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n-Channel Asymmetric Entropy-Constrained Multiple-Description Lattice Vector Quantization

Jan Østergaard, *Member, IEEE,* Richard Heusdens, and Jesper Jensen

Abstract

This paper is about the design and analysis of an index-assignment (IA) based multiple-description coding scheme for the n-channel asymmetric case. We use entropy constrained lattice vector quantization and restrict attention to simple reconstruction functions, which are given by the inverse IA function when all descriptions are received or otherwise by a weighted average of the received descriptions. We consider smooth sources with finite differential entropy rate and MSE fidelity criterion. As in previous designs, our construction is based on nested lattices which are combined through a single IA function. The results are exact under high-resolution conditions and asymptotically as the nesting ratios of the lattices approach infinity. For any n, the design is asymptotically optimal. Moreover, in the case of two descriptions and finite lattice vector dimensions greater than one, the performance is strictly better than that of existing designs. In the case of three descriptions, we show that in the limit of large lattice vector dimensions, points on the inner bound of Pradhan et al. can be achieved. Furthermore, for three descriptions and finite lattice vector dimensions, we show that the IA-based approach yields a smaller rate loss than the recently proposed source-splitting approach.

Index Terms

distributed source coding, high-rate quantization, lattice quantization, multiple description coding, random binning, vector quantization.

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I. INTRODUCTION

Multiple-description coding (MDC) is about (lossy) encoding of information for transmission over an unreliable n-channel communication system. The channels may break down resulting in erasures and a loss of information at the receiving side. The receiver knows which subset of the n channels that are working; the transmitter does not. The problem is then to design an MDC system which, for given channel rates, minimizes the distortions due to reconstruction of the source using information from any subsets of the channels.

The achievable multiple-description (MD) rate-distortion function is completely known for the case of two channels, squared-error fidelity criterion and the memoryless Gaussian source [1], [2]. An extension to colored Gaussian sources was provided in [3]–[5]. Inner and outer bounds to the n-channel quadratic Gaussian rate-distortion region for memoryless sources was presented in [6]–[11].

Practical symmetric multiple-description lattice vector quantization (MD-LVQ) based schemes for two descriptions have been introduced in [12], which in the limit of infinite-dimensional source vectors and under high-resolution assumptions, approach the symmetric MD rate-distortion bound.[1] An extension to $n > 2$ descriptions was presented in [13]. Asymmetric MD-LVQ was considered in [14] for the case of two descriptions. Common for all of the designs [12]–[14] is that a central quantizer is first applied on the source after which an *index-assignment* (IA) algorithm (also known as a labeling function) maps the reconstruction points of the central quantizer to reconstruction points of the side quantizers, which is an idea that was first presented in [15]. These designs are usually referred to as IA based designs.

There also exists non IA based n-channel schemes, which are proven optimal in the two-channel quadratic Gaussian case. In particular, the source-splitting approach of Chen et al. [16] and the delta-sigma quantization approach of Østergaard et al. [17], [18].

While the different designs mentioned above are able to achieve the rate-distortion bounds in the asymptotical limit as the lattice vector quantizer dimension (L) gets arbitrarily large, there is an inherent *rate loss* when finite dimensional vector quantizers are employed.[2] For example, in the two-channel symmetric case and at high resolutions, the rate loss (per description) of the IA based schemes is given by $\frac{1}{4}\log_2(G(\Lambda^{(L)})G(S_L)(2\pi e)^2)$ where $G(\Lambda^{(L)})$ is the dimensionless normalized second moment of the L-

[1]The term symmetric relates to the situation where all channel rates (description rates) are equal and the distortion depends only upon the number of working channels (received descriptions) and as such not on which of the channels that are working. In the asymmetric case, the description rates and side distortions are allowed to be unequal.

[2]The term *rate loss* refers to the excess rate due to using a suboptimal implementation.

dimensional lattice $\Lambda^{(L)}$ and $G(S_L)$ is the dimensionless normalized second moment of an L-dimensional hypersphere [19]. For the source-splitting approach the rate loss is $\frac{1}{4}\log_2(G(\Lambda^{(L)})^3(2\pi e)^3)$ whereas for the delta-sigma quantization approach the rate loss is $\frac{1}{4}\log_2(G(\Lambda^{(L)})^2(2\pi e)^2)$. Since $G(S_L) \leq G(\Lambda^{(L)}), \forall L > 0$, it follows that the IA based approaches yield the smallest rate loss of all existing asymptotically optimal designs.[3]

We will like to point out that there exist a substantial amount of different practical approaches to MDC. For example, the work of [20] on asymmetric vector quantization, the work of [21], [22] on n-channel scalar quantization and the transform based MDC approaches presented in [23]–[26].

In this paper, we are interested in IA based MDC. Specifically, we propose a design of an asymmetric IA based MD-LVQ scheme for the case of $n \geq 2$ descriptions. The design uses a single labeling function and simple reconstruction functions, which are given by the inverse IA function when all descriptions are received or otherwise by a weighted average of the received descriptions. We consider the case of MSE distortion and smooth sources with finite differential entropy rate.[4] To the best of the authors knowledge, the above restrictions (or even less general restriction) are also necessary for the existing IA-based designs proposed in the literature.

The contributions of the paper are summarized below and are valid under high-resolution conditions and asymptotically large nesting ratios:

- We provide a simple construction of the labeling function for the asymmetric case which is optimal for any number $n \geq 2$ of descriptions.
- For $n = 3$ and any $L \geq 1$, we provide closed-form rate-distortion expressions.
- For $n = 3$ and in the limit as $L \to \infty$, the distortion points of our scheme lie on the inner bound provided by Pradhan et al. [7], [8].
- For $n = 2$ and any $1 < L < \infty$, we strictly improve the side distortions over that of the asymmetric design by Diggavi et al. [14].
- For $n = 3$ and $1 \leq L < \infty$, we show that our construction yields a smaller rate loss than that of source-splitting [16].

This paper is organized as follows. In Section II we briefly review some lattice properties, describe the required asymptotical conditions which we will be assuming through-out the work, and introduce the

[3]By use of time-sharing, the rate loss of the source-splitting scheme can be reduced to that of the delta-sigma quantization scheme. Moreover, in the scalar case, the rate loss can be further reduced, see [16] for details.

[4]For each side description, we assume that the sequence of quantized source vectors is jointly entropy coded using an arbitrarily complex entropy coder.

concept of an IA function. The actual design of the MD-LVQ system, which is the main contribution of the paper, is presented in Section III. In Section IV, we compare the proposed design to known inner bounds and existing MD schemes. The conclusion follows in Section V and appendices are reserved for lengthy proofs.

II. PRELIMINARIES

A. Lattice Properties

Let the L-dimensional real lattice $\Lambda \subset \mathbb{R}^L$ form the codewords of the lattice vector quantizer $\mathcal{Q}_\Lambda(\cdot)$ having Voronoi cells. Thus, $\mathcal{Q}_\Lambda(x) = \lambda$ if $x \in V(\lambda)$ where $V(\lambda) \triangleq \{x \in \mathbb{R}^L : \|x-\lambda\|^2 \leq \|x-\lambda'\|^2, \forall\, \lambda' \in \Lambda\}$ is a Voronoi cell. We define $\langle x, x\rangle \triangleq \frac{1}{L}x^t x$ and use $\|x\|^2 = \langle x, x\rangle$. The dimensionless normalized second-moment of inertia $G(\Lambda)$ of Λ is defined as [19]

$$G(\Lambda) \triangleq \frac{1}{\nu^{1+2/L}} \int_{V(0)} \|x\|^2 dx \tag{1}$$

where $V(0)$ is the Voronoi cell around the origin and ν denotes the volume of $V(0)$. Recall that $\frac{1}{12} \geq G(\Lambda) \geq G(S_L) \geq \frac{1}{2\pi e}$ where $G(S_L) = \frac{1}{(L+2)\pi}\Gamma\left(\frac{L}{2}+1\right)^{2/L}$ is the dimensionless normalized second moment of an L-dimensional hypersphere and $\Gamma(\cdot)$ is the Gamma function [19].

Let Λ be a lattice, then a sublattice $\Lambda_s \subseteq \Lambda$ is a subset of the elements of Λ that is itself a lattice. We say that Λ_s is a coarse lattice nested within the fine lattice Λ. Let ν and ν_s be the volumes of $V(0)$ and $V_s(0)$, respectively, where the subscript s indicates the sublattice. Then the index value N_s of Λ_s with respect to Λ is $N_s = \nu_s/\nu$ and the nesting ratio N_s' is given by $N_s' = \sqrt[L]{N_s}$.

Let $\{\Lambda^{(L)}\}$ be a sequence of lattices indexed by their dimension L. Then, $\Lambda^{(L)}$ is said to be asymptotically good for quantization (under MSE) if and only if for any $\epsilon > 0$ and sufficiently large L [27]

$$\log_2(2\pi e G(\Lambda^{(L)})) < \epsilon. \tag{2}$$

B. The Existence of Lattices and Sublattices for MD coding

We need a central lattice (central quantizer) Λ_c with Voronoi cell $V_c(0)$ of volume ν_c and n sublattices (side quantizers) $\Lambda_i \subset \Lambda_c$ with Voronoi cells $V_i(0)$ of volumes ν_i, where $i = 0, \ldots, n-1$. Finally, we need a sublattice $\Lambda_\pi \subset \Lambda_i$ which we will refer to as a product lattice. The Voronoi cell $V_\pi(0)$ of Λ_π has volume $\nu_\pi = N_\pi \nu_c$ where N_π is the index value of Λ_π with respect to Λ_c.

Previous work on two-description IA based MD coding focused on the existence and construction of nested lattices for a few low dimensional (root) lattices cf. [12], [14]. The techniques of [12], [14] was extended to the case of n descriptions for the symmetric case in [13]. While some of the root lattices

are considered to be among the best of all lattices (of the same dimensions) for quantization, they are not good for quantization in the sense of (2). Furthermore, their index values belong to some discrete sets of integers and since they are finite dimensional, arbitrary nesting ratios cannot be achieved.

Let us first clarify the requirements of the lattices to be used in this work:

1) The central lattice $\Lambda_c \in \mathbb{R}^L$, is asymptotically good for quantization as $L \to \infty$.
2) The central lattice $\Lambda_c \in \mathbb{R}^L$ admits sublattices $\Lambda_i \subset \Lambda_c$ of arbitrary nesting ratios $1 \leq N_i' \in \mathbb{R}$.
3) There exists a product lattice $\Lambda_\pi \subset \Lambda_i, i = 0, \ldots, n-1$, with arbitrary nesting ratio N_π' (with respect to Λ_c) where $N_i' < N_\pi' \in \mathbb{R}$ for all $i = 0, \ldots, n-1$.

That there exists a sequence of lattices which are asymptotically good for quantization was established in [28]. It is also known that there exists nested lattices $\Lambda^{(L)} \subset \Lambda_c^{(L)}$ where the coarse lattice ($\Lambda^{(L)}$) is asymptotically good for quantization and the fine lattice ($\Lambda_c^{(L)}$) is asymptotically good for channel coding [29]. Moreover, in recent work [30], it has been established that there exists a sequence of nested lattices where the coarse lattice as well as the fine lattice are simultaneously good for quantization.

Interestingly, we do not require $\{\Lambda_i\}_{i=0}^{n-1}$ nor Λ_π to be good for quantization. This is because we are able to construct a labeling function which, asymptotically as $N_i \to \infty, \forall i$, results in a distortion that becomes independent of the type of sublattices being used. Furthermore, Λ_π is used to provide a simple construction of a shift invariant region $V_\pi(0)$ and its quantization performance is therefore irrelevant.

We have yet to show the existence of $\Lambda_\pi^{(L)} \subset \Lambda_i^{(L)}$ for $i = 0, \ldots, n-1$. Towards that end, we refer to the construction of nested lattices provided in [30]. Here a coarse lattice $\Lambda_s^{(L)}$ is first fixed and then a fine lattice $\Lambda_c^{(L)}$ is constructed such that $\Lambda_s^{(L)} \subseteq \Lambda_c^{(L)}$ with an arbitrary nesting ratio. Without loss of generality, let $N_0' \leq N_1' \leq \cdots \leq N_{n-1}' < N_\pi'$. Moreover, let the set of integers $\mathbb{Z}^L$ form a product lattice $\Lambda_\pi^{(L)}$. Now let $\Lambda_\pi^{(L)}$ be the coarse lattice and construct a fine lattice $\Lambda_{n-1}^{(L)}$ so that the nesting ratio is N_π'/N_{n-1}' by using the method of [30]. Next, let $\Lambda_{n-1}^{(L)}$ be the coarse lattice and construct a fine lattice $\Lambda_{n-2}^{(L)}$ with a nesting ratio of N_{n-1}'/N_{n-2}'. This procedure is repeated until the sublattice $\Lambda_0^{(L)}$ is constructed as the fine lattice of $\Lambda_1^{(L)}$. At this point, the central lattice $\Lambda_c^{(L)}$ is finally constructed by using $\Lambda_0^{(L)}$ as the coarse lattice and making sure that the nesting ratio is N_0'. It should be clear that we end up with a sequence of nested lattices, i.e. $\Lambda_\pi^{(L)} \subset \Lambda_{n-1}^{(L)} \subseteq \cdots \subseteq \Lambda_0^{(L)} \subset \Lambda_c^{(L)}$ with the desired nesting ratios with respect to Λ_c, i.e. $N_\pi', N_{n-1}', \cdots, N_0'$. Without loss of generality, we can take $N_\pi' = \prod_{i=0}^{n-1} N_i'$.[5]

In the limit as $L \to \infty$ it is guaranteed that $\Lambda_c^{(L)}$ becomes asymptotically good for quantization.

[5]If $1 < m < n$ nesting ratios are identical, we keep only one of them when forming the product lattice. If all nesting ratios are identical, we form the product lattice based on the product of any two of them, see [13] for details.

Furthermore, the sublattices $\Lambda_i^{(L)}$ can be shaped so that they are also good for quantization or they can, for example, be shaped like the cubic lattice. This is not important for the design proposed in this work.

C. Lattice Asymptotics

As is common in IA based MD-LVQ, we will in this work require a number of asymptotical conditions to be satisfied in order to guarantee the prescribed rate-distortion performance. Specifically, we require high-resolution conditions, i.e. we will be working near the limit where the rates of the central and side quantizers diverge towards infinity, or equivalently, in the limit where the volumes of the Voronoi cells of the lattices in question become asymptotically small. This condition makes it possible to assume an approximately uniform source distribution over small regions so that standard high-resolution lattice quantization results become valid [31]. Let $\Lambda \subset \mathbb{R}^L$ be a real lattice and let $\nu = \det(\Lambda)$ be the volume of a fundamental region of Λ. Moreover, let $\tilde{V} \subset \mathbb{R}^L$ be a connected region of volume $\tilde{\nu}$. Then, the high-resolution assumption makes it possible to approximate the number of lattice points in $\tilde{V}$ by $\tilde{\nu}/\nu$, which is an approximation that becomes exact as the number of lattice shells within $\tilde{V}$ goes to infinity. To be more specific, let $S(c, r)$ be a sphere in $\mathbb{R}^L$ of radius r and center $c \in \mathbb{R}^L$. Then, according to Gauss' counting principle, the number $A_{\mathbb{Z}}$ of integer lattice points in a convex body $\mathcal{C}$ in $\mathbb{R}^L$ equals the volume $\text{Vol}(\mathcal{C})$ of $\mathcal{C}$ with a small error term [32]. In fact if $\mathcal{C} = S(c, r)$ then by use of a theorem due to Minkowski it can be shown that, for any $c \in \mathbb{R}^L$ and asymptotically as $r \to \infty$, $A_{\mathbb{Z}}(r) = \text{Vol}(S(c, r)) = \omega_L r^L$, where ω_L is the volume of the L-dimensional unit sphere [33]. It is also known that the number of lattice points $A_\Lambda(j)$ in the first j shells (i.e., the j shells nearest the origin) of the lattice Λ satisfies, asymptotically as $j \to \infty$, $A_\Lambda(j) = \omega_L j^{L/2}/\nu$ [12].

In addition to the high-resolution assumption, we also require that the index values of the sublattices become asymptotically large. With this, it follows that the number of central lattice points within a Voronoi cell of a sublattice becomes arbitrarily large. Furthermore, to guarantee that the sublattices satisfy the high-resolution quantization properties, we must force the volume of their Voronoi cells to be small. In other words, we require that $N_i \to \infty$ and $\nu_i \to 0$ where $\nu_i = \nu N_i$ is the volume of a Voronoi cell of the ith sublattice. We also note that, in order to avoid that some subset of the sublattices asymptotically dominate the overall distortion, we will require that their index values grow at the same rate, i.e. $N_i/N_j = c_{i,j}$ for some constant $c_{i,j} \in \mathbb{R}$.

Finally, as mentioned in the previous section, we require the existence of good lattices for quantization. We therefore require that the lattice vector dimension L tends towards infinity.

We note that the above asymptotical conditions are only required to guarantee exact results. In fact,

at some point, we relax the requirement on L and provide exact results for any $L \geq 1$. Moreover, the proof technique is constructive in the sense that in non-asymptotical situations, i.e. for finite N_i and R_i, the results are approximately true. This is interesting from a practical perspective, since, in practice, the asymptotical conditions will never be truly satisfied.

D. Index Assignments

In the MDC scheme considered in this paper, a source vector x is quantized to the nearest reconstruction point λ_c in the central lattice Λ_c. Hereafter follows IAs (mappings), which uniquely map all λ_c's to reproduction points in each of the sublattices Λ_i. This mapping is done through a labeling function α, and we denote the individual component functions of α by α_i. In other words, the function α that maps Λ_c into $\Lambda_0 \times \cdots \times \Lambda_{n-1}$, is given by $\alpha(\lambda_c) = (\alpha_0(\lambda_c), \alpha_1(\lambda_c), \ldots, \alpha_{n-1}(\lambda_c)) = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$, where $\alpha_i(\lambda_c) = \lambda_i \in \Lambda_i$ and $i = 0, \ldots, n-1$. Each n-tuple $(\lambda_0, \ldots, \lambda_{n-1})$ is used only once when labeling points in Λ_c so that λ_c can be recovered unambiguously when all n descriptions are received.

Since lattices are infinite arrays of points, we adopt the procedure first used in [12] and construct a shift invariant labeling function, so we only need to label a finite number of points. We generalize the approach of [14] and construct a product lattice Λ_π which has N_π central lattice points and N_π/N_i sublattice points from the ith sublattice in each of its Voronoi cells. The Voronoi cells $V_\pi(\lambda_\pi)$ of the product lattice Λ_π are all similar so by concentrating on labeling only central lattice points within one Voronoi cell of Λ_π, the rest of the central lattice points may be labeled simply by translating this Voronoi cell throughout $\mathbb{R}^L$. We will therefore only label central lattice points within $V_\pi(0)$, which is the Voronoi cell of Λ_π around the origin. With this we get

$$\alpha(\lambda_c + \lambda_\pi) = \alpha(\lambda_c) + \lambda_\pi \tag{3}$$

for all $\lambda_\pi \in \Lambda_\pi$ and all $\lambda_c \in \Lambda_c$.

III. Construction of the Labeling Function

This section focuses on the labeling problem and is split into several subsections. We begin by Section III-A which shows how to guarantee shift invariance of the labeling function. Then, in Section III-B, we define the cost function to be minimized by an optimal labeling function. In Section III-C we show how to construct an optimal set of n-tuples and the assignment of the n-tuples to central lattice points follows Section III-D. We end by assessing the rate and distortion performances of the labeling function in Section III-E and Section III-F, respectively.

A. Guaranteeing Shift Invariance of the Labeling Function

In order to ensure that α is shift-invariant, we must make sure that an n-tuple is not assigned to more than one central lattice point $\lambda_c \in \Lambda_c$. Notice that two n-tuples which are translates of each other by some $\lambda_\pi \in \Lambda_\pi$ must not both be assigned to central lattice points located within the same region $V_\pi(\lambda_\pi)$, since this causes assignment of an n-tuple to multiple central lattice points.

The region $V_\pi(0)$ will be translated through-out $\mathbb{R}^L$ and centered at $\lambda_\pi \in \Lambda_\pi$. Let us for now assume that Λ_π is clean[6] with respect to Λ_0. Then no points of Λ_0 will be inside more than one $V_\pi(\lambda_\pi)$ region. This is the key insight required to guarantee shift invariance. Let us now construct an n-tuple, say $(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$, where the first element is inside $V_\pi(0)$, i.e. $\lambda_0 \in V_\pi(0)$. Once we shift the n-tuple by a multiple of Λ_π, the first element of the shifted n-tuple will never be inside $V_\pi(0)$ and the n-tuple is therefore shift invariant. In other words, $(\lambda_0 + \lambda_\pi) \notin V_\pi(0)$ for $0 \neq \lambda_\pi \in \Lambda_\pi$.

If Λ_π is not clean, it is still fairly simple to guarantee shift invariance. Let $\mathcal{C} \triangleq \{\lambda_0 \in \Lambda_0 : \lambda_0 \in V_\pi(0)\}$, i.e., the set of points of Λ_0 which are inside $V_\pi(0)$. Moreover, let $\mathcal{C}_I$ and $\mathcal{C}_B$ denote the set of interior and boundary points of $\mathcal{C}$, respectively. Thus, $\mathcal{C}_I \triangleq \{\lambda_0 \in \mathcal{C} : (\lambda_0 + \lambda_\pi) \notin \mathcal{C}, 0 \neq \lambda_\pi \in \Lambda_\pi\}$ and $\mathcal{C}_B = \mathcal{C} \backslash \mathcal{C}_I$. If $\mathcal{C}_B \neq \emptyset$ then it contains subsets of points which are multiple-λ_π of each other for some non-zero λ_π. We will identify such subsets of points by their cosets $[\lambda_0] \triangleq \{(\lambda_0 + \lambda_\pi) \in \mathcal{C}_B : \lambda_\pi \in \Lambda_\pi\}, \forall \lambda_0 \in \mathcal{C}_B$, where λ_0 is said to be the coset leader of $[\lambda_0]$. We say that the cosets $[\lambda_0]$ and $[\lambda_0']$ are identical if $\lambda_0' \in [\lambda_0]$ which implies that $\lambda_0 \in [\lambda_0']$. Only one of each set of identical cosets should be used. With this, we define $\mathcal{C}_B^u$, to be a set of distinct coset leaders, whose cosets do not overlap but cover $\mathcal{C}_B$. Thus, for any pair $\lambda_0, \lambda_0' \in \mathcal{C}_B^u$ where $\lambda_0 \neq \lambda_0'$ it holds that $\lambda_0 \notin [\lambda_0']$ and $\lambda_0' \notin [\lambda_0]$, i.e., $[\lambda_0] \cap [\lambda_0'] = \emptyset$. Moreover, $\bigcup_{\lambda_0 \in \mathcal{C}_B^u} [\lambda_0] = \mathcal{C}_B$. Finally, let $\mathcal{C}^u \triangleq \mathcal{C}_I \cup \mathcal{C}_B^u$. Now when we construct the n-tuples to be assigned to the central lattice points within $V_\pi(0)$ we simply have to make sure that the first coordinate of the n-tuples belongs to $\mathcal{C}^u$. Clearly, this guarantees shift invariance.

In the above design we restricted λ_0 to be in $V_\pi(0)$ so that the map α is shift-invariant. However, this also means that all n-tuples (for $\lambda_c \in V_\pi(0)$) have their first element (i.e. λ_0) inside $V_\pi(0)$. This restriction is easily removed by considering all cosets of each n-tuple. This is completely analogous to the coset construction defined above for the boundary points of Λ_0. Specifically, let us define the coset

[6]A sublattice $\Lambda_s \subset \Lambda$ is said to be clean with respect to Λ if no points of Λ falls on the boundary of the Voronoi cells of Λ_s. In other words, the set $\{\lambda \in \Lambda : \lambda \in V_s(\lambda_s) \cap V_s(\lambda_s')\}$ is empty for all $\lambda_s \neq \lambda_s' \in \Lambda_s$. We note that it is an open problem to construct a sequence of nested lattices which are asymptotically good for quantization and where the coarse lattice is clean.

of an n-tuple modulo Λ_π to be

$$\bar{\mathcal{C}}_{\Lambda_\pi}(\lambda_0,\ldots,\lambda_{n-1}) \triangleq \{(\lambda'_0,\ldots,\lambda'_{n-1}) \in \Lambda_0 \times \cdots \times \Lambda_{n-1} : \lambda'_i = \lambda_i + \lambda_\pi, \lambda_\pi \in \Lambda_\pi\}. \tag{4}$$

The n-tuples in a coset are equivalent modulo Λ_π. So since that only one member from each coset is used when assigning n-tuples to central lattice points within $V_\pi(0)$, the shift-invariance property is preserved.

To summarize, if Λ_π is clean, we first construct a set of shift invariant n-tuples by making sure that $\lambda_0 \in V_\pi(0)$ for all n-tuples. Then, we extend the set of possible n-tuples by allowing one member from each coset (as given by (4)) to be used when labeling central lattice points in $V_\pi(0)$. If Λ_π is not clean, then as before we first construct a set of shift invariant n-tuples by making sure that $\lambda_0 \in V_\pi(0)$ for all n-tuples. Then we resolve the conflicting boundary points, simply by reducing the set of candidate n-tuples. Specifically, we make sure that only one member of each coset of C^u is used as first element in the n-tuples.[7] Finally, we extend the set of candidate n-tuples by allowing any one member of each coset as given by (4) to be used.

B. Defining the Cost Function for the Labeling Problem

We will treat the asymmetric problem where the individual descriptions are weighted and the distortions due to reception of subsets of descriptions are also weighted. There are in general several ways of receiving κ out of n descriptions. Let $\mathcal{L}^{(n,\kappa)}$ denote an index set consisting of all possible κ combinations out of $\{0,\ldots,n-1\}$ so that $|\mathcal{L}^{(n,\kappa)}| = \binom{n}{\kappa}$. For example, for $n = 3$ and $\kappa = 2$ we have $\mathcal{L}^{(3,2)} = \{\{0,1\},\{0,2\},\{1,2\}\}$. Furthermore, let $0 < \mu_i \in \mathbb{R}$ be the weight for the ith description.

Recall that α takes a single vector λ_c and maps it to a set of vectors $\{\lambda_i\}, i = 0,\ldots,n-1$, where $\lambda_i \in \Lambda_i$. The mapping is invertible so that we have $\lambda_c = \alpha^{-1}(\lambda_0,\ldots,\lambda_{n-1})$. Thus, if all n descriptions are received we reconstruct using the inverse map α^{-1} and obtain λ_c. If no descriptions are received, we reconstruct using the statistical mean of the source. In all other cases, we reconstruct using a weighted average of the received elements.

We define the reconstruction formula when receiving the set of κ out of n descriptions indexed by $\ell \in \mathcal{L}^{(n,\kappa)}$ to be

$$\hat{x}_\ell \triangleq \frac{1}{\kappa}\sum_{i\in\ell} \mu_i \alpha_i(\lambda_c) \tag{5}$$

[7]Similar approach is used to reduce the set of central lattice points in the case where Λ_π is not clean with respect to Λ_c.

where $1 \leq \kappa < n$ and where $\lambda_c = \mathcal{Q}_{\lambda_c}(x)$, i.e. x is quantized to $\lambda_c \in \Lambda_c$. The distortion d_ℓ due to approximating x by $\hat{x}_\ell$ is then given by

$$d_\ell = \left\| x - \frac{1}{\kappa} \sum_{i \in \ell} \mu_i \alpha_i(\lambda_c) \right\|^2 \tag{6}$$

and the expected distortion with respect to X is given by $\bar{D}_\ell = \mathbb{E} d_\ell$.

Lemma 1 ([12]): For any $1 \leq \kappa < n$, $\ell \in \mathcal{L}^{(n,\kappa)}$, asymptotically as $\nu_c \to 0$ and independently of α

$$\bar{D}_\ell = \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} f_X(x) \left\| X - \lambda_c \right\|^2 dx + \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} f_X(X) \left\| \lambda_c - \frac{1}{\kappa} \sum_{i \in \ell} \mu_i \alpha_i(\lambda_c) \right\|^2 dx. \tag{7}$$

Proof: The lemma was proved in [12] for the symmetric case and two descriptions. The extension to the asymmetric case and n descriptions is straight forward. See [34] for details. ■

Notice that only the second term of (7) is affected by the labeling function. We will make use of this fact and therefore define

$$D_\ell \triangleq \left\| \lambda_c - \frac{1}{\kappa} \sum_{i \in \ell} \mu_i \alpha_i(\lambda_c) \right\|^2. \tag{8}$$

The cost function to be minimized by the labeling function must take into account the entire set of distortions due to reconstructing from different subsets of descriptions. With this in mind, we combine the distortions through a set of scalar (Lagrangian) weights. Specifically, let $\gamma_\ell \in \mathbb{R}, \ell \in \mathcal{L}^{(n,\kappa)}$ be the weight for the distortion D_ℓ due to reconstructing using the set of descriptions indexed by ℓ. With this, we define the cost function $\mathcal{J}^n$ for the n-description labeling problem to be:

$$\begin{aligned} \mathcal{J}^n \triangleq \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} f_X(x) \Bigg\{ & \sum_{i=0}^{n-1} \gamma_i \left\| \lambda_c - \mu_i \alpha_i(\lambda_c) \right\|^2 + \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \gamma_{i,j} \left\| \lambda_c - \frac{\mu_i \alpha_i(\lambda_c) + \mu_j \alpha_j(\lambda_c)}{2} \right\|^2 \\ & + \sum_{i=0}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \gamma_{i,j,k} \left\| \lambda_c - \frac{\mu_i \alpha_i(\lambda_c) + \mu_j \alpha_j(\lambda_c) + \mu_k \alpha_k(\lambda_c)}{3} \right\|^2 + \cdots \Bigg\} dx \end{aligned} \tag{9}$$

which can be written more compactly as

$$\mathcal{J}^n \triangleq \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} f_X(x) \sum_{\kappa=1}^{n-1} \sum_{\ell \in \mathcal{L}^{(n,\kappa)}} \gamma_\ell D_\ell \, dx. \tag{10}$$

For example, using the fact that $\lambda_i = \alpha_i(\lambda_c)$, we can write $\mathcal{J}^n$ for the $n = 3$ case as

$$\mathcal{J}^3 = \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} f_X(x) \sum_{i=0}^{2} \gamma_i \left\| \lambda_c - \mu_i \lambda_i \right\|^2 dx + \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} f_X(x) \sum_{i=0}^{1} \sum_{j=i+1}^{2} \gamma_{i,j} \left\| \lambda_c - \frac{\mu_i \lambda_i + \mu_j \lambda_j}{2} \right\|^2 dx. \tag{11}$$

Since we are considering the high-resolution regime, we can make the following simplifications

$$\mathcal{J}^n = \sum_{\lambda_c\in\Lambda_c}\sum_{\kappa=1}^{n-1}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}} \gamma_\ell \int_{V_c(\lambda_c)} f_X(x) D_\ell \, dx \tag{12}$$

$$= \sum_{\lambda_c\in\Lambda_c} P(X\in V_c(\lambda_c)) \sum_{\kappa=1}^{n-1}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}} \gamma_\ell D_\ell \tag{13}$$

$$\approx \sum_{\lambda_\pi\in\Lambda_\pi} \frac{P(X\in V_\pi(\lambda_\pi))}{N_\pi} \sum_{\lambda_c\in V_\pi(\lambda_\pi)}\sum_{\kappa=1}^{n-1}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}} \gamma_\ell D_\ell \tag{14}$$

$$= \frac{1}{N_\pi}\sum_{\lambda_c\in V_\pi(0)}\sum_{\kappa=1}^{n-1}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}} \gamma_\ell D_\ell \tag{15}$$

where $P(X\in V_c(\lambda_c))$ is the probability that X will be mapped (or quantized) to λ_c. The approximation follows by substituting $P(X\in V_c(\lambda_c)) \approx P(X\in V_\pi(\lambda_\pi))/N_\pi$ for $\lambda_\pi\in\Lambda_\pi$ which becomes exact as $\nu_i\to 0$. In (14), we also exploited that α is shift invariant in order to decompose the sum $\sum_{\lambda_c\in\Lambda_c}$ into the double sum $\sum_{\lambda_\pi\in\Lambda_\pi}\sum_{\lambda_c\in V_\pi(\lambda_\pi)}$ as follows from (3).

We would like to simplify $\mathcal{J}^n$ even further. In order to do so, we introduce the following notation. Let $\mathcal{L}_i^{(n,\kappa)}$ indicate the set of all $\ell\in\mathcal{L}^{(n,\kappa)}$ that contains the index i, i.e., $\mathcal{L}_i^{(n,\kappa)} = \{\ell\in\mathcal{L}^{(n,\kappa)} : i\in\ell\}$. Similarly, $\mathcal{L}_{i,j}^{(n,\kappa)} = \{\ell\in\mathcal{L}^{(n,\kappa)} : i,j\in\ell\}$. Moreover, let $\bar\gamma(\mathcal{L}^{(n,\kappa)}) = \sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell$, $\bar\gamma(\mathcal{L}_i^{(n,\kappa)}) = \sum_{\ell\in\mathcal{L}_i^{(n,\kappa)}}\gamma_\ell$ and $\bar\gamma(\mathcal{L}_{i,j}^{(n,\kappa)}) = \sum_{\ell\in\mathcal{L}_{i,j}^{(n,\kappa)}}\gamma_\ell$. Thus, $\bar\gamma(\mathcal{L}^{(3,2)}) = \gamma_{0,1}+\gamma_{0,2}+\gamma_{1,2}$ and $\bar\gamma(\mathcal{L}_1^{(3,2)}) = \gamma_{0,1}+\gamma_{1,2}$.

Theorem 1: Let $1\le\kappa<n<\infty$. Given a set of distortion weights $\{\gamma_\ell\in\mathbb{R} : \ell\in\mathcal{L}^{(n,\kappa)}, 1\le\kappa\le n-1\}$, a set of description weights $\{0<\mu_i\in\mathbb{R} : i=0,\dots,n-1\}$ and any $\lambda_c\in\Lambda_c$ we have

$$\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell D_\ell = \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\hat\gamma_{i,j}^{(n,\kappa)}\left\|\mu_i\lambda_i-\mu_j\lambda_j\right\|^2 + \bar\gamma(\mathcal{L}^{(n,\kappa)})\left\|\lambda_c - \frac{1}{\kappa\bar\gamma(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\mu_i\lambda_i\right\|^2 \tag{16}$$

where $\lambda_i = \alpha_i(\lambda_c)$ and

$$\hat\gamma_{i,j}^{(n,\kappa)} = \frac{1}{\kappa^2}\left(\frac{\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\bar\gamma(\mathcal{L}_j^{(n,\kappa)})}{\bar\gamma(\mathcal{L}^{(n,\kappa)})} - \bar\gamma(\mathcal{L}_{i,j}^{(n,\kappa)})\right). \tag{17}$$

Proof: See Appendix A. ∎

From (16) we make the observation that whenever λ_i appears, it is multiplied by μ_i. Without loss of generality, we can therefore scale the lattice Λ_i by μ_i and consider the scaled lattice $\tilde\Lambda_i = \mu_i\Lambda_i$ instead. This simplifies the notation. For example, $\hat x_\ell = \frac{1}{\kappa}\sum_{i\in\ell}\tilde\lambda_i$ where $\tilde\lambda_i = \mu_i\lambda_i$ for $i=0,\dots,n-1$. Clearly, scaling the sublattices affects the side description rates. We address this issue in Section III-E.

By use of Theorem 1 we can rewrite the cost function to be minimized by the labeling function as

$$\mathcal{J}^n = \frac{1}{N_\pi}\sum_{\lambda_c\in V_\pi(0)}\sum_{\kappa=1}^{n-1}\left\{\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\hat\gamma_{i,j}^{(n,\kappa)}\left\|\tilde\lambda_i-\tilde\lambda_j\right\|^2 + \bar\gamma(\mathcal{L}^{(n,\kappa)})\left\|\lambda_c - \frac{1}{\kappa\bar\gamma(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\tilde\lambda_i\right\|^2\right\} \tag{18}$$

where $\hat{\gamma}_{i,j}^{(n,\kappa)}$ is given by (17).

The following theorem allows us to simplify the construction of the labeling function:

Theorem 2: Let $1 < n \in \mathbb{N}$. The cost function $\mathcal{J}^n$ is asymptotically separable in the sense that, as $N_i \to \infty$ and $\nu_i \to 0, \forall i$, an optimal set $\mathcal{T}^*$ of N_π distinct and shift invariant n-tuples satisfies

$$\mathcal{T}^* = \arg\min_{\mathcal{T}} \sum_{(\lambda_0,\ldots,\lambda_{n-1})\in\mathcal{T}} \sum_{\kappa=1}^{n-1} \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \hat{\gamma}_{i,j}^{(n,\kappa)} \left\| \tilde{\lambda}_i - \tilde{\lambda}_j \right\|^2 \tag{19}$$

where $\mathcal{T} = \{(\lambda_0, \ldots, \lambda_{n-1}) \in \Lambda_0 \times \cdots \times \Lambda_{n-1} : (\lambda_0, \ldots, \lambda_{n-1}) \text{ is shift invariant}\}, |\mathcal{T}| = N_\pi$ and where $\hat{\gamma}_{i,j}^{(n,\kappa)}$ is given by (17).

Proof: See Appendix B. ■

Theorem 2 provides a guideline for the construction of n-tuples. One should first find a set of N_π distinct and shift invariant n-tuples which satisfies (19). These n-tuples (or members of their cosets) should then be assigned to central lattice points in $V_\pi(0)$ such that

$$\sum_{\lambda_c\in V_\pi(0)} \sum_{\kappa=1}^{n-1} \left\| \lambda_c - \frac{1}{\kappa\bar{\gamma}(\mathcal{L}^{(n,\kappa)})} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2 \tag{20}$$

is minimized.

Remark 1: Notice that we have not claimed that $\mathcal{T}^*$ is unique. Thus, there might be several sets of n-tuples which all satisfy (19) but yield different distortions when inserted in (20). However, Theorem 2 states that the asymptotically (as $N_i \to \infty$) dominating distortion is due to that of (19). Thus, any set of n-tuples satisfying (19) will be asymptotically optimal. Interestingly, we show in Section III-D that $\mathcal{T}^*$ is, in certain cases, indeed asymptotically unique (up to translations by coset members).

C. Constructing n-Tuples

In order to construct n-tuples which are shift invariant we extend the technique previously proposed for the symmetric n-description MD problem [13].

We first center a sphere $\tilde{V}$ at all sublattice points $\lambda_0 \in V_\pi(0)$ and construct n-tuples by combining sublattice points from the other sublattices (i.e. $\Lambda_i, i = 1, \ldots, n-1$) within $\tilde{V}(\lambda_0)$ in all possible ways and select the ones that minimize (19). For each $\lambda_0 \in V_\pi(0)$ it is possible to construct $\prod_{i=1}^{n-1} \tilde{N}_i$ n-tuples, where $\tilde{N}_i$ is the number of sublattice points from the ith sublattice within the region $\tilde{V}$. This gives a total of $(N_\pi/N_0)\prod_{i=1}^{n-1} \tilde{N}_i$ n-tuples when all $\lambda_0 \in V_\pi(0)$ are used. The number $\tilde{N}_i$ of lattice points within $\tilde{V}$ may be approximated by $\tilde{N}_i \approx \tilde{\nu}/\nu_i$ where $\tilde{\nu}$ is the volume of $\tilde{V}$.[8]

[8]This approximation becomes exact in the usual asymptotical sense of $N_i \to \infty$ and $\nu_i \to 0$.

Since $\tilde{N}_i \approx \tilde{\nu}/(\nu N_i)$ and we need N_0 n-tuples for each $\lambda_0 \in V_\pi(0)$ we see that

$$N_0 \leq \prod_{i=1}^{n-1} \tilde{N}_i \approx \frac{\tilde{\nu}^{n-1}}{\nu^{n-1}} \prod_{i=1}^{n-1} N_i^{-1},$$

so in order to obtain at least N_0 n-tuples the volume of $\tilde{V}$ must satisfy (asymptotically as $N_i \to \infty$)

$$\tilde{\nu} \geq \nu_c \prod_{i=0}^{n-1} N_i^{1/(n-1)}. \tag{21}$$

For the symmetric case, i.e. $N = N_i$, $i = 0, \ldots, n-1$, we have $\tilde{\nu} \geq \nu_c N^{n/(n-1)}$, which is in agreement with the results obtained in [13].

The design procedure can be outlined as follows:

1) Center a sphere $\tilde{V}$ at each $\lambda_0 \in V_\pi(0)$ and construct all possible n-tuples $(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ where $\lambda_i \in \tilde{V}(\lambda_0)$ and $i = 1, \ldots, n-1$. This makes sure that all n-tuples have their first element (λ_0) inside $V_\pi(0)$ and they are therefore shift-invariant.
2) Keep only n-tuples whose elements satisfy $\|\lambda_i - \lambda_j\|^2 \leq r^2/L, \forall i, j \in 0, \ldots n-1$, where r is the radius of $\tilde{V}$.
3) Make $\tilde{V}$ large enough so at least N_0 distinct n-tuples are found for each λ_0.

The restriction $\|\lambda_i - \lambda_j\|^2 \leq r^2/L$ in step 2 above, is imposed to to avoid bias towards any of the sublattices. At this point, one might wonder why we wish to avoid such bias. After all, the expression to be minimized, i.e. (19), includes weights $\hat{\gamma}_{i,j}^{(n,\kappa)}$ (which might not be equal) for every pair of sublattices. In otherwords, why not use spheres $\tilde{V}_{i,j}$ of different sizes to guarantee that $\|\lambda_i - \lambda_j\|^2 \leq r_{i,j}^2/L$ where the radius $r_{i,j}$ now depends on the particular pair of sublattices under consideration. This is illustrated in Fig. 3 where $r_{i,j}$ denotes the radius of the sphere $\tilde{V}_{i,j}$. Here we center $\tilde{V}_{0,1}$ at some $\lambda_0 \in V_\pi(0)$ as illustrated in Fig. 3 by the solid circle. Then, for any n-tuples having this λ_0 point as first element, we only include λ_1 points which are inside $\tilde{V}_{0,1}(\lambda_0)$. This guarantees that $\|\lambda_0 - \lambda_1\|^2 \leq r_{0,1}/L$. Let us now center a sphere $\tilde{V}_{1,2}$ at some λ_1 which is inside $\tilde{V}_{0,1}(\lambda_0)$. This is illustrated by the dotted sphere of radius $r_{1,2}$ in the figure. We then only include λ_2 points which are in the intersection of $\tilde{V}_{1,2}(\lambda_1)$ and $\tilde{V}_{0,2}(\lambda_0)$. This guarantees that $\|\lambda_i - \lambda_j\|^2 \leq r_{i,j}/L$ for all (i, j) pairs.

Clearly, the radius $r_{i,j}$ must grow at the same rate for any pair (i, j) so that, without loss of generality, $r_{0,1} = a_2 r_{0,2} = a_1 r_{1,2}$ for some fixed $a_1, a_2 \in \mathbb{R}$. Interestingly, from Fig. 3 we see that $r_{0,2}$ cannot be greater than $r_{0,1} + r_{1,2}$ which effectively upper bounds a_2. Thus, the ratio $r_{i,j}/r_{k,l}$ cannot be arbitrary. Furthermore, it is important to see that the asymmetry in distortions between the descriptions, is not dictated by $r_{i,j}$ but instead by how the n-tuples are assigned to the central lattice points. Recall from (20) that the assignment is such that the distances between the central lattice points and the weighted centroids

of the n-tuples are minimized. In other words, if we wish to reduce the distortion due to receiving description i we assign the n-tuples so that the ith element of the n-tuples is closer (on average) to the associated central lattice points. Obviously, the remaining elements of the n-tuples will then be further away from the assigned central lattice points.

In the following we first consider the case where $r = r_{i,j}$ for any (i,j). We later show that this is indeed the optimal choice in the symmetric distortion case. It is trivially also optimal in the two-channel asymmetric case, since there is only a single weight $\hat{\gamma}_{0,1}^{(2,1)}$. In general, we can always scale the radii such that

$$\sum_{(\lambda_0,\ldots,\lambda_{n-1})\in\mathcal{T}} \sum_{\kappa=1}^{n-1} \hat{\gamma}_{i,j}^{(n,\kappa)} \|\lambda_i - \lambda_j\|^2 = \sum_{(\lambda_0,\ldots,\lambda_{n-1})\in\mathcal{T}} c_{k,l} \sum_{\kappa=1}^{n-1} \hat{\gamma}_{k,l}^{(n,\kappa)} \|\lambda_k - \lambda_l\|^2 \tag{22}$$

for any $(i,j) \neq (k,l)$ where $\mathcal{T}$ indicates the set of N_π n-tuples and $c_{k,l} \in \mathbb{R}$. The resulting distortions weights (as given by (34) and (35)) should then include the additional set of scaling factors $\{c_{k,l}\}$. This case is treated by Lemma 2.

We now proceed to find the optimal $\tilde{\nu}$, i.e. the smallest volume which (asymptotically for large N_i) leads to exactly N_0 tuples satisfying step 2. In order to do so, we adopt the approach of [13] and introduce a dimensionless expansion factor $\psi_{n,L}$. The expansion factor $\psi_{n,L}$ describe how much $\tilde{V}$ needs to be expanded (per dimension) from the theoretical lower bound (21), to make sure that exactly N_0 optimal n-tuples can be constructed by combining sublattice points within a region $\tilde{V}$. With this approach, we have that

$$\tilde{\nu} = \psi_{n,L}^L \nu_c \prod_{i=0}^{n-1} N_i^{1/(n-1)}. \tag{23}$$

In practice, it is straight-forward to determine $\psi_{n,L}$. One can simply start at $\psi_{n,L} = 1$ and iteratively increase $\psi_{n,L}$ in small steps until exactly N_0 n-tuples are found which all satisfy $\|\lambda_i - \lambda_j\|^2 \leq r/L$. For volumes containing a large number of lattice points, i.e. asymptotically as $N_i \to \infty$, such an approach determines $\psi_{n,L}$ to arbitrary accuracy. Furthermore, in this asymptotical case, $\psi_{n,L}$ becomes independent of the type of lattice (and also N_i), since it then only depends on the number of lattice points within a large volume. Thus, it should be clear that for any $1 < n \in \mathbb{N}$ and $1 \leq L \in \mathbb{N}$, and asymptotically as $N_i \to \infty, \forall i$, there exist a unique $1 \leq \psi_{n,L} \in \mathbb{R}$.

In general, it is complicated to find an analytical expression for $\psi_{n,L}$. However, we have previously been able to do it for the symmetric MD problem in some interesting cases. It turns out that the proof technique and solutions provided for the symmetric case, carry over to the asymmetric case. To see this, we sketch the proof technique here for the asymmetric case and $n = 3$.

Recall that we seek 3-tuples such that any two members of the 3-tuple is distanced no more than r^2/L apart. Specifically, we require $\|\lambda_i - \lambda_j\|^2 \leq r^2/L$ where r is the radius of $\tilde{V}$. Essentially, this is a counting problem. We first center a sphere $\tilde{V}$ at some $\lambda_0 \in V_\pi(0) \cap \Lambda_0$. Then we pick a single $\lambda_1 \in \tilde{V}(\lambda_0) \cap \Lambda_1$. Finally, we center an equivalent sphere $\tilde{V}$ at this λ_1 and count the number, say $\#_{\lambda_1}$, of $\lambda_2 \in \tilde{V}(\lambda_0) \cap \tilde{V}(\lambda_1) \cap \Lambda_2$. Thus, there is $\#_{\lambda_1}$ 3-tuples having the same pair (λ_0, λ_1) as first and second element. The procedure is now repeatedly applied for all $\lambda_1 \in \tilde{V}(\lambda_0) \cap \Lambda_1$ yielding the total number of 3-tuples to be $\sum_{\lambda_1 \in \tilde{V}(\lambda_0)} \#_{\lambda_1}$ (all having the same λ_0 as first element).

For large volumes, the number of lattice points in a region S is given by $\text{Vol}(S)/\nu_2$ where $\text{Vol}(S)$ is the volume of S and ν_2 is the volume of the Voronoi cell of the sublattice points $\lambda_2 \in \Lambda_2$. Thus, given the pair (λ_0, λ_1), the number of λ_2 sublattice points is approximately given by $\text{Vol}(S)/\nu_2$ where $S = \tilde{V}(\lambda_0) \cap \tilde{V}(\lambda_1)$. It follows that we need to find the radius (or actually the volume $\tilde{\nu}$ of $\tilde{V}$) such that $\sum_{\lambda_1 \in \tilde{V}(\lambda_0)} \#_{\lambda_1} = N_0$, since we need exactly N_0 3-tuples for each $\lambda_0 \in V_\pi(0) \cap \Lambda_0$. To find the optimal $\tilde{\nu}$, we derive the volume of intersecting L-dimensional spheres distanced $0 < b \in \mathbb{R}$ apart. We then let b_k be a sequence of increasing distances which yields a sequence of volumes $\{\text{Vol}(S_k)\}$ of the partial intersections $S_k = \tilde{V}(0) \cap \tilde{V}(b_k)$. We finally form the equality $\sum_{k=1}^{r} \#_{S_k} \text{Vol}(S_k)/\nu_2 = N_0$ where $\#_{S_k}$ denotes the number of times each S_k occurs. By solving for r, we find the unique volume $\tilde{\nu}$ which leads to exactly N_0 n-tuples. It can be shown that this procedure yields the optimal $\tilde{\nu}$ and is asymptotically exact for large volumes. Furthermore, it is essentially equivalent to the symmetric case the only exception being that the index values forming the product (23) are allowed to be different in the asymmetric case. We therefore refer the reader to [13], [34] for the rigorous proof and quote some results below.

In the case of $n = 2$, it trivially follows that $\psi_{2,L} = 1$ for all L. For the case of $n = 3$ and L odd we have the following theorem.

Theorem 3 ([13, Thm. 3.2]): Let $n = 3$. Asymptotically as $N_i \to \infty, \nu_i \to 0, \forall i$, $\psi_{3,L}$ for L odd is given by

$$\psi_{3,L} = \left(\frac{\omega_L}{\omega_{L-1}}\right)^{\frac{1}{2L}} \left(\frac{L+1}{2L}\right)^{\frac{1}{2L}} \beta_L^{-\frac{1}{2L}} \tag{24}$$

where ω_L denotes the volume of an L-dimensional unit-sphere and β_L only depends on L and is given by

$$\beta_L = \sum_{m=0}^{\frac{L+1}{2}} \binom{\frac{L+1}{2}}{m} 2^{\frac{L+1}{2}-m} (-1)^m \sum_{k=0}^{\frac{L-1}{2}} \frac{(\frac{L+1}{2})_k (\frac{1-L}{2})_k}{(\frac{L+3}{2})_k \, k!} \times \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{2}\right)^{k-j} (-1)^j \left(\frac{1}{4}\right)^j \frac{1}{L+m+j} \tag{25}$$

where $(\cdot)_k$ is the Pochhammer symbol.[9] ▲

[9]The Pochhammer symbol is defined as $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

Theorem 4 ([13], [34]): Let $n = 3$. Asymptotically as $N_i \to \infty, \nu_i \to 0, \forall i$, and $L \to \infty$

$$\psi_{3,\infty} = \left(\frac{4}{3}\right)^{\frac{1}{4}}. \tag{26}$$

▲

Remark 2: The proposed construction also provides a shift invariant set of n-tuples in the non-asymptotical case where N_i is finite. Thus, the design is useful in practice.

D. Assigning n-Tuples to Central Lattice Points

At this point, we may assume that we have a set $\mathcal{T}$ containing N_π shift invariant n-tuples. These n-tuples need to be assigned to the N_π central lattice points within $V_\pi(0)$. However, before doing so, we first construct the coset of each n-tuple of $\mathcal{T}$. Recall that the coset of an n-tuple is given by (4).

As first observed by Diggavi et al. [14], assignment of n-tuples (or more correctly cosets of n-tuples) to central lattice points, is a standard linear assignment problem where only one member from each coset is assigned. This guarantees that the labeling function is shift invariant. The cost measure to be minimized by the linear assignment problem is given by (20). Thus, the sum of distances between the weighted centroids of the n-tuples and the central lattice points should be minimized.

Remark 3: Notice that we have shown that there exists a set of n-tuples and an assignment that satisfy the desired set of distortions. However, there might exist several assignments (for the same set of n-tuples) all yielding the same overall Lagrangian cost. Thus, in practice, when solving the bipartite matching problem one might need to search through the complete set of solutions (assignments) in order to find one that leads to the desired set of distortions. Alternatively, one can pick different solutions (assignments) and use each of them a certain amount of time so that on average the desired set of distortions are satisfied.

Remark 4: It might appear that the shift invariance restriction enforced by using only one member from each coset will unfairly penalize Λ_0. However, the following theorems prove that, asymptotically as $N_i \to \infty$, there is no bias towards any of the sublattices. We will consider here the case of $n > 2$ (for $n = 2$ we can use similar arguments as given in [14]).

Theorem 5: Let $n > 2$. Asymptotically as $N_i \to \infty, \forall i$, the number of n-tuples that includes sublattice points outside $V_\pi(0)$ becomes negligible compared to the number of n-tuples which have all there sublattice points inside $V_\pi(0)$.

Proof: See Appendix C. ■

Theorem 6: Let $n > 2$. Asymptotically as $N_i \to \infty, \forall i$, the set of N_π n-tuples that is constructed by centering $\tilde{V}$ at each $\lambda_0 \in V_\pi(0) \cap \Lambda_0$ becomes identical to the set constructed by centering $\tilde{V}$ at each $\lambda_i \in V_\pi(0) \cap \Lambda_i$, where $i \in \{1, \ldots, n-1\}$.

Proof: See Appendix D. ■

Remark 5: Notice that the above theorems imply that the set of n-tuples which satisfies (19) and is constructed so that $r_{i,j} = a_{i,j} r, \forall (i,j)$ and $a_{i,c} \in \mathbb{R}^L$, is unique (at least up to translations by members of their cosets). The assignment of the n-tuples to central lattice points, however, might not be unique.

E. Description Rates

The single-description rate R_c, i.e. the rate of the central quantizer, is given by

$$R_c = -\frac{1}{L} \sum_{\lambda_c \in \Lambda_c} \left(\int_{V_c(\lambda_c)} f_X(x) dx \right) \log_2 \left(\int_{V_c(\lambda_c)} f_X(x) dx \right).$$

Using the fact that each Voronoi cell $V_c(\lambda_c)$ has identical volume ν_c and assuming that $f_X(x)$ is approximately constant over Voronoi cells of the central lattice Λ_c, it can be shown that [31]

$$R_c \approx \frac{1}{L} h(X) - \frac{1}{L} \log_2(\nu_c), \tag{27}$$

where $h(X)$ is the differential entropy of a source vector and the approximation becomes asymptotically exact in the high resolution limit where $\nu_c \to 0$.

The side descriptions are based on a coarser lattice obtained by scaling the Voronoi cells of the central lattice by a factor of $N_i \mu_i$. Assuming the pdf of X is roughly constant within a sublattice cell, the rates of the side descriptions are given by

$$R_i \approx \frac{1}{L} h(X) - \frac{1}{L} \log_2(N_i \mu_i \nu_c) \tag{28}$$

where the approximation becomes exact asymptotically as $N_i \nu_c = \nu_i \to 0$ for a fixed weight $0 < \mu_i \in \mathbb{R}$. The description rates are related to the single-description rate by

$$R_i \approx R_c - \frac{1}{L} \log_2(N_i \mu_i).$$

It follows that, given description rates R_i and description weights μ_i for $i = 0, \ldots, n-1$, the index values are given by

$$N_i = \frac{1}{\nu_c \mu_i} 2^{h(X) - L R_i} \tag{29}$$

and the nesting ratios by $N_i' = N_i^{\frac{1}{L}}$.

F. Distortions

We now provide analytical expressions for the expected distortions in the case of $n = 2$ and $n = 3$ descriptions.

Theorem 7: Let $n = 2$ and $1 \leq L \in \mathbb{N}$. Furthermore, fix the weights $0 < \mu_i \in \mathbb{R}$ and $\gamma_i \in \mathbb{R}$ where $i = 0, 1$. Given an optimal labeling function α, then, asymptotically as $N_i \to \infty$ and $\nu_i \to 0$, the expected distortion $\bar{D}_i = \mathbb{E}\|X - \hat{X}_i\|^2$ where $\hat{X}_i = \mu_i \lambda_i$ satisfies

$$\bar{D}_i = \frac{\gamma_j^2}{(\gamma_0 + \gamma_1)^2} G(S_L) \nu_c^{2/L} (N_0 N_1)^{2/L} (\mu_0 \mu_1)^{2/L} \tag{30}$$

$$= \frac{\gamma_j^2}{(\gamma_0 + \gamma_1)^2} G(S_L) 2^{\frac{2}{L} h(X)} 2^{2(R_c - (R_0 + R_1))} \tag{31}$$

where $i, j \in \{0, 1\}$ and $i \neq j$.

Proof: Follows by applying the proof technique of Diggavi et al. [14] and using the fact that we are here optimizing over L-dimensional spheres rather than Voronoi cells as was the case in [14]. ■

Theorem 8: Let $n = 3$ and $1 \leq L \in \mathbb{N}$. Given the set of distortion weights $\{\gamma_\ell \in \mathbb{R} : \ell \in \mathcal{L}^{(n,\kappa)}, 1 \leq \kappa \leq n-1\}$, and set of description weights $\{0 < \mu_i \in \mathbb{R} : i = 0, \ldots, n-1\}$ and an optimal labeling function α. Then, for any $1 \leq \kappa < n$, any $\ell \in \mathcal{L}^{(n,\kappa)}$, and asymptotically as $N_i \to \infty$ and $\nu_i \to 0$, the expected distortion $\bar{D}_\ell = \mathbb{E}\|X - \hat{X}_\ell\|^2$ where $\hat{X}_\ell = \sum_{i \in \ell} \mu_i \lambda_i$ is given by

$$\bar{D}_\ell = \hat{\gamma}_\ell \Phi_L G(S_L) \nu_c^{2/L} (\mu_0 \mu_1 \mu_2)^{1/L} (N_0 N_1 N_2)^{1/L} \tag{32}$$

$$= \hat{\gamma}_\ell \Phi_L G(S_L) 2^{\frac{2}{L} h(X)} 2^{R_c - (R_0 + R_1 + R_2)} \tag{33}$$

where the weights $\hat{\gamma}_\ell \in \mathbb{R}$ for $\kappa = 1$ is given by

$$\hat{\gamma}_i = \frac{\gamma_j^2 + \gamma_k^2 + \gamma_j \gamma_k}{(\gamma_0 + \gamma_1 + \gamma_2)^2} \tag{34}$$

and for $\kappa = 2$ by

$$\hat{\gamma}_{i,j} = \frac{1}{4} \frac{\gamma_{i,k}^2 + \gamma_{j,k}^2 + \gamma_{i,k} \gamma_{j,k}}{(\gamma_{0,1} + \gamma_{0,2} + \gamma_{1,2})^2} \tag{35}$$

where $k \neq i, k \neq j$, and $j \neq i$ and $\Phi_L = \frac{L+2}{L} \frac{\tilde{\beta}_L}{\beta_L} \psi_{3,L}^2$ where $\psi_{3,L}$ is given by (24), β_L is given by (25) and

$$\tilde{\beta}_L = \sum_{m=0}^{\frac{L+1}{2}} \binom{\frac{L+1}{2}}{m} 2^{\frac{L+1}{2} - m} (-1)^m \sum_{k=0}^{\frac{L-1}{2}} \frac{(\frac{L+1}{2})_k (\frac{1-L}{2})_k}{(\frac{L+3}{2})_k \, k!} \times \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{2}\right)^{k-j} (-1)^j \left(\frac{1}{4}\right)^j \frac{1}{L + m + j + 2}. \tag{36}$$

Proof: See Appendix E. ■

For large L, we can simplify the term Φ_L appearing in Theorem 8, which we for later reference put into the following corollary:

Corollary 1: Asymptotically as $N_i \to \infty$ and $L \to \infty$, $\Phi_L = \left(\frac{4}{3}\right)^{\frac{1}{2}}$. ▲

If we in the construction of the n-tuples use the additional set of weights $\{c_{i,j}\}$ as given by (22), then $\hat{\gamma}_\ell$ is given by the following lemma:

Lemma 2: For any $n > 1, 1 \leq \kappa < n$ and $\ell \in \mathcal{L}^{(n,\kappa)}$ we have

$$\hat{\gamma}_\ell = \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa^2} \Bigg(\bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \sum_{j \in \ell} \sum_{\substack{i=0 \\ i \neq j}}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) c_{i,j} - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \sum_{i=0}^{\kappa-2} \sum_{j=i+1}^{\kappa-1} c_{i,j} - \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) c_{i,j} \Bigg) \tag{37}$$

where if $c_{i,j} = 1$ and $n = 3$, (37) reduces to (34) and (35) for $\kappa = 1$ and $\kappa = 2$, respectively.

Proof: Follows by inserting the additional weights $\{c_{i,j}\}$ in (82). ■

Notice also that, for any $n \geq 1$ and asymptotically as $\nu_c \to 0$, the expected central distortion is trivially given by

$$\bar{D}_c = \mathbb{E} D_c = \mathbb{E} \|X - \mathcal{Q}_{\Lambda_c}(X)\|^2 = G(\Lambda_c) \nu_c^{2/L}. \tag{38}$$

We end this section by establishing an interesting result for the n-channel IA based MD problem.

Corollary 2: Let $n > 1$ and $1 \leq L < \infty$. Given the set of distortion weights $\{\gamma_\ell \in \mathbb{R} : \ell \in \mathcal{L}^{(n,\kappa)}, 1 \leq \kappa \leq n-1\}$, and set of description weights $\{0 < \mu_i \in \mathbb{R} : i = 0, \ldots, n-1\}$ and an optimal labeling function α. Then, for any $1 \leq \kappa < n$, any $\ell \in \mathcal{L}^{(n,\kappa)}$, and asymptotically as $N_i \to \infty$ and $\nu_i \to 0$, the expected distortion $\bar{D}_\ell = \mathbb{E}\|X - \hat{X}_\ell\|^2$ where $\hat{X}_\ell = \sum_{i \in \ell} \mu_i \lambda_i$ is linearly proportional to $\bar{D}_{\ell'}$ where $\ell' \in \{\mathcal{L}^{(n,\kappa)} : 1 \leq \kappa < n\}$. In particular

$$\bar{D}_\ell = \hat{\gamma}_\ell c_\ell 2^{\frac{2}{L} h(X)} 2^{\frac{2}{n-1}(R_c - \sum_{i=0}^{n-1} R_i)} \tag{39}$$

where $\hat{\gamma}_\ell$ is given by (37) and $0 < c_\ell < \infty, \forall \ell$.

Proof: Follows from the proof of Theorem 8. ■

Remark 6: We have not been able to find the set of constants $\{c_\ell\}$ in (39) for the case of $n > 3$. However, since $0 < c_\ell < \infty$ it follows that, for any $n > 1$, the side distortions for different subsets of descriptions are linearly related, independently of the description rates. This observation has an interesting consequence. Let the growth of $N_\pi = \prod_i N_i$ as a function of the rates be given by $N_\pi = 2^{La(n-1)\sum_i R_i}$ where $0 < a < 1$. Moreover, since $R_i = R_c - \frac{1}{L}\log_2(N_i)$ we also have that $N_\pi = 2^{L(nR_c - \sum_i R_i)}$.

Equating the two expressions for N_π and solving for R_c yields $R_c = \frac{1}{n}\sum_i R_i(a(n-1)+1)$. Inserting this into (39) and (38) leads to

$$\lim_{\sum_i R_i \to \infty} \bar{D}_\ell \, 2^{\frac{2}{n}(1-a)\sum_i R_i} = c' \, 2^{\frac{2}{L}h(X)} \tag{40}$$

for any $\ell \in \mathcal{L}^{(n,\kappa)}$ and

$$\lim_{\sum_i R_i \to \infty} \bar{D}_c \, 2^{\frac{2}{n}(1+a(n-1))\sum_i R_i} = c \, 2^{\frac{2}{L}h(X)}, \tag{41}$$

where $c' \in \mathbb{R}$ depends on ℓ, $c \in \mathbb{R}$ is independent of ℓ and a controls the rate trade-offs between the central and the side descriptions. Thus, the product of the central distortion $\bar{D}_c$ (41) and an arbitrary set of $(n-1)$ side distortions $\bar{D}_\ell$ (40) is independent of a. This observation agrees with the symmetric n-channel product considered in [35].

IV. Comparison to Existing Schemes

We first assess the two-channel performance. This is interesting partly because it is the only case where the complete achievable MD rate-distortion region is known and partly because it makes it possible to compare the performance to that of existing schemes.

A. Two-Channel Performance

The side distortions $\bar{D}_0'$ and $\bar{D}_1'$ of the two-channel asymmetric MD-LVQ system presented in [14] satisfy (under identical asymptotical conditions as that of the proposed design)

$$\bar{D}_i' \approx \frac{\gamma_j^2}{(\gamma_0+\gamma_1)^2} G(\Lambda_\pi) 2^{2h(X)} 2^{-2(R_0+R_1-R_c)} \tag{42}$$

where $i,j \in \{0,1\}$ and $i \neq j$ and the central distortion is given by $\bar{D}_c' \approx G(\Lambda_c)2^{2(h(X)-R_c)}$. Notice that the only difference between (42) and (31) is that the former depends on $G(\Lambda_\pi)$ and the latter on $G(S_L)$. For the two dimensional case it is known that $G(S_2) = 1/(4\pi)$ whereas if Λ_π is similar to Z^2 we have $G(\Lambda_\pi) = 1/12$ which is approximately 0.2 dB worse than $G(S_2)$. Fig. 1 shows the performance when quantizing $2\cdot 10^6$ zero-mean unit-variance independent Gaussian vectors constructed by blocking an i.i.d. scalar Gaussian process into two-dimensional vectors and using the Z^2 quantizer for the design of [14] as well as for the proposed system. In this setup we have fixed $R_0 = 5$ bit/dim. but R_1 is varied in the range 5 – 5.45 bit/dim. We have fixed the ratio $\gamma_0/\gamma_1 = 1.55$ and we keep the side distortions fixed and change the central distortion. Since the central distortion is the same for the two schemes we have not shown it. Notice that $\bar{D}_0$ (resp. $\bar{D}_1$) is strictly smaller (about 0.2 dB) than $\bar{D}_0'$ (resp. $\bar{D}_1'$).

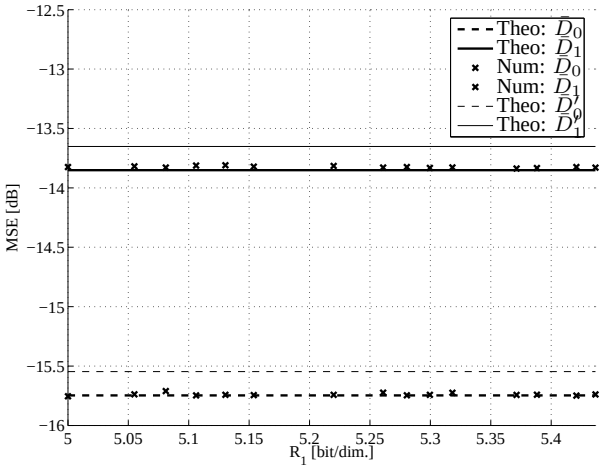

Fig. 1. The side distortions are here kept fixed as the rate is increased. Notice that the numerically obtained side distortions $\bar{D}_0$ and $\bar{D}_1$ are strictly smaller than the theoretical $\bar{D}'_0$ and $\bar{D}'_1$.

B. Three Channel Performance

In this section we compare the rate-distortion properties of the proposed design to the inner bound provided in [7], [8]. In order to do this, we first define an MD distortion product for the three channel case. Then, we show that by random binning one can further reduce the description rates. Finally, we assess the rate loss when finite-dimensional quantizers are used but no binning.

1) Three Channel Distortion Product: To assess the performance of the three channel design it is convenient to define the distortion product D^π which in the symmetric distortion case (i.e. for $\bar{D}_0 = \bar{D}_1 = \bar{D}_2$ and $\bar{D}_{0,1} = \bar{D}_{0,2} = \bar{D}_{1,2}$) takes the form $D^\pi = \bar{D}_c \bar{D}_i \bar{D}_{i,j}$. This is similar in spirit to Vaishampayan's widely used symmetric two-channel distortion product [36].

Let $n = 3$ and consider the symmetric case where $\mu_i = 1, \gamma_i = c_1$ and $\gamma_{i,j} = c_2$ for $i, j = 0, 1, 2$ where c_1, c_2 are some constants. Moreover, $R_i = R$ and $N_i = N$ for $i = 0, 1, 2$. It follows from (34) that $\hat{\gamma}_i = \frac{1}{3}$ and from (35) that $\hat{\gamma}_{i,j} = \frac{1}{12}$ so that by (33) we see that the one-channel distortion is given by

$$\bar{D}_i = \frac{1}{3} \Phi_L G(S_L) 2^{\frac{2}{L} h(X) + R_c - 3R} \tag{43}$$

and the two-channel distortion is given by

$$\bar{D}_{i,j} = \frac{1}{12} \Phi_L G(S_L) 2^{\frac{2}{L} h(X) + R_c - 3R}. \tag{44}$$

We also recall that the central distortion is given by

$$\bar{D}_c = G(\Lambda_c) 2^{\frac{2}{L} h(X) - 2R_c}. \tag{45}$$

This leads to the following distortion product

$$D^{\pi} = \frac{1}{36}\Phi_L^2 G(S_L)^2 G(\Lambda_c) 2^{\frac{6}{L}h(X)-6R} \tag{46}$$

which is independent of R_c and only depends upon the description rate R.

Recall that in the Gaussian case, $h(X) = \frac{L}{2}\log_2(2\pi e\sigma_X^2)$ and for $L \to \infty$ we have $G(S_L) = G(\Lambda_c) = 1/(2\pi e)$ and (by Corollary 1) $\Phi_\infty^2 = \frac{4}{3}$ so that the distortion product reduces to

$$D^{\pi} = \frac{1}{27}\sigma_X^6 2^{-6R}. \tag{47}$$

The following lemma shows that the proposed design is able to achieve a distortion product based on the inner bound of [7], [8].

Lemma 3: The high-resolution distortion product D^{π} of the three-channel achievable quadratic Gaussian rate-distortion region of Pradhan et al. [7], [8] is identical to (47).

Proof: See Appendix F. ■

Remark 7: Thus, for any rate trade-offs between central and side descriptions, the distortion product of the proposed MDLVQ achieves a distortion product based on the inner bound of [7], [8]. This inner bound is not always tight as shown in [37]. However, in the case where we are only interested in the one-channel distortion D_i and the central distortion D_c, optimality was recently proven in [35]. In particular, independently of our work, Zhang et al. proposed a distortion product based on the outer bound of [6]. Moreover, Zhang et al. showed that in the three-channel case, the product $D_i^2 D_c$ of our MDLVQ construction achieves their distortion product. We show next that in the case where we are only interested in the two-channel distortion $D_{i,j}$ and the central distortion D_c, we are in fact also optimal.

2) Random Binning on the Labeling Function: It was shown in [7], [8] that the achievable rate region can be enlarged by using random binning arguments on the random codebooks. Interestingly, we can show that it also makes sense to apply random binning on the labeling function proposed in this work. For example, in the case of three descriptions, we can utilize the universality of random binning so that one can faithfully decode on reception of e.g. at least two of the three descriptions. With such a strategy, it is then possible to reduce the effective description rate, since the binning rate is smaller than the codebook rate. The price to pay is that one cannot faithfully decode if e.g. only a single description is received.

In order to understand how we apply random binning on the labeling function, recall that every $\lambda_i \in \Lambda_i$ is combined with the set of λ_j's given by $\mathcal{T}_j(\lambda_i) \triangleq \{\lambda_j \in \Lambda_j : \lambda_i = \alpha_i(\lambda_c), \lambda_j = \alpha_j(\lambda_c), \lambda_c \in \Lambda_c\}$. The trick is now to randomly assign members of $\mathcal{T}_j(\lambda_i)$ to a set of bins in such a way that it is very unlikely that two or more members of $\mathcal{T}_j(\lambda_i)$ fall into the same bin. When encoding, we first apply the central

quantizer $\mathcal{Q}_{\Lambda_c}$ on the source variable X in order to obtain the central lattice point $\lambda_c = \mathcal{Q}_{\Lambda_c}(X)$. We then map the given λ_c to the triplet $(\lambda_0, \lambda_1, \lambda_2) = \alpha(\lambda_c)$. We finally find and transmit the bin indices of $\lambda_i, i = 0, 1, 2$, rather than their codebook indices. On reception of at least (any) two bin indices, we search through all the elements in the two bins in order to find a pair of sublattice points which are elements of the same n-tuple. If the binning rate is large enough, there will (with high probability) be only one such pair of sublattice points for any two bin indices.

Theorem 9: Let $n = 3$ and let α be an optimal labeling function. Moreover, assume we apply random binning on the labeling function such that one can faithfully (and uniquely) decode on reception of any two descriptions. Then, asymptotically, as $N_i \to \infty, \nu_i \to 0$, and $L \to \infty$, the binning rate R_b must satisfy

$$R_b > \frac{1}{2}R + \frac{1}{2}\log_2(\psi_{3,L}\sqrt{N'}) \tag{48}$$

where R is the description rate.

Proof: The proof is essentially similar to the technique presented in [7].[10] ■

We can further show that the binning rate, as given by (48), coincide with that of [7] for this particular case where we can only decode on reception of at least two out of three descriptions. To show this, note that when we get arbitrarily close to the binning rate in (48), it follows that

$$R = 2R_b - \frac{1}{2}\log_2(\psi_{3,L}^2) - \frac{1}{2}\log_2(N'). \tag{49}$$

In this case, the two-channel distortion $\bar{D}_{i,j}$, as given by (44), can be written as

$$\begin{aligned}
\bar{D}_{i,j} &= \frac{1}{12}\Phi_L G(S_L) 2^{\frac{2}{L}h(X)} 2^{R_c - 3R} \\
&\overset{(a)}{=} \frac{1}{12}\psi_{3,\infty}^2 2^{Rc-3R} \\
&\overset{(b)}{=} \frac{N'}{12}\psi_{3,\infty}^2 2^{-2R} \\
&\overset{(c)}{=} \frac{(N')^2}{12}\psi_{3,\infty}^4 2^{-4R_b}
\end{aligned} \tag{50}$$

where (a) is valid for (unit-variance) Gaussian sources, in the limit as $L \to \infty$ so that $\Phi_\infty = \psi_{3,\infty}^2$ and $2^{\frac{2}{L}h(X)} = G(S_L)^{-1}$. (b) follows since $R_c = R + \log_2(N')$ and (c) follows by inserting (49). Similarly, in the limit as $L \to \infty$, the three-channel distortion (central distortion D_c) is given by

$$D_c = 2^{-Rc} \tag{51}$$

$$= \frac{1}{N'}\psi_{3,\infty}^2 2^{-4R_b}. \tag{52}$$

[10]The complete proof for the asymmetric case can be found in [34].

On the other hand, from [7], see also Appendix F, it follows that the two-channel distortion $D'_{i,j}$ of Pradhan et al., is given by

$$D'_{i,j} = \frac{1}{2}\sigma_q^2(1+\rho) \tag{53}$$

where ρ is defined in Appendix F and

$$\sigma_q^2 = 2(1-\rho)^{-1/3}(1+2\rho)^{-2/3}2^{-4R_b}. \tag{54}$$

Moreover, the three-channel distortion $D'_{i,j,k}$ is given by

$$D'_{i,j,k} = \frac{1}{3}\sigma_q^2(1+2\rho). \tag{55}$$

Let us equate the pair of two-channel distortions, i.e. $\bar{D}_{i,j} = D'_{i,j}$, from which we obtain

$$(1+2\rho)^{1/3} = \left(\frac{1}{12}\psi_{3,\infty}^4(N')^2\right)^{-1/2}(1+\rho)^{1/2}(1-\rho)^{-1/6}. \tag{56}$$

Inserting (56) and (54) into (55) yields

$$D'_{i,j,k} = \frac{2}{3}\left(\frac{1}{12}\psi_{3,\infty}^4(N')^2\right)^{-1/2}(1+\rho)^{1/2}(1-\rho)^{-1/2}2^{-4R_b} \tag{57}$$

$$= \frac{1}{N'}\left(\frac{4}{3}\right)^{\frac{1}{2}}2^{-4R_b} \tag{58}$$

where the last equality follows by inserting $\psi_{3,\infty}^2 = \left(\frac{4}{3}\right)^{\frac{1}{2}}$ and letting $\rho \to -\frac{1}{2}$, which corresponds to the asymptotical case where $N' \to \infty$. It follows that the resulting two and three-channel distortions are identical (the ratio of (52) and (58) is one) for the the proposed design and the bounds of Pradhan et al. [7].

3) Rate Loss: A rate loss $R_{\text{loss}} = R - R^{\text{opt}}$ (per description) in finite dimensions and based on the inner bound of Pradhan et al., can easily be derived from the distortion product by isolating the rates in (46) and (47) and forming their difference, that is

$$R_{\text{loss}} = \frac{1}{6}\log_2(\Phi_L^2) + \frac{1}{6}\log_2\left(\frac{3}{4}\right) + \frac{1}{6}\log_2(G(S_L)^2G(\Lambda_c)(2\pi e)^3) \tag{59}$$

which clearly goes to zero for large L since $\Phi_\infty^2 = \frac{4}{3}$. With this definition of rate loss, the scalar rate loss (i.e. for $L=1$) is $R_{\text{loss}} = 0.2358$ bit/dim. whereas for $L=3$ and using the BCC lattice, the rate loss is 0.1681 bit/dim. Furthermore, we have numerically evaluated the terms $\log_2(G(S_L)2\pi e)$ and $\log_2(\Phi_L^2\,\frac{3}{4})$ for $1 \le L \le 21$ (and L odd) as shown in Fig. 2. It may be noticed that $\log_2(\Phi_L^2\,\frac{3}{4})$ is strictly smaller than $\log_2(G(S_L)2\pi e)$. It follows that, at least for this range of dimensions, the overall description rate loss, as given by (59), is less than the space-filling loss of the lattice in question. This is in contrast to, for example, the MD scheme presented in [16] where the description rate loss is larger than the space-filling loss of the lattices being used. At high dimensions, the rate loss vanishes for both schemes.

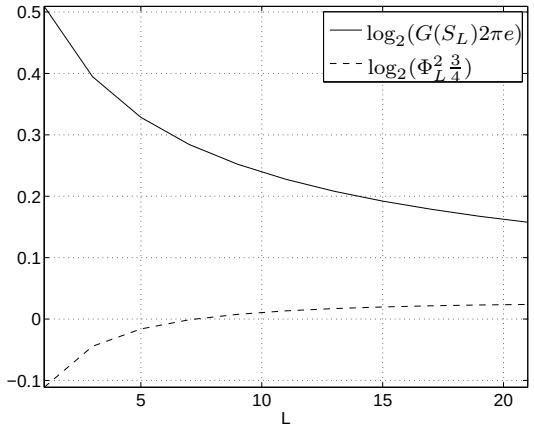

Fig. 2. The terms $\log_2(G(S_L)2\pi e)$ and $\log_2(\Phi_L^2 \frac{3}{4})$ as a function of the dimension L. Both terms converge to 0 in the limit as $L \to \infty$. Notice that $\log_2(G(S_L)2\pi e) > \log_2(\Phi_L^2 \frac{3}{4})$ in the range shown.

V. Conclusion

We proposed a simple method for constructing IA based n-channel asymmetric MD-LVQ schemes. The design was shown to be asymptotical optimal for any number of descriptions. For two descriptions, the rate loss was smaller than that of existing IA based designs whereas for three descriptions, the rate loss was smaller than that of source splitting. It was finally shown that the rate-distortion performance achieves points on the inner bound proposed by Pradhan et al.

Appendix A
Proof of Theorem 1

To prove Theorem 1 we need the following results.

Lemma 4: For $1 \leq \kappa \leq n$ and any $i \in \{0, \ldots, n-1\}$ we have

$$\sum_{\substack{j=0 \\ j \neq i}}^{n-1} \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) = \kappa \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}).$$

Proof: Since $|\mathcal{L}_j^{(n,\kappa)}| = \binom{n-1}{\kappa-1}$ the sum $\sum_{j=0}^{n-1} \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)})$ contains $n\binom{n-1}{\kappa-1}$ terms. However, the number of distinct terms is $|\mathcal{L}^{(n,\kappa)}| = \binom{n}{\kappa}$ and each term is then used κ times, since

$$\frac{n\binom{n-1}{\kappa-1}}{\binom{n}{\kappa}} = \kappa.$$

Subtracting the terms for $j = i$ proves the lemma. ■

Lemma 5: For $1 \leq \kappa \leq n$ and any $i, j \in \{0, \ldots, n-1\}$ we have

$$\sum_{j=0}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) = \kappa \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}).$$

Proof: It is true that $\mathcal{L}_{i,i}^{(n,\kappa)} = \mathcal{L}_{i}^{(n,\kappa)}$ and since $|\mathcal{L}_{i}^{(n,\kappa)}| = \binom{n-1}{\kappa-1}$ and $|\mathcal{L}_{i,j}^{(n,\kappa)}| = \binom{n-2}{\kappa-2}$ the sum $\sum_{j=0}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)})$ contains $(n-1)\binom{n-2}{\kappa-2} + \binom{n-1}{\kappa-1}$ terms. However, the number of distinct $l \in \mathcal{L}_{i}^{(n,\kappa)}$ terms is $|\mathcal{L}_{i}^{(n,\kappa)}| = \binom{n-1}{\kappa-1}$ and each term is then used κ times, since

$$\frac{(n-1)\binom{n-2}{\kappa-2} + \binom{n-1}{\kappa-1}}{\binom{n-1}{\kappa-1}} = \kappa.$$

■

Lemma 6: For $1 \leq \kappa \leq n$ we have

$$\sum_{\ell \in \mathcal{L}^{(n,\kappa)}} \gamma_\ell \left\langle \lambda_c, \sum_{i \in \ell} \lambda_i \right\rangle = \left\langle \lambda_c, \sum_{i=0}^{n-1} \lambda_i \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \right\rangle.$$

Proof: Follows immediately since $\mathcal{L}_{i}^{(n,\kappa)}$ denotes the set of all ℓ-terms that contains the index i. ■

Lemma 7: For $1 \leq \kappa \leq n$ we have

$$\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) \|\lambda_i - \lambda_j\|^2 = \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \left(\kappa \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \right) \|\lambda_i\|^2 - 2 \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) \langle \lambda_i, \lambda_j \rangle.$$

Proof: We have that

$$\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) \|\lambda_i - \lambda_j\|^2 = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) (\|\lambda_i\|^2 + \|\lambda_j\|^2) - 2 \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) \langle \lambda_i, \lambda_j \rangle.$$

Furthermore, it follows that

$$\begin{aligned}
&\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) (\|\lambda_i\|^2 + \|\lambda_j\|^2) \\
&\quad = \sum_{i=0}^{n-2} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \|\lambda_i\|^2 \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) + \sum_{j=1}^{n-1} \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) \|\lambda_j\|^2 \sum_{i=0}^{j-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \\
&\quad = \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \|\lambda_i\|^2 \underbrace{\sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)})}_{0 \text{ for } i=n-1} + \sum_{j=0}^{n-1} \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) \|\lambda_j\|^2 \underbrace{\sum_{i=0}^{j-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)})}_{0 \text{ for } j=0} \\
&\quad = \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_{i}^{(n,\kappa)}) \|\lambda_i\|^2 \left(\sum_{j=0}^{i-1} \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) + \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{j}^{(n,\kappa)}) \right)
\end{aligned}$$

$$= \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\lambda_i\|^2 \sum_{\substack{j=0 \\ j\neq i}}^{n-1} \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)})$$

$$= \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\lambda_i\|^2 \left(\kappa \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \right),$$

where the last equality follows by use of Lemma 4. ■

Lemma 8: For $1 \leq \kappa \leq n$ we have

$$\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \|\lambda_i - \lambda_j\|^2 = (\kappa - 1) \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\lambda_i\|^2 - 2 \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \langle \lambda_i, \lambda_j \rangle.$$

Proof: We have that

$$\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \|\lambda_i - \lambda_j\|^2 = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) (\|\lambda_i\|^2 + \|\lambda_j\|^2) - 2 \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \langle \lambda_i, \lambda_j \rangle.$$

Furthermore, it follows that

$$\begin{aligned}
\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) (\|\lambda_i\|^2 + \|\lambda_j\|^2) &= \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \|\lambda_i\|^2 + \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \|\lambda_j\|^2 \\
&= \sum_{i=0}^{n-2} \|\lambda_i\|^2 \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) + \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \|\lambda_j\|^2 \\
&= \sum_{i=0}^{n-1} \|\lambda_i\|^2 \underbrace{\sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)})}_{0 \text{ for } i=n-1} + \sum_{j=0}^{n-1} \|\lambda_j\|^2 \underbrace{\sum_{i=0}^{j-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)})}_{0 \text{ for } j=0} \\
&= \sum_{i=0}^{n-1} \|\lambda_i\|^2 \left(\sum_{j=0}^{i-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) + \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \right) \\
&= \sum_{i=0}^{n-1} \|\lambda_i\|^2 \left(\sum_{j=0}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \right) \\
&\overset{(a)}{=} \sum_{i=0}^{n-1} \|\lambda_i\|^2 \left(\kappa \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \right) \\
&= (\kappa - 1) \sum_{i=0}^{n-1} \|\lambda_i\|^2 \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}),
\end{aligned}$$

where (a) follows by use of Lemma 5. ■

Lemma 9: For $1 \leq \kappa \leq n$ we have

$$\sum_{\ell \in \mathcal{L}^{(n,\kappa)}} \gamma_\ell \left\| \sum_{i \in \ell} \lambda_i \right\|^2 = \kappa \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\lambda_i\|^2 - \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)}) \|\lambda_i - \lambda_j\|^2.$$

Proof: The set of all elements ℓ of $\mathcal{L}^{(n,\kappa)}$ that contains the index i is denoted by $\mathcal{L}_i^{(n,\kappa)}$. Similar the set of all elements that contains the indices i and j is denoted by $\mathcal{L}_{i,j}^{(n,\kappa)}$. From this we see that

$$\begin{aligned}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell\left\|\sum_{i\in\ell}\lambda_i\right\|^2 &= \sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell\left(\sum_{i\in\ell}\|\lambda_i\|^2+2\sum_{i=0}^{\kappa-2}\sum_{j=i+1}^{\kappa-1}\langle\lambda_{l_i},\lambda_{l_j}\rangle\right)\\ &= \sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\|\lambda_i\|^2+2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\bar\gamma(\mathcal{L}_{i,j}^{(n,\kappa)})\langle\lambda_i,\lambda_j\rangle.\end{aligned}$$

By use of Lemma 8 it follows that

$$\begin{aligned}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell\left\|\sum_{i\in\ell}\lambda_i\right\|^2 &= \sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\|\lambda_i\|^2+(\kappa-1)\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\|\lambda_i\|^2-\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\bar\gamma(\mathcal{L}_{i,j}^{(n,\kappa)})\|\lambda_i-\lambda_j\|^2\\ &= \kappa\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\|\lambda_i\|^2-\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\bar\gamma(\mathcal{L}_{i,j}^{(n,\kappa)})\|\lambda_i-\lambda_j\|^2\end{aligned}$$

■

We are now in a position to prove the following result:

Lemma 10: For $1\le\kappa\le n$ we have

$$\begin{aligned}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell\left\|\lambda_c-\frac{1}{\kappa}\sum_{i\in\ell}\lambda_i\right\|^2 &= \bar\gamma(\mathcal{L}^{(n,\kappa)})\left\|\lambda_c-\frac{1}{\kappa\bar\gamma(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\|^2\\ &\quad+\frac{1}{\kappa^2}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\left(\frac{\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\bar\gamma(\mathcal{L}_j^{(n,\kappa)})}{\bar\gamma(\mathcal{L}^{(n,\kappa)})}-\bar\gamma(\mathcal{L}_{i,j}^{(n,\kappa)})\right)\|\lambda_i-\lambda_j\|^2.\end{aligned}\tag{60}$$

Proof: Expansion of the norm on the left-hand-side in (60) leads to

$$\begin{aligned}&\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell\left\|\lambda_c-\frac{1}{\kappa}\sum_{i\in\ell}\lambda_i\right\|^2 = \sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell\left(\|\lambda_c\|^2-2\left\langle\lambda_c,\frac{1}{\kappa}\sum_{i\in\ell}\lambda_i\right\rangle+\frac{1}{\kappa^2}\left\|\sum_{i\in\ell}\lambda_i\right\|^2\right)\\
&\overset{(a)}{=}\bar\gamma(\mathcal{L}^{(n,\kappa)})\|\lambda_c\|^2-2\left\langle\lambda_c,\frac{1}{\kappa}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\rangle+\frac{1}{\kappa^2}\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\gamma_\ell\left\|\sum_{i\in\ell}\lambda_i\right\|^2\\
&=\bar\gamma(\mathcal{L}^{(n,\kappa)})\left\|\lambda_c-\frac{1}{\kappa\bar\gamma(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\|^2-\frac{1}{\kappa^2\bar\gamma(\mathcal{L}^{(n,\kappa)})}\left\|\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\|^2+\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\frac{\gamma_\ell}{\kappa^2}\left\|\sum_{i\in\ell}\lambda_i\right\|^2\\
&=\bar\gamma(\mathcal{L}^{(n,\kappa)})\left\|\lambda_c-\frac{1}{\kappa\bar\gamma(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\|^2+\sum_{\ell\in\mathcal{L}^{(n,\kappa)}}\frac{\gamma_\ell}{\kappa^2}\left\|\sum_{i\in\ell}\lambda_i\right\|^2\\
&\quad-\frac{1}{\kappa^2\bar\gamma(\mathcal{L}^{(n,\kappa)})}\left(\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})^2\|\lambda_i\|^2+2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\bar\gamma(\mathcal{L}_j^{(n,\kappa)})\langle\lambda_i,\lambda_j\rangle\right)\\
&\overset{(b)}{=}\bar\gamma(\mathcal{L}^{(n,\kappa)})\left\|\lambda_c-\frac{1}{\kappa\bar\gamma(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\|^2+\frac{1}{\kappa}\sum_{i=0}^{n-1}\bar\gamma(\mathcal{L}_i^{(n,\kappa)})\|\lambda_i\|^2\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\kappa^2}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)})\|\lambda_i-\lambda_j\|^2 - \frac{1}{\kappa^2\bar{\gamma}(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})^2\|\lambda_i\|^2 \\
&+ \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\frac{\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})\bar{\gamma}(\mathcal{L}_j^{(n,\kappa)})}{\kappa^2\bar{\gamma}(\mathcal{L}^{(n,\kappa)})}\|\lambda_i-\lambda_j\|^2 - \frac{1}{\kappa^2\bar{\gamma}(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})(\kappa\bar{\gamma}(\mathcal{L}^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}))\|\lambda_i\|^2 \\
&= \bar{\gamma}(\mathcal{L}^{(n,\kappa)})\left\|\lambda_c - \frac{1}{\kappa\bar{\gamma}(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\|^2 \\
&+ \frac{1}{\kappa^2}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\left(\frac{\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})\bar{\gamma}(\mathcal{L}_j^{(n,\kappa)})}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})} - \bar{\gamma}(\mathcal{L}_{i,j}^{(n,\kappa)})\right)\|\lambda_i-\lambda_j\|^2,
\end{aligned}
$$

where (a) follows by use of Lemma 6 and (b) by use of Lemmas 7 and 9. ■

APPENDIX B
PROOF OF THEOREM 2

Without loss of generality, let $\mu_i = 1, \forall i$. Furthermore, let

$$f = \sum_{\lambda_c\in V_\pi(0)}\sum_{\kappa=1}^{n-1}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\hat{\gamma}_{i,j}^{(n,\kappa)}\left\|\lambda_i-\lambda_j\right\|^2 \tag{61}$$

and

$$g = \sum_{\lambda_c\in V_\pi(0)}\sum_{\kappa=1}^{n-1}\left\|\lambda_c - \frac{1}{\kappa\bar{\gamma}(\mathcal{L}^{(n,\kappa)})}\sum_{i=0}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})\lambda_i\right\|^2. \tag{62}$$

We prove the theorem by constructing a labeling function which lower bounds f independently of g. We then show that with this choice of f we have $f\to\infty$ and $g\to\infty$ but $g/f\to 0$ as $N_i\to\infty, \nu_i\to 0, \forall i$. Furthermore, we show that this holds for any admissible choice of g. Since $\mathcal{J}^n = c_0 f + c_1 g$ for some constants $c_0, c_1 \in \mathbb{R}$ it follows that in order to minimize $\mathcal{J}^n$ an optimal labeling function must jointly minimize f and g. However, a jointly optimal labeling function can never improve upon the lower bound on f which occur when f is independently minimized. Furthermore, g can only be reduced if taking into account during the optimization. Thus, for any optimal labeling function we must have $g/f\to 0$. It follows that f is asymptotically dominating and therefore must be minimized in order to minimize $\mathcal{J}^n$.

Let $\mathcal{T}$ denote the set of n-tuples assigned to central lattice points in $V_\pi(0)$ so that $|\mathcal{T}| = N_\pi$ and let $\mathcal{T}_i$ be the set of ith elements (i.e. a set of sublattice points all from Λ_i). Moreover, let $\mathcal{T}(\lambda_i)$ be the set of n-tuples containing a specific λ_i as the ith element. Finally, let $\mathcal{T}_j(\lambda_i)$ be the set of $\lambda_j\in\Lambda_j$ sublattice points which are the jth elements in the n-tuples that has the specific λ_i as the ith element. With this, for any fixed $\lambda_0\in\mathcal{T}_0$, the sum $\sum_{\lambda_1\in\mathcal{T}_1(\lambda_0)}\|\lambda_0-\lambda_1\|^2$ runs over the set of λ_1 points which are in the same n-tuples as the given λ_0. Notice that this sum can be written as $\sum_{\lambda_1\in\mathcal{T}_1^u(\lambda_0)}\#_{\lambda_1}\|\lambda_0-\lambda_1\|^2$ where the

superscript ^{u} denotes the unique λ_1 elements of $\mathcal{T}_1(\lambda_0)$ and $\#_{\lambda_1}$ denotes the number of times the given λ_1 is used. Clearly, this sum is minimized if the unique λ_1 points are as close as possible to the given λ_0. In other words, for any given "distribution" $\{\#_{\lambda_1}\}$, the sum is minimized if the λ_1's are contained within the smallest possible sphere around λ_0. In fact, this holds for any $\lambda_0 \in \mathcal{T}_0$. On the other hand, keeping the set of λ_1's fixed we can also seek the minimizing distribution $\{\#_{\lambda_j}\}$. A good choice appears to be that the λ_j points that are closer to the given λ_0 should be used more frequently than those further way.

We pause to make the following observation. Due to the shift-invariance property of the labeling function, we can restrict attention to the n-tuples which are assigned to central lattice points within $\Lambda_\pi(0)$. Thus, we have a total of N_π n-tuples. Recall that we guarantee the shift-invariance property by restricting λ_0 to be inside $V_\pi(0)$ (a restriction which we later relax by considering cosets). Furthermore, to avoid possible bias towards any $\lambda_0 \in V_\pi(0)$, we require that each λ_0 is used an equal amount of times. Since there are N_π/N_0 distinct λ_0 points in $V_\pi(0)$ it follows that each λ_0 must be used N_0 times.

Let us for the moment being consider the case of $n = 3$, i.e. we need to construct a set of N_π triplets $\mathcal{T} = \{(\lambda_0, \lambda_1, \lambda_2)\}$. If we fix some λ_0, we can construct a set of pairs of sublattice points by centering a sphere $\tilde{V}$ at λ_0 and forming the set of distinct pairs $\mathcal{S} = \{(\lambda_0, \lambda_1) : \lambda_1 \in \tilde{V}(\lambda_0) \cap \Lambda_1\}$. For each pair $(\lambda_0', \lambda_1') \in \mathcal{S}$ we can form a triplet $(\lambda_0', \lambda_1', \lambda_2)$ by combining the given pair with some λ_2. It is important that λ_2 is close to λ_0' as well as λ_1' in order to reduce the distances $\|\lambda_0' - \lambda_2\|^2$ and $\|\lambda_1' - \lambda_2\|^2$. This can be done by guaranteeing that $\lambda_2 \in \tilde{V}(\lambda_0)$ and $\lambda_2 \in \tilde{V}(\lambda_1)$. In other words, $\lambda_2 \in \tilde{V}(\lambda_0) \cap \tilde{V}(\lambda_1)$. With this strategy, fix some $\lambda_0 \in V_\pi(0)$ and start by using some "small" $\tilde{V}$ in order to construct the set of pairs $\tilde{S} = \{(\lambda_0, \lambda_1) : \lambda_1 \in \tilde{V}(\lambda_0) \cap \Lambda_1\}$. Then for each pair $s \in \tilde{S}$ we construct the set of triplets $\{(s, \lambda_2) : \lambda_2 \in \tilde{V}(\lambda_0) \cap \tilde{V}(\lambda_1)\}$. Recall that we need N_0 triplets for each λ_0. However, since $\tilde{V}$ was chosen "small" we end up with too few triplets. The trick is now to increase the volume of $\tilde{V}$ in small steps until we end up with exactly N_0 triplets (keep in mind that for large N_0 we work with large volumes).

In the n-description case, we require that $\lambda_k \in \tilde{V}(\lambda_0) \cap \tilde{V}(\lambda_1) \cap \cdots \cap \tilde{V}(\lambda_{k-1})$. If we let r be the radius of $\tilde{V}$, then with the above procedure it is guaranteed that $\|\lambda_i - \lambda_j\|^2 \leq r^2/L$ for all (i, j) where $i, j = 0, \ldots, n-1$.

Notice that f, i.e. the expression to be minimized as given by (61), includes weights $\hat{\gamma}_{i,j}^{(n,\kappa)}$ (which might not be equal) for every pair of sublattices. In otherwords, we might use spheres $\tilde{V}_{i,j}$ of different sizes to guarantee that $\|\lambda_i - \lambda_j\|^2 \leq r_{i,j}^2/L$ where the radius $r_{i,j}$ now depends on the particular pair of sublattices under consideration. This is illustrated in Fig. 3 where $r_{i,j}$ denotes the radius of the sphere

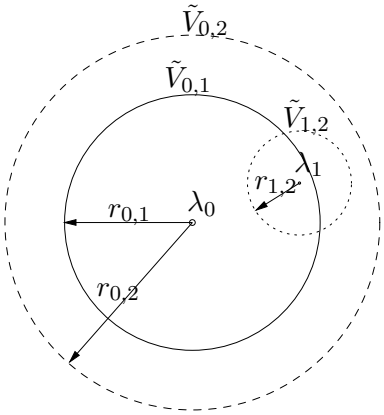

Fig. 3. Different sizes of the spheres.

$\tilde{V}_{i,j}$. Here we center $\tilde{V}_{0,1}$ at some $\lambda_0 \in V_\pi(0)$ as illustrated in Fig. 3 by the solid circle. Then, for any n-tuples having this λ_0 point as first element, we only include λ_1 points which are inside $\tilde{V}_{0,1}(\lambda_0)$. This guarantees that $\|\lambda_0 - \lambda_1\|^2 \leq r_{0,1}/L$. Let us now center a sphere $\tilde{V}_{1,2}$ at some λ_1 which is inside $\tilde{V}_{0,1}(\lambda_0)$. This is illustrated by the dotted sphere of radius $r_{1,2}$ in the figure. We then only include λ_2 points which are in the intersection of $\tilde{V}_{1,2}(\lambda_1)$ and $\tilde{V}_{0,2}(\lambda_0)$. This guarantees that $\|\lambda_i - \lambda_j\|^2 \leq r_{i,j}/L$ for all (i,j) pairs. Interestingly, from Fig. 3 we see that $r_{0,2}$ cannot be greater than $r_{0,1} + r_{1,2}$. Thus, the radius $r_{i,j}$ must grow at the same rate for any pair (i,j) so that, without loss of generality, $r_{0,1} = a_2 r_{0,2} = a_1 r_{1,2}$ for some fixed $a_1, a_2 \in \mathbb{R}$.

Recall that, the number $\tilde{N}_i$ of lattice points of Λ_i within a connected region $\tilde{V}$ of $\mathbb{R}^L$ may be approximated by $\tilde{N}_i \approx \tilde{\nu}/\nu_i$ where $\tilde{\nu}$ is the volume of $\tilde{V}$. Moreover, the number of λ_0 points within $V_\pi(0)$ is given by $\#_{\lambda_0} \approx \text{Vol}(V_\pi(0))/\nu_0 = \nu_c N_\pi/\nu_0$. Since we need to construct a total of N_π n-tuples to label the N_π central lattice points, it follows that each λ_0 is used $N_\pi/\#_{\lambda_0} = N_0$ times. Let us now center a sphere $\tilde{V}$ of volume $\tilde{\nu}$ at some $\lambda_0 \in \Lambda_0$. The number of λ_i points inside this sphere is asymptotically given by $\tilde{\nu}/\nu_i$. Thus, the number of distinct n-tuples we can construct by forming all combinations of sublattice points from $\Lambda_i, i = 1, \ldots, n-1$ within $\tilde{V}$ and using λ_0 as first element is given by $\tilde{\nu}^{n-1}/(\prod_{i=1}^{n-1} \nu_i)$. Recall that we need N_0 n-tuples for each λ_0. Thus, we obtain $N_0 = \tilde{\nu}^{n-1}/(\prod_{i=1}^{n-1} \nu_i)$ from which we see that the volume of the sphere $\tilde{V}$ must satisfy

$$\tilde{\nu} \geq \nu_c \prod_{i=0}^{n-1} N_i^{\frac{1}{n-1}}. \tag{63}$$

We previously argued that we need to make $\tilde{V}$ large enough so as to be able to create exactly N_0 n-tuples for each λ_0 which satisfy $\|\lambda_i - \lambda_j\|^2 \leq r^2/L$. Having equality in (63) guarantees that $\|\lambda_0 - \lambda_j\|^2 \leq r^2/L$ for $j = 1, \ldots, n-1$ but then we must have $\|\lambda_i - \lambda_j\|^2 > r^2/L$ for some $i \neq 0$. However, since we

are aiming at lower bounding f we may indeed proceed by assuming that $\|\lambda_i - \lambda_j\|^2 \leq r^2/L$ for all i. Furthermore, the different radii $r_{i,j}$ are related through a multiplicative constant which will not affect the rate of growth of the volumes of $\tilde{V}_{i,j}$ as $N_i \to \infty$. Thus, we proceed by assuming $r_{i,j} = r$ so that $\tilde{V}_{i,j} = \tilde{V}$.

We are now in a position to evaluate the following sum

$$\sum_{\lambda_j \in \mathcal{T}_j(\lambda_i)} \|\lambda_i - \lambda_j\|^2 \stackrel{(a)}{=} \sum_{\lambda_j \in \mathcal{T}_j(0)} \|\lambda_j\|^2 \tag{64}$$

$$= \sum_{\lambda_j \in \mathcal{T}_j^u(0)} \#\lambda_j \|\lambda_j\|^2. \tag{65}$$

The volume of the sphere $\tilde{V}$ is independent of which sublattice point it is centered at, so we may take $\lambda_i = 0$ from which (a) follows. Notice that for a fixed λ_0 and for $n > 2$, the set $\mathcal{T}_j(\lambda_i)$ contains several identical λ_j elements. We therefore use the notation $\mathcal{T}_j^u(\lambda_i)$ to indicate the unique set of λ_j elements. Furthermore, we use the notation $\#_{\lambda_j}$ to indicate the number of times the given λ_j is used. Since $\sum_{\lambda_j \in T_j^u(0)} \#_{\lambda_j} = N_0$ it follows that $\sum_{\lambda_j \in T_j^u(0)} \min_j\{\#_{\lambda_j}\} \leq N_0$ so that $\min_j\{\#_{\lambda_j}\} \leq N_0 / \sum_{\lambda_j \in T_j^u(0)}$. Moreover, $|\lambda_j \in T_j^u(0)| = \tilde{\nu}/\nu_j$ which implies that $\sum_{\lambda_j \in T_j^u(0)} = \tilde{\nu}/\nu_j$ and we therefore have that $\min_j\{\#_{\lambda_j}\} \leq N_0 \nu_j/\tilde{\nu}$. By similar reasoning it is easy to show that $\max_j\{\#_{\lambda_j}\} \geq N_0 \nu_j/\tilde{\nu}$.

We have previously shown that the intersection of any number of (large) spheres of equal radii r which are distanced no further apart than r, is positively bounded away from zero [13]. In fact, the volume of the smallest intersection can be lower bounded by the volume of a regular L-simplex having side lengths r [13]. Recall that the volume $\mathrm{Vol}(\mathfrak{S})$ of a regular L-simplex $\mathfrak{S}$ with side length r is given by [38]

$$\mathrm{Vol}(\mathfrak{S}) = \frac{r^L}{L!}\sqrt{\frac{L+1}{2^L}} = c_L r^L \tag{66}$$

where c_L is constant for fixed L. It follows that, in the three channel case, $\#_{\lambda_j}$ is lower bounded by $c_L r^L/\nu_k$ where ν_k is the volume of sublattice with the largest index value. Moreover, for $n \geq 3$ we have that $\#_{\lambda_j}$ is lower bounded by $(c_L r^L/\nu_k)^{n-2}$.

Interestingly, $\#_{\lambda_j}$ is obviously upper bounded by $(\omega_L r^L/\nu_{k'})^{n-2}$, i.e. ratio of the volume of an L-dimensional sphere of radius r and the volume of a Voronoi cell of $\Lambda_{k'}$, where k' denotes the sublattice with the smallest index value. Notice that the lower bound is proportional to the upper bound and we have the following sandwhich

$$\left(\frac{\omega_L r^L}{\nu_{k'}}\right)^{n-2} \geq \max_j\{\#_{\lambda_j}\} \geq N_0 \nu_j/\tilde{\nu} \geq \min_j\{\#_{\lambda_j}\} \geq \left(\frac{c_L r^L}{\nu_k}\right)^{n-2} \tag{67}$$

where the left and right hand sides of (67) differ by a constant for any n which implies that there exists a positive constant $c > 0$ such that $\min_j\{\#_{\lambda_j}\} \geq c N_0 \nu_j/\tilde{\nu}$.

Using the above in (65) leads to

$$\begin{aligned}\sum_{\lambda_j\in\mathcal{T}_j(\lambda_i)}\|\lambda_i-\lambda_j\|^2\nu_j &> \frac{cN_i\nu_j}{\tilde{\nu}}\sum_{\lambda_j\in\mathcal{T}_j^u(0)}\|\lambda_j\|^2\nu_j\\ &\overset{(a)}{\approx} \frac{cN_i\nu_j}{\tilde{\nu}}\int_{x\in\tilde{V}}\|x\|^2dx\\ &= \frac{cN_i\nu_j}{\tilde{\nu}}LG(S_L)\tilde{\nu}^{1+2/L}\\ &= cN_i\nu_jLG(S_L)\nu_c^{2/L}\prod_{i=0}^{n-1}N_i^{\frac{2}{(n-1)L}}\end{aligned}$$

where $G(S_L)$ is the dimensionless normalized second moment of an L-dimensional hypersphere and (a) follows by replacing the sum by an integral (standard Riemann approximation). This approximations becomes exact asymptotically as $N_i\to\infty$ and $\nu_i\to 0$.

We finally see that

$$\begin{aligned}\frac{1}{L}\sum_{\lambda_c\in V_\pi(0)}\left\|\lambda_i-\lambda_j\right\|^2 &= \frac{1}{L}\sum_{\lambda_i\in\mathcal{T}_i}\sum_{\lambda_j\in\mathcal{T}_j(\lambda_i)}\|\lambda_i-\lambda_j\|^2\\ &= \frac{1}{L}\frac{N_\pi}{N_i}\sum_{\lambda_j\in\mathcal{T}_j(\lambda_i)}\|\lambda_i-\lambda_j\|^2\\ &> cG(S_L)N_\pi\nu_c^{2/L}\prod_{i=0}^{n-1}N_i^{\frac{2}{(n-1)L}}\end{aligned}$$

so that

$$f=\Omega\left(N_\pi\nu_c^{2/L}\prod_{i=0}^{n-1}N_i^{\frac{2}{(n-1)L}}\right). \tag{68}$$

We now upper bound g. Notice that g describes the sum of distances between the central lattice points and the weighted average of their associated n-tuples. By construction, these weighted averages will be distributed evenly through-out $V_\pi(0)$. Thus, the distance of a central lattice point and the weighted average of its associated n-tuple can be upper bounded by the covering radius of the sublattice with the largest index value, say N_k. This is a conservative upper bound but will do for the proof.[11] The rate of growth of the covering radius of the kth sublattice is proportional to $\nu_k^{1/L}=(N_k\nu_c)^{1/L}$. Thus

$$g=\mathcal{O}\left(N_\pi\nu_c^{2/L}N_k^{2/L}\right). \tag{69}$$

[11]The worst case situation occur if the weighted centroids are distributed such that the minimal distance between any two centroids is maximized. Notice that the weighted centroids form convex combinations of the sublattice points. Since the weights are less than one, the worst case situation occurs if the weighted centroids are distributed on a lattice with an index value equal to the sublattice with the maximum index value (and therefore also the maximum covering radius). Thus, the bound is indeed valid for an arbitrary set of n-tuples and not tied to the specific construction of n-tuples used so far in the proof.

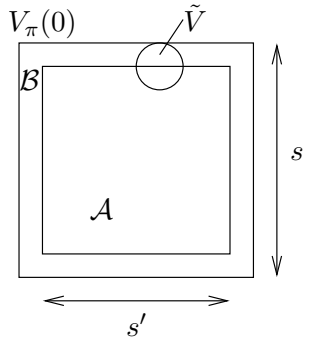

Fig. 4. $V_\pi(0)$ is a Voronoi cell of Λ_π. $\mathcal{A}$ is a scaled and centered version of $V_\pi(0)$ and $\mathcal{B}$ is the "strip" surrounding $\mathcal{A}$, i.e., $\mathcal{B} = V_\pi(0) \backslash \mathcal{A}$.

It follows that

$$\frac{g}{f} = \Theta\left(\frac{N_k^{2/L}}{\prod_{i=0}^{n-1} N_i^{\frac{2}{(n-1)L}}}\right) = \Theta\left(N_k^{-\frac{2}{L(n-1)}}\right) \tag{70}$$

where the last equality follows since the index values are growing at the same rate so that $N_i = N_k/b_i$ for some constant $b_i \in \mathbb{R}$. The theorem is proved by noting that $N_k^{-\frac{2}{L(n-1)}} \to 0$ as $N_k \to \infty$. ■

APPENDIX C
PROOF OF THEOREM 5

We restrict attention to the case where $V_\pi(0)$ is the Voronoi cell of a product lattice generated by the approach outlined in Section II-B. In this case, the shape of $V_\pi(0)$ can be either hyper cubic, or as the dimension increases, the shape can become more and more spherical.

Let us first assume that that $V_\pi(0)$ forms a hyper cube having side lengths s as shown in Fig. 4. The n-tuples are constructed by centering a sphere $\tilde{V}$ of volume $\tilde{\nu}$ around each $\lambda_0 \in V_\pi(0)$ and taking all combinations of lattice points within this region (keeping λ_0 as first coordinate). From Fig. 4 it may be seen that any λ_0 which is contained in the region denoted $\mathcal{A}$ will always be combined with sublattice points that are also contained in $V_\pi(0)$. On the other hand, any λ_0 which is contained in region $\mathcal{B}$ will occasionally be combined with points outside $V_\pi(0)$. Therefore, we need to show that the volume $V_\mathcal{A}$ of $\mathcal{A}$ approaches the volume of $V_\pi(0)$ as $N_\pi \to \infty$ or similarly that the ratio of $V_\mathcal{B}/V_\mathcal{A} \to 0$ as $N_\pi \to \infty$, where $V_\mathcal{B}$ denotes the volume of the region $\mathcal{B}$.

Let $\mathcal{A}$ be the centered hyper cube having side lengths $s' = s - 2\tilde{r}$ where $\tilde{r}$ is the radius of $\tilde{V}$, see Fig. 4. Since the volume of $V_\pi(0)$ is $\nu_\pi = \nu N_\pi$ it follows that $s = \nu_\pi^{1/L} = (\nu N_\pi)^{1/L}$. Moreover, the volume $V_\mathcal{A}$ of $\mathcal{A}$ is

$$V_\mathcal{A} = (s')^L$$

$$
\begin{aligned}
&= \left(s - 2\tilde{r}\right)^L \\
&= \left(\nu_\pi^{1/L} - 2\left(\frac{\tilde{\nu}}{\omega_L}\right)^{1/L}\right)^L \\
&= \left((\nu N_\pi)^{1/L} - 2\left(\frac{\psi_{n,L}^L \nu N_\pi^{1/(n-1)}}{\omega_L}\right)^{1/L}\right)^L \\
&= \nu\left(N_\pi^{1/L} - 2\left(\frac{\psi_{n,L}^L}{\omega_L}\right)^{1/L} N_\pi^{1/L(n-1)}\right)^L,
\end{aligned}
$$

where $\tilde{\nu} = \psi_{L,n}^L \nu N_\pi^{1/(n-1)}$ is the volume of $\tilde{V}$ and $\tilde{r} = (\tilde{\nu}/\omega_L)^{1/L}$, where ω_L is the volume of an L-dimensional unit sphere. Since the volume $V_\mathcal{B}$ of $\mathcal{B}$ is given by $V_\mathcal{B} = \nu_\pi - V_\mathcal{A}$ we find the ratio

$$
\lim_{N_\pi \to \infty} \frac{V_\mathcal{B}}{V_\mathcal{A}} = \lim_{N_\pi \to \infty} \frac{N_\pi}{\left(N_\pi^{1/L} - 2\left(\frac{\psi_{n,L}^L}{\omega_L}\right)^{1/L} N_\pi^{1/L(n-1)}\right)^L} - 1 \tag{71}
$$

$$
= 0, \tag{72}
$$

where the second equality follows since $n > 2$.

At this point, we note that the hyper cubic region as used above is actually the worst shape to consider. Specifically, it is the one that yields the minimum $V_\mathcal{A}$ and thus the maximum $V_\mathcal{B}$, since ν_π is constant. To see this, note that we can always pick the region $\mathcal{A}$ to be a centered scaled version of $V_\pi(0)$. Thus, since the boundary of the inscribed region $\mathcal{A}$ will be farthest away from the boundary of $V_\pi(0)$ at corner points it follows that the more spherical $V_\pi(0)$, the larger $V_\mathcal{A}$ compared to ν_π. This proves the claim. ■

APPENDIX D
PROOF OF THEOREM 6

We only prove it for Λ_0 and Λ_1. Then by symmetry it must hold for any pair. Define the set $\mathcal{S}_{\lambda_0}$ as the set of n-tuples constructed by centering $\tilde{V}$ at some $\lambda_0 \in V_\pi(0) \cap \Lambda_0$. Hence, $s \in \mathcal{S}_{\lambda_0}$ has λ_0 as first coordinate and the distance between any two elements of s is less than r, the radius of $\tilde{V}$. We will assume[12] that $S_{\lambda_0} \neq \emptyset, \forall \lambda_0$.

[12]This is always the case if $r \geq \max_i r(\Lambda_i)$ where $r(\Lambda_i)$ is the covering radius of the ith sublattice. The covering radius depends on the lattice and is maximized if Λ_i is geometrically similar to Z^L, in which case we have [19]

$$
r(\Lambda_i) = \frac{1}{2}\sqrt{2}\nu^{1/L} N_i^{1/L}.
$$

Since $r = \psi_{n,L}\nu^{1/L} N_\pi^{1/L(n-1)}/\omega_L^{1/L}$ it follows that in order to make sure that $\mathcal{S}_{\lambda_0} \neq \emptyset$ the index values must satisfy

$$
N_i \leq (\sqrt{2}\psi_{n,L})^L \omega_L N_\pi^{1/(n-1)}, \quad i = 0, \ldots, n-1. \tag{*}
$$

Similarly, define the set $\mathcal{S}_{\lambda_1} \neq \emptyset$ by centering $\tilde{V}$ at some $\lambda_1 \in V_\pi(0) \cap \Lambda_1$. Recall from Theorem 5 that, asymptotically as $N_i \to \infty, \forall i$, all elements of the n-tuples are in $V_\pi(0)$. Then it must hold that for any $s \in \mathcal{S}_{\lambda_1}$ we have $s \in \bigcup_{\lambda_0 \in V_\pi(0)\cap\Lambda_0} \mathcal{S}_{\lambda_0}$. But it is also true that for any $s' \in \mathcal{S}_{\lambda_0}$ we have $s' \in \bigcup_{\lambda_1 \in V_\pi(0)\cap\Lambda_1} \mathcal{S}_{\lambda_1}$. Hence, since the n-tuples in $\bigcup_{\lambda_0 \in V_\pi(0)\cap\Lambda_0} \mathcal{S}_{\lambda_0}$ are distinct and the n-tuples in $\bigcup_{\lambda_1 \in V_\pi(0)\cap\Lambda_1} \mathcal{S}_{\lambda_1}$ are also distinct, it follows that the two sets $\bigcup_{\lambda_0 \in V_\pi(0)\cap\Lambda_0} \mathcal{S}_{\lambda_0}$ and $\bigcup_{\lambda_1 \in V_\pi(0)\cap\Lambda_1} \mathcal{S}_{\lambda_1}$ must be equivalent. ■

APPENDIX E
PROOF OF THEOREM 8

We notice from Lemma 1 and (14) that $\bar{D}_\ell = \mathbb{E}\|X - \hat{X}_\ell\|^2$ can be written as

$$\bar{D}_\ell = \frac{1}{N_\pi} \sum_{\lambda_c \in V_\pi(0)} \left\| \lambda_c - \frac{1}{\kappa} \sum_{i \in \ell} \mu_i \alpha_i(\lambda_c) \right\|^2 + \left\{ \sum_{\lambda_c \in \Lambda_c} \int_{V_c(\lambda_c)} f_X(x) \|X - \lambda_c\|^2 \, dx \right\} \tag{73}$$

where from (38) we know that the last term is $G(\Lambda_c)\nu_c^{2/L}$. In the following we therefore focus on finding a closed-form solution to the first term in (73). This we do by taking the following three steps (which are valid in the usual asymptotical sense):

1) We first show, by Proposition 1, that

$$\sum_{\lambda_c \in V_\pi(0)} \left\| \lambda_c - \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j \right\|^2 = \sum_{\lambda_c \in V_\pi(0)} \left\| \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2 . \tag{74}$$

2) Then, by Lemma 14, we show that

$$\sum_{\lambda_c \in V_\pi(0)} \left\| \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2 = \sum_{\lambda_c \in V_\pi(0)} \sum_{k} \sum_{i=0}^{n_k-2} \sum_{i=0}^{n_k-1} c_k \|\tilde{\lambda}_i - \tilde{\lambda}_j\|^2 \tag{75}$$

for some $c_k \in \mathbb{R}$ and $n_k \leq n$.

3) Finally, we show by Proposition 2 that for the case of $n = 3$, we have

$$\sum_{\lambda_c \in V_\pi(0)} \|\tilde{\lambda}_i - \tilde{\lambda}_j\|^2 = c\nu_c^{2/L} N_\pi \prod_{m=0}^{2} N_m^{1/L} \tag{76}$$

for some $c \in \mathbb{R}$.

In order to establish step 1, we need the following results.

Lemma 11: For any $1 \leq \kappa < n$ and $\ell \in \mathcal{L}^{(n,\kappa)}$ we have

$$\sum_{\lambda_c \in V_\pi(0)} \left\| \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\| = \mathcal{O}\left(\nu_c^{1/L} N_\pi \prod_{m=0}^{n-1} N_m^{1/L(n-1)} \right).$$

Through-out this work we therefore require (and implicitly assume) that (*) is satisfied.

Proof: Recall that the sublattice points λ_i and λ_j satisfy $\|\lambda_i - \lambda_j\| \leq r/\sqrt{L}$, where $r = (\tilde{\nu}/\omega_L)^{1/L}$ is the radius of $\tilde{V}$. Without loss of generality, we let $\tilde{\lambda}_j = r$ and $\tilde{\lambda}_i = 0$, which leads to

$$\sum_{\lambda_c \in V_\pi(0)} \left\| \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\| \leq c \frac{r N_\pi}{\sqrt{L}} = \mathcal{O}\left(\nu_c^{1/L} N_\pi \prod_{m=0}^{n-1} N_m^{1/L(n-1)} \right)$$

where $0 < c \in \mathbb{R}$ and $\tilde{\nu} = \psi_{n,L}^L \nu_c \prod_{m=0}^{n-1} N_m^{1/(n-1)}$. ■

Proposition 1: For $1 \leq \kappa < n$, $\ell \in \mathcal{L}^{(n,\kappa)}$, $N_i \to \infty$ and $\nu_i \to 0$ we have

$$\sum_{\lambda_c \in V_\pi(0)} \left\| \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j - \lambda_c \right\|^2 = \sum_{\lambda_c \in V_\pi(0)} \left\| \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2$$

where $\tilde{\lambda}_j = \mu_j \alpha_j(\lambda_c)$.

Proof: Let $\bar{\lambda} = \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i$ and $\lambda' = \frac{1}{\kappa} \sum_{j \in \ell} \tilde{\lambda}_j$. After some manipulations similar to [14, Eqs. (67) – (72)] we obtain the following inequalities:

$$\left(\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\|^2 \right) \left(1 - 2 \frac{\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\| \|\bar{\lambda} - \lambda_c\|}{\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\|^2} \right) \leq \sum_{\lambda_c \in V_\pi(0)} \|\lambda_c - \lambda'\|^2 \tag{77}$$

$$\leq \left(\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\|^2 \right) \left(1 + \frac{\sum_{\lambda_c \in V_\pi(0)} \|\bar{\lambda} - \lambda_c\|^2}{\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\|^2} + 2 \frac{\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\| \|\bar{\lambda} - \lambda_c\|}{\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\|^2} \right). \tag{78}$$

We now use the fact that $\|\bar{\lambda} - \lambda_c\| = \mathcal{O}(N_k \nu_c)^{1/L}$, i.e. we can upper this distance by the covering radius of the sublattice with the largest index value, say N_k. By use of Lemma 11, it is possible to upper bound the numerator of the fraction in (77) by

$$\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\| \|\bar{\lambda} - \lambda_c\| = \mathcal{O}\left((N_k \nu_c)^{1/L} N_\pi \nu_c^{1/L} \prod_{m=0}^{n-1} N_m^{1/L(n-1)} \right). \tag{79}$$

At this point we recall that the growth of the denominator in (77) is at least as great as (68), which leads to the following lower bound

$$\sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\|^2 = \Omega\left(\nu_c^{2/L} N_\pi \prod_{m=0}^{n-1} N_m^{2/L(n-1)} \right). \tag{80}$$

By comparing (79) to (80), it follows that the fractions in (77) and (78) go to zero asymptotically as $N_i \to \infty$. It follows that

$$\sum_{\lambda_c \in V_\pi(0)} \|\lambda_c - \lambda'\|^2 = \sum_{\lambda_c \in V_\pi(0)} \|\lambda' - \bar{\lambda}\|^2 \tag{81}$$

which completes the proof. ■

In order to establish step 2, we need the following results.

Lemma 12: For $1 \leq \kappa < n$ and any $\ell \in \mathcal{L}^{(n,\kappa)}$ we have

$$\left\| \sum_{j\in\ell} \tilde{\lambda}_j \right\|^2 = \kappa \sum_{j\in\ell} \|\tilde{\lambda}_j\|^2 - \sum_{i=0}^{\kappa-2} \sum_{j=i+1}^{\kappa-1} \|\tilde{\lambda}_{l_j} - \tilde{\lambda}_{l_i}\|^2.$$

Proof: We expand the norm as follows

$$\left\| \sum_{j\in\ell} \tilde{\lambda}_j \right\|^2 = \sum_{j\in\ell} \|\tilde{\lambda}_j\|^2 + 2\sum_{i=0}^{\kappa-2} \sum_{j=i+1}^{\kappa-1} \langle \tilde{\lambda}_{l_j}, \tilde{\lambda}_{l_i} \rangle = \kappa \sum_{j\in\ell} \|\tilde{\lambda}_j\|^2 - \sum_{i=0}^{\kappa-2} \sum_{j=i+1}^{\kappa-1} \|\tilde{\lambda}_{l_j} - \tilde{\lambda}_{l_i}\|^2.$$

■

Lemma 13: For $1 \leq \kappa < n$ and any $\ell \in \mathcal{L}^{(n,\kappa)}$ we have

$$2\left\langle \sum_{j\in\ell} \tilde{\lambda}_j, \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\rangle = \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \kappa \sum_{j\in l} \|\tilde{\lambda}_j\|^2 + \kappa \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_i\|^2 - \sum_{j\in l} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_j - \tilde{\lambda}_i\|^2.$$

Proof:

$$\begin{aligned}
2\left\langle \sum_{j\in\ell} \tilde{\lambda}_j, \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\rangle &= 2\sum_{j\in\ell} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \langle \tilde{\lambda}_j, \tilde{\lambda}_i \rangle \\
&= -\sum_{j\in\ell} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_j - \tilde{\lambda}_i\|^2 + \sum_{j\in\ell} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \left(\|\tilde{\lambda}_j\|^2 + \|\tilde{\lambda}_i\|^2 \right) \\
&= -\sum_{j\in\ell} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_j - \tilde{\lambda}_i\|^2 + \kappa \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \sum_{j\in\ell} \|\tilde{\lambda}_j\|^2 + \kappa \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_i\|^2
\end{aligned}$$

where the last equality follows from Lemma 4. ■

Lemma 14: For any $1 \leq \kappa < n$ and $\ell \in \mathcal{L}^{(n,\kappa)}$ we have

$$\begin{aligned}
\left\| \frac{1}{\kappa} \sum_{j\in\ell} \tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2 = \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa^2} \Bigg(& \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \sum_{j\in\ell} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_j - \tilde{\lambda}_i\|^2 \\
& - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \sum_{i=0}^{\kappa-2} \sum_{j=i+1}^{\kappa-1} \|\tilde{\lambda}_{l_i} - \tilde{\lambda}_{l_j}\|^2 - \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) \|\tilde{\lambda}_i - \tilde{\lambda}_j\|^2 \Bigg).
\end{aligned}$$

Proof: We have that

$$\begin{aligned}
\left\| \frac{1}{\kappa} \sum_{j\in\ell} \tilde{\lambda}_j - \frac{1}{\kappa\bar{\gamma}(\mathcal{L}^{(n,\kappa)})} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2 = \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa^2} \Bigg(& \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \left\| \sum_{j\in\ell} \tilde{\lambda}_j \right\|^2 + \left\| \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2 \\
& - 2\bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \left\langle \sum_{j\in\ell} \tilde{\lambda}_j, \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\rangle \Bigg)
\end{aligned}$$

which by use of Lemmas 12 and 13 leads to

$$\left\| \frac{1}{\kappa} \sum_{j\in\ell} \tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \tilde{\lambda}_i \right\|^2 = \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa^2} \Bigg(\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa \sum_{j\in\ell} \|\tilde{\lambda}_j\|^2 \tag{82}$$

$$
\begin{aligned}
&- \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \sum_{i=0}^{\kappa-2} \sum_{j=i+1}^{\kappa-1} \|\tilde{\lambda}_{l_i} - \tilde{\lambda}_{l_j}\|^2 + \bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_i\|^2 \\
&- \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) \|\tilde{\lambda}_i - \tilde{\lambda}_j\|^2 - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa \sum_{j\in\ell} \|\tilde{\lambda}_j\|^2 - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_i\|^2 \\
&+ \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \sum_{j\in\ell} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_j - \tilde{\lambda}_i\|^2 \Bigg) \\
= \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa^2} &\Bigg(\bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \sum_{j\in\ell} \sum_{i=0}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \|\tilde{\lambda}_j - \tilde{\lambda}_i\|^2 - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \sum_{i=0}^{\kappa-2} \sum_{j=i+1}^{\kappa-1} \|\tilde{\lambda}_{l_i} - \tilde{\lambda}_{l_j}\|^2 \\
&- \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) \|\tilde{\lambda}_i - \tilde{\lambda}_j\|^2 \Bigg).
\end{aligned}
$$

■

In order to establish step 3, we extend the proof technique previously used to find $\psi_{n,L}$ in [13]. Let a_m denote the number of λ_1 points at distance m from some $\lambda_0 \in V_\pi(0)$. It follows that a fixed $\lambda_0 \in V_\pi(0)$ is paired with a_m distinct λ_1 points when forming the N_π n-tuples. Furthermore, let b_m denote the number of λ_2 points which are paired with a fixed (λ_0, λ_1) tuple. The total number of n-tuples (having λ_0 as first element) is given by $\sum_{m=1}^{r} a_m b_m$ where r is the radius of $\tilde{V}$. It was shown in [13] that this procedure is asymptotically exact for large index values (and thereby large r).

For a given $\lambda_0 \in V_\pi(0)$, we seek to find an expression for $\sum_{\lambda_j \in \mathcal{T}_j(\lambda_i)} \|\lambda_i - \lambda_j\|^2$ where $\mathcal{T}_j(\lambda_i)$ was previously defined in the proof of Theorem 2 to be the set of $\lambda_j \in \Lambda_j$ which is in n-tuples having the specific λ_i as the ith element.

Proposition 2: For $n = 3$, any $1 \leq L \in \mathbb{N}$, and asymptotically as $N_i \to \infty, \nu_i \to 0, \forall i$, we have

$$
\sum_{\lambda_0 \in V_\pi(0)} \sum_{\lambda_j \in \mathcal{T}_j(\lambda_i)} \|\lambda_i - \lambda_j\|^2 = \frac{L+2}{L} G(S_L) \psi_{3,L}^2 \nu_c^{2/L} N_\pi^{1/L} N_\pi \frac{\tilde{\beta}_L}{\beta_L} \tag{83}
$$

where $\tilde{\beta}_L$ is given by (36), β_L by (25) and $\psi_{3,L}$ by (24).

Proof: Without loss of generality, we let $\lambda_0 = 0$ so that

$$
\begin{aligned}
\sum_{\lambda_j \in \mathcal{T}_j(\lambda_i)} \|\lambda_i - \lambda_j\|^2 &= \sum_{\lambda_j \in \mathcal{T}_j(0)} \|\lambda_j\|^2 \\
&= \frac{1}{L} \sum_{m=1}^{r} a_m b_m m^2
\end{aligned}
$$

where we used the fact that $m^2 = \|\lambda_j\|^2 / L$.

The first part of the proof follows now by results of [13]. Specifically, from (65) in [13] (see also [34, (H.43)]) it follows that

$$\frac{1}{L}\sum_{m=1}^{r} a_m b_m = 2\frac{\omega_L\omega_{L-1}}{\nu_1\nu_2}\frac{1}{L+1}\tilde{\beta}_L r^{2L} \tag{84}$$

where it is easy to show that we can replace $a_m b_m$ by $a_m b_m m^2$ and obtain

$$\begin{aligned}
\frac{1}{L}\sum_{m=1}^{r} a_m b_m m^2 &= 2\frac{\omega_L\omega_{L-1}}{\nu_1\nu_2}\frac{1}{L+1}\tilde{\beta}_L r^{2L+2} \\
&\overset{(a)}{=} 2\frac{\omega_L\omega_{L-1}}{\nu_1\nu_2}\frac{1}{L+1}\tilde{\beta}_L\tilde{\nu}^{2/L}\tilde{\nu}^2\frac{1}{\omega_L^{2+2/L}} \\
&\overset{(b)}{=} \frac{1}{L}\frac{1}{\omega_L^{2/L}}\tilde{\nu}^{2/L}N_0\frac{\tilde{\beta}_L}{\beta_L} \\
&\overset{(c)}{=} \frac{L+2}{L}G(S_L)\tilde{\nu}^{2/L}N_0\frac{\tilde{\beta}_L}{\beta_L} \\
&= \frac{L+2}{L}G(S_L)\psi_{n,L}^2\nu_c^{2/L}N_\pi^{1/L}N_0\frac{\tilde{\beta}_L}{\beta_L}
\end{aligned}$$

where (a) follows by use of (23), i.e. $\tilde{\nu} = \omega_L r^L = \psi_{n,L}^L\nu_c\sqrt{N_\pi}$ and (b) follows by use of (24). Finally, (c) follows since $\omega_L^{-2/L} = (L+2)G(S_L)$.

The proof now follows by using the fact that (84) is independent of λ_0 so that, since there are N_π/N_0 distinct λ_0's in $V_\pi(0)$, we get

$$\frac{1}{L}\sum_{\lambda_0\in V_\pi(0)}\sum_{m=1}^{r} a_m b_m m^2 = \frac{L+2}{L}G(S_L)\psi_{n,L}^2\nu_c^{2/L}N_\pi^{1/L}N_\pi\frac{\tilde{\beta}_L}{\beta_L}.$$

■

We are now in a position to prove the theorem.

Proof of Theorem 8: Let $\tilde{\lambda}_i = \mu_i\alpha_i(\lambda_c)$, then asymptotically as $N_i\to\infty, \nu_i\to 0, \forall i$, we have that

$$\begin{aligned}
&\frac{1}{N_\pi}\sum_{\lambda_c\in V_\pi(0)}\left\|\lambda_c - \frac{1}{\kappa}\sum_{i\in\ell}\tilde{\lambda}_i\right\|^2 \overset{(a)}{=} \frac{1}{N_\pi}\sum_{\lambda_c\in V_\pi(0)}\left\|\frac{1}{\kappa}\sum_{j\in\ell}\tilde{\lambda}_j - \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\kappa}\sum_{i=0}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})\tilde{\lambda}_i\right\|^2 \\
&\quad\overset{(b)}{=} \frac{1}{N_\pi}\sum_{\lambda_c\in V_\pi(0)}\frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2\kappa^2}\Bigg(\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\sum_{j\in\ell}\sum_{i=0}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})\|\tilde{\lambda}_j - \tilde{\lambda}_i\|^2 \\
&\qquad - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2\sum_{i=0}^{\kappa-2}\sum_{j=i+1}^{\kappa-1}\|\tilde{\lambda}_{l_i} - \tilde{\lambda}_{l_j}\|^2 - \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)})\bar{\gamma}(\mathcal{L}_j^{(n,\kappa)})\|\tilde{\lambda}_i - \tilde{\lambda}_j\|^2\Bigg) \\
&\quad\overset{(c)}{=} \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2\kappa^2}\Bigg(\bar{\gamma}(\mathcal{L}^{(n,\kappa)})\sum_{j\in\ell}\sum_{\substack{i=0\\ i\neq j}}^{n-1}\bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2\binom{\kappa}{2}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) \Bigg) \frac{L+2}{L} G(S_L) \psi_{3,L}^2 \nu_c^{2/L} N_\pi^{1/L} \frac{\tilde{\beta}_L}{\beta_L} \\
&= \frac{1}{\bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \kappa^2} \Bigg(\kappa^2 \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 - \bar{\gamma}(\mathcal{L}^{(n,\kappa)}) \sum_{j \in \ell} \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) - \bar{\gamma}(\mathcal{L}^{(n,\kappa)})^2 \binom{\kappa}{2} \\
&- \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \bar{\gamma}(\mathcal{L}_i^{(n,\kappa)}) \bar{\gamma}(\mathcal{L}_j^{(n,\kappa)}) \Bigg) \frac{L+2}{L} G(S_L) \psi_{3,L}^2 \nu_c^{2/L} N_\pi^{1/L} \frac{\tilde{\beta}_L}{\beta_L}
\end{aligned}
$$

where (a) follows by Proposition 1, (b) follows by Lemma 14, and (c) follows by Proposition 2. The proof now follows by observing that the second term of (73) is negligible compared to the first term of (73). ■

APPENDIX F
PROOF OF LEMMA 3

We consider a zero-mean and unit-variance Gaussian source X and define three random variables $Y_i \triangleq X + Q_i, i = 0, 1, 2$ where the Q_i's are identically distributed jointly Gaussian random variables (independent of X) with variance σ_q^2 and covariance matrix Q given by

$$
Q = \sigma_q^2 \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}
$$

where the correlation coefficient satisfies $-\frac{1}{2} < \rho \leq \frac{1}{2}$. It is easy to show that the MMSE when estimating X from any set of m Y_i's is given by [6], [7]

$$
\text{MMSE}_m = \frac{\sigma_q^2(1 + (m-1)\rho)}{m + \sigma_q^2(1 + (m-1)\rho)}.
$$

In the high-resolution case where $\sigma_q^2 \ll 1$ it follows that we have

$$
\text{MMSE}_1 = \sigma_q^2
$$

$$
\text{MMSE}_2 = \frac{1}{2}\sigma_q^2(1 + \rho)
$$

and

$$
\text{MMSE}_3 = \frac{1}{3}\sigma_q^2(1 + 2\rho).
$$

It was shown in [7] that, the description rate R is given by

$$
R = \frac{1}{2} \log_2 \left(\frac{1 + \sigma_q^2}{\sigma_q^2(1-\rho)} \right) \left(\frac{1-\rho}{1+2\rho} \right)^{\frac{1}{3}}
$$

so that

$$\begin{aligned}\sigma_q^2 &= \left((1-\rho)2^{2R}\left(\frac{1+2\rho}{1-\rho}\right)^{\frac{1}{3}} - 1\right)^{-1} \\ &\approx (1-\rho)^{-\frac{2}{3}}(1+2\rho)^{-\frac{1}{3}}2^{-2R}\end{aligned}$$

where the approximation becomes exact at high resolution (i.e. for $R \gg 1$). We can now form the high-resolution distortion product

$$\begin{aligned}D^{\pi} &= \frac{\sigma_q^6}{6}(1+\rho)(1+2\rho) \\ &= \frac{1}{6}(1+\rho)(1-\rho)^{-2}2^{-6R} \\ &= \frac{1}{27}2^{-6R}\end{aligned}$$

where the last inequality follows by inserting $\rho \to -\frac{1}{2}$ which corresponds to having a high side-to-central distortion ratio, i.e. it resembles the asymptotical condition of letting $N_i \to \infty$ in the IA based approach. This proves the lemma. ■

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