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Sequences of Operators, Monotone in the Sense of Contractive Domination

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Abstract

A sequence of operators T_n from a Hilbert space $\mathfrak{H}$ to Hilbert spaces $\mathfrak{K}_n$ which is nondecreasing in the sense of contractive domination is shown to have a limit which is still a linear operator T from $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Moreover, the closability or closedness of T_n is preserved in the limit. The closures converge likewise and the connection between the limits is investigated. There is no similar way of dealing directly with linear relations. However, the sequence of closures is still nondecreasing and then the convergence is governed by the monotonicity principle. There are some related results for nonincreasing sequences.

Keywords Domination of linear relations · Nondecreasing sequences of linear relations in the sense of domination · Monotonicity principle

Mathematics Subject Classification Primary 47A30, 47A63, 47B02; Secondary 47B25, 47B65

Dedicated to the memory of V. E. Katsnelson.

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1 Introduction

Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, $n \in \mathbb{N}$, be a sequence of linear operators from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}_n$, which satisfy

$$\operatorname{dom} T_{n+1} \subset \operatorname{dom} T_n \quad \text{and} \quad \|T_n f\| \leq \|T_{n+1} f\|, \quad f \in \operatorname{dom} T_{n+1}. \tag{1.1}$$

Here and elsewhere the notation $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ indicates the class of all linear relations between the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$. It will be shown that there exists a limit of this sequence, namely a linear operator $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, whose domain is given by

$$\operatorname{dom} T = \left\{ \varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} T_n : \sup_{n \in \mathbb{N}} \|T_n \varphi\| < \infty \right\},$$

while, furthermore,

$$\|T_n f\| \nearrow \|T f\| \quad \text{for all} \quad f \in \operatorname{dom} T.$$

The limit is uniquely determined up to partial isometries. Moreover, it will be shown that closability and closedness of the operators T_n are preserved in the limit. The main idea about the existence of the limit is the notion of a representing map that was described by Szymański [14]. In the present paper the emphasis is on how to construct the limit of the sequence of operators and to discuss analogous sequences of linear relations. There is a close connection with similar convergence results in the context of nonnegative forms by Simon [13] (see also [12]), but the details will be left for a treatment in [9] in terms of Lebesgue decompositions and Lebesgue type decompositions of semibounded forms.

The monotonicity in (1.1) can also be discussed for the case of linear relations $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ by requiring that T_{n+1} contractively dominates T_n, i.e., there are contractions $C_n \in \mathbf{L}(\mathfrak{H}_{n+1}, \mathfrak{H}_n)$ which satisfy

$$C_n T_{n+1} \subset T_n. \tag{1.2}$$

Likewise, this kind of monotonicity is preserved under closures T_n^{**} and under taking regular parts $T_{n,\text{reg}}$ of the relations T_n (see below). In general there is no convergence result as for operators. However, the regular parts $T_{n,\text{reg}}$ form a nondecreasing sequence of closable operators (as in (1.1)) and one may apply the above mentioned results for operators. Thanks to the condition (1.2) the sequence of nonnegative selfadjoint relations $T_n^* T_n^{**}$ is nondecreasing in the usual sense and the monotonicity principle may be applied. This connects the various forms of convergence.

As mentioned above, in the present paper regular parts of operators or relations play an important role. The regular part T_{reg} of a linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ shows up in its Lebesgue decomposition, as follows

$$T = T_{\text{reg}} + T_{\text{sing}} \quad \text{with} \quad T_{\text{reg}} = (I - P)T, \; T_{\text{sing}} = PT,$$

where P stands for the orthogonal projection from $\mathfrak{K}$ onto $\operatorname{mul} T^{**}$; see [4, 5]. Hence $T_{\rm reg}$ is a closable operator, while $T_{\rm sing}$ is singular in the sense that its closure in the graph sense is the product of closed linear subspaces; note in particular that $\operatorname{ran} T_{\rm reg} \perp \operatorname{mul} T^{**}$. The regular part $T_{\rm reg}$ is the largest closable operator that is dominated by T in the sense of contractive domination. There is an interplay with the closure T^{**} of T, given by the formula

$$(T^{**})_{\rm reg} = (T_{\rm reg})^{**}, \tag{1.3}$$

see Sect. 1. If the relation T is closed, then $\operatorname{mul} T^{**} = \operatorname{mul} T$ and $T_{\rm reg}$ is the usual closed orthogonal operator part of T, often denoted by $T_{\rm op}$. In this case, clearly, $T_{\rm reg} \subset T$ and T has the decomposition

$$T = T_{\rm reg} \mathbin{\widehat{+}} (\{0\} \times \operatorname{mul} T),$$

where the sum is componentwise. Note that the left-hand side of the identity (1.3) stands for the orthogonal operator part of T^{**}. In the general case the following identity

$$T^* T^{**} = (T_{\rm reg})^* (T_{\rm reg})^{**}$$

expresses the nonnegative selfadjoint relation on the left-hand side in terms of a similar product of closable operators.

The case of a sequence of nonincreasing linear operators will also be discussed with the same methods. Now closability is not preserved so that the main result is about a nonincreasing sequence of closed linear operators.

The paper is organized as follows. In Sect. 2 there is brief review of the notion of contractive domination for relations and operators. For the convenience of the reader the relevant facts for the monotonicity principle are reviewed in Sect. 3. The representing map is discussed in Sect. 4 in an appropriate context. The convergence results are treated next. The general case of sequences of linear operators can be found in Sect. 5, the special case of sequences of closable operators is treated in Sect. 6, and the general case of sequences of linear relations is given in Sect. 7. In this last section one can also find the connection with the monotonicity principle. In Sect. 8 a simple example shows the different behaviours of the various sequences that have been considered. The approximation of closed linear operators is considered in Sect. 9. A brief discussion about nonincreasing sequences of linear operators or relations can be found in Sect. 10. Finally, in Sect. 11 there is a collection of facts concerning the regular part of the relations T^*T and T^*T^{**} which are used throughout this paper.

In the present paper the interest is in monotone sequences of linear operators or relations in a Hilbert space. The above mentioned results have a close connection to work on sequences of operators in the literature; see [11], [12, Supplement to VIII.7], and [13]. The present work also connects sequences which are monotone in the sense of contractive domination with the monotonicity principle in its version for semibounded selfadjoint relations [2]. Related results in the context of sequences of semibounded quadratic forms will be discussed in [9] (including the connections to [13] and [1]).

2 Contractive Domination for Linear Relations

The notion of domination for linear relations was introduced in [6]. The definition and some basic properties are given here. The notation $\mathbf{B}(\mathfrak{H}, \mathfrak{K})$ will be used to indicate the class of all bounded everywhere defined linear operators between the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$.

Definition 2.1 Let $\mathfrak{H}_A$, $\mathfrak{H}_B$, and $\mathfrak{H}$ be Hilbert spaces, let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_A)$ and let $B \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_B)$. Then B is said to contractively dominate A, denoted by $A \prec_c B$, if there exists a contraction $C \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that

$$CB \subset A. \tag{2.1}$$

It follows from $C \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ that $CB = \{\{f, Cf'\} : \{f, f'\} \in B\}$. Therefore, (2.1) implies

$$\begin{cases} \operatorname{dom} B \subset \operatorname{dom} A, & \ker B \subset \ker A, \\ C(\operatorname{ran} B) \subset \operatorname{ran} A, & C(\operatorname{mul} B) \subset \operatorname{mul} A. \end{cases} \tag{2.2}$$

Observe that Definition 2.1 implies that the contraction $C \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ is only fixed as a mapping form ran B to ran A. In fact, the boundedness of C implies that C takes $\overline{\operatorname{ran}}\, B$ into $\overline{\operatorname{ran}}\, A$. Hence, it may and will be assumed that

$$C((\operatorname{ran} B)^{\perp}) = \{0\}.$$

Note that if A and B are linear relations which satisfy $B \subset A$, then B contractively dominates A with $C = I_{\operatorname{ran} B}$. In particular, A contractively dominates A^{**}. Finally, the notion of contractive domination is transitive:

$$A \prec_c B \quad \text{and} \quad B \prec_c C \quad \Rightarrow \quad A \prec_c C.$$

If $A \prec_c B$ with a contraction $C \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$, then it follows from (2.1) and [2, Proposition 1.3.9] that

$$A^* \subset B^* C^* \quad \text{and} \quad CB^{**} \subset A^{**}. \tag{2.3}$$

In other words, the second inclusion in (2.3) shows that the contractive domination in (2.1) is preserved with the same operator C. In particular, if $A \prec_c B$, then the following inclusions are valid: ran $A^* \subset$ ran B^* and dom $B^{**} \subset$ dom A^{**}. Recall that in the particular case when A and B in Definition 2.1 are linear operators it is possible to give an equivalent characterization of contractive domination: $A \prec_c B$ if and only if

$$\operatorname{dom} B \subset \operatorname{dom} A \quad \text{and} \quad \|Af\| \le \|Bf\|, \quad f \in \operatorname{dom} B.$$

The following result shows that contractive domination is preserved by the regular parts. This observation goes back to [13] for the case of nonnegative forms and to [4]. Furthermore, it is shown that there is a converse statement in the case of closed linear relations.

Lemma 2.2 *Let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_A)$ and $B \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_B)$ be linear relations. Then*

$$A \prec_c B \quad \Rightarrow \quad A_{\text{reg}} \prec_c B_{\text{reg}}.$$

Moreover, if the linear relations A and B are closed, then

$$A \prec_c B \quad \Leftrightarrow \quad A_{\text{reg}} \prec_c B_{\text{reg}}.$$

Proof Assume that $CB \subset A$ with a contraction $C \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$. By (2.2) the operator C maps $\text{mul}\, B^{**}$ into $\text{mul}\, A^{**}$. Let P_B be the orthogonal projection onto $\text{mul}\, B^{**}$ and let P_A be the orthogonal projection onto $\text{mul}\, A^{**}$. Let $\{f, f'\} \in B$ and write $\{f, f'\} = \{f, (I - P_B)f' + P_B f'\}$ (i.e., the Lebesgue decomposition of B). Here $P_B f' \in \text{mul}\, B^{**}$ and one concludes that

$$\{f, Cf'\} = \{f, C(I - P_B)f' + CP_B f'\} \in A,$$

where $CP_B f' \in \text{mul}\, A^{**}$. Now observe that

$$\{f, (I - P_B)f'\} \in B_{\text{reg}} \quad \text{and} \quad \{f, (I - P_A)C(I - P_B)f'\} \in A_{\text{reg}}.$$

Equivalently, this leads to $[(I-P_A)C]B_{\text{reg}} \subset A_{\text{reg}}$, and since $(I-P_A)C$ is a contraction this implies $A_{\text{reg}} \prec_c B_{\text{reg}}$.

Let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_A)$ and $B \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_B)$ be closed linear relations. Then A_{reg} and B_{reg}, belonging to $\mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$, are the closed linear operator parts. Assume the inequality $A_{\text{reg}} \prec B_{\text{reg}}$. Then there exists a contraction $C \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that $CB_{\text{reg}} \subset A_{\text{reg}}$. Without loss of generality one may take $C \restriction (\text{ran}\, B_{\text{reg}})^\perp = 0$. Then, in particular, $C \restriction \text{ran}\, P_B = \{0\}$ and it follows from the Lebesgue decomposition $B = B_{\text{reg}} + B_{\text{sing}}$ that

$$CB = CB_{\text{reg}} \subset A_{\text{reg}}.$$

Since A is closed, one sees that $A_{\text{reg}} \subset A$. Therefore, $CB \subset A$ and $A \prec_c B$. □

The equivalence in the above theorem is restricted to closed linear relations. By modifying the notion of domination the condition that the relations are closed can be relaxed by introducing a weaker form of the Lebesgue decomposition; cf. [4, 10].

Contractive domination of closed linear relations can be characterized in terms of the corresponding nonnegative selfadjoint relations; see [6, Theorem 4.4]. Recall from [2, Definition 5.2.8] that two nonnegative relations H_1 and H_2 in $\mathbf{L}(\mathfrak{H})$ satisfy $H_1 \leq H_2$ when

$$\text{dom}\, H_2^{\frac{1}{2}} \subset \text{dom}\, H_1^{\frac{1}{2}} \quad \text{and} \quad \|(H_{1,\text{reg}})^{\frac{1}{2}} f\| \leq \|(H_{2,\text{reg}})^{\frac{1}{2}} f\|, \quad f \in \text{dom}\, H_2^{\frac{1}{2}}. \quad (2.4)$$

With this definition the following theorem is clear.

Theorem 2.3 *Let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_A)$ and $B \in \mathbf{L}(\mathfrak{H}, \mathfrak{H}_B)$ be closed linear relations. Then the following statements are equivalent*

(i) $A^*A \leq B^*B$;
(ii) $A \prec_c B$ *or, equivalently,* $A_{\rm reg} \prec_c B_{\rm reg}$.

Proof Let $H_1 = A^*A$ and $H_2 = B^*B$. By Lemma 11.2 it follows that there exist partial isometries $U_1 \in \mathbf{L}(\mathfrak{H}_A, \mathfrak{H})$ and $U_2 \in \mathbf{L}(\mathfrak{H}_B, \mathfrak{H})$, such that.

$$\begin{cases} \operatorname{dom} H_1^{\frac{1}{2}} = \operatorname{dom} A, & (H_{1,\rm reg})^{\frac{1}{2}} = U_1 A_{\rm reg}, \\ \operatorname{dom} H_2^{\frac{1}{2}} = \operatorname{dom} B, & (H_{2,\rm reg})^{\frac{1}{2}} = U_2 B_{\rm reg}. \end{cases}$$

Therefore by means of (2.4) this shows that $A^*A \leq B^*B$, i.e., $H_1 \leq H_2$, is equivalent to the assertions

$$\begin{cases} \operatorname{dom} B \subset \operatorname{dom} A, \\ \|A_{\rm reg} h\| \leq \|B_{\rm reg} h\|, \quad h \in \operatorname{dom} B. \end{cases}$$

In other words, the inequality $A^*A \leq B^*B$ in (i) is equivalent to the inequality $A_{\rm reg} \prec_c B_{\rm reg}$ in (ii). □

This characterization makes it possible to apply the monotonicity principle in the next section.

3 The Monotonicity Principle

A linear relation $H \in \mathbf{L}(\mathfrak{H})$ is called the *strong graph limit* of a sequence of linear relations $H_n \in \mathbf{L}(\mathfrak{H})$, $n \in \mathbb{N}$, if for each $\{h, h'\} \in H$ there exists a sequence $\{h_n, h_n'\} \in H_n$ such that $\{h_n, h_n'\} \to \{h, h'\}$; see [2, Definition 1.9.1]. The strong graph limit is automatically closed, see [2, p. 80]. Clearly, if all H_n are symmetric, then H is symmetric. In particular, if all H_n are nonnegative, then H is nonnegative.

Lemma 3.1 *Let $H_n \in \mathbf{L}(\mathfrak{H})$ be a sequence of nonnegative selfadjoint relations and let its strong graph limit H_∞ be nonnegative and selfadjoint. Then for every $f \in \operatorname{dom} H_\infty$ there exists a sequence $f_n \in \operatorname{dom} H_n$ such that*

$$f_n \to f \quad and \quad \|(H_{n,\rm reg})^{\frac{1}{2}} f_n\| \to \|(H_{\infty,\rm reg})^{\frac{1}{2}} f\|.$$

Proof Let $A \in \mathbf{L}(\mathfrak{H})$ be any nonnegative selfadjoint relation with square root $A^{\frac{1}{2}}$. Recall that $\operatorname{mul} A^{\frac{1}{2}} = \operatorname{mul} A$, so that $(A^{\frac{1}{2}})_{\rm reg} = (A_{\rm reg})^{\frac{1}{2}}$. If $\{f, f'\} \in A$, then there exists an element $h \in \mathfrak{H}$ such that $\{f, h\} \in A^{\frac{1}{2}}$ and $\{h, f'\} \in A^{\frac{1}{2}}$, which gives

$$(f', f) = \|h\|^2. \tag{3.1}$$

Since $h \in \operatorname{dom} A^{\frac{1}{2}} \subset (\operatorname{mul} A)^{\perp}$, one sees that $h = (A_{\rm reg})^{\frac{1}{2}} f$. Therefore, it is clear that (3.1) may be written as

$$(f', f) = (A_{\rm reg} f, f) = \|(A_{\rm reg})^{\frac{1}{2}} f\|^2. \tag{3.2}$$

Now let $f \in \operatorname{dom} H_\infty$, then $\{f, f'\} \in H_\infty$ for some $f' \in \mathfrak{H}$. By the strong graph convergence there exists a sequence $\{f_n, f_n'\} \in H_n$ such that $f_n \to f$ and $f_n' \to f'$. Therefore, by definition, there exist elements $h_n \in \mathfrak{H}$ such that

$$\{f_n, h_n\} \in (H_n)^{\frac{1}{2}} \quad \text{and} \quad \{h_n, f_n'\} \in (H_n)^{\frac{1}{2}},$$

and, likewise, there exists an element $h \in \mathfrak{H}$ such that

$$\{f, h\} \in (H_\infty)^{\frac{1}{2}} \quad \text{and} \quad \{h, f'\} \in (H_\infty)^{\frac{1}{2}}.$$

Then clearly

$$\|h_n\|^2 = (f_n', f_n) \to (f', f) = \|h\|^2,$$

or, equivalently, using (3.2),

$$\|(H_{n,\rm reg})^{\frac{1}{2}} f_n\| \to \|(H_{\infty,\rm reg})^{\frac{1}{2}} f\|.$$

□

In the case of a nondecreasing sequence of nonnegative selfadjoint relations H_n there is a much stronger result. First observe that

$$H_m \le H_n \quad \Leftrightarrow \quad (H_m)^{\frac{1}{2}} \prec_c (H_n)^{\frac{1}{2}},$$

due to Theorem 2.3, so that if H_n is nondecreasing, one also has

$$(H_{m,\rm reg})^{\frac{1}{2}} \prec_c (H_{n,\rm reg})^{\frac{1}{2}}.$$

The following monotonicity principle will be recalled from [3, Theorem 3.5], [2, Theorem 5.2.11].

Theorem 3.2 *Let $H_n \in \mathbf{L}(\mathfrak{H})$ be a sequence of nonnegative selfadjoint relations and assume they satisfy*

$$H_m \le H_n, \quad m \le n.$$

Then there exists a nonnegative selfadjoint relation $H_\infty \in \mathbf{L}(\mathfrak{H})$ with

$$H_n \le H_\infty, \quad n \in \mathbb{N}.$$

In fact, $H_n \to H_\infty$ in the strong resolvent sense or, equivalently, in the strong graph sense. Moreover, the square root of H_∞ satisfies

$$\operatorname{dom}(H_\infty)^{\frac{1}{2}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom}(H_n)^{\frac{1}{2}} : \sup_{n\in\mathbb{N}} \|(H_{n,\mathrm{reg}})^{\frac{1}{2}}\varphi\| < \infty \right\} \tag{3.3}$$

and, furthermore,

$$\|(H_{n,\mathrm{reg}})^{\frac{1}{2}}\varphi\| \nearrow \|(H_{\infty,\mathrm{reg}})^{\frac{1}{2}}\varphi\|, \quad \varphi \in \operatorname{dom}(H_\infty)^{\frac{1}{2}}. \tag{3.4}$$

Note that the multivalued parts of the relations H_n in Theorem 3.2 form a nondecreasing sequence. Of course, if all relations H_n in Theorem 10.1 are operators, then the limit H_∞ may still be a linear relation with a nontrivial multivalued part; see the example below.

Example 3.3 Let $A \in \mathbf{L}(\mathfrak{H})$ be a nonnegative selfadjoint operator or relation. Then it is clear that the sequence $H_n = nA$ of nonnegative selfadjoint operators or relations is nondecreasing. Hence there exists a nonnegative selfadjoint relation H_∞ such that $H_n \to H_\infty$ is the strong graph sense. To determine H_∞, let $\{f, g\} \in H_\infty$, then there exists a sequence $\{f_n, g_n\} \in H_n$ such that $f_n \to f$ and $g_n \to g$. Here $g_n = nh_n$ with $\{f_n, h_n\} \in A$ and, clearly, $h_n \to 0$. Since A is closed, this implies $\{f, 0\} \in A$. Furthermore, note that $h_n \in \operatorname{ran} A \subset (\ker A)^\perp$. Hence $g_n \in (\ker A)^\perp$ which implies $g \in (\ker A)^\perp$. Therefore, it follows that

$$H_\infty = \ker A \times (\ker A)^\perp,$$

since both relations are selfadjoint. Furthermore one has $\operatorname{dom}(H_\infty)^{\frac{1}{2}} = \ker A$ and $(H_\infty)_{\mathrm{reg}} = \ker A \times \{0\}$ (as in (3.3) and (3.4)).

For sequences of closed relations which are nondecreasing in the sense of domination there are close connections with Theorem 3.2 via Theorem 2.3.

4 Semi-inner Products and Representing Maps

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and let $\mathfrak{D} \subset \mathfrak{H}$ be a linear subspace which is provided with a semi-inner product $(\cdot,\cdot)_+$. In the following lemma it will be shown that such a subspace is generated by a so-called representing map. The assertion is inspired by [14].

Lemma 4.1 *Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. Let $\mathfrak{D} \subset \mathfrak{H}$ be a linear subspace which is provided with a semi-inner product $(\cdot,\cdot)_+$. Then there exists a representing map $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{K}$ is a Hilbert space, such that*

$$(\varphi, \psi)_+ = (T\varphi, T\psi)_{\mathfrak{K}}, \quad \varphi, \psi \in \mathfrak{D} = \operatorname{dom} T.$$

If $T' \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}')$, where $\mathfrak{K}'$ is a Hilbert space, is another representing map with $\operatorname{dom} T' = \mathfrak{D}$*, then there exists a partial isometry $V \in \mathbf{B}(\mathfrak{K}, \mathfrak{K}')$ with initial space* $\overline{\operatorname{ran}}\, T$ *and final space* $\overline{\operatorname{ran}}\, T'$*, such that $T' = VT$.*

Proof Let $\mathfrak{N}$ be the set of neutral elements in $\mathfrak{D}$:

$$\mathfrak{N} = \left\{\varphi \in \mathfrak{D} : (\varphi, \varphi)_+ = 0\right\}.$$

Due to the Cauchy-Schwarz inequality the space $\mathfrak{N}$ is linear. Hence, one may introduce an inner product on the quotient space $\mathfrak{D}/\mathfrak{N}$ by

$$[\varphi + \mathfrak{N}, \psi + \mathfrak{N}] = (\varphi, \psi)_+, \quad \varphi, \psi \in \mathfrak{D}.$$

The completion of this quotient space is indicated by $\mathfrak{K}$, so that $\mathfrak{K}$ is a Hilbert space. Denote the inner product on $\mathfrak{K}$ by $(\cdot, \cdot)_{\mathfrak{K}}$, so that $(\varphi + \mathfrak{N}, \psi + \mathfrak{N})_{\mathfrak{K}} = [\varphi + \mathfrak{N}, \psi + \mathfrak{N}]$ for $\varphi, \psi \in \mathfrak{D}$. Next define the operator T from $\mathfrak{D} \subset \mathfrak{H}$ to $\mathfrak{K}$ by

$$T\varphi = \varphi + \mathfrak{N}, \quad \varphi \in \mathfrak{D}.$$

Then it follows that

$$(T\varphi, T\psi)_{\mathfrak{K}} = [\varphi + \mathfrak{N}, \psi + \mathfrak{N}] = (\varphi, \psi)_+, \quad \varphi, \psi \in \mathfrak{D},$$

which is the first assertion of the lemma.

If $T' \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}')$, where $\mathfrak{K}'$ is a Hilbert space, is another representing map with $\operatorname{dom} T' = \mathfrak{D}$, then

$$(T'\varphi, T'\psi) = (\varphi, \psi)_+, \quad \varphi, \psi \in \mathfrak{D} = \operatorname{dom} T'.$$

Then the linear relation V from $\mathfrak{K}$ to $\mathfrak{K}'$, defined by

$$\left\{\{T\varphi, T'\varphi\} : \varphi \in \mathfrak{D}\right\},$$

is an isometric operator from $\operatorname{ran} T$ onto $\operatorname{ran} T'$, which can be extended as an isometric operator from $\overline{\operatorname{ran}}\, T$ onto $\overline{\operatorname{ran}}\, T'$, such that $T'f = VTf$ holds for all $f \in \mathfrak{D}$. To get the desired partial isometry V it remains to continue the isometric map to $(\operatorname{ran} T)^\perp$ as a zero mapping. This gives the desired result. □

Let $\mathfrak{D} \subset \mathfrak{H}$ be a linear subspace as in Lemma 4.1. A sequence $\varphi_n \in \mathfrak{D}$ is said to converge to $\varphi \in \mathfrak{H}$ in the sense of $\mathfrak{D}$, in notation $\varphi_n \to_{\mathfrak{D}} \varphi$, if

$$\varphi_n \to \varphi \quad \text{in} \quad \mathfrak{H} \quad \text{and} \quad \|\varphi_n - \varphi_m\|_+ \to 0.$$

Then $\mathfrak{D}$ is called *closable* if for any sequence $\varphi_n \in \mathfrak{D}$ one has

$$\varphi_n \to_{\mathfrak{D}} 0 \quad \Rightarrow \quad \|\varphi_n\|_+ \to 0,$$

and, likewise, $\mathfrak{D}$ is called *closed* if for any sequence $\varphi_n \in \mathfrak{D}$ one has

$$\varphi_n \to_{\mathfrak{D}} \varphi \quad \Rightarrow \quad \varphi \in \mathfrak{D} \quad \text{and} \quad \|\varphi_n - \varphi\|_+ \to 0.$$

These definitions take a more familiar form in terms of the representing map T in Lemma 4.1 One sees immediately for a sequence $\varphi_n \in \mathfrak{D}$ that

$$\varphi_n \to_{\mathfrak{D}} \varphi \quad \Leftrightarrow \quad \varphi_n \to \varphi \quad \text{in} \quad \mathfrak{H} \quad \text{and} \quad \|T(\varphi_n - \varphi_m)\| \to 0.$$

Therefore, $\mathfrak{D}$ is *closable* if and only if T is closable, and, likewise, $\mathfrak{D}$ is closed if and only if T is closed.

An example of a representing map appears in the following construction that will be used in [8]. Let $A \in \mathbf{B}(\mathfrak{K})$ be a nonnegative contraction in a Hilbert space $\mathfrak{K}$. The range space $\mathfrak{A} = \operatorname{ran} A^{\frac{1}{2}}$, as a subspace of $\mathfrak{K}$, is provided with the semi-inner product

$$(A^{\frac{1}{2}}h, A^{\frac{1}{2}}k)_{\mathfrak{A}} = (\pi h, \pi k)_{\mathfrak{K}}, \quad h, k \in \mathfrak{K}, \tag{4.1}$$

where π is the orthogonal projection in $\mathfrak{K}$ onto $\overline{\operatorname{ran}}\, A^{\frac{1}{2}} = (\ker A^{\frac{1}{2}})^{\perp}$. Then it is clear that the operator $T \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ defined by

$$A^{\frac{1}{2}}h \mapsto \pi h, \quad h \in \mathfrak{H},$$

with $\operatorname{dom} T = \mathfrak{A}$, is actually a representing map as follows from (4.1).

5 Nondecreasing Sequences of Linear Operators

It will be shown that a sequence of linear operators, that is nondecreasing in the sense of contractive domination, as in Definition 2.1, has a linear operator as limit. The limit will be constructed by means of representing maps. Moreover, it will be shown that closability and closedness of the operators are preserved in the limit.

Theorem 5.1 *Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, be a sequence of linear operators which satisfy*

$$T_m \prec_c T_n, \quad m \leq n. \tag{5.1}$$

Then there exists a linear operator $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{K}$ is a Hilbert space, such that

$$\operatorname{dom} T = \left\{ \varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} T_n \,:\, \sup_{n \in \mathbb{N}} \|T_n \varphi\| < \infty \right\} \tag{5.2}$$

and which satisfies

$$T_n \prec_c T \quad \textit{and} \quad \|T_n \varphi\| \nearrow \|T\varphi\|, \quad \varphi \in \operatorname{dom} T. \tag{5.3}$$

Moreover, the following statements hold:

(a) if T_n is closable for all $n \in \mathbb{N}$, then T is closable;
(b) if T_n is closed for all $n \in \mathbb{N}$, then T is closed.

Proof Let T_n be a sequence of operators that satisfies (5.1). Then it is seen by Cauchy's inequality that the right-hand side $\mathfrak{D}$ in (5.2) is a linear space. Next the existence of the operator T will be shown. For each $\varphi \in \mathfrak{D}$ define

$$\|\varphi\|_+ = \sup_{n \in \mathbb{N}} \|T_n \varphi\|.$$

Then $\|\cdot\|_+$ is clearly a well defined seminorm on $\mathfrak{D}$ and let $(\cdot, \cdot)_+$ be the corresponding semi-inner product. By Lemma 4.1 there exists a linear operator T defined on $\operatorname{dom} T = \mathfrak{D} \subset \mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ such that

$$(\varphi, \psi)_+ = (T\varphi, T\psi), \quad \varphi \in \mathfrak{D}.$$

This shows the assertion in (5.3).

(a) Assume that T_n, $n \in \mathbb{N}$, is closable. To show that T is closable, it suffices to show that $T = T_{\mathrm{reg}}$. By (5.3) one has

$$T_n \prec_c T.$$

Hence there exist contractions $C_n \in \mathbf{B}(\mathfrak{K}, \mathfrak{K}_n)$, such that $C_n T \subset T_n$ for all $n \in \mathbb{N}$. This implies that

$$C_n T^{**} \subset T_n^{**};$$

see (2.3). In particular, if $\{0, \varphi\} \in T^{**}$, then $\{0, C_n\varphi\} \in T_n^{**}$, so that $C_n\varphi = 0$. Thus one concludes that $\operatorname{mul} T^{**} \subset \ker C_n$. Let P be the orthogonal projection from $\mathfrak{K}$ onto $\operatorname{mul} T^{**}$, then $C_n P = 0$. By means of the Lebesgue decomposition $T = (I - P)T + PT$, this leads to

$$C_n T_{\mathrm{reg}} = C_n(I - P)T = C_n[(I - P)T + PT] = C_n T \subset T_n.$$

Hence, $C_n T_{\mathrm{reg}} \subset T_n$ for all $n \in \mathbb{N}$ and thus

$$\|T_n\varphi\| = \|C_n T_{\mathrm{reg}}\varphi\| \leq \|T_{\mathrm{reg}}\varphi\| \leq \|T\varphi\|, \quad \varphi \in \operatorname{dom} T. \tag{5.4}$$

Taking the supremum over $n \in \mathbb{N}$ in (5.4) and combining with (5.3) gives

$$\|T\varphi\| = \|T_{\mathrm{reg}}\varphi\|, \quad \varphi \in \operatorname{dom} T.$$

This implies that $T_{\mathrm{sing}} = 0$ and hence T is closable.

(b) Assume that T_n, $n \in \mathbb{N}$, is closed. To show that T is closed, let φ_n be a sequence in $\operatorname{dom} T$ such that

$$\varphi_n \to \varphi \text{ in } \mathfrak{H} \quad \text{and} \quad T(\varphi_n - \varphi_m) \to 0 \text{ in } \mathfrak{K}. \tag{5.5}$$

Due to (5.3) one sees that $T_k(\varphi_n - \varphi_m) \to 0$. Since for each $k \in \mathbb{N}$ the operator T_k is closed one obtains that $\varphi \in \operatorname{dom} T_k$ and $T_k(\varphi_n - \varphi) \to 0$ as $n \to \infty$. In particular, $\varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} T_n$. In order to verify that $\varphi \in \operatorname{dom} T$, first observe that the inequality

$$|\|T\varphi_n\| - \|T\varphi_m\|| \le \|T(\varphi_n - \varphi_m)\|,$$

implies, via (5.5), that $\sup_{m \in \mathbb{N}} \|T\varphi_m\| < \infty$. Now it follows from $T_n\varphi_m \to T_n\varphi$ and (5.3) that

$$\|T_n\varphi\| = \lim_{m\to\infty} \|T_n\varphi_m\| \le \lim_{m\to\infty} \|T\varphi_m\| \le \sup_{m\in\mathbb{N}} \|T\varphi_m\| < \infty.$$

Since this holds for all $n \in \mathbb{N}$, one concludes that $\sup_{n \in \mathbb{N}} \|T_n\varphi\| < \infty$. Therefore, $\varphi \in \operatorname{dom} T$. Since by (a) the operator T is closable, it now follows from (5.5) that T is closed.

□

The existence of the limit in Theorem 6.1 has been established; however it is clear that there is no uniqueness. In fact, this question has been already addressed in Lemma 4.1. The corollary below is easily verified directly.

Corollary 5.2 *Assume the conditions from Theorem 5.1 and let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be the limit. If $T' \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}')$, where $\mathfrak{K}'$ is a Hilbert space, is another limit with* $\operatorname{dom} T' = \operatorname{dom} T$, *then there exists a partial isometry $V \in \mathbf{B}(\mathfrak{K}, \mathfrak{K}')$ with initial space* $\overline{\operatorname{ran}}\, T$ *and final space* $\overline{\operatorname{ran}}\, T'$, *such that $T' = VT$.*

The following simple result is that an operator that dominates the sequence also dominates the limit. This fact will have important consequences.

Corollary 5.3 *Assume the conditions from Theorem 5.1 and let $T' \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}')$, where $\mathfrak{K}'$ is a Hilbert space, be a linear operator. Then*

$$T_n \prec_c T', \quad n \in \mathbb{N} \quad \Rightarrow \quad T \prec_c T'.$$

Proof The inequality $T_n \prec_c T'$ implies that $\operatorname{dom} T' \subset \operatorname{dom} T_n$ and $\|T_n\varphi\| \le \|T'\varphi\|$ for $\varphi \in \operatorname{dom} T'$. Since this holds for all $n \in \mathbb{N}$, one sees that

$$\operatorname{dom} T' \subset \operatorname{dom} T \quad \text{and} \quad \|T\varphi\| = \sup_{n\in\mathbb{N}} \|T_n\varphi\| \le \|T'\varphi\|, \quad \varphi \in \operatorname{dom} T',$$

in other words $T \prec T'$. □

6 Nondecreasing Sequences of Closable Operators

It is a consequence of Theorem 5.1 that a sequence of closable linear operators which satisfy (5.1) has a closable limit. The description of the limit of the closures is of interest.

Proposition 6.1 *Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, be a sequence of linear operators for which (5.1) holds and assume that T_n, $n \in \mathbb{N}$, is closable. Let T be the closable limit of T_n in (5.2) and (5.3). Then the closures $T_n^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ of T_n satisfy*

$$T_m^{**} \prec_c T_n^{**}, \quad m \leq n, \quad \textit{and} \quad T_n^{**} \prec_c T^{**}. \tag{6.1}$$

Consequently, there exists a closed linear operator $S \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$, where $\mathfrak{K}_c$ is a Hilbert space, such that

$$\operatorname{dom} S = \left\{ \varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} T_n^{**} : \sup_{n \in \mathbb{N}} \|T_n^{**}\varphi\| < \infty \right\} \tag{6.2}$$

and which satisfies

$$T_n^{**} \prec_c S \prec_c T^{**} \quad \textit{and} \quad \|T_n^{**}\varphi\| \nearrow \|S\varphi\|, \quad \varphi \in \operatorname{dom} S. \tag{6.3}$$

In fact, $\operatorname{dom} T^{**} \subset \operatorname{dom} S$, *while* $\|S\varphi\| = \|T^{**}\varphi\|$ *for all* $\varphi \in \operatorname{dom} T^{**}$.

Proof The sequence T_n is assumed to satisfy (5.1), thus it follows that $T_m^{**} \prec_c T_n^{**}$ for $m \leq n$, by (2.3). Moreover, by Theorem 5.1 one has $T_n \prec_c T$, so that also $T_n^{**} \prec_c T^{**}$ by (2.3). Hence (6.1) holds and, in particular, Theorem 5.1 may be applied to the sequence of closed operators T_n^{**}.

Recall from Theorem 5.1 that the right-hand side in (6.2) is a linear space. Moreover, by the same theorem there exists a closed linear operator S defined on dom S in (6.2) for which (6.3) holds; observe that $S \prec_c T^{**}$ by Corollary 5.3.

Now it follows from (5.3) and (6.3) that $\|T\varphi\| = \|S\varphi\|$ for all $\varphi \in \operatorname{dom} T$. Here the operator S is closed and T is closable, and $S \prec_c T^{**}$ means that $CT^{**} \subset S$ for some contraction $C \in \mathbf{B}(\mathfrak{K}, \mathfrak{K}_c)$. One concludes that $\|S\varphi\| = \|CT^{**}\varphi\| = \|T^{**}\varphi\|$ holds in fact for all $\varphi \in \operatorname{dom} T^{**}$. □

A special case of Theorem 5.1, where all T_n are bounded everywhere defined operators, is worth mentioning separately.

Corollary 6.2 *Let $T_n \in \mathbf{B}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, such that*

$$\|T_m\varphi\| \leq \|T_n\varphi\|, \quad \varphi \in \mathfrak{H}, \quad m \leq n.$$

Then there exists a closed linear operator $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_c)$, where $\mathfrak{K}_c$ a Hilbert space, such that

$$\operatorname{dom} T = \left\{ \varphi \in \mathfrak{H} : \sup_{n \in \mathbb{N}} \|T_n\varphi\| < \infty \right\} \tag{6.4}$$

and which satisfies

$$T_n \prec_c T \quad \textit{and} \quad \|T_n\varphi\| \nearrow \|T\varphi\|, \quad \varphi \in \operatorname{dom} T. \tag{6.5}$$

Proof This is just an application of Theorem 5.1, as $\bigcap_{n=1}^{\infty} \operatorname{dom} T_n = \mathfrak{H}$. Hence there exists a linear operator $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ for which (6.4) and (6.5) hold. Since $T_n \in \mathbf{B}(\mathfrak{H}, \mathfrak{K}_n)$ one observes that T_n, $n \in \mathbb{N}$, is closed, which implies that T is closed. □

Remark 6.3 If in Corollary 6.2 one has $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$, then $\operatorname{dom} T = \mathfrak{H}$ and $T \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ by the closed graph theorem. However, if $\sup_{n\in\mathbb{N}} \|T_n\| = \infty$, then by the uniform boundedness principle there is an element $\varphi \in \mathfrak{H}$ for which $\sup_{n\in\mathbb{N}} \|T_n\varphi\| = \infty$ and $\operatorname{dom} T$ is a proper subset of $\mathfrak{H}$. Note that $\operatorname{dom} T$ is closed if and only if T is a bounded operator.

7 Nondecreasing Sequences of Linear Relations

In this section the emphasis will be on nondecreasing sequences of linear relations in the general case, i.e., the relations are not necessarily operators or not necessarily closed. However, also the regular parts and the closures form nondecreasing sequences. In particular, one may apply Theorem 2.3, which leads to a connection with the monotonicity principle in Theorem 3.2.

Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, be a sequence of linear relations which satisfy

$$T_m \prec_c T_n \quad m \leq n. \tag{7.1}$$

Observe that the regular parts $T_{n,\mathrm{reg}} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ of the relations T_n are closable operators which satisfy

$$T_{m,\mathrm{reg}} \prec_c T_{n,\mathrm{reg}}, \quad m \leq n, \tag{7.2}$$

see Lemma 2.2. Hence, by Theorem 5.1, there exists a closable linear operator $T_{\mathrm{r}} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_{\mathrm{r}})$, where $\mathfrak{K}_{\mathrm{r}}$ is a Hilbert space, such that

$$\operatorname{dom} T_{\mathrm{r}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} T_n : \sup_{n\in\mathbb{N}} \|T_{n,\mathrm{reg}}\varphi\| < \infty \right\} \tag{7.3}$$

and which satisfies

$$T_{n,\mathrm{reg}} \prec_c T_{\mathrm{r}} \quad \text{and} \quad \|T_{n,\mathrm{reg}}\varphi\| \nearrow \|T_{\mathrm{r}}\varphi\|, \quad \varphi \in \operatorname{dom} T_{\mathrm{r}}. \tag{7.4}$$

Moreover, the closures $(T_{n,\mathrm{reg}})^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ are closed linear operators which satisfy

$$(T_{m,\mathrm{reg}})^{**} \prec_c (T_{n,\mathrm{reg}})^{**}, \quad m \leq n, \quad \text{and} \quad (T_{n,\mathrm{reg}})^{**} \prec_c (T_{\mathrm{r}})^{**}, \tag{7.5}$$

see Proposition 6.1. By the same proposition, there exists a closed linear operator $S_{\mathrm{r}} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_{\mathrm{c}})$, where $\mathfrak{K}_{\mathrm{c}}$ is a Hilbert space, such that

$$\operatorname{dom} S_{\mathrm{r}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} T_n^{**} : \sup_{n\in\mathbb{N}} \|(T_{n,\mathrm{reg}})^{**}\varphi\| < \infty \right\} \tag{7.6}$$

and which satisfies

$$(T_{n,\mathrm{reg}})^{**} \prec_c S_{\mathrm{r}} \prec_c (T_{\mathrm{r}})^{**} \quad \text{and} \quad \|(T_{n,\mathrm{reg}})^{**}\varphi\| \nearrow \|S_{\mathrm{r}}\varphi\|, \quad \varphi \in \operatorname{dom} S_{\mathrm{r}}. \tag{7.7}$$

In fact, $\operatorname{dom}(T_{\mathrm{r}})^{**} \subset \operatorname{dom} S_{\mathrm{r}}$, while $\|S_{\mathrm{r}}\varphi\| = \|(T_{\mathrm{r}})^{**}\varphi\|$ for $\varphi \in \operatorname{dom}(T_{\mathrm{r}})^{**}$.

It follows from (7.1) that the closures $T_n^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ of T_n are closed relations which satisfy

$$T_m^{**} \prec_c T_n^{**} \quad m \le n, \tag{7.8}$$

see (2.3). Of course, by Lemma 2.2 also the regular parts of T_n satisfy such an inequality; but this gives again (7.5), due to the identity

$$((T_n)^{**})_{\mathrm{reg}} = (T_{n,\mathrm{reg}})^{**},$$

see (1.3). Since the relation $T_n^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ is closed, it follows that the product

$$H_n = T_n^* T_n^{**} \in \mathbf{L}(\mathfrak{H})$$

is a nonnegative selfadjoint relation and by Theorem 2.3 one sees that (7.8) implies

$$H_m \le H_n, \quad m \le n.$$

Thus according to Theorem 3.2 there exists a nonnegative selfadjoint relation $H_\infty \in \mathbf{L}(\mathfrak{H})$ which is the limit of the relations H_n in the strong resolvent sense or, equivalently, in the strong graph sense.

Theorem 7.1 *Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, be a sequence of linear relations which satisfy (7.1). Let $H_\infty \in \mathbf{L}(\mathfrak{H})$ be the nonnegative selfadjoint relation, which is the limit of the nondecreasing sequence of nonnegative selfadjoint relations $T_n^* T_n^{**} \in \mathbf{L}(\mathfrak{H})$. Then H_∞ satisfies*

$$\operatorname{dom}(H_\infty)^{\frac{1}{2}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} T_n^{**} : \sup_{n\in\mathbb{N}} \|(T_{n,\mathrm{reg}})^{**}\varphi\| < \infty \right\} \tag{7.9}$$

and, furthermore,

$$(T_{n,\mathrm{reg}})^{**}\varphi\| \nearrow \|(H_{\infty,\mathrm{reg}})^{\frac{1}{2}}\varphi\|, \quad \varphi \in \operatorname{dom}(H_\infty)^{\frac{1}{2}}. \tag{7.10}$$

Moreover, the limit $S_{\mathrm{r}} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_{\mathrm{c}})$ *of the sequence* $(T_{n,\mathrm{reg}})^{**} \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ *in (7.6) and (7.7) satisfies*

$$\|(T_{n,\mathrm{reg}})^{**}\varphi\| \nearrow \|S_{\mathrm{r}}\varphi\|, \quad \varphi \in \mathrm{dom}\, S_{\mathrm{r}} = \mathrm{dom}\,(H_\infty)^{\frac{1}{2}}. \tag{7.11}$$

Consequently, there exists a partial isometry $U \in \mathbf{L}(\mathfrak{K}_{\mathrm{c}}, \mathfrak{H})$ *such that*

$$(H_{\infty,\mathrm{reg}})^{\frac{1}{2}} = US_{\mathrm{r}} \quad \textit{and} \quad H_{\infty,\mathrm{reg}} = (S_{\mathrm{r}})^* S_{\mathrm{r}}. \tag{7.12}$$

Proof It is clear that the product $H_n = T_n^* T_n^{**} \in \mathbf{L}(\mathfrak{H})$ is a nonnegative selfadjoint relation. Furthermore, the closures T_n^{**} of T_n satisfy the inequalities (7.8). Therefore, the nonnegative selfadjoint relations $H_n = T_n^* T_n^{**} \in \mathbf{L}(\mathfrak{H})$ form a nondecreasing sequence thanks to Theorem 2.3. Thus by Theorem 3.2 there exists a nonnegative selfadjoint relation H_∞ such that (3.3) and (3.4) hold. Remember that

$$H_n = T_n^* T_n^{**} = (T_{n,\mathrm{reg}})^*(T_{n,\mathrm{reg}})^{**},$$

so that there exists a partial isometry $U_n \in \mathbf{L}(\mathfrak{K}_n, \mathfrak{H})$, such that

$$(H_{n,\mathrm{reg}})^{\frac{1}{2}} = U_n\,(T_{n,\mathrm{reg}})^{**}.$$

In other words, (3.3) and (3.4) lead to (7.9) and (7.10). Similarly, a comparison of (7.3) and (7.4) with (7.9) and (7.10) shows that (7.11) holds. Therefore, there exists a partial isometry $U \in \mathbf{L}(\mathfrak{L}, \mathfrak{H})$ such that $(H_{\infty,\mathrm{reg}})^{\frac{1}{2}} = US_{\mathrm{r}}$, which is the first assertion in (7.12). This identity shows that also the second assertion in (7.12) holds. □

Assume that the sequence $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ in Theorem 7.1 has an upper bound, i.e., there exists a linear relation $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{K}$ is a Hilbert space, such that

$$T_n \prec_c T, \quad m \leq n. \tag{7.13}$$

For instance, if the sequence T_n consists of operators then T may be taken as the limit of T_n by Theorem 5.1. It follows from (7.13) that

$$T_{n,\mathrm{reg}} \prec_c T_{\mathrm{reg}} \quad \text{and} \quad (T_{n,\mathrm{reg}})^{**} \prec_c (T_{\mathrm{reg}})^{**}.$$

With these upper bounds it follows for the closable limit T_{r} of $T_{n,\mathrm{reg}}$ that

$$T_{\mathrm{r}} \prec_c T_{\mathrm{reg}} \quad \text{and hence} \quad (T_{\mathrm{r}})^{**} \prec_c (T_{\mathrm{reg}})^{**}.$$

Consequently, for the closed limit S_{r} of $(T_{n,\mathrm{reg}})^{**}$ one has via (7.5)

$$S_{\mathrm{r}} \prec_c (T_{\mathrm{r}})^{**} \prec_c (T_{\mathrm{reg}})^{**}.$$

8 An Example of a Nondecreasing Sequence

In order to illustrate the various possibilities of convergence a simple example of a nondecreasing sequence will be presented. Let $R \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a linear operator and define the sequence of linear operators $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $n \in \mathbb{N}$, by

$$T_n = \sqrt{n}\, R. \tag{8.1}$$

Then it is clear from (8.1) that

$$\bigcap_{n=1}^{\infty} \operatorname{dom} T_n = \operatorname{dom} R \quad \text{and} \quad T_n \prec_c T_{n+1}, \quad n \in \mathbb{N},$$

so that (5.1) is satisfied. Hence one can apply Theorem 5.1 to determine the limit T of the sequence T_n. It follows from (5.2) and (5.3) that

$$\operatorname{dom} T = \ker R \quad \text{and} \quad T = O_{\ker R}. \tag{8.2}$$

In fact, it is clear that T is closable and singular, simultaneously, and that

$$T^{**} = O_{\overline{\ker}\, R}. \tag{8.3}$$

Moreover, observe that it follows from (8.2) and (8.3) that

$$T^*T = \ker R \times (\ker R)^{\perp} \quad \text{and} \quad T^*T^{**} = \overline{\ker}\, R \times (\ker R)^{\perp}.$$

Note that in the special case where $R \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ this illustrates [2, Corollary 5.2.13]. If, in addition, the operator $R \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is closable, then all T_n in (8.1) are closable. The closures T_n^{**} of T_n are given by

$$T_n^{**} = \sqrt{n}\, R^{**},$$

and it is clear that (6.1) is satisfied. Hence one can apply Proposition 6.1 to obtain the closed limit S of the sequence T_n^{**}. It follows from (6.2) and (6.3) that

$$\operatorname{dom} S = \ker R^{**} \quad \text{and} \quad S = O_{\ker R^{**}}. \tag{8.4}$$

One sees directly from (8.3) that $T^{**} \subset S$, which illustrates the situation in Proposition 6.1. The inclusion $T^{**} \subset S$ is strict precisely when $\overline{\ker}\, R \subset \ker R^{**}$ is strict. As an example where the inclusion is strict, let R be an operator such that R^{-1} is an operator that is not closable, in which case $\ker R = \{0\}$ and $\ker R^{**} \neq \{0\}$. Note that the nonnegative selfadjoint relation S^*S is given by

$$S^*S = \ker R^{**} \times (\ker R^{**})^{\perp}.$$

as follows from (8.4).

Next consider the Lebesgue decomposition of R which is given by

$$R = R_{\rm reg} + R_{\rm sing}, \quad R_{\rm reg} = (I - P)R, \quad R_{\rm sing} = PR,$$

where P be the orthogonal projection form $\mathfrak{K}$ onto mul R^{**}. Then the regular parts $T_{n,\rm reg}$ of T_n in (8.1) are given by

$$T_{n,\rm reg} = \sqrt{n}\, R_{\rm reg},$$

and it is clear that (7.2) is satisfied. For the closable limit $T_{\rm r}$ of the sequence $T_{n,\rm reg}$ it follows from (7.3) and (7.4) that

$$\mathrm{dom}\, T_{\rm r} = \ker R_{\rm reg} \quad \text{and} \quad T_{\rm r} = O_{\ker R_{\rm reg}}.$$

Since $T_{\rm reg} = O_{\ker \rm R}$ one sees directly that $T_{\rm r} \prec_c T_{\rm reg}$, which is the general situation. The inequality is strict precisely when $\ker R \subset \ker R_{\rm reg}$ is strict. Observe that

$$(T_{\rm r})^* T_{\rm r} = \ker R_{\rm reg} \times (\ker R_{\rm reg})^\perp \quad \text{and} \quad (T_{\rm r})^*(T_{\rm r})^{**} = \overline{\ker}\, R_{\rm reg} \times (\ker R_{\rm reg})^\perp.$$

The closures of $T_{n,\rm reg}$ are given by

$$(T_{n,\rm reg})^{**} = \sqrt{n}(R_{\rm reg})^{**},$$

and it is clear that (7.5) is satisfied. For the closed limit $S_{\rm r}$ of the sequence $(T_{n,\rm reg})^{**}$ it follows from (7.6) and (7.7) that

$$\mathrm{dom}\, S_{\rm r} = \ker (R_{\rm reg})^{**} \quad \text{and} \quad S_{\rm r} = O_{\ker (R_{\rm reg})^{**}}.$$

Therefore, one sees that

$$(S_{\rm r})^* S_{\rm r} = \ker (R_{\rm reg})^{**} \times (\ker (R_{\rm reg})^{**})^\perp. \tag{8.5}$$

Finally consider T_n as in (8.1) with a general operator $R \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$. Then the product relation $H_n = T_n^* T_n^{**}$ is given by

$$H_n = nR^* R^{**} = n(R_{\rm reg})^*(R_{\rm reg})^{**}.$$

Since $\ker (R_{\rm reg})^*(R_{\rm reg})^{**} = \ker (R_{\rm reg})^{**}$, it follows from Example 3.3 that the limit H_∞ of H_n is given by $H_\infty = \ker (R_{\rm reg})^{**} \times (\ker (R_{\rm reg})^{**})^\perp$, which agrees with (8.5).

9 A Description of Closed Linear Operators

Let $T_n \in \mathbf{B}(\mathfrak{H}, \mathfrak{K}_n)$ be a sequence of operators that satisfy (5.1). According to Corollary 6.2 there is a closed limit $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ which satisfies (6.4) and (6.5). This section contains some variations on this theme.

First it is shown that any nonnegative selfadjoint operator is rougly speaking the limit of a certain class of nonnegative bounded linear operators.

Lemma 9.1 *Let $A \in \mathbf{L}(\mathfrak{H})$ be a nonnegative selfadjoint operator. Then there exists a sequence of nonnegative selfadjoint operators $A_n \in \mathbf{B}(\mathfrak{H})$ such that*

$$(A_m\varphi, \varphi) \le (A_n\varphi, \varphi), \quad \varphi \in \mathfrak{H}, \quad m \le n, \tag{9.1}$$

and

$$(A_n\varphi, \varphi) \nearrow \|A^{\frac{1}{2}}\varphi\|^2, \quad \varphi \in \operatorname{dom} A^{\frac{1}{2}}. \tag{9.2}$$

Proof Consider the spectral representation of the nonnegative selfadjoint operator A the Hilbert space $\mathfrak{H}$:

$$A = \int_0^\infty \lambda \, dE_\lambda.$$

By means of this representation let the nonnegative selfadjoint operators $A_n \in \mathbf{B}(\mathfrak{H})$ be defined by

$$A_n = \int_0^n \lambda \, dE_\lambda, \quad n \in \mathbb{N}.$$

Then is is clear that $(A_m\varphi, \varphi) \le (A_n\varphi, \varphi)$, $m \le n$, for all $\varphi \in \mathfrak{H}$. This gives (9.1). By the construction of the sequence A_n one obtains

$$(A_n\varphi, \varphi) \nearrow \|A^{\frac{1}{2}}\varphi\|^2, \quad \varphi \in \operatorname{dom} A^{\frac{1}{2}},$$

which gives (9.2). □

As a consequence of Lemma 9.1 there is some kind of converse of Corollary 6.2.

Proposition 9.2 *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a closed linear operator. Then there exists a sequence of linear operators $T_n \in \mathbf{B}(\overline{\operatorname{dom}}\, T, \mathfrak{H})$, such that*

$$\|T_m\varphi\| \le \|T_n\varphi\|, \quad \varphi \in \overline{\operatorname{dom}}\, T, \quad m \le n, \tag{9.3}$$

and

$$\|T_n\varphi\| \nearrow \|T\varphi\|, \quad \varphi \in \operatorname{dom} T. \tag{9.4}$$

Proof The product relation $H = T^*T$ is nonnegative and selfadjoint in $\mathfrak{H}$ with $\operatorname{mul} H = \operatorname{mul} T^* = (\operatorname{dom} T)^\perp$. Then $H = A \mathbin{\widehat{\oplus}} (\{0\} \times (\operatorname{dom} T)^\perp$, where $A = H_{\rm reg}$ is a nonnegative selfadjoint operator in $\overline{\operatorname{dom}}\, T$. Then there exists a sequence of nonnegative selfadjoint operators $A_n \in \mathbf{B}(\overline{\operatorname{dom}}\, T)$ such that

$$(A_m\varphi, \varphi) \le (A_n\varphi, \varphi), \quad \varphi \in \overline{\operatorname{dom}}\, T, \quad m \le n,$$

and

$$(A_n\varphi,\varphi)\to \|A^{\frac{1}{2}}\varphi\|^2,\quad \varphi\in \operatorname{dom} A^{\frac{1}{2}}=\operatorname{dom} T\subset \overline{\operatorname{dom}}\, T.$$

Due to $H=T^*T$ and $A=H_{\rm reg}$ one sees that

$$\|A^{\frac{1}{2}}\varphi\|=\|T\varphi\|,\quad \varphi=\operatorname{dom} A^{\frac{1}{2}}=\operatorname{dom} T,$$

see Lemma 11.2. Finally define $T_n=A_n^{\frac{1}{2}}$, so that (9.3) and (9.4) are satisfied. □

The last result in this section is a direct consequence of Proposition 9.2; it describes the closability of an operator in terms of a sequence of bounded linear operators; see for the original statement [4, Theorem 8.8, Theorem 8.9].

Corollary 9.3 *Let $S\in \mathbf{L}(\mathfrak{H},\mathfrak{K})$ be a linear operator. Then the following statements are equivalent:*

(i) *S is closable;*
(ii) *there exists a sequence of linear operators $T_n\in \mathbf{B}(\overline{\operatorname{dom}}\, S,\mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, such that*

$$\|T_m\varphi\|\le \|T_n\varphi\|,\quad \varphi\in \overline{\operatorname{dom}}\, S,\quad m\le n, \tag{9.5}$$

and

$$\|T_n\varphi\|\nearrow \|S\varphi\|,\quad \varphi\in \operatorname{dom} S. \tag{9.6}$$

Proof (i) ⇒ (ii) Let $S\in \mathbf{L}(\mathfrak{H},\mathfrak{K})$ be a closable operator and denote its closure by T. Then $T\in \mathbf{L}(\mathfrak{H},\mathfrak{K})$ is a closed operator which extends S, such that $\overline{\operatorname{dom}}\, T=\overline{\operatorname{dom}}\, S$. Now apply Proposition 9.2.

(ii) ⇒ (i) Let $T_n\in \mathbf{B}(\overline{\operatorname{dom}}\, S,\mathfrak{K}_n)$ be a sequence for which (9.5) holds. Then by Corollary 6.2 there exists a closed linear operator $T\in \mathbf{L}(\overline{\operatorname{dom}}\, S,\mathfrak{K})$, such that

$$\operatorname{dom} T=\left\{\varphi\in \overline{\operatorname{dom}}\, S:\ \sup_{n\in\mathbb{N}}\|T_n\varphi\|<\infty\right\},$$

and which satisfies

$$\|T_n\varphi\|\nearrow \|T\varphi\|,\quad \varphi\in \overline{\operatorname{dom}}\, S.$$

Thanks to (9.6) one has $\|S\varphi\|=\|T\varphi\|$ for all $\varphi\in \operatorname{dom} S$. Since T is closed if follows that S is closable. □

An application of these results can be found in [7, Theorem 6.4], where pairs of bounded linear operators are classified in terms of almost domination.

10 Nonincreasing Sequences of Linear Operators

In this section there is a brief review for the situation of nonincreasing sequences of linear operators in the sense of contractive domination. First recall the analog of the monotonicity principle in Theorem 3.2 for nonincreasing sequences; see [3, Theorem 3.7].

Theorem 10.1 *Let $K_n \in \mathbf{L}(\mathfrak{H})$ be a sequence of nonnegative selfadjoint relations and assume they satisfy*

$$K_n \leq K_m, \quad m \leq n.$$

Then there exists a nonnegative selfadjoint relation $K_\infty \in \mathbf{L}(\mathfrak{H})$ with

$$K_\infty \leq K_n, \quad n \in \mathbb{N}. \tag{10.1}$$

In fact, $K_n \to K_\infty$ in the strong resolvent sense or, equivalently, in the strong graph sense. Moreover, the square root of K_∞ satisfies

$$\operatorname{ran}(K_\infty)^{\frac{1}{2}} = \left\{ \varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{ran}(K_n)^{\frac{1}{2}} : \lim_{n\to\infty} \|((K_n)^{-\frac{1}{2}})_{\mathrm{reg}}\varphi\| < \infty \right\} \tag{10.2}$$

and, furthermore,

$$\|((K_n)^{-\frac{1}{2}})_{\mathrm{reg}}\varphi\| \nearrow \|((K_\infty)^{-\frac{1}{2}})_{\mathrm{reg}}\varphi\|, \quad \varphi \in \operatorname{ran}(K_\infty)^{\frac{1}{2}}. \tag{10.3}$$

Proof A short proof is included for completeness. By antitonicity, the sequence $(K_n)^{-1} \in \mathbf{L}(\mathfrak{H})$ is nondecreasing; cf. [2, Corollary 5.2.8]. Hence, by Theorem 3.2, there exists a nonnegative selfadjoint relation, say, $(K_\infty)^{-1} \in \mathbf{L}(\mathfrak{H})$, such that $(K_\infty)^{-1}$ is the limit of the sequence $(K_n)^{-1} \in \mathbf{L}(\mathfrak{H})$ in the strong resolvent sense or, equivalently, in the strong graph sense, and $(K_n)^{-1} \leq (K_\infty)^{-1}$. Then, again by antitonicity, $K_\infty \leq K_n$ and, moreover, K_∞ is the limit of the sequence K_n in the strong graph sense. The rest of the statements is a direct translation of similar statements in Theorem 3.2. □

Note that the multivalued parts of the relations K_n in Theorem 3.2 form a nonincreasing sequence. If one of the relations K_n in Theorem 10.1 is an operator, then all of its successors are operators and, ultimately, the limit K_∞ is an operator.

Example 10.2 Let $A \in \mathbf{L}(\mathfrak{H})$ be a nonnegative selfadjoint operator or relation. Then it is clear that the sequence $K_n = \frac{1}{n}A$ of nonnegative selfadjoint operators or relations is nonincreasing. Hence there exists a nonnegative selfadjoint relation $K_\infty \in \mathbf{L}(\mathfrak{H})$ such that $K_n \to K_\infty$ is the strong graph sense. By means of Example 3.3 one sees immediately that

$$K_\infty = \overline{\operatorname{dom}} A \times \operatorname{mul} A.$$

The following result is the analog of Theorem 5.1 for nonincreasing sequences of linear operators. Due to the sequence being nonincreasing there are no further convergence restrictions for the limit as in Theorem 5.1.

Theorem 10.3 *Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, be a sequence of linear operators which satisfy*

$$T_n \prec_c T_m, \quad m \leq n. \tag{10.4}$$

Then there exists a linear operator $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{K}$ is a Hilbert space, such that

$$\operatorname{dom} T = \bigcup_{n \in \mathbb{N}} \operatorname{dom} T_n, \tag{10.5}$$

and which satisfies

$$T \prec_c T_n \quad \textit{and} \quad \|T_n \varphi\| \searrow \|T\varphi\|, \quad \varphi \in \operatorname{dom} T. \tag{10.6}$$

Proof Denote the right-hand side of (10.5) by $\mathfrak{D}$. Now let $\varphi \in \mathfrak{D}$, then clearly $\varphi \in \operatorname{dom} T_N$ for some $N \in \mathbb{N}$. For all $n \geq N$ one has $T_n \prec_c T_N$, which implies that $\varphi \in \operatorname{dom} T_n$ for all $n \geq N$ and $\lim_{n \to \infty} \|T_n \varphi\|$ exists by (10.4). Hence for each $\varphi \in \mathfrak{D}$ one may define

$$\|\varphi\|_+ = \lim_{n \to \infty} \|T_n \varphi\|.$$

Then $\| \cdot \|_+$ generates a well-defined seminorm on the linear subspace $\mathfrak{D}$. Let $(\cdot, \cdot)_+$ be the corresponding semi-inner product. By Lemma 4.1 there exists a linear operator T defined on $\operatorname{dom} T = \mathfrak{D} \subset \mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ such that

$$(\varphi, \psi)_+ = (T\varphi, T\psi), \quad \varphi \in \mathfrak{D}.$$

This shows the assertion in (10.6). □

Now Theorem 10.3 will be applied under the assumption that the linear operators $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$ are closed. Then the corresponding relations $K_n = T_n^* T_n \in \mathbf{L}(\mathfrak{H})$ are nonnegative and selfadjoint.

Theorem 10.4 *Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{K}_n)$, where $\mathfrak{K}_n$ are Hilbert spaces, be a sequence of closed linear operators which satisfy* (10.4) *and let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{K}$ is a Hilbert space, be the limit operator satisfying* (10.5) *and* (10.6). *Let $K_\infty \in \mathbf{L}(\mathfrak{H})$ be the nonnegative selfadjoint relation, which is the limit of the nonincreasing sequence of nonnegative selfadjoint relations $K_n = T_n^* T_n \in \mathbf{L}(\mathfrak{H})$, so that K_∞ satisfies* (10.2) *and* (10.3). *Then K_∞ and T are connected via*

$$K_\infty = T^* T^{**}. \tag{10.7}$$

Consequently, there exists a partial isometry $U \in \mathbf{B}(\mathfrak{K}, \mathfrak{H})$ *such that*

$$(K_{\infty,\mathrm{reg}})^{\frac{1}{2}} = U(T_{\mathrm{reg}})^{**} \quad or \quad (T_{\mathrm{reg}})^{**} = U^*(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}. \tag{10.8}$$

Moreover, for the limit $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ *one has*

(a) T *is closable if and only if* $T \subset U^*(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}$*;*
(b) T *is closed if and only if* $T = U^*(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}$*;*
(c) T *is singular if and only if* K_∞ *is singular.*

Proof Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be the limit operator in (10.5) and (10.6). Recall from (10.6) that $T \prec_c T_n$. This leads to $T^{**} \prec_c (T_n)^{**} = T_n$, which gives $T^*T^{**} \le T_n^*T_n = K_n$ by Theorem 2.3. Since this holds for all $n \in \mathbb{N}$ one obtains

$$T^*T^{**} \le K_\infty. \tag{10.9}$$

Moreover, recall from (10.1) that $K_\infty \le K_n$, so that $(K_\infty)^{\frac{1}{2}} \prec_c T_n$ by Theorem 2.3. In particular, it follows that $(K_{\infty,\mathrm{reg}})^{\frac{1}{2}} \prec_c T_n$. Hence one has

$$\mathrm{dom}\, T_n \subset \mathrm{dom}\, (K_{\infty,\mathrm{reg}})^{\frac{1}{2}} \quad \text{and} \quad \|(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}\varphi\| \le \|T_n\varphi\|, \quad \varphi \in \mathrm{dom}\, T_n.$$

Clearly, with (10.5) this now leads to

$$\mathrm{dom}\, T \subset \mathrm{dom}\, (K_{\infty,\mathrm{reg}})^{\frac{1}{2}} \quad \text{and} \quad \|(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}\varphi\| \le \inf_{n\in\mathbb{N}} \|T_n\varphi\|, \quad \varphi \in \mathrm{dom}\, T.$$

Thanks to (10.6) this reads

$$\mathrm{dom}\, T \subset \mathrm{dom}\, (K_{\infty,\mathrm{reg}})^{\frac{1}{2}} \quad \text{and} \quad \|(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}\varphi\| \le \|T\varphi\|, \quad \varphi \in \mathrm{dom}\, T,$$

or equivalently, $(K_{\infty,\mathrm{reg}})^{\frac{1}{2}} \prec_c T$. Since closures and regular parts are preserved under the inequality, this gives $(K_{\infty,\mathrm{reg}})^{\frac{1}{2}} \prec_c (T^{**})_{\mathrm{reg}}$ or $(K_\infty)^{\frac{1}{2}} \prec_c T^{**}$ by Lemma 2.2. Therefore, one obtains

$$K_\infty \le T^*T^{**}. \tag{10.10}$$

Combining the inequalities (10.9) and (10.10) leads to the inequalities

$$T^*T^{**} \le K_\infty \le T^*T^{**},$$

or, equivalently,

$$(T^*T^{**} - \lambda)^{-1} \le (K_\infty - \lambda)^{-1} \le (T^*T^{**} - \lambda)^{-1}, \quad \lambda < 0.$$

This shows that (10.7) holds. Next (10.8) follows thanks to Lemma 11.2.

Finally, the last assertions concerning the relationship between T and K_∞ will be discussed.

(a) If $T \subset U^*(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}$, then T is closable. Conversely, if T is closable, then $T = T_{\mathrm{reg}} \subset (T_{\mathrm{reg}})^{**} = U^*(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}$.
(b) If $T = U^*(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}$, then T is closed. Conversely, if T is closed, then $T = (T_{\mathrm{reg}})^{**} = U^*(K_{\infty,\mathrm{reg}})^{\frac{1}{2}}$.
(c) If T is singular, then $T^* = \mathfrak{A} \times \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are closed linear subspaces of $\mathfrak{K}$ and $\mathfrak{H}$, respectively. Hence $T^{**} = \mathfrak{B}^\perp \times \mathfrak{A}^\perp$, so that $T^*T^{**} = \mathfrak{B}^\perp \times \mathfrak{B}$ and K_∞ is singular. Conversely, let $K_\infty = T^*T^{**}$ be singular. Then $T^*T^{**} = \mathfrak{B}^\perp \times \mathfrak{B}$ with a closed linear subspace $\mathfrak{B}$ in $\mathfrak{H}$. Hence it follows that

$$\begin{cases} \mathrm{mul}\, T^* = \mathrm{mul}\, T^*T^{**} = \mathfrak{B}, \\ \ker T^{**} = \ker T^*T^{**} = \mathfrak{B}^\perp. \end{cases}$$

Therefore $\overline{\mathrm{ran}}\, T^* = (\ker T^{**})^\perp = \mathrm{mul}\, T^*$, i.e. T^* and, hence, also T is singular.

□

In the present circumstances there is in general no preservation of closedness in Theorem 10.3. This will be shown in the following example; it is a simple adaptation of [3, Example 4.5] or [12, p. 374].

Example 10.5 Let $T_n \in \mathbf{L}(\mathfrak{H}, \mathfrak{H} \oplus \mathbb{C})$ with $\mathfrak{H} = L^2(0,1)$ be given as a column operator by $T_n = \mathrm{col}\,(T_n^1, T_n^2)$ (see [8]) with the operators T_n^1 and T_n^2 given by

$$T_n^1 f = \frac{1}{\sqrt{n}} iDf, \quad f(1) = 0, \quad \text{and} \quad T_n^2 f = f(0).$$

Here D stands for the maximal differentiation operator in $L^2(0,1)$. Then T_n^1 is closed, T_n^2 is singular, while the column T_n is closed. It is clear that $T_n \prec_c T_m$, $m \le n$, and the limit $T \in \mathbf{L}(\mathfrak{H})$ is given by $Tf = f(0)e$, where the function $e \in \mathfrak{H} = L^2(0,1)$ is defined by $e(x) = 1$. Moreover, the corresponding nonnegative selfadjoint relation $K_n = T_n^* T_n$ is the operator in $\mathfrak{H} = L^2(0,1)$ given by

$$K_n f = -\frac{1}{n} D^2 f, \quad f'(0) = nf(0), \ f(1) = 0.$$

The relations K_n form a sequence that is nonincreasing with the nonnegative selfadjoint limit K_∞ and, by Theorem 10.4, one has

$$K_\infty = T^*T^{**}.$$

Now observe that $T^* = (\mathrm{span}\,\{e\})^\perp \times \{0\}$ and $T^{**} = \mathfrak{H} \times \mathrm{span}\,\{e\}$, so that T is a singular operator and, in fact $T^*T^{**} = \mathfrak{H} \times \{0\}$. Hence it follows that

$$K_\infty = T^*T^{**} = \mathfrak{H} \times \{0\}.$$

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11. Appendix: On the Products T^*T and T^*T^{**}

This appendix contains a number of properties of the relations T^*T and T^*T^{**} when $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ is a linear relation. The main emphasis is on the interplay with the regular parts of these relations. For the convenience of the reader, the arguments are included.

Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, so that $T^* \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$ is a closed linear relation. The product $T^*T \in \mathbf{L}(\mathfrak{H})$ is defined as

$$T^*T = \left\{ \{f, f'\} \in \mathfrak{H} \times \mathfrak{H} : \{f, h\} \in T, \ \{h, f'\} \in T^* \text{ for some } h \in \mathfrak{K} \right\}. \tag{11.1}$$

Hence, for the elements in the right-hand side of (11.1) it is clear that

$$(f', f) = \|h\|^2. \tag{11.2}$$

It follows immediately from (11.1) and (11.2) that the relation T^*T is nonnegative. Moreover, it also follows from (11.1) and (11.2) that

$$\operatorname{mul} T^*T = \operatorname{mul} T^*. \tag{11.3}$$

It is clear from $T \subset T^{**}$ that the nonnegative relation T^*T has a nonnegative extension T^*T^{**}. Since T^{**} is closed the product T^*T^{**} is selfadjoint; cf. [2, Lemma 1.5.8]). Moreover one sees that

$$T^*T \subset T^*T^{**} \subset (T^*T)^*. \tag{11.4}$$

In particular, it follows from (11.4) that the closure of T^*T satisfies

$$(T^*T)^{**} \subset T^*T^{**}. \tag{11.5}$$

However, in general, even when T is closable, there is no equality in (11.5).

Recall the definition of the regular part $T_{\rm reg}$: $T_{\rm reg} = (I - P)T$ where P is the orthogonal projection from $\mathfrak{K}$ onto $\operatorname{mul} T^{**}$, so that also $(T^{**})_{\rm reg} = (I - P)T^{**}$. This gives $(T_{\rm reg})^* = ((T^{**})_{\rm reg})^*$, which by taking adjoints leads to the formal identity $(T_{\rm reg})^{**} = ((T^{**})_{\rm reg})^{**}$. Note that $(T^{**})_{\rm reg}$ is closed, so that $(T^{**})_{\rm reg} = (T_{\rm reg})^{**}$ in (1.3) is clear.

There is an interesting interplay between linear relations and their regular parts when forming quadratic combinations. Let $\{f, f'\} \in T^{**}$ and $\{g, g'\} \in T^*$, then by definition there is the identity

$$(g', f) = (g, f'). \tag{11.6}$$

Recall that the orthogonal projection P maps $\mathfrak{K}$ onto $\operatorname{mul} T^{**} = \overline{\operatorname{dom}}\, T^*$, and let Q be the orthogonal projection from $\mathfrak{H}$ onto $\operatorname{mul} T^* = \overline{\operatorname{dom}}\, T^{**}$. Therefore the identity (11.6) reads

$$(g', (I - Q)f) = ((I - P)g, f'), \tag{11.7}$$

which can be rewritten in terms of the regular parts

$$((T^*)_{\rm reg} g, f) = (g, (T_{\rm reg})^{**} f), \quad f \in \operatorname{dom} T^{**}, \ g \in \operatorname{dom} T^*, \tag{11.8}$$

where the equality (1.3) has been used. Likewise, there is the identity

$$((T^*)_{\rm reg} g, f) = (g, T_{\rm reg} f), \quad f \in \operatorname{dom} T, \ g \in \operatorname{dom} T^*, \tag{11.9}$$

which also follows from (11.6) and (11.7). The following lemma shows the various interrelationships.

Lemma 11.1 *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a linear relation. Then*

$$\begin{cases} \big\{\{\varphi, h\} \in T : h \in \operatorname{dom} T^*\big\} \subset T_{\rm reg}, \\ \big\{\{\varphi, h\} \in T^{**} : h \in \operatorname{dom} T^*\big\} \subset (T^{**})_{\rm reg} = (T_{\rm reg})^{**}, \end{cases} \tag{11.10}$$

and

$$\begin{cases} T^*T \subset T^*T_{\rm reg} = (T_{\rm reg})^* T_{\rm reg}, \\ T^*T^{**} = T^*(T_{\rm reg})^{**} = (T_{\rm reg})^*(T_{\rm reg})^{**}. \end{cases} \tag{11.11}$$

Moreover, the multivalued parts in (11.11) satisfy

$$\operatorname{mul} T^* = \operatorname{mul} (T_{\rm reg})^*, \tag{11.12}$$

and, consequently,

$$\begin{cases} (T^*T)_{\rm reg} \subset (T^*)_{\rm reg} T_{\rm reg} = \big((T_{\rm reg})^* T_{\rm reg}\big)_{\rm reg}, \\ (T^*T^{**})_{\rm reg} = (T^*)_{\rm reg}(T_{\rm reg})^{**} = \big((T_{\rm reg})^*(T_{\rm reg})^{**}\big)_{\rm reg}. \end{cases} \tag{11.13}$$

In particular,

$$\begin{cases} \big(((T_{\mathrm{reg}})^* T_{\mathrm{reg}}\big)_{\mathrm{reg}}\varphi, \psi\big) = (T_{\mathrm{reg}}\varphi, T_{\mathrm{reg}}\psi), \\ \qquad\qquad\qquad\qquad \varphi \in \mathrm{dom}\, T^*T_{\mathrm{reg}},\ \ \psi \in \mathrm{dom}\, T, \\ \big(((T_{\mathrm{reg}})^*(T_{\mathrm{reg}})^{**}\big)_{\mathrm{reg}}\varphi, \psi\big) = ((T_{\mathrm{reg}})^{**}\varphi, (T_{\mathrm{reg}})^{**}\psi), \\ \qquad\qquad\qquad\qquad \varphi \in \mathrm{dom}\, T^*T^{**},\ \ \psi \in \mathrm{dom}\, T^{**}. \end{cases} \tag{11.14}$$

Proof Due to $\mathrm{dom}\, T^* \subset \overline{\mathrm{dom}}\, T^* = (\mathrm{mul}\, T^{**})^\perp$ and (1.3) one sees that (11.10) holds. Hence it is clear that $T^*T \subset T^*T_{\mathrm{reg}}$. With the orthogonal projection P from $\mathfrak{K}$ onto $\mathrm{mul}\, T^{**}$, one sees that

$$T^*(I-P)T = T^*(I-P)^2T = ((I-P)T)^*(I-P)T,$$

which completes the proof of the first part of (11.11). Furthermore, replacing T by T^{**} in the first part of (11.11) leads with (1.3) to the second part; the original inclusion is now an identity since T^*T^{**} is selfadjoint. The identity (11.12) is a consequence of (11.11) due to (11.3). The consequence in (11.13) is obtained from (11.12) together with (11.3).

It follows from (11.9) with $f = \psi$ and $g = T_{\mathrm{reg}}\varphi$ that

$$((T^*)_{\mathrm{reg}}T_{\mathrm{reg}}\varphi, \psi) = (T_{\mathrm{reg}}\varphi, T_{\mathrm{reg}}\psi), \quad \varphi, \psi \in \mathrm{dom}\, T, \quad T_{\mathrm{reg}}\varphi \in \mathrm{dom}\, T^*.$$

Note that the conditions $\varphi \in \mathrm{dom}\, T$ and $T_{\mathrm{reg}}\varphi \in \mathrm{dom}\, T^*$ are equivalent to the condition $\varphi \in \mathrm{dom}\, T^*T_{\mathrm{reg}}$. Thus, with (11.13), the first assertion in (11.14) has been shown. Likewise, it follows from (11.8) with $f = \psi$ and $g = (T_{\mathrm{reg}})^{**}\varphi$ that

$$\begin{aligned} ((T^*)_{\mathrm{reg}}(T_{\mathrm{reg}})^{**}\varphi, \psi) &= ((T_{\mathrm{reg}})^{**}\varphi, (T_{\mathrm{reg}})^{**}\psi), \\ &\qquad \varphi, \psi \in \mathrm{dom}\, T^{**}, \quad (T_{\mathrm{reg}})^{**}\varphi \in \mathrm{dom}\, T^*. \end{aligned}$$

Note that the conditions $\varphi \in \mathrm{dom}\, T^{**}$ and $(T_{\mathrm{reg}})^{**}\varphi \in \mathrm{dom}\, T^*$ are equivalent to the condition $\varphi \in \mathrm{dom}\, T^*T$, thanks to (11.11). Thus, with (11.13), the second assertion in (11.14) has been shown. □

There is a special, useful, case of Lemma 11.1 that deserves attention. It is about the orthogonal operator part of $H = T^*T$ when T is closed.

Lemma 11.2 *Let $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a closed linear relation and let $H \in \mathbf{L}(\mathfrak{H})$ be the nonnegative selfadjoint relation defined by $H = T^*T$. Then*

$$(H_{\mathrm{reg}}\varphi, \psi) = (T_{\mathrm{reg}}\varphi, T_{\mathrm{reg}}\psi), \quad \varphi \in \mathrm{dom}\, T^*T,\ \ \psi \in \mathrm{dom}\, T, \tag{11.15}$$

and there exists a partial isometry $U \in \mathbf{B}(\mathfrak{K}, \mathfrak{H})$ such that

$$(H_{\mathrm{reg}})^{\frac{1}{2}} = UT_{\mathrm{reg}}.$$

Proof Recall that $H = T^*T \in \mathbf{L}(\mathfrak{H})$ is nonnegative and selfadjoint and that mul $H =$ mul T^*. It follows from Lemma 11.1 that the identity (11.15) is satisfied. Therefore

$$\|(H_{\text{reg}})^{\frac{1}{2}}\varphi\| = \|T_{\text{reg}}\varphi\|, \quad \varphi \in \text{dom}\, H_{\text{reg}} = \text{dom}\, H = \text{dom}\, T^*T = \text{dom}\, (T_{\text{reg}})^* T_{\text{reg}}.$$

It is clear that dom $H_{\text{reg}} = \text{dom}\, (T_{\text{reg}})^* T_{\text{reg}}$ is a core for $(H_{\text{reg}})^{\frac{1}{2}}$ and for T_{reg}; cf. [2, Lemma 1.5.10]. Hence the assertion follows. □

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