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Complementation and Lebesgue-type decompositions of linear operators and relations

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Abstract

In this paper, a new general approach is developed to construct and study Lebesgue-type decompositions of linear operators or relations T in the Hilbert space setting. The new approach allows to introduce an essentially wider class of Lebesgue-type decompositions than what has been studied in the literature so far. The key point is that it allows a nontrivial interaction between the closable and the singular components of T. The motivation to study such decompositions comes from the fact that they naturally occur in the corresponding Lebesgue-type decomposition for pairs of quadratic forms. The approach built in this paper uses so-called complementation in Hilbert spaces, a notion going back to de Branges and Rovnyak.

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1 | INTRODUCTION

The usual Lebesgue decomposition of measures has inspired the study of similar decompositions of, for instance, pairs of positive operators and semibounded forms; see [1, 2, 13, 33]. In the context of linear operators or linear relations such decompositions can be seen as the source for all the other Lebesgue-type decompositions. It should be noted that the standard Lebesgue decomposition of a pair of positive measures can be obtained as a special case of the Lebesgue decomposition of a pair of nonnegative forms; for details, see [13].

In this paper, a new general type of decomposition of linear operators and, more generally, of linear relations is introduced and explained which allows a nontrivial interaction between the regular (closable) component and the singular component. This work is inspired by the Lebesgue-type decompositions for a pair of forms that have been studied in [13]. It turned out that in the Lebesgue-type decompositions of a nonnegative form $\mathfrak{t}$ written as the additive sum $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$, where

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} \mathfrak{t}_1 = \operatorname{dom} \mathfrak{t}_2,$$

while $\mathfrak{t}_1$ is a regular (closable) form and $\mathfrak{t}_2$ is a singular form, the components $\mathfrak{t}_1$ and $\mathfrak{t}_2$ need not in general be singular with respect to each other. In the setting of measures this corresponds to the situation, where the absolutely continuous and the singular components are not mutually singular with respect to each other. On the other hand, all Lebesgue-type decompositions of a nonnegative quadratic form can be derived by introducing a so-called representing map $Q : \mathfrak{H} \to \mathfrak{K}$ (here $\mathfrak{H}$ and $\mathfrak{K}$ are Hilbert spaces) for the form $\mathfrak{t}$: $\mathfrak{t}[h,k] = (Qh, Qk)$, $h, k \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q$, where one can assume that $\overline{\operatorname{ran}}\, Q = \mathfrak{K}$; a detailed study of representing maps will appear in [18]. A key fact in the connection of quadratic forms is that the components $\mathfrak{t}_1$ and $\mathfrak{t}_2$ generate a nonnegative contraction K acting in the space $\mathfrak{K}$, such that

$$\mathfrak{t}_1[h,k] = \left((I-K)^{\frac{1}{2}} h, (I-K)^{\frac{1}{2}} k\right), \mathfrak{t}_2[h,k] = \left(K^{\frac{1}{2}} h, K^{\frac{1}{2}} k\right), h,k \in \operatorname{dom}\mathfrak{t},$$

and then one can prove the following general formula

$$(\mathfrak{t}_1 : \mathfrak{t}_2)[h,k] = (((I-K) : K)Qh, Qk), \quad h,k \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} Q, \tag{1.1}$$

where “:” stands for the parallel sum of the involved components; see [18]. Recall from [13, Proposition 2.10] that the forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are mutually singular precisely when $\mathfrak{t}_1 : \mathfrak{t}_2 = 0$, while $(I-K) : K = 0$ if and only if K is an orthogonal projection, which is equivalent to the intersection $\operatorname{ran}(I-K) \cap \operatorname{ran} K = \{0\}$ being nontrivial; further details for this special case of forms can be found in [18]. Closely related is a recent treatment of Lebesgue-type decompositions via the technique of reproducing kernel Hilbert spaces; see [3].

The new approach developed here to analyze this phenomenon on the side of linear operators or relations and their Lebesgue-type decompositions allowing such an interaction between the regular and singular parts is built in this paper by using the notion of complementation going back to de Branges and Rovnyak. This leads to several new results on Lebesgue-type decompositions of unbounded operators and linear relations. In particular, the results generalize recent results obtained in the case of orthogonal operator range decompositions in [14, 17]. For instance, among the set of all such Lebesgue-type decompositions there is still a unique decomposition, whose regular part in this new setting continues to be maximal; it is called the Lebesgue decomposition of linear operators, see [17].

Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, that is, T is a linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ (note that all linear operators from $\mathfrak{H}$ to $\mathfrak{K}$ belong to this class); if $\mathfrak{H} = \mathfrak{K}$ the shorter notation $T \in \mathbf{L}(\mathfrak{K})$ is used. Denote by T^{**} the closure of T (as a graph in the Cartesian product $\mathfrak{H} \times \mathfrak{K}$); moreover, $\operatorname{mul} T^{**}$ stands for the linear space of all $g \in \mathfrak{K}$ for which $\{0, g\} \in T^{**}$. For T^{**} there are two extreme cases: the *regular* (closable) case is defined by the equality $\operatorname{mul} T^{**} = \{0\}$, that is, the closure T^{**} is an operator, and the *singular* case is defined by the equality $T^{**} = \operatorname{dom} T^{**} \times$

mul T^{**}, that is, T^{**} is the Cartesian product of closed linear subspaces of $\mathfrak{H}$ and $\mathfrak{K}$. Clearly, T is singular if and only if $\operatorname{dom} T^{**} \subset \ker T^{**}$ or $\operatorname{ran} T^{**} \subset \operatorname{mul} T^{**}$. In particular, T is singular if and only if T^* is singular; see [4, 14] for further details. In general, a linear relation is neither closable nor singular. However, every $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ has a sum decomposition $T = T_1 + T_2$ of the form

$$T = \big\{\{f, g\} \in \mathfrak{H} \times \mathfrak{K} : \ g = g_1 + g_2, \{f, g_1\} \in T_1, \{f, g_2\} \in T_2\big\},$$

where $T_1, T_2 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ with $\operatorname{dom} T = \operatorname{dom} T_1 = \operatorname{dom} T_2$, while T_1 is closable and T_2 is singular. Such a sum decomposition $T = T_1 + T_2$ is called an *orthogonal Lebesgue-type decomposition* of T if the Hilbert space $\mathfrak{K}$ is the orthogonal sum of the closed linear subspaces $\mathfrak{X}$ and $\mathfrak{Y}$ of $\mathfrak{K}$, such that $\operatorname{ran} T_1 \subset \mathfrak{X}$ and $\operatorname{ran} T_2 \subset \mathfrak{Y}$. The usual Lebesgue decomposition of T is an example of an orthogonal Lebesgue-type decomposition. In the case of an orthogonal Lebesgue-type decomposition, it is clear that $\overline{\operatorname{ran}}\, T_1 \cap \overline{\operatorname{ran}}\, T_2 = \{0\}$. Orthogonal Lebesgue-type decompositions have been studied in [14, 17], extending earlier work of Izumino [20–22].

In this paper more general, pseudo-orthogonal, decompositions $T = T_1 + T_2$ will be introduced. The decomposition of the space $\mathfrak{K}$ will be based on a pair of complemented operator range spaces $\mathfrak{X}$ and $\mathfrak{Y}$ with inner products $(\cdot, \cdot)_{\mathfrak{X}}$ and $(\cdot, \cdot)_{\mathfrak{Y}}$ that are contained contractively in $\mathfrak{K}$; such spaces were introduced by de Branges and Rovnyak, see [5, 7, 8, 10]. It is assumed that $\mathfrak{X}$ and $\mathfrak{Y}$ are generated by nonnegative contractions $X, Y \in \mathbf{B}(\mathfrak{K})$, that is the linear space of all bounded linear operators from $\mathfrak{K}$ to itself, for which

$$\|h\|^2_{\mathfrak{K}} = \|Xh\|^2_{\mathfrak{X}} + \|Yh\|^2_{\mathfrak{Y}}, \quad h \in \mathfrak{K},$$

and this is equivalent to the condition $X + Y = I$; moreover, this condition automatically leads to $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$. For the sum decomposition $T = T_1 + T_2$ that satisfies $\operatorname{dom} T = \operatorname{dom} T_1 = \operatorname{dom} T_2$, while T_1 closable and T_2 singular, one now requires that $\operatorname{ran} T_1 \subset \mathfrak{X}$ and $\operatorname{ran} T_2 \subset \mathfrak{Y}$, in which case for every $\{f, g\} \in T$ with $\{f, g_1\} \in T_1$, $\{f, g_2\} \in T_2$, $g = g_1 + g_2$, one has the inequality

$$\|g\|^2_{\mathfrak{H}} \leqslant \|g_1\|^2_{\mathfrak{X}} + \|g_1\|^2_{\mathfrak{Y}}.$$

If, instead of this inequality, one has the Pythagorean equality

$$\|g\|^2_{\mathfrak{K}} = \|g_1\|^2_{\mathfrak{X}} + \|g_2\|^2_{\mathfrak{Y}},$$

then one speaks of a *pseudo-orthogonal Lebesgue-type decomposition* of T. In the orthogonal case, the closed linear subspaces $\mathfrak{X}$ and $\mathfrak{Y}$ are isometrically contained in $\mathfrak{K}$ and $\operatorname{ran} T_1 \perp \operatorname{ran} T_2$, so that the Pythagorean equality is automatically satisfied. The new feature with complemented operator range spaces $\mathfrak{X}$ and $\mathfrak{Y}$ that are contractively contained in the original Hilbert space $\mathfrak{K}$ is that there is an overlapping space $\mathfrak{X} \cap \mathfrak{Y}$; moreover $\mathfrak{X} \cap \mathfrak{Y}$ contains the intersection $\operatorname{ran} X \cap \operatorname{ran} Y$. This overlapping space has consequences for the interaction of the components T_1 and T_2: it may now happen that $\overline{\operatorname{ran}}\, T_1 \cap \overline{\operatorname{ran}}\, T_2$ is nontrivial, or even that $\operatorname{ran} T_1 \cap \operatorname{ran} T_2$ is nontrivial. This interaction can be also explained in measure-theoretic terms; there is no interaction between the components T_1 and T_2 when they are mutually singular, that is, singular with respect to each other. This will be shown by studying a linear relation $L(T_1, T_2)$ that is generated by T_1 and T_2; see Definition 3.11.

It will be shown that the pseudo-orthogonal Lebesgue-type decompositions of a linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ are all of the form

$$T = T_1 + T_2 \quad \text{with} \quad T_1 = (I - K)T, \quad T_2 = KT,$$

where $K \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction for which $(I - K)T$ is closable and KT is singular. The pseudo-orthogonal Lebesgue-type decomposition is orthogonal precisely if K is an orthogonal projection. The usual Lebesgue decomposition $T = T_{\mathrm{reg}} + T_{\mathrm{sing}}$ is orthogonal and it is uniquely defined by the property that T_{reg} is the largest closable part of T among all pseudo-orthogonal decompositions of T. Furthermore, there is a characterization of the situation where the Lebesgue decomposition is the only pseudo-orthogonal Lebesgue-type decomposition of T. As a consequence, one can always find in the nonunique case a pseudo-orthogonal Lebesgue-type decomposition that is not orthogonal; see Lemma 5.5.

In the special case that T is an operator range relation, that is,

$$T = \left\{ \{\Phi\eta, \Psi\eta\} : \eta \in \mathfrak{E} \right\}, \tag{1.2}$$

where $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and $\mathfrak{E}$, $\mathfrak{H}$, and $\mathfrak{K}$ are Hilbert spaces, the pseudo-orthogonal Lebesgue-type decompositions of T translate into the so-called pseudo-orthogonal Lebesgue-type decompositions of the operator Ψ in terms of Φ; for the orthogonal case and the notion of Radon–Nikodym derivative, see [17]. It should be mentioned that so far there are only partial results for Lebesgue-type decompositions of operator range relations T of the above form (1.2) when either Φ or Ψ is unbounded.

The contents of the paper are now described. A short introduction to pairs of complemented operator range spaces can be found in Section 2. This section is modeled on the relevant appendix in [7]. General pseudo-orthogonal decompositions of a linear relation are introduced in Section 3, where some material on the occurrence of overlapping in a pseudo-orthogonal decomposition can be found. There also the linear relation $L(T_1, T_2)$ is introduced as an operator-theoretic analog for the parallel sum in (1.1). For a relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and a selfadjoint operator $R \in \mathbf{B}(\mathfrak{K})$ there are a number of criteria in Section 4 under which the product relation RT is regular or singular. In Section 5, the previous characterizations are used to study the pseudo-orthogonal Lebesgue-type decompositions of a linear relation T. The particular case where T is an operator range relation is briefly reviewed in Section 6; see [17] for the orthogonal case and the corresponding Radon–Nikodym derivatives.

The Lebesgue decomposition for measures and the associated Radon–Nikodym derivatives for their absolutely continuous parts have seen many generalizations to more abstract settings. At this stage it suffices to mention the work of Dye [9] and Henle [19]. The second half of the seventies saw the work of Ando [1] for pairs of nonnegative operators and the work of Simon [33] for nonnegative forms. This lead to many papers devoted to related contexts, such as C^*-algebras and the theory of positive maps; see, for instance, the references in [2, 12, 13, 26], and note also, more recently, [6, 41], and, for example, the construction of Lebesgue decomposition of noncommutative measures in multi-variable setting into absolutely continuous and singular parts via Lebesgue decompositions for quadratic forms and via reproducing kernel space techniques; see [3, 24, 25]. Shortly after the papers of Ando and Simon appeared the work of Jorgensen [23] and Ôta [27–29], which was devoted to the decompositions of linear operators. This context (linear operators and also linear relations) was taken up in [15] and later in [14, 17]. The Lebesgue-type decompositions in those papers were orthogonal, whereas in the present paper the pseudo-orthogonal case is dealt with. Decomposition results in the context of forms, based on a Hilbert space decomposition similar to the de Branges–Rovnyak decomposition (as worked out at the end of Section 2), will appear in [18] and include the results in Simon [33]. In the pseudo-orthogonal decompositions the notion of overlapping spaces appears in a natural way. Furthermore, the pseudo-orthogonal situation for a pair of nonnegative bounded operators (as in [1]) and for a pair of forms on a linear space (as in [13]) can also be treated in the context of the associated linear relations; this is connected to the recent work by Sebestyén, Szücs, Tarcsay, and Titkos [31, 32, 34–40].

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2 | PSEUDO-ORTHOGONAL DECOMPOSITIONS

This section provides a short review of pseudo-orthogonal decompositions of a Hilbert space. These decompositions involve linear subspaces of such a Hilbert space, which are generated by a pair of nonnegative contractions. In such decompositions there is in general an overlapping of the summands; see [5, 7, 8, 10] and also [11]. At the end of this section there is a brief discussion of an analogous overlapping decomposition; see [18].

This review begins with the notion of an operator range space. Let $\mathfrak{K}$ be a Hilbert space and let $A \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction. Provide the range $\mathfrak{A} = \operatorname{ran} A^{\frac{1}{2}}$, a subspace of $\mathfrak{K}$, with the inner product

$$\left(A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi\right)_{\mathfrak{A}} = (\pi\varphi, \pi\psi)_{\mathfrak{K}}, \quad \varphi, \psi \in \mathfrak{K}, \tag{2.1}$$

where π is the orthogonal projection in $\mathfrak{K}$ onto $\overline{\operatorname{ran}}\, A^{\frac{1}{2}} = (\ker A^{\frac{1}{2}})^{\perp}$. Note that it follows from (2.1) that the mapping

$$\varphi \mapsto A^{\frac{1}{2}}\varphi, \quad \varphi \in \overline{\operatorname{ran}}\, A^{\frac{1}{2}}, \tag{2.2}$$

is unitary from $\overline{\operatorname{ran}}\, A^{\frac{1}{2}}$ onto $\mathfrak{A}$. Clearly, $\mathfrak{A}$ with this inner product is a Hilbert space. As A is a contraction, one has

$$\|A^{\frac{1}{2}}\varphi\|_{\mathfrak{K}} = \|A^{\frac{1}{2}}\pi\varphi\|_{\mathfrak{K}} \leqslant \|\pi\varphi\|_{\mathfrak{K}}, \quad \varphi \in \mathfrak{K},$$

and, hence, the identity (2.1) shows that

$$\|A^{\frac{1}{2}}\varphi\|_{\mathfrak{A}} \geqslant \|A^{\frac{1}{2}}\varphi\|_{\mathfrak{K}}, \quad \varphi \in \mathfrak{H}. \tag{2.3}$$

It is a consequence of (2.1) that

$$\left(A^{\frac{1}{2}}\varphi, A\psi\right)_{\mathfrak{A}} = (A^{\frac{1}{2}}\varphi, \psi)_{\mathfrak{K}}, \quad \varphi, \psi \in \mathfrak{K}, \tag{2.4}$$

which shows that the linear space ran A is dense in the Hilbert space $\mathfrak{A}$. Moreover, (2.4) leads to the useful identities

$$(A\varphi, A\psi)_{\mathfrak{A}} = (A\varphi, \psi)_{\mathfrak{K}} \quad \text{and} \quad \left(A^{\frac{1}{2}}\varphi, AA^{\frac{1}{2}}\psi\right)_{\mathfrak{A}} = \left(A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi\right)_{\mathfrak{K}}, \quad \varphi, \psi \in \mathfrak{K}. \tag{2.5}$$

It is clear that A maps $\mathfrak{A} = \operatorname{ran} A^{\frac{1}{2}}$ into itself; in fact, it can be seen from (2.5) that A is nonnegative in $\mathfrak{A}$ and that A maps $\mathfrak{A}$ contractively into itself. Note that if A is an orthogonal projection in $\mathfrak{K}$, then $\pi = A$ and

$$\left(A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi\right)_{\mathfrak{A}} = \left(A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi\right)_{\mathfrak{K}}, \quad \varphi, \psi \in \mathfrak{K}, \tag{2.6}$$

so that the inner product $(\cdot,\cdot)_{\mathfrak{A}}$ on $\operatorname{ran} A^{\frac{1}{2}} = \operatorname{ran} A$ coincides with the inner product of $\mathfrak{K}$.

The equality (2.1) and the inequality (2.3) can be formalized. Recall that a linear subspace $\mathfrak{M}$ of a Hilbert space $\mathfrak{K}$ is called a *contractive operator range space*, when $\mathfrak{M}$ has an inner product $(\cdot,\cdot)_{\mathfrak{M}}$, such that

(a) $\|\varphi\|_{\mathfrak{K}} \leqslant \|\varphi\|_{\mathfrak{M}}$, $\varphi \in \mathfrak{M}$;
(b) $\mathfrak{M}$ with the inner product $(\cdot,\cdot)_{\mathfrak{M}}$ is a Hilbert space.

It is clear that the space $\mathfrak{A}$ above is an example of a contractive operator range space. In fact, it is the only example; see [17].

The interest in this section is in pairs of nonnegative contractions $X, Y \in \mathbf{B}(\mathfrak{K})$ with the connecting property $X + Y = I$. For the convenience of the reader some simple, but useful facts are presented.

Lemma 2.1. *Let $\mathfrak{K}$ be a Hilbert space and let $X, Y \in \mathbf{B}(\mathfrak{K})$ be nonnegative contractions with $X + Y = I$. Then $XY = YX \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction with*

$$\ker XY = \ker X \oplus \ker Y. \tag{2.7}$$

Moreover, the following identities hold:

$$\begin{cases} \operatorname{ran} X \cap \operatorname{ran} Y = \operatorname{ran} XY, \\ \operatorname{ran} X^{\frac{1}{2}} \cap \operatorname{ran} Y^{\frac{1}{2}} = \operatorname{ran} X^{\frac{1}{2}} Y^{\frac{1}{2}}, \\ \overline{\operatorname{ran}}\, X \cap \overline{\operatorname{ran}}\, Y = \overline{\operatorname{ran}}\, XY. \end{cases} \tag{2.8}$$

Consequently, each of the following statements

$$\operatorname{ran} X \cap \operatorname{ran} Y = \{0\}, \quad \operatorname{ran} X^{\frac{1}{2}} \cap \operatorname{ran} Y^{\frac{1}{2}} = \{0\},$$

(or, similarly, with the closures of the ranges) and, in particular $\operatorname{ran} X \perp \operatorname{ran} Y$ *or* $\operatorname{ran} X^{\frac{1}{2}} \perp \operatorname{ran} Y^{\frac{1}{2}}$, *is equivalent to the nonnegative contractions X and Y being orthogonal projections.*

Proof. The commutativity of X and Y and of their square roots is clear. Hence, the nonnegativity of the product XY follows from $XY = X^{\frac{1}{2}} Y X^{\frac{1}{2}}$. Note that $\ker X$ and $\ker Y$ are perpendicular in $\mathfrak{K}$. It is clear that the right-hand side of (2.7) is contained in the left-hand side. To show the remaining inclusion let $h \in \ker XY$. Then $h = Yh + Xh$ with $Yh \in \ker X$ and $Xh \in \ker Y$.

To see the first identity in (2.8), let $g \in \operatorname{ran} X \cap \operatorname{ran} Y$. Then clearly one has $g = Xh = Yk$ for some $h, k \in \mathfrak{K}$. Hence, $h = Y(h + k) \in \operatorname{ran} Y$ and $g \in \operatorname{ran} XY$. The reverse inclusion is clear. For the second identity in (2.8), let $g \in \operatorname{ran} X^{\frac{1}{2}} \cap \operatorname{ran} Y^{\frac{1}{2}}$. Then one has, similarly, $g = X^{\frac{1}{2}} h = Y^{\frac{1}{2}} k$ for some $h, k \in \mathfrak{K}$. Hence,

$$h = Yh + X^{\frac{1}{2}} Y^{\frac{1}{2}} k \in \operatorname{ran} Y^{\frac{1}{2}} \quad \text{and} \quad g \in \operatorname{ran} X^{\frac{1}{2}} Y^{\frac{1}{2}}.$$

The reverse inclusion is clear.

By taking orthogonal complements in (2.7) one obtains the third identity in (2.8) for the closures of the ranges. □

Let $\mathfrak{K}$ be a Hilbert space and let $X, Y \in \mathbf{B}(\mathfrak{K})$ be nonnegative contractions with $X + Y = I$. Let $\mathfrak{X} = \operatorname{ran} X^{\frac{1}{2}}$ and $\mathfrak{Y} = \operatorname{ran} Y^{\frac{1}{2}}$ be the corresponding operator range spaces; see (2.1). Then the Hilbert space has a decomposition of the form

$$\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}. \tag{2.9}$$

This can be seen as follows. By definition one has $\mathfrak{X} \subset \mathfrak{K}$ and $\mathfrak{Y} \subset \mathfrak{K}$, so that the right-hand side of (2.9) is contained in the left-hand side. For the converse, observe that for all $h \in \mathfrak{K}$ one has $h = Xh + Yh$ with $Xh \in \mathfrak{X}$ and $Yh \in \mathfrak{Y}$, which gives $\mathfrak{K} \subset \mathfrak{X} + \mathfrak{Y}$. The intersection $\mathfrak{L} = \mathfrak{X} \cap \mathfrak{Y}$ is called the *overlapping space* of the Hilbert spaces $\mathfrak{X}$ and $\mathfrak{Y}$ with respect to the decomposition (2.9). It is characterized in the following lemma.

Lemma 2.2. *Let $\mathfrak{K}$ be a Hilbert space and let $X, Y \in \mathbf{B}(\mathfrak{K})$ be nonnegative contractions with $X + Y = I$. The overlapping space $\mathfrak{L} = \mathfrak{X} \cap \mathfrak{Y}$ is an operator range space associated with $X^{\frac{1}{2}} Y^{\frac{1}{2}}$, whose inner product satisfies*

$$(\varphi, \psi)_{\mathfrak{L}} = (\varphi, \psi)_{\mathfrak{X}} + (\varphi, \psi)_{\mathfrak{Y}}, \quad \varphi, \psi \in \mathfrak{X} \cap \mathfrak{Y}. \tag{2.10}$$

Proof. The overlapping $\mathfrak{L} = \mathfrak{X} \cap \mathfrak{Y}$ in (2.9) is a linear space given by $\mathfrak{L} = \operatorname{ran} X^{\frac{1}{2}} Y^{\frac{1}{2}}$, as follows from Lemma 2.1. To see (2.10) first observe for $h, k \in \mathfrak{K}$ that the identity $X + Y = I$ gives

$$(h, k)_{\mathfrak{K}} = (Yh, k)_{\mathfrak{K}} + (Xh, k)_{\mathfrak{K}} = (Y^{\frac{1}{2}} h, Y^{\frac{1}{2}} k)_{\mathfrak{K}} + (X^{\frac{1}{2}} h, X^{\frac{1}{2}} k)_{\mathfrak{K}}. \tag{2.11}$$

Note that $Y^{\frac{1}{2}} \overline{\operatorname{ran}}\, X^{\frac{1}{2}} Y^{\frac{1}{2}} \subset \overline{\operatorname{ran}}\, X^{\frac{1}{2}}$ and $X^{\frac{1}{2}} \overline{\operatorname{ran}}\, X^{\frac{1}{2}} Y^{\frac{1}{2}} \subset \overline{\operatorname{ran}}\, Y^{\frac{1}{2}}$. Hence, if in (2.11) one takes $h, k \in \overline{\operatorname{ran}}\, X^{\frac{1}{2}} Y^{\frac{1}{2}}$, then it follows that

$$\left(X^{\frac{1}{2}} Y^{\frac{1}{2}} h, X^{\frac{1}{2}} Y^{\frac{1}{2}} k \right)_{\mathfrak{L}} = \left(X^{\frac{1}{2}} Y^{\frac{1}{2}} h, X^{\frac{1}{2}} Y^{\frac{1}{2}} k \right)_{\mathfrak{X}} + \left(Y^{\frac{1}{2}} X^{\frac{1}{2}} h, Y^{\frac{1}{2}} X^{\frac{1}{2}} k \right)_{\mathfrak{Y}}.$$

Moreover, it is clear that the last identity holds for all $h, k \in \mathfrak{K}$. Therefore, the inner product on $\mathfrak{L}$ satisfies (2.10). □

Let $\mathfrak{K}$ be a Hilbert space and let $X, Y \in \mathbf{B}(\mathfrak{K})$ be nonnegative contractions with $X + Y = I$. Provide the Cartesian product $\mathfrak{X} \times \mathfrak{Y}$ with the inner product generated by $\mathfrak{X}$ and $\mathfrak{Y}$, respectively.

Define the column operator V from $\mathfrak{K}$ to $\mathfrak{X} \times \mathfrak{Y}$ by

$$V = \operatorname{col}(X, Y) = \left\{ \left\{ h, \begin{pmatrix} Xh \\ Yh \end{pmatrix} \right\} : h \in \mathfrak{K} \right\}. \tag{2.12}$$

The operator V is clearly isometric, as

$$\|Xh\|_{\mathfrak{X}}^2 + \|Yh\|_{\mathfrak{Y}}^2 = (Xh, h)_{\mathfrak{K}} + (Yh, h)_{\mathfrak{K}} = ((X + Y)h, h), \quad h \in \mathfrak{K}, \tag{2.13}$$

see (2.5). Hence, V is a closed operator and $\operatorname{ran} V$ is closed. In general, the isometry V does not map onto $\mathfrak{X} \times \mathfrak{Y}$.

Proposition 2.3. *Let $\mathfrak{K}$ be a Hilbert space, let $X, Y \in \mathbf{B}(\mathfrak{K})$ be nonnegative contractions, and assume that $X + Y = I$. Let the column operator V be given by (2.12). Then the adjoint mapping V^* from $\mathfrak{X} \times \mathfrak{Y}$ to $\mathfrak{K}$ is a partial isometry, given by*

$$V^* \begin{pmatrix} f \\ g \end{pmatrix} = f + g, \quad f \in \mathfrak{X}, \ g \in \mathfrak{Y}. \tag{2.14}$$

Consequently, for all $f \in \mathfrak{X}$ and $g \in \mathfrak{Y}$, there is the inequality

$$\|f + g\|_{\mathfrak{K}}^2 \leqslant \|f\|_{\mathfrak{X}}^2 + \|g\|_{\mathfrak{Y}}^2, \tag{2.15}$$

with equality in (2.15) if and only if $f = Xh$ and $g = Yh$ for some $h \in \mathfrak{K}$, namely, $h = f + g$.

Proof. A simple calculation gives for all $f \in \mathfrak{X}$, $g \in \mathfrak{Y}$, and $h \in \mathfrak{K}$ that

$$\begin{aligned} \left(V^* \begin{pmatrix} f \\ g \end{pmatrix}, h \right)_{\mathfrak{K}} &= \left(\begin{pmatrix} f \\ g \end{pmatrix}, Vh \right)_{\mathfrak{X} \times \mathfrak{Y}} = \left(\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} Xh \\ Yh \end{pmatrix} \right)_{\mathfrak{X} \times \mathfrak{Y}} \\ &= (f, Xh)_{\mathfrak{X}} + (g, Yh)_{\mathfrak{Y}} = (f, h)_{\mathfrak{K}} + (g, h)_{\mathfrak{K}} \\ &= (f + g, h)_{\mathfrak{K}}, \end{aligned}$$

which follows from (2.1); the identity shows (2.14). As V is an isometry, V^* is partially isometric and, in particular, V^* is contractive. Moreover, according to (2.9), the mapping V^* is onto. Thus, (2.14) implies (2.15). Finally, there is equality in (2.15) if and only if $\binom{f}{g} \in \operatorname{ran} V$. □

The connection between the overlapping space $\mathfrak{L} = \mathfrak{X} \cap \mathfrak{Y}$ and the range of the isometry V is now clear.

Proposition 2.4. *The isometry V satisfies*

$$(\operatorname{ran} V)^{\perp} = \left\{ \begin{pmatrix} -X^{\frac{1}{2}} Y^{\frac{1}{2}} k \\ X^{\frac{1}{2}} Y^{\frac{1}{2}} k \end{pmatrix} : k \in \overline{\operatorname{ran}} \, XY \right\}. \tag{2.16}$$

Moreover, V is surjective if and only if X and Y are orthogonal projections.

Proof. It is clear that $(\operatorname{ran} V)^{\perp} = \ker V^*$ and (2.14) shows that

$$\ker V^* = \left\{ \begin{pmatrix} \varphi \\ -\varphi \end{pmatrix} : \varphi \in \mathfrak{L} \right\}. \tag{2.17}$$

Now apply Lemma 2.2 to obtain the assertion (2.16). In particular, (2.17) and the isometric property of V, see (2.13), show that V is surjective if and only if $\mathfrak{L} = \{0\}$. The conclusion follows from Lemma 2.2. □

Recall that X and Y act as nonnegative contractions in $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. The next corollary presents the orthogonal projection VV^* as a common dilation in the Hilbert space $\mathfrak{X} \times \mathfrak{Y}$ for this pair of nonnegative contractions; see [30].

Corollary 2.5. *The orthogonal projection VV^* onto* $\operatorname{ran} V$ *is given by*

$$VV^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} X & X \\ Y & Y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \quad f \in \mathfrak{X}, \quad g \in \mathfrak{Y}.$$

The terminology in the following definition will be used in the rest of this paper.

Definition 2.6. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be linear subspaces of the Hilbert space $\mathfrak{K}$. Then $\mathfrak{K}$ is said to have a *pseudo-orthogonal decomposition* $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$ if

(a) $\mathfrak{X}$ and $\mathfrak{Y}$ are contractive operator range spaces that are contractively contained in $\mathfrak{K}$;
(b) the corresponding nonnegative contractions X and Y satisfy $X + Y = I$.

Recall that the condition $X + Y = I$ is equivalent to the condition

$$\|h\|_{\mathfrak{K}}^2 = \|Xh\|_{\mathfrak{X}}^2 + \|Yh\|_{\mathfrak{Y}}^2, \quad h \in \mathfrak{K}, \tag{2.18}$$

see (2.13) and Proposition 2.3. Moreover, if X and Y in Definition 2.6 are orthogonal projections, then the definition reduces to the usual orthogonal decomposition as the contractive operator range spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are closed linear subspaces of $\mathfrak{K}$; see Lemma 2.1 and (2.6).

At the end of the section a closely related situation will be reviewed for nonnegative contractions $X, Y \in \mathbf{B}(\mathfrak{K})$ that satisfy $X + Y = I$. Provide the closed linear subspaces $\mathfrak{K}_1 = \overline{\operatorname{ran}}\, X$ and $\mathfrak{K}_2 = \overline{\operatorname{ran}}\, Y$ with the inner product inherited from $\mathfrak{K}$. It is clear that the Hilbert space $\mathfrak{K}$ has the decomposition

$$\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2. \tag{2.19}$$

The intersection $\mathfrak{K}_1 \cap \mathfrak{K}_2$ is called the *overlapping space* of the Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$ with respect to the decomposition (2.19). It is characterized by

$$\mathfrak{K}_1 \cap \mathfrak{K}_2 = \overline{\operatorname{ran}}\, X \cap \overline{\operatorname{ran}}\, Y = \overline{\operatorname{ran}}\, XY;$$

see Lemma 2.1.

Note that the column operator $W = \operatorname{col}\left(X^{\frac{1}{2}}, Y^{\frac{1}{2}}\right)$, defined by

$$Wh := \left\{ \left\{ h, \begin{pmatrix} X^{\frac{1}{2}}h \\ Y^{\frac{1}{2}}h \end{pmatrix} \right\} : h \in \mathfrak{K} \right\}, \tag{2.20}$$

is a closed isometric mapping from $\mathfrak{K}$ to $\mathfrak{K}_1 \times \mathfrak{K}_2$. The mapping W in (2.20) is closely related to the mapping V in (2.12). To see this, first observe that the operator matrix

$$U = \begin{pmatrix} X^{\frac{1}{2}} & 0 \\ 0 & Y^{\frac{1}{2}} \end{pmatrix} : \begin{matrix} \mathfrak{K}_1 \\ \times \\ \mathfrak{K}_2 \end{matrix} \to \begin{matrix} \mathfrak{X} \\ \times \\ \mathfrak{Y} \end{matrix} \tag{2.21}$$

between the indicated Hilbert spaces is a unitary mapping; compare this with the property (2.2) of the operator $A \in \mathbf{B}(\mathfrak{K})$ in (2.1). Next observe that U connects the operators W and V via

$$UW = V. \tag{2.22}$$

Hence, the following result is a consequence of Proposition 2.3.

Proposition 2.7. *Let $\mathfrak{K}$ be a Hilbert space and let $X, Y \in \mathbf{B}(\mathfrak{K})$ be nonnegative contractions with $X + Y = I$. Let the column operator W be given by (2.20). Then the adjoint mapping W^* from $\mathfrak{K}_1 \times \mathfrak{K}_2$ to $\mathfrak{K}$ is a partial isometry, given by*

$$W^* \begin{pmatrix} f \\ g \end{pmatrix} = X^{\frac{1}{2}} f + Y^{\frac{1}{2}} g, \quad f \in \mathfrak{K}_1,\ g \in \mathfrak{K}_2. \tag{2.23}$$

Consequently, for all $f \in \mathfrak{K}_1$ and $g \in \mathfrak{K}_2$, there is the inequality

$$\|X^{\frac{1}{2}} f + Y^{\frac{1}{2}} g\|_{\mathfrak{K}}^2 \leqslant \|f\|_{\mathfrak{K}}^2 + \|g\|_{\mathfrak{K}}^2, \tag{2.24}$$

with equality in (2.24) if and only if $f = X^{\frac{1}{2}}h$ and $g = Y^{\frac{1}{2}}h$ for some $h \in \mathfrak{K}$, namely $h = X^{\frac{1}{2}} f + Y^{\frac{1}{2}} g$.

Furthermore, the isometry W is not onto in general and the intersection of $\overline{\operatorname{ran}}\, X$ and $\overline{\operatorname{ran}}\, Y$ comes into play.

Proposition 2.8. *The isometry W satisfies*

$$(\operatorname{ran} W)^{\perp} = \ker W^* = \left\{ \begin{pmatrix} -Y^{\frac{1}{2}}k \\ X^{\frac{1}{2}}k \end{pmatrix} : k \in \overline{\operatorname{ran}}\, XY \right\}. \tag{2.25}$$

Moreover, W is surjective if and only if X and Y are orthogonal projections.

Proof. The operator U in (2.21) maps $\ker W^*$ in (2.25) onto $\ker V^*$ in (2.16). Hence, the assertion (2.25) follows from Proposition 2.4. The characterization of surjectivity follows from (2.22) and Proposition 2.4. □

Note that X and Y act as nonnegative contractions in $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively. The following corollary presents the orthogonal projection WW^* as a common dilation in the Hilbert space $\mathfrak{K}_1 \times \mathfrak{K}_2$ for this pair of nonnegative contractions; see [30]. It can be seen as a consequence of Corollary 2.5, as $WW^* = U^*VV^*U$.

Corollary 2.9. *The orthogonal projection WW^* onto* $\operatorname{ran} W$ *is given by*

$$WW^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} X & X^{\frac{1}{2}}Y^{\frac{1}{2}} \\ Y^{\frac{1}{2}}X^{\frac{1}{2}} & Y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \quad f \in \mathfrak{K}_1,\ g \in \mathfrak{K}_2.$$

The model involving $\mathfrak{K}_1 = \overline{\operatorname{ran}}\, X$ and $\mathfrak{K}_2 = \overline{\operatorname{ran}}\, Y$ is connected to the de Branges–Rovnyak model involving $\mathfrak{X} = \operatorname{ran} X^{\frac{1}{2}}$ and $\mathfrak{Y} = \operatorname{ran} Y^{\frac{1}{2}}$ via the unitary mapping (2.21), and the overlapping spaces satisfy $\mathfrak{X} \cap \mathfrak{Y} \subset \mathfrak{K}_1 \cap \mathfrak{K}_2$. The present model and the mapping W in (2.20) and its properties will play a role in the Lebesgue-type decompositions of a single semibounded form [18].

3 | PSEUDO-ORTHOGONAL DECOMPOSITIONS

In this section, one can find a brief introduction to sum decompositions of linear operators or relations from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ with respect to a so-called pseudo-orthogonal decomposition of $\mathfrak{K}$. First some preliminary properties about sums of relations are discussed.

Let T_1 and T_2 belong to $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$. The sum $T_1 + T_2 \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is defined by

$$T_1 + T_2 = \big\{ \{f, f' + f''\} : \{f, f'\} \in T_1, \{f, f''\} \in T_2 \big\}. \tag{3.1}$$

With the sum $T = T_1 + T_2$ it is clear that for the domains one has

$$\operatorname{dom} T = \operatorname{dom} T_1 \cap \operatorname{dom} T_2,$$

while it is straightforward to check for the ranges that there is an inclusion

$$\operatorname{ran} T \subset \operatorname{ran} T_1 + \operatorname{ran} T_2.$$

However, for the multivalued parts there is equality

$$\operatorname{mul} T = \operatorname{mul} T_1 + \operatorname{mul} T_2, \tag{3.2}$$

so that $\operatorname{mul} T_1 \subset \operatorname{mul} T$ and $\operatorname{mul} T_2 \subset \operatorname{mul} T$.

Definition 3.1. The sum in (3.1) is said to be *strict* if the sum in (3.2) is direct, that is,

$$\operatorname{mul} T_1 \cap \operatorname{mul} T_2 = \{0\}.$$

In other words, the sum in (3.1) is strict precisely when the elements f' and f'' in (3.1) are uniquely determined by the sum $f' + f''$.

In particular, the sum $T = T_1 + T_2$ is strict if either T_1 or T_2 is an operator. A variation on the theme of sums is given in the following lemma.

Lemma 3.2. *Let T, T_1, and T_2 belong to $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$. Assume the domain equality* $\operatorname{dom} T = \operatorname{dom} T_1 = \operatorname{dom} T_2$ *and the inclusion*

$$T \subset T_1 + T_2. \tag{3.3}$$

Then there is equality $T = T_1 + T_2$ in (3.3) if and only if

$$\operatorname{mul} T = \operatorname{mul} T_1 + \operatorname{mul} T_2. \tag{3.4}$$

Consequently, there is equality in (3.3) if and only if

$$\operatorname{mul} T_1 \subset \operatorname{mul} T \quad \text{and} \quad \operatorname{mul} T_2 \subset \operatorname{mul} T.$$

Proof. By assumption one has $\operatorname{dom} T = \operatorname{dom} (T_1 + T_2)$ and it follows from the inclusion (3.3) that $\operatorname{mul} T \subset \operatorname{mul} (T_1 + T_2)$. Hence, by an observation that goes back to Arens (see [4, Corollary 1.1.3]), there is equality $T = T_1 + T_2$ if and only if $\operatorname{mul} (T_1 + T_2) \subset \operatorname{mul} T$, that is, (3.4) holds. □

The next corollary illustrates a situation that will be of interest in the rest of the paper; see [14].

Corollary 3.3. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and let $X, Y \in \mathbf{B}(\mathfrak{K})$ be nonnegative contractions such that $X + Y = I$. Then* $\operatorname{dom} T = \operatorname{dom} XT = \operatorname{dom} YT$ *and, in addition,*

$$T \subset XT + YT. \tag{3.5}$$

There is equality $T = XT + YT$ in (3.5) if and only if

$$\operatorname{mul} T = X \operatorname{mul} T + Y \operatorname{mul} T.$$

Consequently, there is equality in (3.5) if and only if

$$X \operatorname{mul} T \subset \operatorname{mul} T \quad \text{or, equivalently,} \quad Y \operatorname{mul} T \subset \operatorname{mul} T. \tag{3.6}$$

Moreover, in this case

$$X \operatorname{mul} T \cap Y \operatorname{mul} T = XY \operatorname{mul} T; \tag{3.7}$$

thus the sum $T = XT + YT$ is strict in the sense of Definition 3.1 if and only if $\operatorname{mul} T \subset \ker XY$.

Proof. These assertions follow from Lemma 3.2 except the identity (3.7). To see (3.7) let $h \in X \operatorname{mul} T \cap Y \operatorname{mul} T$, so that $h = X\varphi = Y\psi$ where $\varphi, \psi \in \operatorname{mul} T$. Now it follows from $(I - Y)\varphi = Y\psi$ that $\varphi = Y(\varphi + \psi)$ with $\varphi + \psi \in \operatorname{mul} T$. Thus, $h \in XY \operatorname{mul} T$, which shows that $X \operatorname{mul} T \cap Y \operatorname{mul} T \subset XY \operatorname{mul} T$. The reverse inclusion follows immediately from (3.6). □

For a linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, it has been shown in Corollary 3.3 that with $T_1 = XT$ and $T_2 = YT$ one has $T = T_1 + T_2$ if and only if the linear subspace $\operatorname{mul} T$ is invariant under X or Y. Under these circumstances, it is clear that

$$\operatorname{ran} T_1 \cap \operatorname{ran} T_2 \subset \operatorname{ran} X \cap \operatorname{ran} Y = \operatorname{ran} XY.$$

To give a characterization for the intersection $\operatorname{ran} T_1 \cap \operatorname{ran} T_2$, it is convenient to introduce the maximal linear subspace $\mathfrak{M}$ of $\operatorname{ran} T$, which is mapped back into $\operatorname{ran} T$ by X or by Y:

$$\mathfrak{M} = \left\{ \eta \in \operatorname{ran} T : X\eta \in \operatorname{ran} T \right\} = \left\{ \eta \in \operatorname{ran} T : Y\eta \in \operatorname{ran} T \right\}. \tag{3.8}$$

Note that $\operatorname{mul} T \subset \mathfrak{M}$ if $T = T_1 + T_2$.

Theorem 3.4. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ have a decomposition $T = T_1 + T_2$, where $T_1 = XT$ and $T_2 = YT$ for some nonnegative contractions $X, Y \in \mathbf{B}(\mathfrak{K})$ with $X + Y = I$. Then $\operatorname{ran} T_1 \cap \operatorname{ran} T_2$ is given by*

$$\operatorname{ran} T_1 \cap \operatorname{ran} T_2 = XY\,\mathfrak{M}, \tag{3.9}$$

where $\mathfrak{M}$ is given in (3.8). Consequently, the intersection $\operatorname{ran} T_1 \cap \operatorname{ran} T_2$ is nontrivial if and only if $\mathfrak{M} \not\subset \ker XY$. In particular, if X or Y is an orthogonal projection, then $\operatorname{ran} T_1 \cap \operatorname{ran} T_2 = \{0\}$.

Proof. For the inclusion ($\subset$) in (3.9), assume that $\omega \in \operatorname{ran} T_1 \cap \operatorname{ran} T_2$. Then for some $\varphi, \psi \in \operatorname{ran} T$ one has

$$\omega = X\varphi = Y\psi. \tag{3.10}$$

This shows $\psi = X\eta$, where $\eta = \varphi + \psi$; hence, $\eta \in \operatorname{ran} T$. As $\psi = X\eta \in \operatorname{ran} T$, one sees that $\eta \in \mathfrak{M}$. Moreover, it follows from (3.10) that

$$\omega = Y\psi = YX\eta \in XY\,\mathfrak{M},$$

which gives $\operatorname{ran} T_1 \cap \operatorname{ran} T_2 \subset XY\,\mathfrak{M}$.

For the inclusion ($\supset$) in (3.9), assume that $\eta \in \mathfrak{M}$. Then, by (3.8), $\eta \in \operatorname{ran} T$, $X\eta \in \operatorname{ran} T$, and $Y\eta \in \operatorname{ran} T$. It follows that $XY\eta \in Y \operatorname{ran} T = \operatorname{ran} T_1$ and that $XY\eta \in X \operatorname{ran} T = \operatorname{ran} T_2$. Therefore, one sees that $XY\eta \in \operatorname{ran} T_1 \cap \operatorname{ran} T_2$. Thus, $XY\,\mathfrak{M} \subset \operatorname{ran} T_1 \cap \operatorname{ran} T_2$.

The final statement follows directly from the identity (3.9). In particular, if X or Y is an orthogonal projection then $XY = 0$. □

There is a similar result for the intersection of $\overline{\operatorname{ran}}\, T_1 \cap \overline{\operatorname{ran}}\, T_2$ in the presence of a minimality condition.

Lemma 3.5. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ have a decomposition $T = T_1 + T_2$, where $T_1 = XT$ and $T_2 = YT$ for some nonnegative contractions $X, Y \in \mathbf{B}(\mathfrak{K})$ with $X + Y = I$. Assume in addition that $\overline{\operatorname{ran}}\, T = \mathfrak{K}$. Then*

$$\overline{\operatorname{ran}}\, T_1 \cap \overline{\operatorname{ran}}\, T_2 = \overline{\operatorname{ran}}\, XY. \tag{3.11}$$

Consequently, $\overline{\operatorname{ran}}\, T_1 \cap \overline{\operatorname{ran}}\, T_2 = \{0\}$ if and only if X or Y are orthogonal projections.

Proof. Assume $\overline{\operatorname{ran}}\, T = \mathfrak{K}$. To see (3.11) observe the identities $\overline{\operatorname{ran}}\, T_1 = \overline{\operatorname{ran}}\, X$, $\overline{\operatorname{ran}}\, T_2 = \overline{\operatorname{ran}}\, Y$, and $\overline{\operatorname{ran}}\, (XY)T = \overline{\operatorname{ran}}\, XY$. It remains to apply (2.8), which shows that $\overline{\operatorname{ran}}\, X \cap \overline{\operatorname{ran}}\, Y = \overline{\operatorname{ran}}\, XY$. For the last statement, see also Lemma 2.1. □

The interest in this paper is in decompositions $T = T_1 + T_2$ with linear relations or operators going from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$, which have a pseudo-orthogonal decomposition $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$. Before the formal definition is given, note that any element $\{f, g\} \in T$ can be written as

$$\{f, g\} = \{f, g_1 + g_2\}, \quad \{f, g_1\} \in T_1, \quad \{f, g_2\} \in T_2, \quad g = g_1 + g_2.$$

If $\operatorname{ran} T_1 \subset \mathfrak{X}$ and $\operatorname{ran} T_2 \subset \mathfrak{Y}$, then by Proposition 2.3 there is the general inequality

$$\|g\|^2_{\mathfrak{K}} \leqslant \|g_1\|^2_{\mathfrak{X}} + \|g_2\|^2_{\mathfrak{Y}}. \tag{3.12}$$

In the following definition, a special class of such sum decompositions is introduced, involving a Pythagorean equality in (3.12).

Definition 3.6. Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and assume that $\mathfrak{K}$ has a *pseudo-orthogonal decomposition* $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$ with associated nonnegative contractions X and Y such that $X + Y = I$. Let T_1 and T_2 belong to $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$, then the sum

$$T = T_1 + T_2 \quad \text{with} \quad \operatorname{dom} T = \operatorname{dom} T_1 = \operatorname{dom} T_2, \tag{3.13}$$

is said to be a pseudo-orthogonal decomposition of T connected with the pseudo-orthogonal decomposition $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$ (or, equivalently, with the pair of nonnegative contractions X and Y in $\mathbf{B}(\mathfrak{K})$ with $X + Y = I$) if

(a) $\operatorname{ran} T_1 \subset \mathfrak{X}$ and $\operatorname{ran} T_2 \subset \mathfrak{Y}$;
(b) for every $\{f, g\} \in T$ with $\{f, g_1\} \in T_1$, $\{f, g_2\} \in T_2$, $g = g_1 + g_2$, one has

$$\|g\|^2_{\mathfrak{K}} = \|g_1\|^2_{\mathfrak{X}} + \|g_2\|^2_{\mathfrak{Y}}.$$

The definition of pseudo-orthogonal decompositions has an important consequence for the sum (3.13); see Definition 3.1.

Lemma 3.7. *Let $T = T_1 + T_2$ in (3.13) be a pseudo-orthogonal decomposition. Then the sum is strict in the sense of Definition 3.1.*

Proof. Let $\varphi \in \operatorname{mul} T_1 \cap \operatorname{mul} T_2$. Then $\{0, \varphi\} \in T_1$, $\{0, -\varphi\} \in T_2$, and for the sum one sees that $g = \varphi - \varphi = 0$. It follows that $\|\varphi\|^2_{\mathfrak{X}} + \|-\varphi\|^2_{\mathfrak{Y}} = 0$. This shows that $\varphi = 0$. Therefore, $\operatorname{mul} T_1 \cap \operatorname{mul} T_2 = \{0\}$ and the sum $T = T_1 + T_2$ is strict. □

The pseudo-orthogonal decompositions in Definition 3.6 will now be characterized by means of nonnegative contractions in $\mathbf{B}(\mathfrak{K})$. Observe that the condition (3.14) in Theorem 3.8 is automatically satisfied if $(I - K) \operatorname{mul} T = \{0\}$ or $K \operatorname{mul} T = \{0\}$.

Theorem 3.8. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a linear relation. Assume that $K \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction that satisfies*

$$\operatorname{mul} T = (I-K)\operatorname{mul} T + K \operatorname{mul} T, \quad \textit{direct sum}, \tag{3.14}$$

and define

$$T_1 = (I-K)T \quad \textit{and} \quad T_2 = KT. \tag{3.15}$$

Then the sum $T = T_1 + T_2$ in (3.13) is a pseudo-orthogonal decomposition of T, connected with the pair $I-K$ and K in the sense of Definition 3.6.

Conversely, let the sum $T = T_1 + T_2$ in (3.13) be a pseudo-orthogonal decomposition of $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ in the sense of Definition 3.6. Then there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ for which (3.14) and (3.15) are satisfied.

Proof. Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and let $K \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction, such that (3.14) holds. By Corollary 3.3 the relations $T_1 = (I-K)T$ and $T_2 = KT$ in (3.15) satisfy $\operatorname{dom} T_1 = \operatorname{dom} T_2 = \operatorname{dom} T$ and $T \subset T_1 + T_2$. Again by Corollary 3.3 and the identity in (3.14) there is the decomposition $T = T_1 + T_2$. Thus, the identities in (3.13) are satisfied. As the sum in (3.14) is direct, the sum $T = T_1 + T_2$ is strict.

Now let $\mathfrak{X}$ and $\mathfrak{Y}$ be the operator range spaces generated by the nonnegative contractions $X = I-K$ and $Y = K$, respectively. Clearly, $\mathfrak{X}$ and $\mathfrak{Y}$ form a pair of complemented spaces, contractively contained in $\mathfrak{K}$, and furthermore

$$\operatorname{ran} T_1 \subset \operatorname{ran}(I-K) \subset \mathfrak{X} \quad \text{and} \quad \operatorname{ran} T_2 \subset \operatorname{ran} K \subset \mathfrak{Y},$$

which gives (a) in Definition 3.6. To check the Pythagorean equality (b) in Definition 3.6, let $\{f, g\} \in T$. Then $g = g_1 + g_2 = (I-K)g + Kg$, where $\{f, g_1\}, \{f, (I-K)g\} \in T_1$ and $\{f, g_2\}, \{f, Kg\} \in T_2$. By the strictness of the sum $T = T_1 + T_2$ one concludes that $g_1 = (I-K)g$ and $g_2 = Kg$; see Definition 3.1. With (2.5), this implies that

$$\begin{aligned} \|g_1\|_{\mathfrak{X}}^2 + \|g_2\|_{\mathfrak{Y}}^2 &= \|(I-K)g\|_{\mathfrak{X}}^2 + \|Kg\|_{\mathfrak{Y}}^2 \\ &= ((I-K)g, g)_{\mathfrak{K}} + (Kg, g)_{\mathfrak{K}} = \|g\|_{\mathfrak{K}}^2, \end{aligned}$$

and the Pythagorean property has been shown. Hence, the conditions in Definition 3.6 are satisfied.

Conversely, let $T = T_1 + T_2$ be a pseudo-orthogonal decomposition of T of the form (3.13). Let $\{f, g\} \in T$, then by (a) and (b) of Definition 3.6 one has for all $\{f, g_1\} \in T_1$ and $\{f, g_2\} \in T_2$ with $g = g_1 + g_2$, that

$$\|g\|_{\mathfrak{K}}^2 = \|g_1\|_{\mathfrak{X}}^2 + \|g_2\|_{\mathfrak{Y}}^2.$$

Thanks to this Pythagorean identity and Proposition 2.3 one obtains $g_1 = Xg$ and $g_2 = Yg$, which shows $\{f, Xg\} = \{f, g_1\} \in T_1$ and $\{f, Yg\} = \{f, g_2\} \in T_2$. Consequently, one sees the inclusions

$$XT \subset T_1 \quad \text{and} \quad YT \subset T_2. \tag{3.16}$$

By definition, $\operatorname{dom} T = \operatorname{dom} T_1$ and $\operatorname{dom} T = \operatorname{dom} T_2$, and it follows from (3.16) and [4, Proposition 1.1.2] that

$$T_1 = XT \,\widehat{+}\, (\{0\} \times \operatorname{mul} T_1) \quad \text{and} \quad T_2 = YT \,\widehat{+}\, (\{0\} \times \operatorname{mul} T_2); \tag{3.17}$$

here "$\widehat{+}$" stands for the componentwise sum (linear spans) of the graphs. Observe that (3.16) implies $X \operatorname{mul} T \subset \operatorname{mul} T_1$ and $Y \operatorname{mul} T \subset \operatorname{mul} T_2$. Now let $\{0, h\} \in \operatorname{mul} T_1 \subset \operatorname{mul} T$. Then $X + Y = I$ gives

$$h - Xh = Yh \quad \text{with} \quad h - Xh \in \operatorname{mul} T_1 \quad \text{and} \quad Yh \in \operatorname{mul} T_2.$$

By Lemma 3.7, one has $h = Xh$ and thus $(\{0\} \times \operatorname{mul} T_1) \subset XT$. Hence, by (3.17) one sees that $T_1 = XT$ and, likewise, $T_2 = YT$. Consequently, with $K = Y$ one obtains a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ for which (3.14) and (3.15) hold. □

Let T, T_1, and T_2 belong to $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and assume that (3.13) holds. Let the Hilbert space $\mathfrak{K}$ have the orthogonal decomposition $\mathfrak{K} = \mathfrak{X} \oplus \mathfrak{Y}$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are closed subspaces of $\mathfrak{K}$. Then the corresponding nonnegative contractions X and Y, which satisfy $X + Y = I$, are orthogonal projections onto $\mathfrak{X}$ and $\mathfrak{Y}$. Clearly, the condition (a) of Definition 3.6 implies the condition (b). Therefore, the following definition is natural.

Definition 3.9. Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and assume that $\mathfrak{K}$ has an orthogonal decomposition $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$. Then the sum (3.13) is called an orthogonal sum decomposition of T connected with the orthogonal decomposition $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$ (or, equivalently, with the orthogonal projections $X = I - P$ and $Y = P$) if $\operatorname{ran} T \subset \mathfrak{X}$ and $\operatorname{ran} T_2 \subset \mathfrak{Y}$.

The characterization of orthogonal sum decompositions can be given as a corollary of Theorem 3.8; see [14, 17].

Corollary 3.10. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a linear relation. Assume that $P \in \mathbf{B}(\mathfrak{K})$ is an orthogonal projection that satisfies*

$$\operatorname{mul} T = (I - P) \operatorname{mul} T + P \operatorname{mul} T, \tag{3.18}$$

and define

$$T_1 = (I - P)T \quad \text{and} \quad T_2 = PT. \tag{3.19}$$

Then the sum $T = T_1 + T_2$ in (3.13) is an orthogonal sum decomposition of T, connected with the pair $I - P$ and P in the sense of Definition 3.9.

Conversely, let the sum $T = T_1 + T_2$ in (3.13) be an orthogonal sum decomposition of $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ in the sense of Definition 3.9. Then there exists an orthogonal projection $P \in \mathbf{B}(\mathfrak{K})$ for which (3.18) and (3.19) are satisfied.

Note that the nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ in (3.15) is uniquely determined if the relation T is minimal in the sense that $\overline{\operatorname{ran}}\, T = \mathfrak{K}$. In the case of decompositions of semibounded forms via representing maps the minimality may be assumed without loss of generality (see also [18]).

The independence of the components T_1 and T_2 in a decomposition of the form $T = T_1 + T_2$ will be defined in measure-theoretic terms as follows.

Definition 3.11. Let $T = T_1 + T_2$ in (3.13) be a pseudo-orthogonal decomposition of $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$. Define the linear relation $L(T_1, T_2) \in \mathbf{L}(\mathfrak{K})$ by

$$L(T_1, T_2) = \left\{ \{g_1, g_2\} : \{f, g_1\} \in T_1, \{f, g_2\} \in T_2 \text{ with } f \in \operatorname{dom} T \right\}. \tag{3.20}$$

Then the components T_1 and T_2 in the sum $T = T_1 + T_2$ are said to be *mutually singular* if the linear relation $L(T_1, T_2)$ is singular.

If $T = T_1 + T_2$ is connected with the pair of nonnegative contractions $I - K$ and K as in Theorem 3.8, then $L(T_1, T_2)$ can be written as follows

$$L(T_1, T_2) = \left\{ \{(I - K)g, Kg\} : g \in \operatorname{ran} T \right\}, \tag{3.21}$$

where the operators $I - K$ and K are acting on the range of T.

Proposition 3.12. *Let $T = T_1 + T_2$ be a pseudo-orthogonal decomposition of T connected with the pair $I - K$ and K as in Theorem 3.8. Then T_1 and T_2 are mutually singular if and only if*

$$\operatorname{ran} P_T(I - K) \cap \operatorname{ran} P_T K = \{0\}, \tag{3.22}$$

where P_T stands for the orthogonal projection onto $\overline{\operatorname{ran}}\, T$. In particular, if T is minimal then T_1 and T_2 are mutually singular if and only if K is an orthogonal projection in $\mathfrak{K}$.

Proof. The statement is proved via the adjoint of the linear relation $L(T_1, T_2)$ in (3.21). Indeed, note that $\{\varphi, \psi\} \in L(T_1, T_2)^*$ if and only if for all $\{f, g\} \in T$ one has

$$(\psi, (I - K)g) = (\varphi, Kg) \tag{3.23}$$

or, equivalently, $(I - K)\psi - K\varphi \in (\operatorname{ran} T)^\perp = \ker P_T$. Repeating the same argument for the linear relation $L((I - K)P_T, KP_T)$ with $(I - K)P_T, KP_T \in \mathbf{B}(\mathfrak{K})$, one concludes that

$$L(T_1, T_2)^* = L((I - K)P_T, KP_T)^*. \tag{3.24}$$

Therefore, $L(T_1, T_2)$ is singular if and only if $L((I - K)P_T, KP_T)$ is singular. According to [17, Lemma 5.2] this last condition is equivalent to (3.22).

If $\overline{\operatorname{ran}}\, T = \mathfrak{K}$ then $P_T = I_{\mathfrak{K}}$ and hence the condition (3.22) holds if and only if K is an orthogonal projection; see Lemma 2.1. □

Proposition 3.12 shows that for a minimal T the mutual singularity of T_1 and T_2 is equivalent to each of the conditions stated in Lemma 2.1, for instance, the overlapping space $\mathfrak{X} \cap \mathfrak{Y}$ with $X = I - K$, $Y = K$ being trivial; see also Theorem 3.4, Lemma 3.5. In particular, Proposition 3.12 connects mutual singularity of T_1 and T_2 in pseudo-orthogonal sum decompositions $T = T_1 + T_2$ to the orthogonality of the ranges of T_1 and T_2 via Lemma 2.1.

4 | REGULARITY AND SINGULARITY OF SOME PRODUCT RELATIONS

Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and let $R \in \mathbf{B}(\mathfrak{K})$. The interest is in properties of the product

$$RT = \left\{\{f, Rf'\} : \{f, f'\} \in T\right\},$$

so that $RT \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ with $\operatorname{dom} RT = \operatorname{dom} T$. Recall the general fact that

$$\operatorname{mul} RT = R \operatorname{mul} T.$$

In particular, RT is an operator if and only if

$$\operatorname{mul} T \subset \ker R. \tag{4.1}$$

Moreover, one has $(RT)^* = T^*R^*$; see, for example, [4]. In particular, if R is selfadjoint, then $(RT)^* = T^*R$. Thus, if $R \in \mathbf{B}(\mathfrak{K})$ is selfadjoint one has the inclusions

$$RT^{**} \subset (RT)^{**} \quad \text{and} \quad R \operatorname{mul} T^{**} \subset \operatorname{mul} (RT)^{**}. \tag{4.2}$$

Still assuming that $R \in \mathbf{B}(\mathfrak{K})$ is selfadjoint, define the linear subset $\mathcal{D} \subset \overline{\operatorname{ran}}\, R$ by

$$\mathcal{D} = \left\{k \in \overline{\operatorname{ran}}\, R : Rk \in \operatorname{dom} T^*\right\}. \tag{4.3}$$

Then it is clear from the decomposition $\mathfrak{K} = \ker R \oplus \overline{\operatorname{ran}}\, R$ that

$$\operatorname{dom} T^*R = \left\{k \in \mathfrak{K} : Rk \in \operatorname{dom} T^*\right\} = \ker R \oplus \mathcal{D}. \tag{4.4}$$

It follows from (4.4) and the definition in (4.3), respectively, that

$$R(\operatorname{dom} T^*R) = R\mathcal{D} = \operatorname{ran} R \cap \operatorname{dom} T^*. \tag{4.5}$$

The next two lemmas give criteria for the relation RT to be regular (closable) or singular, respectively; see [14] for the case where R is an orthogonal projection. First the characterization of the closable case will be considered.

Lemma 4.1. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and let $R \in \mathbf{B}(\mathfrak{K})$ be selfadjoint. Then RT is closable if and only if*

$$\operatorname{clos} \{k \in \overline{\operatorname{ran}}\, R : Rk \in \operatorname{dom} T^*\} = \overline{\operatorname{ran}}\, R. \tag{4.6}$$

Furthermore, if RT is closable, then

$$\operatorname{clos} (\operatorname{ran} R \cap \operatorname{dom} T^*) = \overline{\operatorname{ran}}\, R, \tag{4.7}$$

and, in particular,

$$\operatorname{ran} R \subset \overline{\operatorname{dom}}\, T^* \quad \text{or, equivalently,} \quad \operatorname{mul} T^{**} \subset \ker R. \tag{4.8}$$

If $\operatorname{ran} R$ *is closed, then the conditions (4.6) and (4.7) are equivalent. Moreover, if* $R \in \mathbf{B}(\mathfrak{K})$ *is invertible, then* RT *is closable if and only if* T *is closable.*

Proof. Recall that RT is closable if and only if its adjoint $(RT)^*$ is densely defined. Thus, it follows from $\operatorname{dom}(RT)^* = \operatorname{dom} T^*R$ and (4.4) that RT is closable if and only if $\mathcal{D}$ is dense in $\overline{\operatorname{ran}}\, R$, that is, if and only if (4.6) is satisfied.

Now assume that RT is closable, that is, (4.6) holds. Then $\mathcal{D}$ is dense in $\overline{\operatorname{ran}}\, R$. As a consequence, also $R\mathcal{D}$ is dense in $\overline{\operatorname{ran}}\, R$. Thanks to (4.5) one sees that (4.7) holds.

The assertion $\operatorname{mul} T^{**} \subset \ker R$ in (4.8) follows directly from (4.2). It is clearly equivalent to $\operatorname{ran} R \subset \overline{\operatorname{dom}}\, T^*$. Both assertions can also be seen as consequences of the identity (4.7).

As to the last assertions, it suffices to show that (4.7) implies (4.6) if $\operatorname{ran} R$ is closed. In this case R maps $\overline{\operatorname{ran}}\, R$ bijectively onto itself and it follows from (4.5) that $\mathcal{D} = R^{-1}(\operatorname{ran} R \cap \operatorname{dom} T^*)$. Thus, if $\operatorname{ran} R \cap \operatorname{dom} T^*$ is dense in $\overline{\operatorname{ran}}\, R$ then $\mathcal{D}$ is dense $\overline{\operatorname{ran}}\, R$. Therefore, (4.7) implies (4.6). □

Note that in the special case when R is an orthogonal projection closability of RT was characterized in [14, Lemmas 2.5 and 3.4] via the condition (4.7).

Corollary 4.2. *With* T *and* R *as in Lemma 4.1, the following statements hold.*

(a) *If* RT *is closable and* $\operatorname{mul} T^{**} \cap \ker R = \{0\}$*, then* T *is closable.*
(b) *If* $\operatorname{dom} T^*$ *is closed, then* RT *is closable if and only if* $\operatorname{ran} R \subset \operatorname{dom} T^*$*. In this case* $(RT)^{**} \in \mathbf{B}(\overline{\operatorname{dom}}\, T, \mathfrak{K})$.

Proof.

(a) If RT be closable, then $\operatorname{mul} T^{**} \subset \ker R$ by Lemma 4.1. An equivalent statement is $\operatorname{mul} T^{**} = \operatorname{mul} T^{**} \cap \ker R$. Thus, (a) is clear.
(b) Assume that $\operatorname{dom} T^*$ is closed. If RT is closable, then $\operatorname{ran} R \subset \operatorname{dom} T^*$ by Lemma 4.1. Conversely, if $\operatorname{ran} R \subset \operatorname{dom} T^*$ then $\operatorname{dom} T^*R = \mathfrak{K}$ and RT is closable. As $\operatorname{dom} T^*R = \mathfrak{K}$ and $(RT)^{**} = (T^*R)^*$, the domain of $(RT)^{**}$ is closed (see [4]) and hence equal to $\overline{\operatorname{dom}}\, T$. Thus, $(RT)^{**} \in \mathbf{B}(\overline{\operatorname{dom}}\, T, \mathfrak{K})$ by the closed graph theorem. □

Next the characterization of the singular case will be considered.

Lemma 4.3. *Let* $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ *and let* $R \in \mathbf{B}(\mathfrak{K})$ *be selfadjoint. Then* RT *is singular if and only if*

$$\operatorname{ran} R \cap \operatorname{dom} T^* \subset \ker T^*. \tag{4.9}$$

If $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ *has a dense range, then* RT *is singular if and only if*

$$\operatorname{ran} R \cap \operatorname{dom} T^* = \{0\}.$$

Proof. Recall that RT is singular if and only if the adjoint $(RT)^* = T^*R$ is singular or, equivalently,

$$\operatorname{dom} T^*R \subset \ker T^*R. \tag{4.10}$$

As by (4.5) one has $R(\operatorname{dom} T^*R) = R\mathcal{D}$, the condition (4.10) holds if and only if $R\mathcal{D} \subset \ker T^*$. By (4.5) this is equivalent to (4.9). □

Remark 4.4. The characterization of closability in Lemma 4.1 has an alternative formulation. If the relation RT is closable then $\operatorname{dom} T^*R$ is dense, which implies $\operatorname{mul} T^{**} \subset \ker R$ (cf. (4.2)), and then

$$\begin{aligned}\operatorname{dom} T^*R &= \{k \in \mathfrak{K} : Rk \in \operatorname{dom} T^*\} \\ &= \operatorname{mul} T^{**} \oplus \left\{k \in \overline{\operatorname{dom}}\, T^* : Rk \in \operatorname{dom} T^*\right\},\end{aligned}$$

where now the orthogonal decomposition $\mathfrak{K} = \overline{\operatorname{dom}}\, T^* \oplus \operatorname{mul} T^{**}$ is used. It is easily seen that the closability of RT is equivalent to

$$\begin{cases}\operatorname{mul} T^{**} \subset \ker R, \\ \operatorname{clos}\left\{k \in \overline{\operatorname{dom}}\, T^* : Rk \in \operatorname{dom} T^*\right\} = \overline{\operatorname{dom}}\, T^*.\end{cases}$$

5 | PSEUDO-ORTHOGONAL LEBESGUE-TYPE DECOMPOSITIONS

In this section, the general notion of a pseudo-orthogonal Lebesgue-type decomposition for linear operators or relations is developed. In [14] the Lebesgue-type decompositions of a linear relation T were always orthogonal. The new notion allows a nontrivial intersection of the components; see Theorem 3.4.

Definition 5.1. Let the relations T, T_1, and T_2 belong to $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$. Then the sum decomposition

$$T = T_1 + T_2 \quad \text{with} \quad \operatorname{dom} T = \operatorname{dom} T_1 = \operatorname{dom} T_2, \tag{5.1}$$

is called a pseudo-orthogonal Lebesgue-type decomposition if it is a pseudo-orthogonal decomposition as in Definition 3.6, such that T_1 is closable and T_2 is singular.

The following characterization of pseudo-orthogonal Lebesgue-type decompositions is a straightforward consequence of Theorem 3.8, Lemma 4.1, and Lemma 4.3. Note that now the condition (3.14) is automatically satisfied.

Theorem 5.2. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a linear relation. Assume that $K \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction that satisfies*

$$\operatorname{clos}\left\{k \in \overline{\operatorname{ran}}\,(I-K) : (I-K)k \in \operatorname{dom} T^*\right\} = \overline{\operatorname{ran}}\,(I-K), \tag{5.2}$$

$$\operatorname{ran} K \cap \operatorname{dom} T^* \subset \ker T^*, \tag{5.3}$$

and define

$$T_1 = (I-K)T \quad \textit{and} \quad T_2 = KT. \tag{5.4}$$

Then the sum $T = T_1 + T_2$ as in (5.1) is a pseudo-orthogonal Lebesgue-type decomposition of T, connected with the pair $I - K$ and K in the sense of Definition 5.1.

Conversely, let the sum $T = T_1 + T_2$ in (5.1) be a pseudo-orthogonal Lebesgue-type decomposition of $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ in the sense of Definition 5.1. Then there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ such that (5.2), (5.3), and (5.4) are satisfied.

Proof. Let $K \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction and assume that (5.2) and (5.3) hold. Then $T_1 = (I - K)T$ is a closable operator and $T_2 = KT$ is a singular relation by Lemmas 4.1 and 4.3. Hence, $\operatorname{mul} T_1 = \{0\}$ so that (3.14) is satisfied. By Theorem 3.8 $T = T_1 + T_2$ is a pseudo-orthogonal decomposition, which is a pseudo-orthogonal Lebesgue-type decomposition according to Definition 5.1.

Conversely, let $T = T_1 + T_2$ be a pseudo-orthogonal Lebesgue-type decomposition. Hence, by definition it is a pseudo-orthogonal decomposition, where T_1 is closable and T_2 is singular. According to Theorem 3.8, there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ for which the identities in (3.14) (trivially, as $\operatorname{mul} T_1 = \{0\}$) and (5.4) hold. In fact, by Lemmas 4.1 and 4.3, the assertions in (5.2) and (5.3) follow. □

The sum decomposition (5.1) in Definition 5.1 is said to be an *orthogonal Lebesgue-type decomposition* if it is an orthogonal decomposition as in Definition 3.9, such that T_1 is closable and T_2 singular. Hence, the following characterization of orthogonal Lebesgue-type decompositions is a direct consequence of Theorem 5.2, Lemma 4.1, and Lemma 4.3; see [14].

Corollary 5.3. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a linear relation. Assume that $P \in \mathbf{B}(\mathfrak{K})$ is an orthogonal projection that satisfies*

$$\operatorname{clos}(\ker P \cap \operatorname{dom} T^*) = \ker P, \tag{5.5}$$

$$\operatorname{ran} P \cap \operatorname{dom} T^* \subset \ker T^*, \tag{5.6}$$

and define

$$T_1 = (I - P)T \quad \text{and} \quad T_2 = PT. \tag{5.7}$$

Then the sum $T = T_1 + T_2$ as in (5.1) is an orthogonal Lebesgue-type decomposition of T, connected with the pair $I - P$ and P.

Conversely, let the sum $T = T_1 + T_2$ in (5.1) be an orthogonal Lebesgue-type decomposition of $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$. Then there exists an orthogonal projection $P \in \mathbf{B}(\mathfrak{K})$ such that (5.5), (5.6), and (5.7) are satisfied.

Let P_0 stand for the orthogonal projection onto $\operatorname{mul} T^{**}$. Then it is clear that the conditions (5.5) and (5.6) in Corollary 5.3 are satisfied and it follows that

$$T = T_{\text{reg}} + T_{\text{sing}}, \quad T_{\text{reg}} = (I - P_0)T, \quad T_{\text{sing}} = P_0 T, \tag{5.8}$$

is an orthogonal Lebesgue-type decomposition of T. Here the *regular part* T_{reg} is closable and the *singular part* T_{sing} is singular. The decomposition in (5.8) is called the *Lebesgue decomposition* of T; see [15, 23, 27-29]. The Lebesgue decomposition in (5.8) shows the existence of Lebesgue-type decompositions of T. Note that T_{reg} is bounded if and only if $\operatorname{dom} T^*$ is closed; see [14].

Among all Lebesgue-type decomposition of a linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ the Lebesgue decomposition in (5.8) is distinguished by the maximality property of its regular part $T_{\rm reg}$. Recall that for linear relations S_1 and S_2 from $\mathfrak{H}$ to $\mathfrak{K}$ one says that S_1 is *dominated* (*contractively dominated*) by S_2, in notation $S_1 \prec S_2$ ($S_1 \prec_c S_2$), if $CS_2 \subset S_1$ for some bounded (contractive) operator $C \in \mathbf{B}(\mathfrak{K})$. When S_1 and S_2 are operators this is equivalent to $\mathrm{dom}\, S_2 \subset \mathrm{dom}\, S_1$ and $\|S_1 h\| \leqslant c\|S_2 h\|$ for all $h \in \mathrm{dom}\, S_2$ for some $0 < c$ ($0 < c \leqslant 1$); see [16] and [14, Definition 8.1, Lemma 8.2]. The next result is a strengthening of the maximality property established earlier for orthogonal Lebesgue-type decompositions in [14] to the wider setting of pseudo-orthogonal Lebesgue-type decompositions of T.

Theorem 5.4. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and let $T = T_1 + T_2$ be a pseudo-orthogonal Lebesgue-type decomposition of T. Then*

$$T_1 \prec_c T_{\rm reg},$$

that is, the regular part $T_{\rm reg}$ of the Lebesgue decomposition is the maximal closable part of T, in the sense of domination, among all pseudo-orthogonal Lebesgue-type decompositions of T.

Proof. In $T = T_1 + T_2$ one has $T_1 = (I - K)T$ for a nonnegative contraction and note that $I - K$ is also a nonnegative contraction. Hence, one concludes $T_1 \prec_c T$. This domination is preserved by their regular parts, see [14, Theorem 8.3]. As T_1 is closable, it is equal to its regular part and it follows that $T_1 \prec_c T_{\rm reg}$. □

For a further consideration of Lebesgue-type decompositions the class of contractions in $\mathbf{B}(\mathfrak{K})$ will now be restricted to contractions of the form

$$K = \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} : \begin{matrix} \mathrm{mul}\, T^{**} \\ \oplus \\ \overline{\mathrm{dom}\, T^*} \end{matrix} \to \begin{matrix} \mathrm{mul}\, T^{**} \\ \oplus \\ \overline{\mathrm{dom}\, T^*} \end{matrix}, \tag{5.9}$$

where $G \in \mathbf{B}\left(\overline{\mathrm{dom}\, T^*}\right)$ is a nonnegative contraction. It follows from Theorem 5.2 that K in (5.9) satisfies (5.2) if and only if

$$\mathrm{clos}\left\{k \in \overline{\mathrm{ran}}\,(I - G) : (I - G)k \in \mathrm{dom}\, T^*\right\} = \overline{\mathrm{ran}}\,(I - G), \tag{5.10}$$

and K satisfies (5.3) if and only if

$$\mathrm{ran}\, G \cap \mathrm{dom}\, T^* \subset \ker\, T^*. \tag{5.11}$$

Conversely, any $G \in \mathbf{B}\left(\overline{\mathrm{dom}\, T^*}\right)$ with the properties (5.10) and (5.11) gives via (5.9) a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ as in Theorem 5.2. The case $G = 0$ corresponds to $K = P_0$, the orthogonal projection onto $\mathrm{mul}\, T^{**}$, and gives the Lebesgue decomposition, while any orthogonal Lebesgue-type decomposition corresponds via (5.9) to an orthogonal projection G that satisfies (5.10) and (5.11).

Now assume that $\operatorname{dom} T^*$ is not closed. Let $\mathfrak{L} \subset \overline{\operatorname{dom}} T^* \setminus \operatorname{dom} T^*$ be a closed linear subspace of $\overline{\operatorname{dom}} T^*$ and decompose this space accordingly:

$$\overline{\operatorname{dom}} T^* = \left(\overline{\operatorname{dom}} T^* \ominus \mathfrak{L}\right) \oplus \mathfrak{L}.$$

This decomposition will be used in the lemma below. As to the existence of such subspaces $\mathfrak{L}$, recall that $\operatorname{dom} T^*$ is an operator range space. Hence, one has $\dim(\overline{\operatorname{dom}} T^* \setminus \operatorname{dom} T^*) = \infty$; see [10, Corollary to Theorem 2.3]. Therefore, one may choose for any $n \in \mathbb{N}$ an n-dimensional linear subspace $\mathfrak{L} \subset \overline{\operatorname{dom}} T^* \setminus \operatorname{dom} T^*$. The following lemmas describe special classes of nonnegative contractions $K \in \mathbf{B}(\mathfrak{K})$ that illustrate several features discussed earlier.

Lemma 5.5. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and assume that $\operatorname{dom} T^*$ is not closed. Let $\mathfrak{L}$ be a nontrivial closed linear subspace of $\overline{\operatorname{dom}} T^* \setminus \operatorname{dom} T^*$. Let $H \in \mathbf{B}(\mathfrak{L})$ be a nonnegative contraction, then the operator G, defined by*

$$G = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix} : \begin{matrix} \overline{\operatorname{dom}} T^* \ominus \mathfrak{L} \\ \oplus \\ \mathfrak{L} \end{matrix} \to \begin{matrix} \overline{\operatorname{dom}} T^* \ominus \mathfrak{L} \\ \oplus \\ \mathfrak{L} \end{matrix}, \tag{5.12}$$

belongs to $\mathbf{B}\left(\overline{\operatorname{dom}} T^*\right)$ and satisfies the condition (5.11). Assume in addition that $(I - H)^{-1} \in \mathbf{B}(\mathfrak{L})$, then the operator G in (5.12) satisfies the condition (5.10). Hence, K in (5.9) satisfies the conditions (5.5) and (5.6). Consequently, the sum $T = T_1 + T_2$ with (5.4) is a pseudo-orthogonal Lebesgue-type decomposition of T.

Proof. Let G be as in (5.12). Then $\operatorname{ran} G \subset \mathfrak{L}$, so that $\operatorname{ran} G \cap \operatorname{dom} T^* = \{0\}$ by the definition of $\mathfrak{L}$. Hence, the condition (5.11) is automatically satisfied. Furthermore, one sees from the condition $(I - H)^{-1} \in \mathbf{B}(\mathfrak{L})$ that $I - G \in \mathbf{B}\left(\overline{\operatorname{dom}} T^*\right)$ is invertible. Thus, the linear space $(I - G)^{-1}\operatorname{dom} T^*$ is dense in $\overline{\operatorname{dom}} T^*$. Therefore, (5.10) is satisfied if $(I - H)^{-1} \in \mathbf{B}(\mathfrak{L})$. □

The next lemma goes back to [14].

Lemma 5.6. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and assume that $\operatorname{dom} T^*$ is not closed. Let $\mathfrak{L}$ be a finite-dimensional linear subspace of $\overline{\operatorname{dom}} T^* \setminus \operatorname{dom} T^*$. Let $H \in \mathbf{B}(\mathfrak{L})$ be an orthogonal projection. Then the operator $K \in \mathbf{B}(\mathfrak{K})$, defined by (5.9) and (5.12), is an orthogonal projection $K = P$ that satisfies the conditions (5.5) and (5.6). Consequently, the sum $T = T_1 + T_2$ with (5.7) is an orthogonal Lebesgue-type decomposition of T.*

Lemmas 5.5 and 5.6 answer questions about the existence of Lebesgue-type decompositions, different from the Lebesgue decomposition: when $\operatorname{dom} T^*$ is not closed there are infinitely many different Lebesgue-type decompositions of T, both pseudo-orthogonal and orthogonal. A necessary and sufficient condition for the uniqueness of the Lebesgue decomposition among all pseudo-orthogonal Lebesgue-type decompositions (thus including the orthogonal ones) is given in the next theorem.

Theorem 5.7. *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, then the following statements are equivalent.*

(i) *The Lebesgue decomposition of T is the only pseudo-orthogonal Lebesgue-type decomposition of T.*
(ii) $\operatorname{dom} T^*$ *is closed.*

Proof. (i) ⇒ (ii) Assume that $\operatorname{dom} T^*$ is not closed. According to Lemmas 5.5 and 5.6, there exist Lebesgue-type decompositions of T, which are pseudo-orthogonal or orthogonal, which are different from the Lebesgue decomposition. This contradiction shows (ii).

(ii) ⇒ (i) Assume that $\operatorname{dom} T^*$ is closed. Let $T = (I - K)T + KT$ have a Lebesgue-type decomposition (5.4), where K is a nonnegative contraction; see Corollary 4.2. Then with the convention (5.9) one has $\operatorname{ran} G \subset \operatorname{dom} T^*$ that, combined with (5.11), leads to $\operatorname{ran} G \subset \ker T^*$ or, equivalently, $\operatorname{ran} T \subset \ker G$. It follows from (5.9) that

$$K - P_0 = \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}.$$

Therefore, the following identities

$$(I - K)T = (I - P_0)T = T_{\text{reg}} \quad \text{and} \quad KT = P_0 T = T_{\text{sing}}$$

are clear. Thus, the pseudo-orthogonal decomposition of T corresponding to K coincides with the Lebesgue decomposition. □

Theorem 5.7 is a strengthening of the corresponding result in [14] from the case of orthogonal Lebesgue-type decompositions to the case of pseudo-orthogonal Lebesgue-type decompositions. The uniqueness condition in (ii) is equivalent to the condition that the operator T_{reg} is bounded, see [14]. To see this equivalence, recall that $\operatorname{dom} T^*$ is closed if and only if $\operatorname{dom} T^{**}$ is closed, while

$$\operatorname{dom} T^{**} = \operatorname{dom} (T^{**})_{\text{reg}} = \operatorname{dom} (T_{\text{reg}})^{**}.$$

The original statement of such a uniqueness result in the setting of pairs of nonnegative bounded operators goes back to Ando [1]. In [35], there is an extensive treatment of the uniqueness question in the context of forms, including a list of the relevant literature.

Finally, it should be observed that Lemma 5.5 provides some concrete examples for nontrivial intersection of the components in a Lebesgue-type decomposition.

Corollary 5.8. *Assume the conditions in Lemma 5.5 and let $T = T_1 + T_2$ be the corresponding Lebesgue-type decomposition. Then the following statements hold.*

(a) *The intersection of $\operatorname{ran} T_1$ and $\operatorname{ran} T_2$ satisfies*

$$\operatorname{ran} T_1 \cap \operatorname{ran} T_2 = \{0\}_{\mathfrak{K} \ominus \mathfrak{L}} \oplus H(I - H)P_{\mathfrak{L}}\mathfrak{M}, \tag{5.13}$$

where $\mathfrak{M}$ is given by (3.8) and $P_{\mathfrak{L}}$ is the orthogonal projection onto $\mathfrak{L}$.

(b) *If $\overline{\operatorname{ran}}\, T = \mathfrak{K}$, then the intersection of $\overline{\operatorname{ran}}\, T_1$ and $\overline{\operatorname{ran}}\, T_2$ satisfies*

$$\overline{\operatorname{ran}}\, T_1 \cap \overline{\operatorname{ran}}\, T_2 = \{0\}_{\mathfrak{K} \ominus \mathfrak{L}} \oplus \overline{\operatorname{ran}}\, H. \tag{5.14}$$

Consequently, if $H \neq 0$ then $\overline{\operatorname{ran}}\, T_1 \cap \overline{\operatorname{ran}}\, T_2 \neq \{0\}$.

Proof. First observe with the matrix representations in (5.9) and (5.12) that

$$(I-K)K = \begin{pmatrix} 0 & 0 \\ 0 & (I-G)G \end{pmatrix} \quad \text{and} \quad (I-G)G = \begin{pmatrix} 0 & 0 \\ 0 & (I-H)H \end{pmatrix}. \tag{5.15}$$

(a) The description (5.13) is obtained directly from Theorem 3.4 by using the block formulae in (5.15).

(b) By assumption $I-H$ is surjective and hence $\overline{\operatorname{ran}}\,(I-H)H = \overline{\operatorname{ran}}\,H$. As T is minimal, the statement in (5.14) follows from Lemma 3.5 again by means of (5.15). □

6 | PAIRS OF BOUNDED LINEAR OPERATORS

Let $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ be bounded linear operators. With these operators, one associates the linear relation $L(\Phi, \Psi) \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, defined by

$$L(\Phi, \Psi) = \left\{ \{\Phi\eta, \Psi\eta\} : \eta \in \mathfrak{E} \right\}, \tag{6.1}$$

so that $L(\Phi, \Psi)$ is an operator range relation in the sense of [4, 17]; see (3.20) in Definition 3.11. It follows directly from the definition of $L(\Phi, \Psi)$ that its domain and range are given by

$$\operatorname{dom} L(\Phi, \Psi) = \operatorname{ran} \Phi \quad \text{and} \quad \operatorname{ran} L(\Phi, \Psi) = \operatorname{ran} \Psi, \tag{6.2}$$

while its kernel and multivalued part are given by

$$\ker L(\Phi, \Psi) = \Phi(\ker \Psi), \quad \operatorname{mul} L(\Phi, \Psi) = \Psi(\ker \Phi). \tag{6.3}$$

This section gives a brief overview of the decompositions $\Psi = \Psi_1 + \Psi_2$ with respect to Φ, with bounded operators Ψ_1 and Ψ_2, in the present context of a pseudo-orthogonal decomposition of the space $\mathfrak{K}$, allowing interaction between the components as in Sections 3 and 4. These decompositions of Ψ with respect to Φ will be obtained via the corresponding decompositions of the corresponding linear relation $L(\Phi, \Psi)$; for the orthogonal case, see [17].

The present interest is in sums $\Psi = \Psi_1 + \Psi_2$ and their interplay with the corresponding linear relations $L(\Phi, \Psi_1 + \Psi_2)$.

Lemma 6.1. *Let $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and assume that $\Psi = \Psi_1 + \Psi_2$ where $\Psi_1, \Psi_2 \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Then there is domain equality*

$$\operatorname{dom} L(\Phi, \Psi) = \operatorname{dom} L(\Phi, \Psi_1) = \operatorname{dom} L(\Phi, \Psi_2),$$

and inclusion of the relations

$$L(\Phi, \Psi) \subset L(\Phi, \Psi_1) + L(\Phi, \Psi_2). \tag{6.4}$$

Moreover, there is equality in (6.4):

$$L(\Phi, \Psi) = L(\Phi, \Psi_1) + L(\Phi, \Psi_2) \tag{6.5}$$

if and only if

$$\Psi(\ker\,\Phi) = \Psi_1(\ker\,\Phi) + \Psi_2(\ker\,\Phi). \tag{6.6}$$

The sum in (6.5) is strict (i.e., the sum in (6.6) is direct) if and only if

$$\Psi_1(\ker\,\Phi) \cap \Psi_2(\ker\,\Phi) = \{0\}. \tag{6.7}$$

Furthermore, if $\Psi_1(\ker\,\Phi) = \{0\}$ *or* $\Psi_2(\ker\,\Phi) = \{0\}$*, then (6.6) and (6.7) are automatically satisfied.*

Proof. From the definition of the relation $L(\Phi, \Psi)$ in (6.1), it is clear that

$$\operatorname{dom} L(\Phi, \Psi) = \operatorname{dom} L(\Phi, \Psi_1) = \operatorname{dom} L(\Phi, \Psi_2) = \operatorname{ran} \Phi,$$

see (6.2). Furthermore, it follows from the definition of the sum in (3.1) that (6.4) holds. Now recall from Lemma 3.2 that there is equality in (6.4) if and only if

$$\operatorname{mul} L(\Phi, \Psi) = \operatorname{mul} L(\Phi, \Psi_1) + \operatorname{mul} L(\Phi, \Psi_2),$$

which is clearly equivalent to (6.6); see (6.3). □

Remark 6.2. Let $\mathfrak{K}$ have a pseudo-orthogonal decomposition $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$ with associated nonnegative contractions X and Y such that $X + Y = I$. Then by Definition 3.6 the decomposition (6.5) of the relation $L(\Phi, \Psi)$ is pseudo-orthogonal if and only if

(a) $\operatorname{ran} \Psi_1 \subset \mathfrak{X}$ and $\operatorname{ran} \Psi_2 \subset \mathfrak{Y}$;
(b) for each $\eta \in \mathfrak{E}$ there exist elements $\eta', \eta'' \in \mathfrak{E}$ with $\Phi\eta = \Phi\eta' = \Phi\eta''$ and $\Psi\eta = \Psi_1\eta' + \Psi_2\eta''$ for which

$$\|\Psi\eta\|^2_{\mathfrak{K}} = \|\Psi_1\eta'\|^2_{\mathfrak{X}} + \|\Psi_2\eta''\|^2_{\mathfrak{Y}}.$$

Note that (b) implies $\eta - \eta', \eta - \eta'' \in \ker\,\Phi$ and $\Psi_1\eta' + \Psi_2\eta'' = \Psi\eta = \Psi_1\eta + \Psi_2\eta$, so that

$$\Psi_1(\eta' - \eta) = \Psi_2(\eta - \eta''). \tag{6.8}$$

By Lemma 3.7, the sum in (6.5) is strict, thus one has (6.7). Therefore, (6.8) gives that $\Psi_1\eta' = \Psi_1\eta$ and $\Psi_2\eta'' = \Psi_2\eta$. Hence, (b) implies that

$$\|\Psi\eta\|^2_{\mathfrak{K}} = \|\Psi_1\eta\|^2_{\mathfrak{X}} + \|\Psi_2\eta\|^2_{\mathfrak{Y}}, \quad \eta \in \mathfrak{E}. \tag{6.9}$$

Note that if (6.9) is satisfied, then (b) holds automatically. Thus, the conditions (b) and (6.9) are equivalent.

Definition 6.3. Let Φ, Ψ, Ψ_1, and Ψ_2 be bounded linear operators in $\mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and assume that Ψ has the decomposition

$$\Psi = \Psi_1 + \Psi_2 \quad \text{with} \quad \Psi(\ker\,\Phi) = \Psi_1(\ker\,\Phi) + \Psi_2(\ker\,\Phi), \quad \text{direct sum}. \tag{6.10}$$

Let $\mathfrak{K}$ have a *pseudo-orthogonal decomposition* $\mathfrak{K} = \mathfrak{X} + \mathfrak{Y}$ with associated nonnegative contractions X and Y such that $X + Y = I$. Then the decomposition (6.10) of Ψ with respect to Φ is called pseudo-orthogonal if Φ, Ψ, Ψ_1, and Ψ_2 satisfy the conditions

(a) $\operatorname{ran}\Psi_1 \subset \mathfrak{X}$ and $\operatorname{ran}\Psi_2 \subset \mathfrak{Y}$;
(b) $\|\Psi\eta\|^2_{\mathfrak{K}} = \|\Psi_1\eta\|^2_{\mathfrak{X}} + \|\Psi_2\eta\|^2_{\mathfrak{Y}}$ for all $\eta \in \mathfrak{E}$.

It is clear from Remark 6.2 that the decomposition (6.10) of Ψ with respect to Φ is pseudo-orthogonal if and only if the corresponding operator range relation $L(\Phi, \Psi)$ in (6.1) is pseudo-orthogonal. The pseudo-orthogonal decompositions of Ψ with respect to Φ in Definition 6.3 can now be characterized by means of nonnegative contractions in $\mathbf{B}(\mathfrak{K})$.

Theorem 6.4. *Let $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$. Assume that $K \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction that satisfies*

$$\Psi(\ker\Phi) = (I-K)\Psi(\ker\Phi) + K\Psi(\ker\Phi), \quad \textit{direct sum}, \tag{6.11}$$

and define

$$\Psi_1 = (I-K)\Psi \quad \textit{and} \quad \Psi_2 = K\Psi. \tag{6.12}$$

Then the sum $\Psi = \Psi_1 + \Psi_2$ as in (6.10) is a pseudo-orthogonal decomposition of Ψ with respect to Φ, connected with the pair $I-K$ and K in the sense of Definition 6.3.

Conversely, let $\Psi = \Psi_1 + \Psi_2$ in (6.10) be a pseudo-orthogonal decomposition of Ψ with respect to Φ in the sense of Definition 6.3. Then there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ such that (6.11) and (6.12) are satisfied.

Proof. Let $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$, $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, and let $K \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction. Define the operators $\Psi_1, \Psi_2 \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$ by (6.12), so that $\Psi = \Psi_1 + \Psi_2$. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the pair of complemented operator range spaces, contractively contained in $\mathfrak{K}$, associated with the nonnegative contractions $X = I - K$ and $Y = K$. By definition

$$\operatorname{ran}\Psi_1 = \operatorname{ran}(I-K)\Psi \subset \mathfrak{X} \quad \text{and} \quad \operatorname{ran}\Psi_2 = \operatorname{ran} K\Psi \subset \mathfrak{Y},$$

so that condition (a) in Definition 6.3 is satisfied. To see (b) in Definition 6.3 observe that

$$\begin{aligned}\|\Psi_1\eta\|^2_{\mathfrak{X}} + \|\Psi_2\eta\|^2_{\mathfrak{Y}} &= \|(I-K)\Psi\eta\|^2_{\mathfrak{X}} + \|K\Psi\eta\|^2_{\mathfrak{Y}} \\ &= ((I-K)\Psi\eta, \Psi\eta)_{\mathfrak{K}} + (K\Psi\eta, \Psi\eta)_{\mathfrak{K}} = \|\Psi\eta\|^2_{\mathfrak{K}},\end{aligned}$$

so that condition (b) in Remark 6.2 is satisfied. Hence, $\Psi = \Psi_1 + \Psi_2$ is a pseudo-orthogonal decomposition of Ψ with respect to Φ in the sense of Definition 6.3.

Conversely, assume that $\Psi = \Psi_1 + \Psi_2$ is a pseudo-orthogonal decomposition with respect to Φ as in Definition 6.3. Then $L(\Phi, \Psi)$ in (6.1) has a pseudo-orthogonal decomposition of the form

$$L(\Phi, \Psi) = L(\Phi, \Psi_1) + L(\Phi, \Psi_2),$$

see Remark 6.2. Therefore, by Theorem 3.8 there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ gives

$$L(\Phi, \Psi_1) = (I-K)L(\Phi, \Psi) \quad \text{and} \quad L(\Phi, \Psi_2) = KL(\Phi, \Psi),$$

which reads

$$L(\Phi, \Psi_1) = L(\Phi, (I-K)\Psi) \quad \text{and} \quad L(\Phi, \Psi_2) = L(\Phi, K\Psi). \tag{6.13}$$

To verify the identities in (6.12) let $\eta \in \mathfrak{E}$. Then due to (6.13) there exist $\eta', \eta'' \in \mathfrak{E}$ such that $\Phi\eta = \Phi\eta' = \Phi\eta''$, while $(I-K)\Psi\eta = \Psi_1\eta'$ and $K\Psi\eta = \Psi_2\eta''$. From $\Psi_1\eta' + \Psi_2\eta'' = \Psi\eta = \Psi_1\eta + \Psi_2\eta$ it follows that $\Psi_1\eta' = \Psi_1\eta$ and $\Psi_2\eta'' = \Psi_2\eta$; see (6.7) and Remark 6.2. Therefore, the identities in (6.12) hold. □

Let Φ and Ψ be in $\mathbf{B}(\mathfrak{K})$ and let $L(\Phi, \Psi)$ be defined as in (6.1). Now consider the case of operator range relations $L(\Phi, \Psi)$ that are closable or singular. Recall that $L(\Phi, \Psi)$ is closable if and only if $\operatorname{mul} L(\Phi, \Psi)^{**} = \{0\}$ or, equivalently, for every sequence $\eta_n \in \mathfrak{E}$ one has

$$\Phi\eta_n \to 0 \quad \text{and} \quad \Psi(\eta_n - \eta_m) \to 0 \quad \Rightarrow \quad \Psi\eta_n \to 0. \tag{6.14}$$

Likewise, $L(\Phi, \Psi)$ is singular if and only if $\operatorname{ran} L(\Phi, \Psi)^{**} \subset \operatorname{mul} L(\Phi, \Psi)^{**}$ (cf. [14, Proposition 2.8]) or, equivalently, for every sequence η_n in $\mathfrak{E}$ there exists a subsequence, denoted by ω_n, such that

$$\Psi(\eta_n - \eta_m) \to 0 \quad \Rightarrow \quad \Phi\omega_n \to 0. \tag{6.15}$$

Note that $L(\Psi, \Phi) = L(\Phi, \Psi)^{-1}$ implies that $L(\Phi, \Psi)$ is singular if and only if $L(\Psi, \Phi)$ is singular.

Remark 6.5. The characterizations in (6.14) and (6.15) of closable and singular operator range relations remain valid if the sequences φ_n are taken from a dense set $\mathfrak{R} \subset \mathfrak{E}$. To see this, observe that for any sequence $\eta_n \in \mathfrak{E}$ there exists an approximating sequence $\eta_n' \in \mathfrak{R}$, such that

$$\|\eta_n - \eta_n'\| \leqslant \frac{1}{n}, \quad \text{in which case} \quad \big|\|L\eta_n\| - \|L\eta_n'\|\big| \leqslant \frac{1}{n}\|L\|,$$

for any $L \in \mathbf{B}(\mathfrak{E}, \mathfrak{L})$, where $\mathfrak{L}$ is a Hilbert space.

The following simple observations about the adjoint relation $L(\Phi, \Psi)^*$ play a role in the rest of this section; see [17]. A direct calculation shows that

$$L(\Phi, \Psi)^* = \big\{ \{k, h\} \in \mathfrak{K} \times \mathfrak{H} : \Psi^* k = \Phi^* h \big\}.$$

Thus, by means of the linear subspaces

$$\mathfrak{D}(\Phi, \Psi) = \{k \in \mathfrak{K} : \Psi^* k \in \operatorname{ran} \Phi^*\}, \quad \mathfrak{R}(\Phi, \Psi) = \{h \in \mathfrak{H} : \Phi^* h \in \operatorname{ran} \Psi^*\},$$

the domain and the range of $L(\Phi, \Psi)^*$ are given by

$$\operatorname{dom} L(\Phi, \Psi)^* = \mathfrak{D}(\Phi, \Psi), \quad \operatorname{ran} L(\Phi, \Psi)^* = \mathfrak{R}(\Phi, \Psi),$$

and, likewise, the kernel and multivalued part of $L(\Phi, \Psi)^*$ are given by

$$\ker L(\Phi, \Psi)^* = \ker \Psi^*, \quad \operatorname{mul} L(\Phi, \Psi)^* = \ker \Phi^*.$$

Definition 6.6. The operator Ψ is called *regular with respect to* Φ if $\mathfrak{D}(\Phi, \Psi)$ is dense in $\mathfrak{K}$, which is the case if and only if the relation $L(\Phi, \Psi)$ is regular. Likewise, the operator Ψ is called *singular with respect to* Φ if $\mathfrak{D}(\Phi, \Psi) \subset \ker \Psi^*$ or, equivalently, $\mathfrak{R}(\Phi, \Psi) \subset \ker \Phi^*$, which is the case if and only if the relation $L(\Phi, \Psi)$ is singular.

Clearly, an equivalent characterization for singularity is that

$$\operatorname{ran}\Phi^* \cap \operatorname{ran}\Psi^* = \{0\}, \tag{6.16}$$

(expressing the symmetry between Φ and Ψ); see also [17, Lemma 5.2].

Remark 6.7. Both notions appearing in Definition 6.6 have equivalent characterizations that resemble their measure-theoretic analogs. In particular, Ψ is regular with respect to Φ if and only if Ψ is almost dominated by Φ. In this case Ψ has a Radon–Nikodym derivative with respect to Φ, which is given by the closed operator

$$R(\Phi,\Psi) = L(\Phi,\Psi)^{**} \in \mathbf{L}(\mathfrak{H},\mathfrak{K}),$$

and then Ψ can be written as $\Psi = R(\Phi,\Psi)\Phi$. Likewise, Ψ is singular with respect to Φ (or Φ is singular with respect to Ψ) precisely if for any $\Xi \in \mathbf{B}(\mathfrak{K})$ one has

$$\Xi \prec \Phi \quad \text{and} \quad \Xi \prec \Psi \quad \Rightarrow \quad \Xi = 0.$$

For the definitions and the arguments, see [17, sections 5 and 6].

Definition 6.8. Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$. Then Ψ is said to have a pseudo-orthogonal Lebesgue-type decomposition

$$\Psi = \Psi_1 + \Psi_2 \tag{6.17}$$

with respect to Φ if the sum (6.17) is a pseudo-orthogonal decomposition of Ψ with respect to Φ as in Definition 6.3, where Ψ_1 is regular with respect to Φ and Ψ_2 is singular with respect to Φ.

The following characterization is now straightforward.

Theorem 6.9. *Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$. Assume that $K \in \mathbf{B}(\mathfrak{K})$ is a nonnegative contraction that satisfies*

$$\operatorname{clos}\left\{k \in \overline{\operatorname{ran}}\,(I-K) \,:\, (I-K)k \in \mathfrak{D}(\Phi,\Psi)\right\} = \overline{\operatorname{ran}}\,(I-K), \tag{6.18}$$

$$\operatorname{ran} K \cap \mathfrak{D}(\Phi,\Psi) \subset \ker \Psi^*, \tag{6.19}$$

and define

$$\Psi_1 = (I-K)\Psi \quad \textit{and} \quad \Psi_2 = K\Psi. \tag{6.20}$$

Then the sum $\Psi = \Psi_1 + \Psi_2$ as in (6.17) is a pseudo-orthogonal Lebesgue-type decomposition of Ψ with respect to Φ, connected with the pair $I-K$ and K in the sense of Definition 6.8.

Conversely, let $\Psi = \Psi_1 + \Psi_2$ in (6.17) be a pseudo-orthogonal Lebesgue-type decomposition of Ψ with respect to Φ in the sense of Definition 6.8. Then there exists a nonnegative contraction $K \in \mathbf{B}(\mathfrak{K})$ such that (6.18), (6.19), and (6.20) are satisfied.

To verify the theorem, recall Theorem 6.4 and apply Definition 6.6 to the components Ψ_1 and Ψ_2 in (6.20); see also Theorem 5.2, or Lemmas 4.1 and 4.3. Note that the condition for the sum in (6.11), as stated in Theorem 6.4, is now absent because this condition automatically follows from the condition (6.18): one has $\Psi(\ker\,\Phi) \subset \ker\,(I-K)$; see Lemma 6.1. Observe, also that the singularity condition (6.19) for the component $\Psi_2 = K\Psi$ is equivalent to $\operatorname{ran}\Psi^*K \cap \operatorname{ran}\Phi^* = \{0\}$; see (6.16).

Furthermore, the components Ψ_1 and Ψ_2 in Theorem 6.9 are mutually singular if and only if the linear relation $L(\Psi_1,\Psi_2)$ is singular, or equivalently,

$$\operatorname{ran}\Psi^*(I-K) \cap \operatorname{ran}\Psi^*K = \{0\}.$$

In particular, if $\operatorname{ran} L(\Phi,\Psi) = \operatorname{ran}\Psi$ is dense in $\mathfrak{K}$, then Ψ_1 and Ψ_2 are mutually singular if and only if K is an orthogonal projection; see Proposition 3.12.

Let P_0 be the orthogonal projection onto $\mathfrak{D}(\Phi,\Psi)^\perp$. Then the pair of operators $\Psi_{\rm reg} = (I-P_0)\Psi$ and $\Psi_{\rm sing} = P_0\Psi$, gives an orthogonal Lebesgue-type decomposition $\Psi = \Psi_{\rm reg} + \Psi_{\rm sing}$ with respect to Φ. It is called the *Lebesgue decomposition* of Ψ with respect to Φ. Note that it follows from Theorem 6.9 that $(I-K)P_0 = 0$, so that $(I-K)(I-P_0) = I-K$. Hence, for the regular part $\Psi_{\rm reg}$ there is the following characterization; see [17].

Corollary 6.10. *Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$. Let $\Psi = \Psi_1 + \Psi_2$ be a pseudo-orthogonal Lebesgue-type decomposition of Ψ with respect to Φ, then*

$$\|\Psi_1 h\| \leqslant \|\Psi_{\rm reg} h\|, \quad h \in \mathfrak{E}.$$

Corollary 6.11. *Let $\Phi \in \mathbf{B}(\mathfrak{E},\mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E},\mathfrak{K})$. Then the following statements are equivalent.*

(i) Ψ *admits a unique pseudo-orthogonal Lebesgue-type decomposition with respect to* Φ.
(ii) $\mathfrak{D}(\Phi,\Psi)$ *is closed.*

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