}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$. □

For the analysis of $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ in general, we have to deal with obstruction hyperrectangles *intersecting* $\partial_+ \downarrow \mathbf{d}$ as well; this may also happen in the case $\mathbf{d} = \mathbf{1}$ for hyperrectangles R^i intersecting the boundary of I^n. For every such "intersection direction" $1 \le j \le n$ with a hyperrectangle intersecting the j-th facet $x_j = d_j$ of the upper boundary of $[\mathbf{0}; \mathbf{d}]$, we apply the new obstruction hyperrectangles R^0_j introduced above.

The result from [25, Proposition 4.3] can then be modified as follows:

Proposition 3.6. *For $M \in M^R_{l',n}(\mathbf{c})$, the following are equivalent:*

1. *M is* not *an object in $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$.*
2. $\vec{T}(X_M)(\mathbf{c}, \mathbf{d}) = \emptyset$.
3. *There is a map $i : [1 : n] \to [0 : l']$ such that $i(j) \neq 0 \Rightarrow m_{i(j),j} = 1$ and such that $\bigcap_{1 \le j \le n} R^{i(j)}_j \neq \emptyset$.*

(4) *There is a map* $i : [1:n] \to [0:l']$ *such that* $i(j) \neq 0 \Rightarrow m_{i(j),j} = 1$ *and such that*
$$\begin{cases} a_j^{i(j)} < b_j^{i(k)} \text{ for } j,k \in [1:n], i(j) > 0; \text{ or } i(j) = 0, a_j^0 = d_j < 1; \\ b_j^{i(k)} = 1 \text{ for } j,k \in [1:n], i(j) = 0, a_j^0 = d_j = 1. \end{cases}$$

Compared to the result [25, Proposition 4.3], remark that further intersections involving hyperrectangles R_j^0 – but only those corresponding to intersection directions – need to be considered.

3.3. **Modified Algorithms.**

3.3.1. *D versus $\mathcal{C}$.* The matrix sets and categories from Raussen [25, Section 4.2.1] need a few modifications: First of all, in both cases, only obstructions intersecting $[\mathbf{c}, \mathbf{d}]$ need to be taken care of, and this may reduce the number of rows from l to l' in the matrices to be considered. For the category $\mathcal{C}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$, one may then proceed as in Raussen [25, Section 4.2.1] – with the simplification that only matrices in $M_{l',n}^R(\mathbf{c})$ – cf (3.1) – need to be considered.

As in [25, Section 4.2], we determine the index category $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ in two steps: First, we calculate the restriction of the map Ψ to the subset $M_{l',n}^C(\mathbf{c})$ consisting of matrices $M(i)$ corresponding to maps $i : [1:n] \to [0, l']$ – cf Section 3.3.2 below – as in Proposition 3.6. As in Raussen [25, Proposition 4.5], one obtains $\Psi(M)$ (cf Section 3.2) for a matrix $M \in M_{l',n}^R(\mathbf{c})$:

Proposition 3.7. *A matrix* $M \in M_{l',n}^R(\mathbf{c})$ *satisfies* $\Psi(M) = 1$ *if and only if there exists a matrix* $N \in M_{l',n}^C(\mathbf{c})$ *with* $\Psi(N) = 1$ *and* $N \leq M$.

In particular, we determine the set of matrices (D for "dead")

$$D(X)(-,-) := \{M \in M_{l',n}^C(\mathbf{c}) |\ \Psi(M) = 1\}. \tag{3.4}$$

Using this set $D(X)(-,-)$ – upward closed under $\leq$ – we will then apply 3.7 to determine the set of matrices

$$\mathcal{C}(X)(-,-) := \{M \in M_{l',n}^R(\mathbf{c}) |\ \Psi(M) = 0\}. \tag{3.5}$$

describing the objects of the relevant index category; the latter is downward closed under $\leq$.

3.3.2. *Determination of $D(X)(\mathbf{c}, \mathbf{d})$.* For a category of type $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$, we replace the matrix set $M_{l,n}^C$ by the set $M_{l',n}^C(\mathbf{c})$ consisting of matrices M has the following properties:

- $a_j^i < c_j \Rightarrow m_{ij} = 0$;
- every column vector $\mathbf{m}_j$ is either a unit vector or the zero vector $\mathbf{0}$;
- if $\mathbf{m}_j = \mathbf{0}$, then j is an intersection direction.

A matrix $M \in M^C_{l',n}(\mathbf{c})$ codes the map

$$i_M : [1:n] \to [0:l'], i_M(j) = \begin{cases} i(j) & \mathbf{m}_j = \mathbf{e}_{i(j)} \\ 0 & \mathbf{m}_j = \mathbf{0} \end{cases}.$$

Vice versa, a (relevant) map $i : [1 : n] \to [0 : l']$ comes with a characteristic matrix $M(i) = (m_{ij}) \in M^C_{l',n}(\mathbf{c}), m_{ij} = 1 \Leftrightarrow i(j) = i \neq 0$.

In order to determine $D(X)(\mathbf{c}, \mathbf{d})$ – cf (3.4), one has to consider both the row set $R(M) \subset [1 : l']$ (cf. [25, Section 4.2.2]) and moreover the *column set* $C(M) \subset [1 : n]$ indexing the *non-zero* rows, resp. columns of M.

The method described in Raussen [25, Lemma 4.6/4.7] has to be extended as follows: for a given non-empty (row) subset $B \subset [1 : l']$, one determines first the bound $\mathbf{b}^B = [b^B_1, \dots, b^B_n]$, $b_j = \min_{i \in B} b^i_j$; the greatest lower bound of the maxima of the hyperrectangles indexed by B. Given $\mathbf{b}^B$, we can determine a *maximal* column set $C(B) := \{j \in [1 : n] | \ i(j) = 0 \Rightarrow d_j < b^{r_j}_j \text{ or } d_j = b^{r_j}_j = 1\} \subset [1 : n]$; this requires checking n (in)equalities.

Next, for every subset of $C \subseteq C(B)$ the sets $R_j(B; C) := \{i \in B, j \notin C \Rightarrow a^i_j < b^B_j\}$ have to be determined – decrementally – as in [25, Lemma 4.7]. As in [25, Lemma 4.6], we end up determining the set of matrices $M \in D(X)(\mathbf{c}, \mathbf{d}) := \{M \in^C_{l',n} (\mathbf{c}) | \ \Psi(M) = 1\}$ as follows:

Lemma 3.8. *A map $i : [1 : n] \to [0 : l']$ gives rise to a matrix $M = M(i) \in D(X)(\mathbf{c}, \mathbf{d})$ if and only if*

(1) $i(j) = 0 \Rightarrow j \in C(i([1 : n])$ *and*
(2) $i(j) > 0 \Rightarrow i(j) \in R_j(i([1 : n]); i^{-1}(0))$.

Having found $D(X)(\mathbf{c}, \mathbf{d})$, we can determine the matrices in $\mathcal{C}(X)(\mathbf{c}, \mathbf{d}) := \{M \in M^R_{l',n}(\mathbf{c}) | \ \Psi(M) = 0\}$ in the same way as described in Raussen [25, Proposition 4.8]; again, only matrices in $M^R_{l',n}(\mathbf{c})$ need to be checked. Alternatively, one can determine the complement of $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ as described in Section 6.

4. The trace category: Varying end points

4.1. **Induced maps.** By concatenation, traces $\sigma \in \vec{T}(X)(\mathbf{d}, \mathbf{d}'), \tau \in \vec{T}(X)(\mathbf{c}', \mathbf{c})$ induce continuous maps $\sigma_\sharp : \vec{T}(X)(\mathbf{c}, \mathbf{d}) \to \vec{T}(X)(\mathbf{c}, \mathbf{d}')$ and $\tau^\sharp : \vec{T}(X)(\mathbf{c}, \mathbf{d}) \to \vec{T}(X)(\mathbf{c}', \mathbf{d})$.

In order to find out "what happens" between $\mathbf{d}$ and $\mathbf{d}'$, one has to study the effect of these induced maps; it suffices to look at d-homotopy classes $[\sigma] \in \vec{\pi}_1(X)(\mathbf{d}, \mathbf{d}')$ of paths/traces $\sigma \in \vec{T}(X)(\mathbf{d}, \mathbf{d}')$, cf Raussen [22] and the discussion of the functor $\vec{T}^X : \vec{D}(X) \to Ho - Top$ from the double category $\vec{D}(X)$ associated to X there. Similarly, one may analyse what happens between $\partial_+ \downarrow \mathbf{d}$ and $\partial_+ \downarrow \mathbf{d}'$; moreover, there are similar contravariant versions "between" $\mathbf{c}'$ and $\mathbf{c}$. For a discussion/determination of so-called components in X ([8, 12, 22]), one would like to know which traces σ induce homotopy equivalences (or at least bijections on sets of path components).

For brevity, we restrict to the first case, the concatenation map $\sigma_\sharp : \vec{T}(X)(\mathbf{c},\mathbf{d}) \to \vec{T}(X)(\mathbf{c},\mathbf{d}')$. Assume that l of the hyperrectangles intersect $[\mathbf{c},\mathbf{d}]$ whereas $l' \geq l$ intersect $[\mathbf{c},\mathbf{d}']$. In order to compare, we will use the *same* larger index set $[1 : l']$ in both cases. A hyperrectangle R^i not intersecting $[\mathbf{c},\mathbf{d}]$ does not pose any conditions to the question $\vec{T}(X_M)(\mathbf{c},\mathbf{d}) \neq \emptyset$ defining index categories; the corresponding i-th row in the matrix M is irrelevant. As a result, the index category $\mathcal{C}(X)(\mathbf{c},\mathbf{d}) \subset M^R_{l,n}$ will be replaced by the pullback

$$\begin{array}{ccc} \mathcal{C}(X)(\mathbf{c},\mathbf{d}) \times M^R_{l'-l,n} & \cong & \begin{array}{ccc} \tilde{\mathcal{C}}(X)(\mathbf{c},\mathbf{d}) & \xrightarrow{\pi} & \mathcal{C}(X)(\mathbf{c},\mathbf{d}) \\ \downarrow{\subseteq} & & \downarrow{\subseteq} \\ M^R_{l',n} & \xrightarrow{\pi} & M^R_{l,n} \end{array} \\ M^R_{l,n} \times M^R_{l'-l,n} & \cong & \end{array}$$

with $\pi : M^R_{l',n} \to M^R_{l,n}$ leaving out superfluous rows.

The pasting scheme corresponding to $\tilde{\mathcal{C}}(X)(\mathbf{c},\mathbf{d})$ gives rise to the prodsimplicial complex $\tilde{\mathbf{T}}(X)(\mathbf{c},\mathbf{d}) = \mathbf{T}(X)(\mathbf{c},\mathbf{d}) \times (\Delta^{n-1})^{l'-l}$ homotopy equivalent to $\mathbf{T}(X)(\mathbf{c},\mathbf{d})$.

The index category $\mathcal{C}(X)(\mathbf{c},\mathbf{d}')$ becomes then a *sub*category of $\tilde{\mathcal{C}}(X)(\mathbf{c},\mathbf{d})$ with certain matrices eliminated; one needs to analyse the effect of the associated *inclusion* of prodsimplicial complexes $\mathbf{T}(X)(\mathbf{c},\mathbf{d}') \hookrightarrow \tilde{\mathbf{T}}(X)(\mathbf{c},\mathbf{d})$.

It is also relevant to ask what happens if one digs an additional forbidden hyperrectangle R out of the state space $X \subset I^n$ to get $X' = X \setminus R$; this is interesting in particular for an inductive determination of index categories and associated prodsimplicial models of trace spaces. Again, the associated map between prodsimplicial models is a combination of a homotopy equivalence (taking the product with a simplex) and an inclusion map reflecting the additional obstruction. The effect of this map (and the maps induced by it on homology etc) have still to be investigated more closely.

4.2. **Homotopy equivalences.** In the following, we give sufficient conditions making sure that the induced maps become homotopy equivalences. This should be useful for the investigation of component categories, cf Fajstrup, Goubault, Haucourt and Raussen [8, 12]:

Proposition 4.1. (1) *Assume that* $\vec{T}(X_M)(\mathbf{c},\mathbf{d}) = \emptyset$ *implies* $\vec{T}(X_M)(\mathbf{c},\mathbf{d}') = \emptyset$ *for all matrices* $M \in M^R_{l,n}$. *Then every trace* $\sigma \in \vec{T}(X)(\mathbf{d},\mathbf{d}')$ *induces a homotopy equivalence* $\sigma_\sharp : \vec{T}(X)(\mathbf{c},\mathbf{d}) \to \vec{T}(X)(\mathbf{c},\mathbf{d}')$.

(2) *Assume that* $\vec{T}(X_M)(\mathbf{c}',\mathbf{d}) = \emptyset \Rightarrow \vec{T}(X_M)(\mathbf{c},\mathbf{d}) = \emptyset$ *for all* $M \in M^R_{l,n}$. *Then every trace* $\tau \in \vec{T}(X)(\mathbf{c},\mathbf{c}')$ *induces a homotopy equivalence* $\tau^\sharp : \vec{T}(X)(\mathbf{c}',\mathbf{d}) \to \vec{T}(X)(\mathbf{c},\mathbf{d})$.

Proof. The maps $\sigma_\sharp$, resp. $\tau^\sharp$ induce always inclusions of the subcategories $\mathcal{C}(X)(\mathbf{c},\mathbf{d}) \hookrightarrow \mathcal{C}(X)(\mathbf{c},\mathbf{d}')$, resp. $\mathcal{C}(X)(\mathbf{c}',\mathbf{d}) \hookrightarrow \mathcal{C}(X)(\mathbf{c},\mathbf{d})$ within $M^R_{l,n}$. The conditions in Proposition

4.1 ensure that these are in fact equalities. As a consequence, the prodsimplicial models agree: $\mathbf{T}(X)(\mathbf{c},\mathbf{d}) = \mathbf{T}(X)(\mathbf{c},\mathbf{d}')$, resp. $\mathbf{T}(X)(\mathbf{c}',\mathbf{d}) = \mathbf{T}(X)(\mathbf{c},\mathbf{d})$. The results follow from Theorem 3.2. $\square$

To check the conditions, one may apply the method from Section 3.3.2 to see whether $D(X)(-,-)$ does (not) change under variation of end points.

A different induced map arises, under certain conditions, from taking the least upper bound with some element $\mathbf{e} \in X$: Suppose that

$$\mathbf{e} \vee \mathbf{y} \in X \text{ for all } \mathbf{y} \text{ satisfying } \vec{T}(X)(\mathbf{c},\mathbf{y}) \neq \emptyset \neq \vec{T}(X)(\mathbf{y},\mathbf{d}); \tag{4.1}$$

ie $\mathbf{y}$ is on a trace connecting $\mathbf{c}$ and $\mathbf{d}$. Then there is an induced map

$$\mathbf{e}\vee : \vec{P}(X)(\mathbf{c},\mathbf{d}) \to \vec{P}(X)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d}),\ (\mathbf{e}\vee p)(t) = \mathbf{e}\vee p(t) \tag{4.2}$$

inducing the map $\mathbf{e}\vee : \vec{T}(X)(\mathbf{c},\mathbf{d}) \to \vec{T}(X)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d})$.

Proposition 4.2. *Assume that* $\mathbf{e} \in \bigcap_{M\in\mathcal{C}(X)(\mathbf{c},\mathbf{d})} X_M$ *and that* $\vec{T}(X_M)(\mathbf{c},\mathbf{d}) = \emptyset \Rightarrow \vec{T}(X_M)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d}) = \emptyset$ *for all* $M \in M^R_{l,n}$.
Then the map $\mathbf{e}\vee : \vec{T}(X)(\mathbf{c},\mathbf{d}) \to \vec{T}(X)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d})$ *is a homotopy equivalence.*

The proof is analogous to that of Proposition 4.1; remark that $\mathbf{e}\vee$ has restrictions on the respective trace spaces in $X_M, M \in \mathcal{C}(X)(\mathbf{c},\mathbf{d})$ since these spaces X_M are closed under $\vee$, cf Raussen [25, Lemma 2.6]. Again, the methods from Section 3.3.2 may be applied to verify the conditions.

Corollary 4.3. *Assume that*

1. $\mathbf{c} \leq \mathbf{e} \in \bigcap_{M\in\mathcal{C}(X)(\mathbf{c},\mathbf{d})} X_M$;
2. $\vec{T}(X_M)(\mathbf{c},\mathbf{e}) \neq \emptyset$ *for* $M \in \mathcal{C}(X)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d})$;
3. $\vec{T}(X_M)(\mathbf{c},\mathbf{d}) = \emptyset \Rightarrow \vec{T}(X_M)(\mathbf{c},\mathbf{e}\vee\mathbf{d}) = \emptyset$ *for all* $M \in M^R_{l,n}$.

Then the map $\mathbf{e}\vee : \vec{T}(X)(\mathbf{c},\mathbf{d}) \to \vec{T}(X)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d})$ *is a homotopy equivalence.*

Proof. For $M \in \mathcal{C}(X)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d})$ choose $\sigma \in \vec{T}(X_M)(\mathbf{c},\mathbf{e})$. Concatenation $\sigma_\sharp$ with σ shows: $\vec{T}(X_M)(\mathbf{e}\vee\mathbf{c},\mathbf{e}\vee\mathbf{d}) \neq \emptyset \Rightarrow \vec{T}(X_M)(\mathbf{c},\mathbf{e}\vee\mathbf{d}) \neq \emptyset$. Condition (3) above implies that $\vec{T}(X_M)(\mathbf{c},\mathbf{d}) \neq \emptyset$, as well. Apply Proposition 4.2. $\square$

Discussing components following [8, 12, 22], one would typically apply Proposition 4.2 to investigate all $\mathbf{c} \leq \mathbf{e}$ for which the map $\mathbf{e}\vee$ from (4.2) is a homotopy equivalence; and then establish a maximal region such that all such maps $\mathbf{e}\vee$ within it yield homotopy equivalences. Details have still to be worked out.

Remark 4.4. Connections to topological complexity as discussed by Farber, cf the book [10] and its list of references, should also be interesting to investigate. Taking account of directions makes matters more complicated since the end point map $ev_{01} : \vec{P}(X) \to X \times X$ is no longer a fibration; it is not even surjective. Nevertheless, one may ask for coverings of $\{(\mathbf{x},\mathbf{y}) \in X \times X |\ \vec{T}(X)(\mathbf{x},\mathbf{y}) \neq \emptyset\}$ by subsets on which there is a continuous section of the restricted map ev_{01}. It is not difficult to produce such a section on sets of type $X_{j_1,\dots,j_l} \times X_{j_1,\dots,j_l} \subset X \times X$ – in the notation of Section 2.1.2.

5. A PARTICULAR CASE: SEMAPHORES OF ARITY ONE

5.1. **Trace spaces are homotopy discrete.** Matters become more specific and combinatorial in nature for a semaphore or PV-model (cf Section 2.1.1) in which *every* semaphore allows only a *single* process to proceed. In this case, the forbidden region F is the union of hyperrectangles of a particular type: Whenever two processes $1 \le j_1 < j_2 \le n$ call the same semaphore h on intervals $I_{j_1} =]a_{j_1}^{h,m_1}, b_{j_1}^{h,m_1}[$ and $I_{j_2} =]a_{j_2}^{h,m_2}, b_{j_2}^{h,m_2}[$, the hyperrectangle $R_h(j_1, m_1; j_2, m_2) := I_{j_1} \times I_{j_2} \times I^{n-2}$ – with I_{j_i} inserted as factor j_i – is added to the forbidden region; remark that all but two subintervals correspond to the full interval I.

Let us assume that semaphore $h,\ 1 \le h \le k$, is called upon by the processes labelled by the subset $J_h \subseteq [1:n]$. For every $j \in J_h$, there is a number r_{hj} of calls Ph by process j, on an interval $]a_j^{h,m}, b_j^{h,m}[,\ 1 \le m \le r_{hj}$. A hyperrectangle $R_h(j_1, m_1; j_2, m_2)$ arises for every semaphore h, every pair $j_1 < j_2, j_i \in J_h$ and every pair of calls corresponding to the two processes j_1, j_2. The total number l of forbidden hyperrectangles is thus

$$l = \sum_{h=1}^{k} \sum_{j_1 < j_2 \in J_h} r_{hj_1} r_{hj_2}. \tag{5.1}$$

Example 5.1. For $k \ge 2$ dining philosophers, cf Dijkstra [6] and Figure 1 in this paper, every semaphore (= fork) is called upon once by exactly two processes (philosophers). Hence $l = k$, and F is the union of k hyperrectangles R^i.

In the following, we will stick to endpoints $\mathbf{c} = \mathbf{0}, \mathbf{d} = \mathbf{1}$ – both for simplicity, and because this is the most interesting case. Apart from the sets of binary matrices introduced in Section 2, we also need the set $M_{l,n}^1 \subset M_{l,n}^R$ consisting of matrices in which *every row* vector is a *unit* vector. Remark that every contributing hyperrectangles $R^i, 1 \le i \le l$ is of type $R_h(j_1(i), k_1; j_2(i), k_2)$ described above.

Proposition 5.2. *Let* $M \in M_{l,n}^R$.

(1) *If* $\vec{T}(X_M)(\mathbf{0}, \mathbf{1})$ *is non-empty, then* $M \in M_{l,n}^1$.
(2) $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ *is homotopy equivalent to a* finite discrete *space; its (contractible) connected components are the non-empty ones among the spaces* $\vec{T}(X_M)(\mathbf{0}, \mathbf{1}), M \in M_{l,n}^1$.

Proof. (1) Suppose $M \notin M_{l,n}^1$, ie $m_{i,j_1} = m_{i,j_2} = 1$ for some $1 \le i \le l; 1 \le j_1 < j_2 \le n$. If, say, $j_1 \notin \{j_1(i), j_2(i)\}$, then $a_{j_1}^i = 0$ and hence trace space is empty by Raussen [25, Proposition 4.3(1)] or by Lemma 3.4 from this paper.
If $\{j_1, j_2\} = \{j_1(i), j_2(i)\}$, we define a map $i : [1:n] \to [0:l]$ with $i(j_1) = i(j_2) = i$ and $i(j) = 0$ for all other j. We check that condition (4) in Proposition 3.6 is

satisfied: $a^i_{j_1} < b^i_{j_1}, a^i_{j_2} < b^i_{j_2}$, and $b^i_j = 1$ for $j_1 \neq j \neq j_2$; hence trace space is empty also in this case.

(2) It follows from (1), that the subposet category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ has no non-trivial morphisms, and hence that the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ has dimension zero.

□

It remains thus to determine for *which* $M \in M^1_{l,n}$ the spaces $\vec{T}(X_M)(\mathbf{0},\mathbf{1})$ are *non-empty*. Remark that there are 2^l such spaces with l as in (5.1) – for every i corresponding to a pair of calls, one may choose either $j_1(i)$ or $j_2(i)$.

5.2. **A single call to a semaphore of arity one.** Let us first consider just a *single* concurrent call to a semaphore of arity one. Without restriction of generality, we assume that all n processes call to it. If only $m < n$ processes call the semaphore, then the forbidden region has type $F = F_m \times I^{n-m}$. Hence the state space is $X = X_m \times I^{n-m}$ with $X_m = I^m \setminus F^m$, and $\vec{T}(X)(\mathbf{0},\mathbf{1}) \simeq \vec{T}(X_m)(\mathbf{0},\mathbf{1})$.

Assume that the semaphore calls are given by intervals $]a_j, b_j[\subset [0,1], 1 \le j \le n$. The associated forbidden region is the union $F = \bigcup_{1 \le j_1 < j_2 \le n} R(j_1, j_2)$ of the $\binom{n}{2}$ hyperrectangles $R(j_1, j_2) = \{\mathbf{x} \in I^n \mid a_{j_i} < x_{j_i} < b_{j_i}, i = 1, 2\}$. As usual, let $X = I^n \setminus F$.

In the proof of the next result, we will also need the extended hyperrectangles $R_{j_1}(j_1, j_2) = \{\mathbf{x} \in I^n \mid 0 \le x_{j_2} < b_{j_2}, a_{j_1} < x_{j_1} < b_{j_1}\}$ and likewise $R_{j_2}(j_1, j_2)$; moreover, as in Section 3.2, the degenerate hyperrectangles $R^0_j = [0,1]^{j-1} \times \{1\} \times [0,1]^{n-j}$, $1 \le j \le n$.

Proposition 5.3. *Trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ is homotopy equivalent to the discrete space whose underlying set is the symmetric group Σ_n.*
A homotopy equivalence $\vec{x} : \Sigma_n \to \vec{T}(X)(\mathbf{0},\mathbf{1})$ is given by $\vec{x}(\pi)(t) = (x_1(t), \dots, x_n(t))$ with

$$x_{\pi(k)}(t) = \begin{cases} 0 & t \le \frac{k-1}{n} \\ (nt - (k-1)) & \frac{k-1}{n} \le t \le \frac{k}{n} \\ 1 & \frac{k}{n} \le t \le n \end{cases}.$$

Proof. Note that every $\vec{x}(\pi)$ describes a d-path on the 1-skeleton of $\vec{I}^n$ that does not intersect the forbidden region F; these are in fact *all* d-paths on the 1-skeleton up to trace equivalence.

Let $\mathcal{P}_2(n)$ denote the set of all 2-element subsets of $[1:n]$ (with $l = \frac{n(n-1)}{2}$ elements indexing the obstruction hyperrectangles), and let $c : \mathcal{P}_2(n) \to [1:n]$ denote a *choice* function with the property $c(\{j_1, j_2\}) \in \{j_1, j_2\}$. For such a choice function c – determining in which order to pass the obstructions $R(j_1, j_2)$ – let $F_c = \bigcup_{1 \le j_1 < j_2 \le n} R_{c(j_1,j_2)}(j_1, j_2)$ and $X_c = I^n \setminus F_c$. By Theorem 3.2 and Proposition 5.2, the (contractible) components of $\vec{T}(X)(\mathbf{0},\mathbf{1})$ correspond to those choice functions c giving rise to *non-empty* trace spaces $\vec{T}(X_c)(\mathbf{0},\mathbf{1})$; more precisely, c corresponds to the matrix $M_c \in M^1_{\binom{n}{2},n}$ – every row a unit vector – with $m_{(j_1,j_2),c(j_1,j_2)} = 1$.

A choice function c gives rise to a relation on the set $[1:n]$ defined by $j_1 \leq_c j_2$ if $c(j_1, j_2) = j_1$ and its reflexive and transitive closure $\preceq_c$. If $\preceq_c$ defines a *total order* on $[1:n]$, then this total order is given by a permutation $\pi \in \Sigma_n$: $\pi(1) \preceq_c \pi(2) \preceq_c \cdots \preceq_c \pi(n)$. On the other hand, every permutation $\pi \in \Sigma_n$ orders the elements of $[1:n]$ and gives thus rise to a choice function: $c(j_1, j_2) = 1 \Leftrightarrow \pi^{-1}(j_1) < \pi^{-1}(j_2)$. We claim:
$\vec{T}(X_c)(\mathbf{0}, \mathbf{1}) \neq \emptyset$ if and only if $\preceq_c$ is a total order.

If $\preceq_c$ is *not* a total order, then there is a chain $j_1 \preceq_c \cdots \preceq_c j_k \preceq_c j_1$ with $k < n$; let $j_{k+1}, \ldots, j_n$ denote the remaining elements of $[1:n]$. The extended, resp. degenerate hyperrectangles $R_{j_1}(j_1, j_2), \ldots, R_{j_k}(j_k, j_1), R^0_{j_{k+1}}, \ldots, R^0_{j_n}$ have a non-empty intersection $\{\mathbf{x} \in I^n |\ a_{j_i} < x_{j_i} < b_{j_i}, i \leq k; x_{j_i} = 1, i > k\}$ giving rise to a deadlock $\mathbf{x} = [x_1, \ldots, x_n]$ with $x_{j_i} = a_{j_i}, i \leq k; x_{j_i} = 1, i > k$. Hence $\vec{T}(X_c)(\mathbf{0}, \mathbf{1}) = \emptyset$.

Suppose now that $\preceq_c$ *is* a total order. No non-degenerate hyperrectangle contributes with an index corresponding to the direction that is maximal under $\preceq_c$. We claim that for every choice of n – one for every direction $j \in [1:n]$ – among the extended and (at least one) degenerate hyperrectangles $R_{j_p}(j_p, j_q), p \preceq_c q$ and $R^0_{j_r}$, their intersection has to be empty: Since there is no loop with respect to $\preceq_c$, the union of all $\{j_p, j_q\}$ corresponding to extended hyperrectangles has at least one element j in common with the set $\{j_r\}$ corresponding to degenerate hyperrectangles.
An element $\mathbf{x} \in \bigcap R_{j_p}(j_p, j_q) \cap \bigcap R^0_{j_r}$ would have to satisfy both $x_j < b_j$ and $x_j = 1$. Hence, all these intersections are empty, there are no deadlocks in X_c, and $\vec{T}(X_c)(\mathbf{0}, \mathbf{1})$ is non-empty. □

To fix notation, we let to every permutation $\pi \in \Sigma_n$ correspond

- the $\binom{n}{2}$ extended hyperrectangles $R_\pi(j_1, j_2) = R_j(j_1, j_2)$ with $j = j_1$ if $\pi(j_1) < \pi(j_2)$ and $j = j_2$ else;
- the forbidden region $F_\pi = \bigcup_{1 \leq j_1 < j_2 \leq n} R_\pi(j_1, j_2)$; and
- the state space $X_\pi = I^n \setminus F_\pi$.

Proposition 5.3 can be reformulated as follows:

Corollary 5.4. *The trace space is a disjoint union* $\vec{T}(X)(\mathbf{0}, \mathbf{1}) = \bigsqcup_{\pi \in \Sigma_n} \vec{T}(X_\pi)(\mathbf{0}, \mathbf{1})$. *All components* $\vec{T}(X_\pi)(\mathbf{0}, \mathbf{1})$, $\pi \in \Sigma_n$, *are contractible.*

Intuitively, a component $\vec{T}(X_\pi)(\mathbf{0}, \mathbf{1})$, $\pi \in \Sigma_n$ consists of all interleaving d-paths that access the semaphore in the order given by the permutation π.

5.3. **Several calls to semaphores of arity one.** Let us now consider a state space corresponding to a collection of k semaphores of arity one. Every semaphore h is called upon by a subset $J_h \subseteq [1:n]$ of processes, and this inclusion induces an inclusion $\Sigma_{J_h} \subseteq \Sigma_n$ of permutation groups: Σ_{J_h} is the stabilizer of $[1:n] \setminus J_h$.

Suppose that semaphore h is locked by process $j \in J_h$ at intervals $]a_j^{m_j(h)}, b_j^{m_j(h)}[$ with $1 \leq m_j(h) \leq r_j(h)$. A concurrent call c consists of the choice of a semaphore $h = h(c)$

with $1 \leq h \leq k$ and an unordered $|J_h|$-tuple $(m_j(h))_{j\in J_h}, 1 \leq m_j(h) \leq r_j(h)$. It gives rise to a forbidden region $F(c) = F(h; m_1(h), \dots, m_{|J_h|(h)})$ determined as in Section 5.2. That forbidden region has extensions $F_\pi(c)$, one for every permutation $\pi \in \Sigma_{J_h}$.

Let C denote the set of calls, ie of tuples of the form $(h; m_1(h), \dots, m_{r_j(h)})$. The total forbidden region is given by $F = \bigcup_{c\in C} F(c) = \bigcup_{1\leq h\leq k} \bigcup_{j\in J_h; 1\leq m_j(h)\leq r_j(h)} F(h; m_1(h), \dots, m_{r_j(h)})$ and the state space is $X = I^n \setminus F$. We need to consider one permutation per call, i.e., elements $\boldsymbol{\pi} = (\pi(c))_{c\in C}$ in the product $\Sigma = \prod_{c\in C} \Sigma_{J_h}$.

Proposition 5.5. *Trace space is a disjoint union* $\vec{T}(X)(\mathbf{0},\mathbf{1}) = \bigsqcup_{\boldsymbol{\pi}\in\Sigma} \vec{T}(X_{\boldsymbol{\pi}})(\mathbf{0},\mathbf{1})$. *Each subspace* $\vec{T}(X_{\boldsymbol{\pi}})(\mathbf{0},\mathbf{1})$ *is either empty or contractible.*

Proof. According to Proposition 5.2, $\vec{T}(X)(\mathbf{0},\mathbf{1})$ is homotopy equivalent to a disjoint union of spaces of the form $\vec{T}(X_M)(\mathbf{0},\mathbf{1}), M \in M^R_{l,n}$; each of those is either empty or contractible. By Proposition 5.3 and Corollary 5.4, only matrices $M \in M^R_{l,n}$ arising from collections of *permutations* can give rise to non-empty trace spaces. $\square$

It remains to study, for *which* collections of permutations $\boldsymbol{\pi} = (\pi_c)_{c\in C} \in \Sigma = \prod_{c\in C} \Sigma_{J_{h(c)}}$ the space $\vec{T}(X_{\boldsymbol{\pi}})(\mathbf{0},\mathbf{1})$ is non-empty: Consider the set of boundary coordinates $a^i_j, b^i_j \in I, 1 \leq j \leq n, i \in \bigcup_{j\in J_h}\{m_j(h)\}$, of *all* calls to semaphores. For every collection $\boldsymbol{\pi} = (\pi_c)_{c\in C} \in \Sigma = \prod_{c\in C} \Sigma_{J_{h(c)}}$, we consider several order relations on subsets of these real numbers:

- The natural order $\leq$, inherited from the reals, on numbers a^i_j, b^i_j with the *same* subscript (direction) j;
- $b^{m_{\pi_c(j)}(h)}_{\pi_c(j)} \preceq a^{m_{\pi_c(j')}(h)}_{\pi_c(j')}$ for $c \in C, j < j' \in J_{h(c)}$
  for the *same* call $c = (h; m_1(h), \dots, m_{r_j(h)}(h)) \in C$.

We call the collection $\boldsymbol{\pi}$ *compatible* if the transitive hull $\sqsubseteq_{\boldsymbol{\pi}}$ of these relations is a *partial order*.

Proposition 5.6. *Let* $X = I^n \setminus F$ *denote the state space corresponding to a collection of* k *semaphores of arity one. Then* $\vec{T}(X)(\mathbf{0},\mathbf{1})$ *is homotopy equivalent to the discrete space* $\{\boldsymbol{\pi} = (\pi_c)_{c\in C} \in \prod_{c\in C} \Sigma_{J_{h(c)}} \mid \boldsymbol{\pi} \text{ compatible}\} \subseteq \prod_{c\in C} \Sigma_{J_{h(c)}} \subseteq (\Sigma_n)^l$.

Proof. We need to show: $\vec{T}(X_{\boldsymbol{\pi}})(\mathbf{0},\mathbf{1}) \neq \emptyset$ if and only if $\boldsymbol{\pi}$ is compatible.

Assume first that $\vec{T}(X_{\boldsymbol{\pi}})(\mathbf{0},\mathbf{1}) \neq \emptyset$. Any d-path $p : \vec{I} \to X_{\boldsymbol{\pi}}$ from $\mathbf{0}$ to $\mathbf{1}$ leads to a *total order* of the boundary coordinates $c^i_j = a^i_j, b^i_j$ given by

$$t_1 \leq t_2 \text{ and } x_{j_1}(t_1) = c^{i_1}_{j_1}, x_{j_2}(t_2) = d^{i_2}_{j_2} \Rightarrow c^{i_1}_{j_1} \leq d^{i_2}_{j_2}$$

compatible with the relations defining $\sqsubseteq_{\boldsymbol{\pi}}$. In particular, $\sqsubseteq_{\boldsymbol{\pi}}$ is a partial order.

Now assume that $\vec{T}(X_{\boldsymbol{\pi}})(\mathbf{0},\mathbf{1}) = \emptyset$, i.e., the forbidden hyperrectangles give rise to a deadlock. A deadlock arises as lower corner of a non-empty intersection of $m \leq n$ hyperrectangles among the $R_{\pi(c)}(j_1, j_2)$, and $n-m$ among the degenerate hyperrectangles $R_j^0, 1 \leq j \leq n$.

At a non-empty intersection of extended hyperrectangles, every contributing a_j^* -coordinate is less (in the sense $\leq$) than every contributing b_j^* -coordinate. Moreover, every contributing b_*^i-coordinate arises from an extended hyperrectangle and does thus not correspond to the *last* pass for that permutation. Hence b_*^i preceeds (in the sense $\preceq$) at least one other a_*^i-coordinate. Hence, one can construct nonconstant chains of arbitrary length on the *finite* set of coordinates using $\sqsubseteq_{\boldsymbol{\pi}}$. Hence there are chains in which a coordinate arises more than once contradicting the partial order condition. $\square$

Remark 5.7. In general, it does not seem easy to check which of the collections $\pi = (\pi_c)_{c\in C} \in \Sigma = \prod_{c\in C} \Sigma_{J_{h(c)}}$ are compatible: The relation $\sqsubseteq_{\boldsymbol{\pi}}$ generated by $\leq$ and by $\preceq$ defines a digraph $G_{\boldsymbol{\pi}}$ with the boundary coordinates a_j^i, b_j^i as vertices; the k-tuple $\boldsymbol{\pi}$ is compatible if and only if $G_{\boldsymbol{\pi}}$ does not contain a directed cycle.

Example 5.8. Let $X_k \subset I^k$ denote the state space corresponding to the PV-model describing k dining philosophers (each protocol of type $PaPbVaVb$; cf Example 5.1, Figure 1 and Dijkstra [6]). Then only two out of 2^k permutations (those in the "diagonal" – all elements the identity or all the nontrivial transposition) in $\Sigma_2^k \subset (\Sigma_k^k)$ lead to a relation with a non-trivial cycle under the relation $\sqsubseteq_{\boldsymbol{\pi}}$; all others give rise to partial orders. As a consequence, $\vec{T}(X_k)(\mathbf{0},\mathbf{1})$ consists of 2^k-2 contractible components: There are 2^k-2 essentially different interleavings of the d-paths corresponding to each individual protocol – indicating who of the two neighbouring philosophers uses a fork first. The number 2^k-2 of schedules is, for $k>3$, considerably smaller than the number $k!$ of ordered k-tuples of philosophers. This is due to the fact that several philosophers can serve themselves concurrently for $k>3$.

6. The complement of a trace space

The aim of this last section is to describe a combinatorial method that yields a prodsimplicial model of the *complement* of $\mathbf{T}(X)(-,-)$ in a product $(S^{n-2})^l$ of spheres. Duality allows to obtain information about $\mathbf{T}(X)(-,-)$ and thus $\vec{T}(X)(-,-)$ itself; cf Remark 6.1. The advantage of this method is, that it is far easier to determine the poset category describing the homotopy type of the complement that that of trace space itself – by upward completion of the set $D(X)(-,-)$ in the category underlying the product of spheres. It is hoped that this will also make implementations easier and faster.

6.1. **A combinatorial description of the complement of trace space.** For simplicity, we restrict to traces from $\mathbf{0}$ to $\mathbf{1}$. First some notation regarding matrices and associated poset categories: Let $\tilde{M}_{l,n}^R \subset M_{l,n}^R \subset M_{l,n}$ consist of the binary matrices such that *every row vector contains at least one 0 and at least one 1.* This subset is a sublattice of the Boolean lattice

$M_{l,n}$ (with least upper bound, greatest lower bound, and coordinatewise involution I switching 0s and 1s). It has the matrices in which every row vector contains *exactly one* 1, resp *exactly one* 0 as minimal, resp. maximal elements. All elements can be written as least upper bounds of minimal matrices and (hence!) as greatest lower bounds of maximal matrices.

In [25, Section 4.2], we introduced the subset $D(X)(\mathbf{0},\mathbf{1}) := \{M \in M^C_{l,n} | \Psi(M) = 1\} \subset M_{l,n}$. Remark that, by [25, Lemma 3.3], matrices in $D(X)(\mathbf{0},\mathbf{1})$ containing a row with only 1s can and will be neglected right away; we let $\tilde{D}(X)(\mathbf{0},\mathbf{1}) \subseteq M^R_{l,n}$ consist of those matrices in $D(X)(\mathbf{0},\mathbf{1})$ without a row vector consisting of 1s only.

We define an upward completion of the matrix set $\tilde{D}(X)(\mathbf{0},\mathbf{1})$ within $\tilde{M}^R_{l,n}$ as follows:

$$\bar{D}(X)(\mathbf{0},\mathbf{1}) := \{M \in \tilde{M}^R_{l,n} |\ \exists N \in \tilde{D}(X)(\mathbf{0},\mathbf{1}) :\ N \leq M\}. \tag{6.1}$$

Obviously, this completed matrix set $\bar{D}(X)(\mathbf{0},\mathbf{1})$ forms an *upward* closed subcategory (with respect to $\leq$) of the poset category $\tilde{M}^R_{l,n}$. Reversing the arrows (using $\geq$ instead of $\leq$ as partial order) yields a *downward* closed subcategory $\bar{D}(X)(\mathbf{0},\mathbf{1})^{op} \subset (M^R_{l,n})^{op}$.

By [25, Lemma 3.3], the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ is contained in the complex $(\partial\Delta^{n-1})^l$ – a product of $(n-1)$-spheres. Let $U(X)(\mathbf{0},\mathbf{1}) := (\partial\Delta^{n-1})^l \setminus \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ denote its *complement* within the latter; this is an open set, which does not have a (prod-)simplicial structure right away.

But it turns out that $U(X)(\mathbf{0},\mathbf{1})$ is homotopy equivalent to a prodsimplicial complex with a pasting scheme construction analogous to (2.6): To every matrix $M \in \bar{D}(X)(\mathbf{0},\mathbf{1})$ we associate the product of simplices $\bar{\Delta}(M) = \prod_{i=1}^l \bar{\Delta}_i(M) \subset (\Delta^{n-1})^l \subset \mathbf{R}^{nl}$ with

$$\bar{\Delta}_i(M) := \{(t_1,\dots,t_n) |\ 0 \leq t_j \leq 1, \sum_{j=1}^n t_j = 1, m_{ij} = 1 \Rightarrow t_j = 0\} \subset \Delta^{n-1}.$$

This time, $M \geq N$ implies $\Delta(M) \subseteq \Delta(N)$. Now, we can define a prod-simplicial complex

$$\mathbf{U}(X)(\mathbf{0},\mathbf{1}) := \bigcup_{M \in \bar{D}(X)(\mathbf{0},\mathbf{1})} \bar{\Delta}(M). \tag{6.2}$$

Remark 6.1. As was pointed out in [25] for trace space, this is a colimit construction: $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$ is the colimit of a functor $\mathcal{F}^l_n : \bar{D}(X)(\mathbf{0},\mathbf{1})^{op} \to \mathbf{Top}$ where

$$\mathcal{F}^l_n : \bar{D}(X)(\mathbf{0},\mathbf{1})^{op} \hookrightarrow (\tilde{M}^R_{l,n})^{op} \xrightarrow{I} \tilde{M}^R_{l,n} \hookrightarrow M^R_{l,n} \xrightarrow{\mathcal{E}^l_n} \mathbf{Top}$$

with the involution I introduced above and the pasting functor $\mathcal{E}^l_n$ from [25, Section 3.1.2].

For a binary matrix $M \in M_{l,n}$, let $o(M)$ denote the number of rows that are zero vectors. Let $r(M) = \sum_{1\leq i\leq l, 1\leq j\leq n} m_{ij} + o(M)$. Remark that $r(M) \geq l$ for $M \in M^R_{l,n}$.

Lemma 6.2. *The complex* $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$ *has dimension* $\dim \mathbf{U}(X)(\mathbf{0},\mathbf{1}) = l(n-1) - \min_{M\in \tilde{D}(X)(\mathbf{0},\mathbf{1})} r(M) \leq l(n-2)$.

Proof. A vertex corresponds to a matrix with $l(n-1)$ ones. A k-face corresponds to a matrix with k additional zeros; zero rows are not considered in $\tilde{M}^R_{l,n}$. □

6.2. **A homotopy equivalence with several consequences.** The proof of the following result is to be found at the end of this article:

Theorem 6.3. *The complement* $U(X)(\mathbf{0},\mathbf{1})$ *is homotopy equivalent to the prodsimplicial complex* $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$.

Remark 6.4. If one knows the homotopy type of $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$ or has calculations of its homology, respectively cohomology at hand, one may apply Poincaré-Alexander-Lefschetz duality (cf eg [2, Theorem VI.8.3]) in order to obtain information about $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ and hence about $\vec{T}(X)(\mathbf{0},\mathbf{1})$:

$$H^p(\mathbf{T}(X)(\mathbf{0},\mathbf{1})) \cong H_{(n-2)l-p}((S^{n-2})^l, \mathbf{U}(X)(\mathbf{0},\mathbf{1}));$$

and, using a compact deformation retract of $U(X)(\mathbf{0},\mathbf{1})$,

$$H^p(\mathbf{U}(X)(\mathbf{0},\mathbf{1})) \cong H_{(n-2)l-p}((S^{n-2})^l, \mathbf{T}(X)(\mathbf{0},\mathbf{1})).$$

From the exact homology sequence of the pair $((S^{n-2})^l, \mathbf{U}(X)(\mathbf{0},\mathbf{1}))$ we conclude in particular:

$$\tilde{H}^0(\vec{T}(X)(\mathbf{0},\mathbf{1})) \cong H_{(n-2)l-1}(\mathbf{U}(X)(\mathbf{0},\mathbf{1}))$$

since, for $\vec{T}(X)(\mathbf{0},\mathbf{1}) \neq \emptyset$, the inclusion $\mathbf{U}(X)(\mathbf{0},\mathbf{1}) \hookrightarrow (S^{n-2})^l$ is not onto; and

$$H^p(\vec{T}(X)(\mathbf{0},\mathbf{1})) \cong H_{(n-2)l-p-1}(\mathbf{U}(X)(\mathbf{0},\mathbf{1})) \text{ for } p \not\equiv 0,1 \bmod n-2.$$

In the remaining cases, one has to study the maps induced on $(n-2)$-dimensional homology by the components $i_k : U \to S^{n-2}$ of inclusion $U \hookrightarrow (S^{n-2})^l$. The maps i_k are geometric realizations corresponding to the functors $\bar{D}(X)(\mathbf{0},\mathbf{1}) \hookrightarrow \tilde{M}^R_{l,n} \downarrow \tilde{M}^R_{1,n}$; the last map projects a matrix to its k-th row.

For $n=2$, both $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ and $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$ are (discrete) complements within $(S^0)^l$; in particular:

$$|\mathbf{T}(X)(\mathbf{0},\mathbf{1})| + |\mathbf{U}(X)(\mathbf{0},\mathbf{1})| = 2^l.$$

Example 6.5. Let X denote the complement of two "adjacent" cubes in I^3; cf Figure 2.

Using Raussen [25, Lemma 4.6], it is not difficult to see that in this case

$$\tilde{D}(X)(\mathbf{0},\mathbf{1}) = \{ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

with completion given by

$$\bar{D}(X)(\mathbf{0},\mathbf{1}) = \tilde{D}(X)(\mathbf{0},\mathbf{1}) \cup \{ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \}.$$

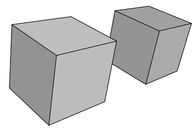

FIGURE 2. Forbidden region within I^3 consisting of two adjacent cubes; figure courtesy to A. Lang [19]

The three matrices in $\tilde{D}(X)(\mathbf{0},\mathbf{1})$ correspond to the *edges*, the last four matrices to the *vertices* of the (prod)simplicial complex $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$. In this case, that $1 = (4-3)$-dimensional complex is just an interval (with four vertices and three edges joining them), which is clearly contractible.

In this case, the complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ homotopy equivalent to trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ is thus homotopy equivalent to the complement of a contractible set within the 2-dimensional torus $(\partial\Delta^2)^2$, i.e., $\vec{T}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq S^1 \vee S^1$.

Proof of Theorem 6.3. It is folklore that the complement of a subcomplex of $\partial\Delta^{n-1}$ can be given a dual simplicial structure; cf Björner and Tancer [1] for a combinatorial proof. We outline a proof using the nerve lemma for subcomplexes of products of simplicial spheres. To this end, we define a cover of the space $U(X)(\mathbf{0},\mathbf{1})$ by open contractible subspaces with contractible or empty intersections and compare the space with the nerve of that cover.

The sphere $\partial\Delta^{n-1}$ can be covered by contractible open neighbourhoods $U(f_j)$ of its $(n-2)$-facets f_j; it is not difficult to write down contractions to the barycenter of the facet that respect the *complements* of subsimplices. The product $(\partial\Delta^{n-1})^l$ is then covered by the contractible open sets $\prod_{i=1}^l U(f^i_{j_i})$ with contractions respecting complements of products of subsimplices. Intersections of neighbourhoods are neighbourhoods of lower dimensional faces, also contractible to their barycenters with contractions respecting complements of subsimplices and their products.

This has consequences for the space $U(X)(\mathbf{0},\mathbf{1})$, the complement of the simplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in $(\partial\Delta^{n-1})^l$: First of all, it is covered by the open sets $\prod_{i=1}^l U(f^i_{j_i})$ corresponding to those matrices $M \in \bar{D}(X)(\mathbf{0},\mathbf{1})$ with exactly one 0 in each row corresponding to the vertices of $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$. Moreover, for such a matrix M with $m_{i,j_i} = 0$, the set $\prod_{i=1}^l U(f^i_{j_i}) \cap U(X)(\mathbf{0},\mathbf{1})$ is (non-empty and) contractible, since the contraction respects the complement of the simplicial complex $T(X)(\mathbf{0},\mathbf{1})$. The same argument holds also for the non-empty intersections with $U(X)(\mathbf{0},\mathbf{1})$ corresponding to matrices in $\bar{D}(X)(\mathbf{0},\mathbf{1})$ with additional zeroes.

Finally, one can argue as in the proof of [25, Theorem 3.5]: Both $U(X)(\mathbf{0},\mathbf{1})$ and $\mathbf{U}(X)(\mathbf{0},\mathbf{1})$ are colimits of functors over $\bar{D}(X)(\mathbf{0},\mathbf{1})^{op}$ homotopy equivalent to the homotopy colimits of these functors; since the functor takes contractible values everywhere, those are in turn homotopy equivalent to the nerve of the category $\bar{D}(X)(\mathbf{0},\mathbf{1})^{op}$. □

Remark 6.6. It should be possible to give an entirely combinatorial proof of Theorem 6.3 along the lines of Björner and Tancer [1] for the complement of a simplicial complex within a sphere (that has to be replaced by a product of spheres).

REFERENCES

1. A. Björner and M. Tancer, *Note: Combinatorial Alexander Duality - A Short and Elementary Proof*, Discrete Comput. Geom. **42** (2009), no. 4, 586–593.
2. G.E. Bredon, *Topology and Geometry*, GTM, vol. 139, Springer-Verlag, 1993.
3. R. Brown and P.J. Higgins, *Colimit theorems for relative homotopy groups*, J. Pure Appl. Algebra **22** (1981), 11–41.
4. ______, *On the algebra of cubes*, J. Pure Appl. Algebra **21** (1981), 233–260.
5. E.W. Dijkstra, *Co-operating sequential processes*, Programming Languages (F. Genuys, ed.), Academic Press, New York, 1968, pp. 43–110.
6. ______, *Hierarchical ordering of sequential processes*, Acta Informatica **1** (1971), no. 2, 115–138.
7. U. Fahrenberg and M. Raussen, *Reparametrizations of continuous paths*, J. Homotopy Relat. Struct. **2** (2007), no. 2, 93–117.
8. L. Fajstrup, É. Goubault, E. Haucourt, and M. Raussen, *Components of the fundamental category*, Appl. Categ. Structures **12** (2004), no. 1, 81–108.
9. L. Fajstrup, É. Goubault, and M. Raussen, *Detecting Deadlocks in Concurrent Systems*, CONCUR '98; Concurrency Theory (Nice, France) (D. Sangiorgi and R. de Simone, eds.), Lect. Notes Comp. Science, vol. 1466, Springer-Verlag, September 1998, 9th Int. Conf., Proceedings, pp. 332 – 347.
10. M. Farber, *Invitation to Topological Robotics*, Zurich Lectures in Advanced Mathematics, EMS Publishing House, 2008.
11. É. Goubault and E. Haucourt, *A Practical Application of Geometric Semantics to Static Analysis of Concurrent Programs*, CONCUR (M. Abadi and L. de Alfaro, eds.), Lecture Notes in Computer Science, vol. 3653, Springer, 2005, pp. 503 – 517.
12. ______, *Components of the Fundamental Category II*, Appl. Categ. Structures **15** (2007), 387–414.
13. M. Grandis, *Directed Homotopy Theory I. The Fundamental Category*, Cahiers Topologie Géom. Différentielle Catégoriques **44** (2003), 281 – 316.
14. ______, *Directed algebraic topology*, New Mathematical Monographs, Cambridge University Press, Cambridge, 2009.
15. J. Gunawardena, *Homotopy and concurrency*, Bulletin of the EATCS **54** (1994), 184–193.
16. M. Herlihy and S. Rajsbaum, *Algebraic topology and distributed computing—a primer*, Computer science today, Lecture Notes in Comput. Sci., vol. 1000, Springer, 1995, pp. 203–217.
17. T. Kaczynski, M. Mrozek, and M. Slusarek, *Homology Computation by Reduction of Chain Complexes*, Computers Math. Applic. **35** (1998), no. 4, 59–70.
18. D. Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, 2008.
19. A. Lang, *Paths in directed spaces*, Tech. report, Ecole Polytechique Paris, CEA LIST/LMeASI, 2010, 26 pages.
20. V. Pratt, *Modelling concurrency with geometry*, Proc. of the 18th ACM Symposium on Principles of Programming Languages. (1991), 311–322.

21. M. Raussen, *On the classification of dipaths in geometric models for concurrency*, Math. Structures Comput. Sci. **10** (2000), no. 4, 427–457.
22. ______, *Invariants of directed spaces*, Appl. Categ. Structures **15** (2007), 355–386.
23. ______, *Reparametrizations with given stop data*, J. Homotopy Relat. Struct. **4** (2009), no. 1, 1–5.
24. ______, *Trace spaces in a pre-cubical complex*, Topology Appl. **156** (2009), no. 9, 1718 – 1728.
25. ______, *Simplicial models for trace spaces*, Algebr. Geom. Topol. **10** (2010), 1683 – 1714.
26. R.J. van Glabbeek, *On the Expressiveness of Higher Dimensional Automata*, Theor. Comput. Sci. **368** (2006), no. 1-2, 168 – 194.
27. G. Winskel and M. Nielsen, *Models for concurrency*, Handbook of Logic in Computer Science, Oxford University Press, Oxford, 1995, pp. 1 – 148.

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