Buying from a Group

Nima Haghpanah, Department of Economics, Pennsylvania State University,

Aditya Kuvalekar, Department of Economics, University of Essex

Elliot Lipnowski, Department of Economics, Columbia University,

Accepted for publication in the American Economic Review.

Buying from a Group

Nima Haghpanah Aditya Kuvalekar Elliot Lipnowski*

May 8, 2024

Abstract

A buyer procures a good owned by a group of sellers whose heterogeneous cost of trade is private information. The buyer must either buy the whole good or nothing, and sellers share the transfer in proportion to their share of the good. We characterize the optimal mechanism: trade occurs if and only if the buyer's benefit of trade exceeds a weighted average of sellers' virtual costs. These weights are endogenous, with sellers who are ex ante less inclined to trade receiving higher weight. This mechanism always outperforms posted-price mechanisms. An extension characterizes the entire Pareto frontier.

JEL Classification: D23, D47, D71, D82, Q15

For developing countries, a key challenge in transitioning from an agricultural economy to a manufacturing economy is land acquisition. Manufacturers often require large parcels of land whose ownership is dispersed among a group of individuals. Acquiring such land from a group of sellers is a challenging problem in the presence of property rights: no individual can be forced to part with his land.[1] In describing the puzzle of empty storefronts in prime areas in Moscow in the post-Soviet era, Heller (1998) terms such a situation "the tragedy of the anticommons": strong property rights lead to underuse of a resource. Several projects in India, for example, have sparked protests around issues of land acquisition largely because of unfair terms offered to the sellers or because they are coerced into selling against their wishes, resulting in those projects' relocation or complete abandonment.[2]

*Haghpanah: Department of Economics, Pennsylvania State University, haghpanah@psu.edu. Kuvalekar: Department of Economics, University of Essex, a.kuvalekar@essex.ac.uk. Lipnowski: Department of Economics, Columbia University, e.lipnowski@columbia.edu. A previous version of this paper was circulated under the title "Selling to a Group." We thank the coeditor, anonymous referees, Nageeb Ali, Dirk Bergemann, Dhruva Bhaskar, Ben Brooks, Yeon-Koo Che, Rahul Deb, Laura Doval, Alex Frankel, Marina Halac, Navin Kartik, George Mailath, Benny Moldovanu, Doron Ravid, Luis Rayo, Ilya Segal, and Ludvig Sinander for helpful discussions.

[1]Sood (2020) argues that frictions associated with land fragmentation have hindered manufacturing growth in India. The effect of land fragmentation has also been studied in agriculture (Chand, Prasanna and Singh (2011); Manjunatha et al. (2013)) and urban development (Gandhi et al. (2021)).

[2]One famous such case is that of protests in Singur in the state of West Bengal, India.

In a similar vein, one can think of redevelopment projects in cities. Redevelopment of apartment complexes typically involves a construction company building a larger apartment complex in place of an old one. The developer usually compensates the existing residents through apartments in the newly built building, where the apartment size is commensurate with the apartment size in the old building. Until recently, some Indian states, such as Gujarat, required the consent of *all* the residents of an apartment complex.

Or consider buying an indivisible asset, such as a business, from a group of siblings who have inherited it. The business has just one price, and owners typically receive a proportional share of the price. And as in the previous two examples, no person can be forced to agree to the offered terms of trade.[3]

While each of the above settings has its own idiosyncratic features, three common features stand out. First, a buyer wishes to purchase an indivisible good collectively owned by a group of sellers (agents). Each agent owns a fraction of the good, and while an agent's share is public information, his valuation of that share is private information. Second, strong property rights give any group member a refusal right—the right not to participate in trading. And third, the buyer can only offer one price for the entire good, and each seller receives a fraction of that price proportional to their ownership share.[4]

Motivated by such settings, we study the problem of acquiring a commonly owned good in a mechanism design setting with private information, voluntary participation, and ex-post transfers that are proportional to agents' shares. We assume that the buyer's valuation of the good is public information, but each seller's type is drawn independently from a publicly known distribution. In a (direct) mechanism, agents first report their types, and then the mechanism specifies the probability of trade and the price of trade (to be divided in proportion to agents' shares) as a function of the entire profile of reported types. Our main goal is to understand the buyer's profit-maximizing mechanism among all Bayesian

The then government of West Bengal used eminent domain provisions to acquire 997 acres of land from farmers to allow Tata Motors to build a factory. The use of eminent domain for an arguably nonpublic project was met with massive protests that, eventually, led to the factory's being shifted out of West Bengal. See https://en.wikipedia.org/wiki/Tata_Nano_Singur_controversy.

[3] Kuran (2004) argues that Islamic inheritance laws have hindered the growth of Middle Eastern countries because they lead to fragmentation of enterprises and therefore prevent the creation of large-scale firms. Kuran documents that, by the nineteenth century, Western enterprises grew in size, but Middle Eastern enterprises did not. He suggests that Islamic inheritance laws played a role.

[4] For example, a recently proposed "Right to Fair Compensation and Transparency in Land Acquisition, Rehabilitation and Resettlement Act, 2013" in India stipulates that the landowners whose land has been acquired for private projects should be paid a specified fraction of the deemed market value of the land parcel. See https://lddashboard.legislative.gov.in/sites/default/files/A2013-30.pdf for details.

incentive-compatible, interim individually rational mechanisms—henceforth the optimal mechanism.[5]

The first step toward understanding optimal mechanisms is to understand the class of implementable allocation rules in our setting. If the buyer could use transfers that need not be proportional to shares, standard arguments à la Myerson (1981) teach us that a given allocation rule would be implementable if and only if its associated interim allocations are nonincreasing. But in our setting, interim transfers are constrained because ex-post transfers must be proportional to shares, and it is a priori unclear what additional constraints this restriction places on the type of implementable mechanisms the buyer can offer. Even with this additional constraint, Lemma 1 shows that the same condition characterizes implementability in our setting. However, because of the proportional-transfers constraint, the agents' average per-share payments must coincide. Hence, the minimal average purchase price that can be attained for a given implementable allocation rule is pinned down by the condition that *one* agent's individual-rationality constraint is binding (and the others' are satisfied). Consequently, we can recast the buyer's problem as a maximin problem in which the maximum is over interim-monotone allocation rules and the minimum is over agents whose individual-rationality constraint is binding.

We solve the buyer's reformulated problem via an analogy to zero-sum games. We view the problem as a two-player zero-sum game in which one player (the Maximizer) chooses an allocation rule and the other player (the Minimizer) chooses an agent but may use a mixed strategy. We characterize the equilibria of this game to establish that the optimal allocation rule is unique and is a weighted allocation rule: the good is sold if and only if the buyer's benefit is larger than the weighted sum of agents' virtual valuations. These weights are endogenously determined, and are characterized by a simple program. These results are summarized in Theorem 1.

Given that the optimal mechanism assigns a weight to each agent, we study which agents are assigned higher weights. At a high level, agents with higher weights have more influence over the outcomes of the mechanism since trade is more sensitive to their reports. Theorem 2 answers this question by giving a condition under which we can rank agents' weights. In short, the optimal mechanism assigns higher weight to agents that have a higher valuation of the good in per-unit terms. More precisely, agent i has a higher valuation than agent j if i's virtual cost is higher in the *reversed hazard-rate order* than j's virtual cost.

[5] The individual-rationality requirement, imposed for each agent, is meant to capture the strong property rights in the above examples. Our model imposes interim incentive constraints—that any agent must find it worthwhile in expectation (being uncertain of others' valuations) to participate in the mechanism and to report his valuation truthfully. In Section VI we discuss ex post versions of both constraints.

The question of which agents have higher weights seems especially relevant to our land-acquisition application. Might the optimal mechanism discriminate against certain agents based on their characteristics? Our ranking result says that in this application, the optimal mechanism assigns more weight to agents with more productive land as measured in per-unit terms. Which agents are more productive depends on the context. Suppose first that there are two sellers who differ in how they use their land. Say agent 1 has a larger plot of land on which he can install a factory, while agent 2 has a smaller plot of land that he can use only for farming. Then, it is conceivable that (per square foot of land) agent 1 typically has higher productivity than agent 2. In this case, the optimal mechanism puts more weight on agent 1. In contrast, suppose that both the agents are small-scale farmers who differ only in their plot sizes. A literature in development economics has documented an inverse relationship between plot size and productivity (e.g., Banerjee et al., 2000). If this relationship were to hold for our two agents, then agent 1—the agent with a larger parcel of land—would have lower productivity (per square foot of land) than agent 2 and so would be granted a smaller weight.

Even though any mechanism in our setting offers all agents a uniform price per share, this price might depend on the entire profile of agents' types. But one might wonder whether the *optimal* mechanism could nonetheless be a posted-price mechanism in which the price is fixed. Such mechanisms are known to be optimal when there is one seller. However, with two or more sellers in the group, posted-price mechanisms are strictly suboptimal (Proposition 2). This result is derived from Theorem 1, which says that the price—even conditional on trade—must be responsive to the profile of reported types by the group members, a property evidently not shared by posted-price mechanisms.

In Section V, we apply our analysis to shed light on the differences between mechanism design for a group versus an individual. To do so, we compare optimal mechanisms in our setting with optimal mechanisms in a benchmark setting in which a *single agent* owns the entire good. We first observe that optimal allocations in both settings have the familiar downward distortion to reduce information rent to the high types. However, trade outcomes are additionally distorted in the group setting in two novel ways. One distortion rotates the trade region in the type space, and the other affects its curvature. The overall effect depends on how the three types of distortions interact, and these distortions might even lead to a form of *over-trading* in which trade occurs when doing so is inefficient. We then compare the efficiency of optimal mechanisms in these two settings. We show that the ranking depends crucially on how large a benefit the good generates to the buyer. If this benefit is low, then the single-agent setting generates greater surplus; and if this benefit is high, then our group setting generates greater surplus.

While most of our analysis focuses on buyer-optimal mechanisms, one might consider alternative bargaining arrangements that give more power to the sellers. A more general notion of efficiency can be especially compelling when the buyer is a government or similar entity that might have a significant concern for nonmonetary welfare outcomes. With this in mind, in Theorem 3 in the Online Appendix we fully characterize the set of all the Pareto-optimal mechanisms. This characterization is facilitated by techniques similar to those we develop en route to Theorem 1. As we show, any Pareto-optimal mechanism allocates the good if and only if the weighted sum of agents' *actual and virtual costs* is lower than the benefit to the buyer.

We impose the assumption that the buyer must pay agents the same price per share as a fairness or institutional requirement that is natural in our main applications. We show additionally that this restriction reduces the agents' incentives to collude in a certain sense. In particular, we study a larger game in which the agents may trade their shares *before* interacting with the mechanism. We assume the sellers have identical distributions of value (i.e., their cost of trade) per unit but may possibly be endowed with nonidentical shares. We show that the sellers optimally choose not to trade shares if they are paid the same price per share, but they may benefit from trading if they are paid different prices per share. The literature on uniform- versus discriminatory-price auctions also argues that uniform-price auctions might reduce agents' incentives to collude (Friedman, 1960). Further, this literature points out that agents might be more likely to participate in uniform-price auctions (Malvey and Archibald, 1998; Ausubel et al., 2014).

Related Work. Because the buyer procures the good from all sellers or none, our work is closely related to the literature on designing mechanisms for the provision of public goods. The canonical model (e.g., d'Aspremont and Gérard-Varet, 1979) allows for arbitrary monetary transfers between agents. Rob (1989) shows that with a large number of agents, profit-maximizing mechanisms are very inefficient, and Mailath and Postlewaite (1990) extend this inefficiency result to all incentive-feasible mechanisms. In a setting in which agents' values for a good are symmetric, and each is initially endowed with a share, Cramton, Gibbons and Klemperer (1987) show efficient and individually rational trading mechanisms exist if and only if agents' shares are sufficiently symmetric. Ekmekci, Kos and Vohra (2016) identify profit-maximizing mechanisms for selling some fraction of a firm owned by a single agent to a single buyer. Güth and Hellwig (1986) identify profit-maximizing mechanisms for public good provision subject to incentive-compatibility and individual-rationality constraints. Hence, our buyer's problem is equivalent (up to a sign change) to that of Güth and Hellwig (1986), with the added restriction that transfers must be proportional to shares. Virtual costs (or values) often appear in the literature that studies profit-maximizing mechanisms, but a special feature of profit-maximizing mechanisms in our setting is that virtual costs are multiplied by endogenous weights that arise because of the proportional-transfers constraint.[6] These weights are interpretable as the degree of influence that the optimal mechanism gives to different agent, and we study how this influence is affected by seller heterogeneity.

Another strand of the literature on public goods studies voting mechanisms without monetary transfers. Starting with Rae (1969), many papers in this literature study mechanisms that maximize utilitarian efficiency. Schmitz and Tröger (2012) and Krishna and Morgan (2015) identify conditions under which a (weighted) majority does or does not maximize efficiency. Azrieli and Kim (2014) show any incentive-compatible mechanism must be a weighted-majority rule, and they characterize the weights that maximize efficiency.[7] The (weighted) majority structure of mechanisms in this literature is typically either assumed or is a property of all incentive compatible mechanisms. In our setting, on the other hand, the weights arise only as a feature of *optimal* mechanisms and are not necessarily a feature of all incentive compatible mechanisms.

Whereas we take a mechanism design approach to our problem, several papers study collective-decision problems in specific bargaining situations. Bergstrom (1978) studies a setting in which each seller of a commonly owned good names a price to sell their share, and he shows that the likelihood of the good being sold approaches zero. Che (2002) studies how the ability to bargain jointly affects a group's bargaining position. The model takes a hybrid approach in which a group cannot commit to which offers to accept but can commit to a mechanism that specifies how the surplus is divided once an offer is accepted. Grossman and Hart (1980) show that takeover of a firm by a buyer might not be profitable when the buyer offers shareholders a uniform price per share even if the takeover increases efficiency. Oliveros and Iaryczower (2022) study coalition formation when a principal bargains sequentially with a group of agents. Naturally, in many of these collective-decision bargaining games, some form of the holdout problem appears. Instead, in our setting, holdout is implicit and is reflected in the constraint that all sellers must be willing to participate in the mechanism.

[6] Cai, Daskalakis and Weinberg (2013) show that virtual values can be constructed to describe optimal mechanisms even in settings with multidimensional types, if agent-specific transfers can be used. Our analysis shows that the appropriate notion of a virtual value/cost is substantially simpler in the context of multidimensional IR constraints than in their setting with multidimensional IC constraints.

[7] Also see Gershkov, Moldovanu and Shi (2017), who further study optimal voting mechanisms for a class of environments with more than two social outcomes.

I. Model

We study the problem of a buyer who wishes to buy one good, such as a plot of land, from a group of sellers who each own some share of the good. We denote the finite set of agents (the sellers) $N = \{1, \ldots, N\}$ and assume $N \geqslant 2$. Each agent i owns a fraction $\sigma_i \in (0, 1)$ of the good, where $\sum_{i \in N} \sigma_i = 1$. The buyer receives a benefit b from purchasing the good. Each agent i's cost of selling his own share of the good (or, equivalently, his valuation for keeping it) is $\sigma_i \boldsymbol{\theta}_i$, where $\boldsymbol{\theta}_i$ denotes the agent's cost per unit of the good.

An outcome of our contracting environment consists of (i) the probability $\mathtt{x} \in [0, 1]$ with which the good is sold to the buyer and (ii) the (signed) transfer $\mathtt{m} \in \mathbb{R}$ paid by the buyer to the group of sellers. This transfer is divided among the agents proportionally to their shares, so each agent i receives a payment of $\sigma_i \mathtt{m}$. The assumption that each agent is paid proportionally to his share is motivated by our application to land acquisition, in which a buyer is often required to offer identical terms to sellers ex post. The buyer must treat the agents identically and cannot offer different prices (per unit) to different agents.

The buyer's payoff for outcome $(\mathtt{x}, \mathtt{m})$ is $b\mathtt{x} - \mathtt{m}$. The payoff of each agent i for this outcome is the amount of money he receives minus his cost for his share if the good is sold, $\sigma_i \mathtt{m} - \sigma_i \boldsymbol{\theta}_i \mathtt{x}$. Since σ_i is a positive constant for each i, we can rescale each agent's payoff to be $\mathtt{m} - \mathtt{x}\boldsymbol{\theta}_i$. Such rescaling leaves the agents' incentives unchanged. We henceforth write agent i's payoff as $\mathtt{m} - \mathtt{x}\boldsymbol{\theta}_i$ and refer to $\boldsymbol{\theta}_i$ as agent i's cost.

Let us now describe our informational assumptions. The benefit b is publicly known. Each agent privately knows his own valuation. We assume that the N random variables $\{\boldsymbol{\theta}_i\}_{i \in N}$ are independent and each takes values in the compact interval $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i] \subset \mathbb{R}$; denote the cumulative distribution function of $\boldsymbol{\theta}_i$ by F_i.[8] All parties know these distributions.

We make the following regularity assumption for each $i \in N$: the cumulative distribution function F_i admits a density f_i which is continuous and strictly positive, and the *virtual cost* $\varphi_i : \Theta_i \to \mathbb{R}$ given by $\varphi_i(\theta_i) := \theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}$ is strictly increasing. Working directly with an agent's virtual cost $\boldsymbol{\varphi}_i := \varphi_i(\boldsymbol{\theta}_i)$, an atomlessly distributed random variable with convex support, is often convenient. To avoid trivialities, we assume every agent i has $\underline{\theta}_i < b < \varphi_i(\bar{\theta}_i)$.[9]

[8] We use the following standard notation throughout. The set of type profiles is $\Theta := \prod_{j \in N} \Theta_j$, and $\Theta_{-i} := \prod_{j \in N \setminus \{i\}} \Theta_j$ for $i \in N$. We also sometimes use a measure and its cumulative distribution function interchangeably, and we use F and F_{-i} to refer to associated product measures on Θ and Θ_{-i}, respectively. Throughout the paper, we use the boldface notation $\boldsymbol{\theta}, \boldsymbol{\theta}_i$, etc. to refer to these random variables, and use the notation θ_i to refer to an element of Θ_i (that is, a potential realization of $\boldsymbol{\theta}_i$).

[9] This assumption reduces casework but is not important for analyzing our model. For exam-

I.A. Mechanisms

An allocation rule is a measurable function $x : \Theta \to [0,1]$; let $\mathcal{X}$ denote the set of all allocation rules. A (collective) transfer rule is a bounded measurable function $m : \Theta \to \mathbb{R}$. A (direct) mechanism is a pair (x, m) consisting of an allocation rule and a transfer rule. For any reported type profile θ, the buyer transfers $m(\theta)$ to the group, and $x(\theta)$ is the probability with which she acquires the good.[10]

Say a mechanism (x, m) is **incentive compatible (IC)** if

$$\theta_i \in \operatorname{argmax}_{\hat{\theta}_i \in \Theta_i} \mathbb{E}\left[m(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - \theta_i x(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\right], \ \forall i \in N, \ \forall \theta_i \in \Theta_i, \qquad \text{(IC)}$$

that is, report $\hat{\theta}_i = \theta_i$ maximizes the expected payoff of type θ_i of agent i over all possible reports in Θ_i, taking the expectation over other agents' types $\boldsymbol{\theta}_{-i}$. Say the mechanism is **individually rational (IR)** if

$$\mathbb{E}\left[m(\theta_i, \boldsymbol{\theta}_{-i}) - \theta_i x(\theta_i, \boldsymbol{\theta}_{-i})\right] \geqslant 0, \ \forall i \in N, \ \forall \theta_i \in \Theta_i, \qquad \text{(IR)}$$

that is, the expected payoff of type θ_i of agent i when reporting truthfully, taking the expectation over type profiles of other agents, is nonnegative. An IC and IR mechanism (x, m) generates a buyer profit of

$$\Pi(x, m) := \mathbb{E}\left[bx(\boldsymbol{\theta}) - m(\boldsymbol{\theta})\right].$$

An **optimal mechanism** is an IC and IR mechanism that generates weakly higher buyer profit than any other IC and IR mechanism. An **optimal allocation rule** is any allocation rule x such that (x, m) is an optimal mechanism for some m.

I.B. Alternative interpretations of our model

Before moving on to our analysis, we discuss some alternative interpretations of our model.

Recall that after normalization, we have a setting in which the buyer's payoff for outcome $(\mathtt{x}, \mathtt{m})$ is $b\mathtt{x} - \mathtt{m}$ and each agent i's payoff is $\mathtt{m} - \boldsymbol{\theta}_i \mathtt{x}$. We can interpret this setting as one in which the agents sell a good they collectively own in exchange for *public funds* (instead of some money that is divided between them). The money $\mathtt{m}$ appears in every agent's payoff function. For example, the agents might be a committee of decision-makers in an organization, such as the high-level executives at a firm, who decide whether they should sell an asset owned by the

ple, without it, Theorem 1 would still hold as stated, except that when the essentially unique allocation rule specifies never trading or always trading, the weights can be non-unique.

[10] Because the payoffs are linear in the transfer, we assume without loss that in a direct mechanism the transfer is a deterministic function of the reported type profile.

organization. In this interpretation, an agent's type $\boldsymbol{\theta}_i$ specifies his marginal rate of substitution between the organization retaining the good and the organization's use of additional funds.

We do not make assumptions about the signs of b or the values $\boldsymbol{\theta}_i$ might take. In particular, we allow them to be negative. In that case, after relabeling the variables appropriately, the problem becomes one of finding optimal mechanisms for a single *seller* who wants to sell a good to a group of buyers. If sold, the good is publicly available to all members of the group. Depending on whether the agents in the group pay for the good with private money or public funds, two interpretations are again available. The first interpretation entails private transfers with a fixed cost-sharing rule. Here, each agent i is responsible for paying a fixed fraction σ_i of the transfer to the seller. So if the good is sold with probability $\mathtt{x}$ and the group pays $\mathtt{m}$ to the seller, then agent i's payoff is $\boldsymbol{v}_i\mathtt{x} - \sigma_i\mathtt{m}$, where $\boldsymbol{v}_i$ denotes agent i's benefit if the good is acquired by the group. For example, the group might be a condo association in which each member pays for a public service proportionally to the size of their unit, or it might be a cartel in which each firm pays proportionally to its market share. Now define $\boldsymbol{\theta}_i := \frac{1}{\sigma_i}\boldsymbol{v}_i$, which allows us to write agent i's payoff as $\sigma_i\boldsymbol{\theta}_i\mathtt{x} - \sigma_i\mathtt{m}$, which can then be normalized to $\boldsymbol{\theta}_i\mathtt{x} - \mathtt{m}$. The second interpretation has the group paying for the product with public funds. Here, if the good is sold with probability $\mathtt{x}$ and the group pays $\mathtt{m}$ to the seller from its collective funds, then agent i's payoff is $\boldsymbol{\theta}_i\mathtt{x} - \mathtt{m}$. Agent i's type $\boldsymbol{\theta}_i$ again denotes his marginal rate of substitution between the public good and the organization's alternative use of its funds.

II. Characterizing the optimal mechanism

In this section, we fully characterize optimal mechanisms. First, we describe which allocation rules are implementable and solve for the buyer's optimal profit from implementing such an allocation rule; doing so requires a reduced-form implementation result for transfers, characterizing exactly which profiles of interim transfer rules can be implemented with some collective transfer rule. Then, the main result of this section establishes that a unique optimal allocation rule exists, and shows it can be described as a weighted allocation rule with (uniquely determined) weights that we explicitly characterize.

We begin by introducing some convenient notation and terminology for standard objects. Just as in the auction setting, the Bayesian incentive properties of our design environment are convenient to discuss in terms of each agent's interim (i.e., conditioning only on his own type) outcomes.

Definition 1: *Fix any agent $i \in N$. Given an allocation rule x, define the* ***interim allocation rule*** *to be $X_i^x : \Theta_i \to \mathbb{R}$ given by $X_i^x(\theta_i) := \mathbb{E}[x(\theta_i, \boldsymbol{\theta}_{-i})]$.*

Similarly, given a transfer rule m, define the ***interim transfer rule*** *to be $M_i^m : \Theta_i \to [0,1]$ given by $M_i^m(\theta_i) := \mathbb{E}[m(\theta_i, \boldsymbol{\theta}_{-i})]$.*

Now, say an allocation rule x is ***interim monotone*** *if X_i^x is weakly decreasing for every $i \in N$. We say that an allocation rule x is* ***implementable*** *if a transfer rule m exists such that (x, m) is IC.*

To characterize optimal mechanisms, we first need to understand which allocation rules are implementable. Classic results (Myerson, 1981; Myerson and Satterthwaite, 1983) would imply that interim monotonicity would fully characterize implementability if the buyer could freely choose the interim transfer rule that each agent faces. However, our buyer is constrained in that different agents' interim transfers must be derived from a common ex post transfer rule. Nevertheless, Lemma 1 below shows that the exact same characterization applies despite the added constraint on the transfers. Using this characterization, we also obtain the maximum buyer profit compatible with implementing an allocation rule x.

Lemma 1: *Let x be some allocation rule.*

(i) Mechanism (x, m) is IC and IR for some transfer rule m if and only if x is interim monotone.

(ii) If some transfer rule m exists such that mechanism (x, m) is IC and IR, then a maximally profitable such mechanism exists, with resulting profit

$$\min_{i \in N} \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right].$$

Part (i) of the lemma combines a standard observation with a novel one. The standard observation is that a given mechanism (x, m) is IC for seller i if and only if X_i is weakly decreasing and some constant $\underline{U}_i$ exists such that the payment identity,

$$M_i(\theta_i) = \underline{U}_i + X_i(\theta_i)\theta_i + \int_{\theta_i}^{\bar{\theta}_i} X_i(\tilde{\theta}_i)\, \mathrm{d}\tilde{\theta}_i \text{ for every } \theta_i, \tag{\$}$$

holds. So the allocation rule x and the constants $\underline{U}_1, \ldots, \underline{U}_N$ pin down the *interim* transfer rules. The novel observation for establishing part (i) is that a profile of interim transfer rules $(M_i)_{i \in N}$ can be implemented via a common ex-post transfer rule m if and only if $\mathbb{E}[M_i(\boldsymbol{\theta}_i)]$ coincide for all i. This condition is obviously necessary, given iterated expectations. Perhaps surprisingly, the condition is also sufficient, and sufficiency has a one-line proof: if $\bar{m}$ is the common expected transfer, then the ex-post transfer rule $m(\theta) := -(N-1)\bar{m} + \sum_{i \in N} M_i(\theta_i)$ generates the desired interim transfer rules.[11]

[11] Gopalan, Nisan and Roughgarden (2018) show that a slight variant of this problem is computationally intractable. In particular, it is computationally hard to decide whether a given profile of interim transfer rules can be implemented via a common ex post transfer *that is constrained to belong to some bounded interval.* Our construction—which settles the question of imple-

To establish part (ii), we use the payment identity ($) to write the buyer's expected payoff as

$$\mathbb{E}\left[bx(\boldsymbol{\theta}) - m(\boldsymbol{\theta})\right] = \mathbb{E}\left[bx(\boldsymbol{\theta})\right] - \mathbb{E}\left[M_i(\boldsymbol{\theta}_i)\right] = \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right] - \underline{U}_i$$

for each seller i. Importantly, because the interim transfers are identical on average, choosing $\underline{U}_i$ for any seller i pins down the entire profile of constants $(\underline{U}_i)_i \in \mathbb{R}^N$. Analogously to how an optimal auction would optimize the transfer rule by setting each agent's IR constraint to be binding, our remaining constant is optimized by requiring that *some* agent's IR constraint binds (and the others' constraints are satisfied). Since $\mathbb{E}[M_i(\boldsymbol{\theta}_i)] = \mathbb{E}[x(\boldsymbol{\theta})\boldsymbol{\varphi}_i] + \underline{U}_i$ coincide for all $i \in N$, an agent i whose IR constraint binds is the one with the highest expected virtual cost of trade $\mathbb{E}\left[x(\boldsymbol{\theta})\boldsymbol{\varphi}_i\right]$. Therefore, the buyer's optimal payoff for a given allocation rule x is

$$\min_{i \in N} \; \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right].$$

With Lemma 1 in hand, our buyer's problem can be recast directly as an optimization over allocation rules. Formally, the buyer's optimization over allocation rules is

$$\begin{aligned} &\max_{x \in \mathcal{X}} \left\{ \min_{i \in N} \; \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right] \right\} \\ &\text{s.t.} \quad x \text{ is interim monotone.} \end{aligned} \tag{BP}$$

Our main result is a complete characterization of the solution to the program (BP). To this end, we define a class of allocation rules that play a special role in our analysis and results.

Definition 2: *Given $\omega \in \Delta N$, the ω-**allocation rule** is the allocation rule $x_\omega := \mathbb{1}_{\omega \cdot \varphi \leq b}$. Say $\omega \in \Delta N$ is **optimal** if the ω-allocation rule is optimal. Say an allocation rule is a **weighted allocation rule** if it is a ω-allocation rule for some $\omega \in \Delta N$.*

We now state our main characterization theorem.

Theorem 1 (Optimal allocation)**:**
A weighted allocation rule is essentially uniquely optimal.[12] *The unique optimal-weight vector ω is characterized by either of the following two equivalent conditions:*

(i) $\omega \in \operatorname{argmin}_{\tilde{\omega} \in \Delta N} \mathbb{E}[(b - \tilde{\omega} \cdot \boldsymbol{\varphi})_+]$.

mentability absent such a constraint—resembles previous constructions in the literature that convert ex-ante budget-balanced mechanisms into ex post budget-balanced mechanisms while preserving the players' interim transfer rules (e.g., Makowski and Mezzetti, 1994; d'Aspremont, Crémer and Gérard-Varet, 2004; Che and Kim, 2006; Börgers and Norman, 2009).

[12] By "essentially uniquely," we mean any alternative optimal allocation rule generates the same trade decision almost surely.

(ii) $\text{supp}(\omega) \subseteq \text{argmax}_{i \in N} \mathbb{E}[\boldsymbol{\varphi}_i \mid \omega \cdot \boldsymbol{\varphi} \leqslant b]$.

Moreover, if $b < \bar{\theta}_j$ *for at least two* $j \in N$*, then every* $i \in N$ *has* $\omega_i < 1$*.*

Theorem 1 says that trade outcomes in optimal mechanisms are given by weighted allocation rules. Notice that a weighted allocation rule is deterministic, so the theorem implies the buyer does not benefit from randomization. Note also that the weights are fixed and do not depend on reports. A higher weight for an agent means the mechanism is in a sense more responsive to that agent's private information. So by comparing agents' weights, we can understand which agents exert greater influence over the outcomes of the mechanism, a topic we will revisit in the next section.

Theorem 1 also characterizes the optimal weights with two equivalent conditions. Condition (i) above is useful for computing the optimal weights numerically and analytically. The function $\omega \mapsto \mathbb{E}[(b - \omega \cdot \boldsymbol{\varphi})_+]$ is convex, and so determining optimal weights corresponds to minimizing a convex objective over a compact convex set. Moreover, in certain cases, as in Example 1 presented later, we can even compute the optimal weights ω analytically using the first-order conditions of the convex optimization problem. Condition (ii) of the theorem facilitates verification of optimality: once a candidate for optimal weights is chosen, one can verify optimality by checking that any agent who has a positive weight is a minimizer of the conditional expected virtual cost term. This condition reflects the fact that every agent who influences the trade outcome should have a binding IR constraint.

Specializing to the case of a single seller, Theorem 1 confirms the classic characterization of optimal mechanisms. In this case, trade happens whenever the benefit to the buyer exceeds the lone seller's virtual cost. The "weight" for this case is trivial, assigning all influence the seller's private information. The optimal allocation is deterministic (trade occurs whenever the agent's type is below a cutoff type at which the benefit is equal to virtual cost), just as in Theorem 1.

The proof of Theorem 1 studies a relaxed program (RBP) in which the interim-monotonicity constraint is ignored. To solve the relaxed program, we consider an auxiliary two-player zero-sum game in which the Maximizer chooses an allocation rule x, the Minimizer chooses an agent i whose IR constraint must bind, and so the Maximizer's objective is $\mathbb{E}[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)]$. Observe that an allocation rule solves (RBP) if and only if it is a cautious optimum for the Maximizer in the auxiliary game—that is, a maximin strategy. Moreover, standard results for zero-sum games imply a maximin strategy is a Nash equilibrium strategy for the Maximizer, and vice versa, as long as some Nash equilibrium exists. Hence, we turn to characterizing Nash equilibria of the auxiliary game.

We first show that if the Minimizer is allowed to choose a mixture, some Nash equilibrium of this auxiliary game exists by an appropriate minimax theorem, and every mixed strategy ω for the Minimizer exhibits a unique (up to almost-sure equivalence) best response for the Maximizer. Indeed, x_ω is the essentially unique maximizer of

$$x \mapsto \sum_i \omega_i \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right] = \mathbb{E}\left[x(\boldsymbol{\theta})(b - \omega \cdot \boldsymbol{\varphi})\right]$$

because it sets $x(\boldsymbol{\theta}) \in [0,1]$ to maximize the integrand $x(\boldsymbol{\theta})(b - \omega \cdot \boldsymbol{\varphi})$ in every realized state. Then, because the set of Nash equilibria of a two-player zero-sum game exhibits a product structure, it follows that an essentially unique allocation rule can be an optimal strategy for the Maximizer of the auxiliary game, and that it takes the form x_ω for any Nash equilibrium choice ω of the Minimizer. The pair of conditions characterizing such ω's are standard to zero-sum games: the mixed strategy ω is a cautious optimum for the Minimizer (condition (i)) if and only if it is a best response to some Maximizer's best response to ω (condition (ii), once the Maximizer's best response to ω is substituted in). Now, observe that the essentially unique Nash equilibrium strategy for the Maximizer is actually interim monotone: because virtual costs are increasing, a cutoff rule for the ω-weighted virtual cost is monotone and hence interim monotone. The result is a characterization of the unique optimal allocation rule, solving not only (RBP) but also (BP). Then, because our assumption that $\underline{\theta}_i < b < \varphi_i(\bar{\theta}_i)$ (for each i) implies every weighted allocation rule has an interior probability of trade, a geometric argument converts uniqueness of the allocation rule into uniqueness of agents' weights. Finally, to verify the last sentence of the theorem, we note that any agents $i \neq j$ have $\mathbb{E}\left[\boldsymbol{\varphi}_i \mid \boldsymbol{\varphi}_i \leqslant b\right] \leqslant b$ and $\mathbb{E}\left[\boldsymbol{\varphi}_j \mid \boldsymbol{\varphi}_i \leqslant b\right] = \mathbb{E}\left[\boldsymbol{\varphi}_j\right] = \bar{\theta}_j$, so that putting all weight on agent i would violate condition (ii) if $\bar{\theta}_j > b$.

To conclude the section, let us specialize our setting to a parametric example. We use the example to illustrate how we can use condition (i) of Theorem 1 to identify the optimal weights analytically. We then give an indirect implementation of the optimal mechanism.

Example 1: *Suppose that there are two sellers. Seller i has a power distribution $F_i(\theta_i) = \theta_i^{\alpha_i}$ over $\theta_i \in [0,1]$ for some power $\alpha_i > 0$.*[13] *Assume that $b \leqslant \min\{\frac{1+\alpha_1}{\alpha_1+\alpha_2}, \frac{1+\alpha_2}{\alpha_1+\alpha_2}\}$—which for instance holds if $\alpha_1, \alpha_2, b \leqslant 1$. Then, as we*

[13] When $\alpha_i \neq 1$, the derivative of F_i at $\underline{\theta}_i = 0$ is either zero or infinite, violating our assumption of a continuous and strictly positive density. However, our main results (in particular, Theorems 1 and 2 and their supporting analysis) apply to the more general version of our model in which the density is only assumed to be continuous and strictly positive on $(\underline{\theta}_i, \bar{\theta}_i)$. In particular, assuming the density f is positive over $(\underline{\theta}_i, \bar{\theta}_i)$, we can define $\varphi_i(\theta_i) := \theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}$ over $(\underline{\theta}_i, \bar{\theta}_i)$, and extend it continuously to the endpoints.

show in the appendix, the optimal weight vector is:

$$\omega^* := \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}, \quad \frac{\alpha_2}{\alpha_1 + \alpha_2} \right).$$

A convenient feature of this example is that virtual costs are linear in costs,

$$\varphi_i(\theta_i) = \left(1 + \frac{1}{\alpha_i}\right) \theta_i.$$

Combining this observation with the optimal weights we have computed, the essentially unique optimal allocation rule results in trade if and only if

$$b \geqslant \omega^* \cdot \boldsymbol{\varphi} = \frac{\alpha_1}{\alpha_1 + \alpha_2} \boldsymbol{\varphi}_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} \boldsymbol{\varphi}_2 = \frac{1 + \alpha_1}{\alpha_1 + \alpha_2} \boldsymbol{\theta}_1 + \frac{1 + \alpha_2}{\alpha_1 + \alpha_2} \boldsymbol{\theta}_2.$$

So trade occurs whenever the benefit is at least a certain positive linear combination of the sellers' costs. This allocation rule, combined with any transfer rule that satisfies the payment identity (\$) with $\underline{U}_1 = \underline{U}_2 = 0$, forms an optimal mechanism.

In addition, we show that the following indirect mechanism is optimal:

- *Both sellers simultaneously send bids, $s_i \in \mathbb{R}_+$.*
- *Trade occurs if and only if $b \geqslant \tau_1 s_1 + \tau_2 s_2$, where $\tau_i := \frac{1+\alpha_{-i}}{(\alpha_1+\alpha_2)\alpha_{-i}}$ for each i.*
- *The price that the buyer pays is $s_1 + s_2 + \kappa b$ if the good is sold, and zero otherwise, where $\kappa = \frac{\alpha_1+\alpha_2}{(1+\alpha_1)(1+\alpha_2)}(1 - \alpha_1\alpha_2)$.*

This indirect mechanism has a "name-your-price" structure. Each seller submits a bid. It is useful to think of the bid as the price the buyer has to pay in exchange for that seller's consent. The buyer pays the sum of the bids (plus a constant) if and only if trade occurs. Trade occurs when the benefit is higher than some linear combination of the bids.

When submitting a bid, each seller faces a trade-off like in a first-price auction. Increasing the bid means the agent is paid more if *trade occurs. But increasing the bid also lowers the probability of trade. We show that the game has an equilibrium with linear strategies in which type θ_i bids $\frac{(1+\alpha_i)\alpha_{-i}}{1+\alpha_{-i}}\theta_i$, and in this equilibrium the buyer obtains her maximum possible payoff.*

As mentioned earlier, determining the optimal weights using condition (i) in Theorem 1 entails solving a convex optimization program. We show in the appendix that ω^* is a local minimum of this program, and hence is optimal. Toward showing the indirect mechanism is optimal, we first establish that the given strategy profile generates the above-described allocation rule and interim allocation rules. Thus, the indirect mechanism and strategy profile constitute an optimal mechanism *if* the strategy profile is an equilibrium. Moreover, because the induced allocation rule is interim monotone and the payment identity ($) holds, it follows that no type of either agent has a profitable deviation to submit another type's bid. Finally, notice that any off-path bid—a bid not submitted by any type on-path—is outcome equivalent to bidding the highest type's bid (generating no trade and zero transfer). Therefore, no type has any profitable deviation—that is, the given strategy profile is indeed an equilibrium.

Let us highlight two important features of the above example. First, notice that $\omega_1^* > \omega_2^*$ whenever $\alpha_1 > \alpha_2$: the seller with a higher α_i receives a higher weight in the optimal mechanism, and in this sense has greater influence over the outcomes of the mechanism. In the bidding-game implementation, this influence ranking is reflected in $\tau_1 > \tau_2$. Section III studies the role of asymmetry more broadly, asking how ex-ante heterogeneity in sellers' characteristics leads to asymmetric treatment by the mechanism beyond this parametric example. Using the characterization of optimal weights in Theorem 1, we show that the optimal mechanism assigns a higher weight to agents who have higher costs ex ante. Theorem 2 formalizes the appropriate sense in which $\alpha_1 \geqslant \alpha_2$ corresponds to seller 1 having higher costs ex ante, and shows that it generates a ranking of weights more generally.

Second, the terms of trade arose in the example from a complex pricing mechanism—complex in the sense that bidding behavior altered the terms of trade, not just whether trade occurred. In particular, the price at which the trade occurs depends on the type profile, unlike the case with $N = 1$ where the optimal mechanism can be implemented with a posted price. In Section IV, we show that this complex pricing aspect of this mechanism is a feature of every optimal mechanism in our setting.

III. The role of agent heterogeneity

In light of our leading application—sale of a large plot of land with dispersed ownership to an industrialist—it is natural to explore how the optimal mechanism treats (ex ante) heterogeneity between agents. Because the optimal mechanism uses a weighted-average allocation rule, this question amounts to understanding how agents' endogenous weights differ.

Specifically, we seek conditions on primitives under which we can rank ω_i and ω_j for two agents i and j. The main result of this section, Theorem 2, provides an interpretable condition under which a ranking of agents' virtual cost distributions implies a ranking on the weights in the optimal mechanism. To state this result, we use the following distributional-ranking definition.

Definition 3: *Given two real random variables* $\mathbf{v}$ *and* $\mathbf{w}$ *with respective cumulative distribution functions given by* G *and* H*,* $\mathbf{v}$ ***is larger than*** $\mathbf{w}$ ***in the re-***

versed hazard-rate order, *denoted by* $\mathbf{v} \succcurlyeq_{\mathrm{rh}} \mathbf{w}$, *if* $\inf[\mathrm{supp}(\mathbf{w})] \leqslant \inf[\mathrm{supp}(\mathbf{v})]$ *and* $\frac{G}{H}$ *is weakly increasing on* $(\inf[\mathrm{supp}(\mathbf{w})], \infty)$.

The above distributional ranking is a useful strengthening of first-order stochastic dominance. Intuitively, the ranking requires that the conditional distributions, when conditioned on lying below any common threshold, are stochastically ranked. This ranking condition has been fruitful in past work in mechanism design. Specifically, in the literature on asymmetric auctions (e.g., Maskin and Riley, 2000; Kirkegaard, 2012), ranking bidders' value distributions via the reversed hazard-rate order has enabled the ranking of equilibrium bidding behavior, which in turn has been used to provide revenue rankings for alternative auction formats. In our setting, as the following theorem shows, a reversed hazard-rate order on agents' *virtual cost* distributions is relevant in designing optimal mechanisms.

Theorem 2 (Ranking allocation weights)**:** *If* $\boldsymbol{\varphi}_i \succcurlyeq_{\mathrm{rh}} \boldsymbol{\varphi}_j + \beta$ *for some* $\beta \geqslant 0$*, then the optimal vector of allocation weights* ω *satisfies* $\omega_i \geqslant \omega_j$*. Moreover,* $\omega_i > \omega_j$ *whenever* $\beta > 0$ *and* $\omega_j > 0$*.*

Theorem 2 follows from more general results (which further provide quantitative bounds on *how* asymmetric the weights are) that we prove in the appendix.[14] The core of the theorem's proof is a result from the theory of stochastic orders that converts a reversed hazard-rate ranking on random variables into a second-order stochastic-dominance ranking of their weighted averages as the weights are made more assortative. More specifically, we work with the convex program given in condition (i) of Theorem 1, and note that its loss function can be written as

$$-\mathbb{E}\left[h(\omega_i \boldsymbol{\varphi}_i + \omega_j \boldsymbol{\varphi}_j)\right]$$

for some increasing and concave function h that depends on $(\omega_k)_{k \neq i,j}$. Suppose $\boldsymbol{\varphi}_i \succcurlyeq_{\mathrm{rh}} \boldsymbol{\varphi}_j + \beta$ for some $\beta \geqslant 0$, and consider any weight vector ω with $\omega_i < \omega_j$. We can then define an alternate weight vector $\tilde{\omega}$ by swapping the i and j coordinates of ω. Because $\boldsymbol{\varphi}_i \succcurlyeq_{\mathrm{rh}} \boldsymbol{\varphi}_j + \beta$ and $\omega_i < \omega_j$, a textbook stochastic ranking result tells us $\omega_i \boldsymbol{\varphi}_i + \omega_j(\boldsymbol{\varphi}_j + \beta)$ is below $\omega_j \boldsymbol{\varphi}_i + \omega_i(\boldsymbol{\varphi}_j + \beta)$ in the sense of second-order stochastic dominance. But then,

$$\mathbb{E}\left[h\left(\omega_i \boldsymbol{\varphi}_i + \omega_j \boldsymbol{\varphi}_j\right)\right] \leqslant \mathbb{E}\left[h\left(\omega_j \boldsymbol{\varphi}_i + \omega_i \boldsymbol{\varphi}_j - (\omega_j - \omega_i)\beta\right)\right] \leqslant \mathbb{E}\left[h\left(\omega_j \boldsymbol{\varphi}_i + \omega_i \boldsymbol{\varphi}_j\right)\right],$$

so that $\tilde{\omega}$ performs at least as well as ω in the convex program. But Theorem 1 then tells us that ω cannot be the unique optimal weight vector.

When types are drawn from power distributions, virtual costs can be ranked in the reversed hazard-rate order, and so Theorem 2 gives a ranking of the weights

[14] More specifically, if $\boldsymbol{\varphi}_i \succcurlyeq_{\mathrm{rh}} \alpha \boldsymbol{\varphi}_j + \beta$, where $0 < \alpha \leqslant 1$ and β exceeds a certain bound, then $\alpha \omega_i \geqslant \omega_j$; and we prove a corresponding strict version of the same.

that matches our closed-form calculations in Example 1. In particular, consider two agents i and j with distributions given by $F_i(\theta_i) = \theta_i^{\alpha_i}$ and $F_j(\theta_j) = \theta_j^{\alpha_j}$ for $\theta_i, \theta_j \in [0,1]$, where $\alpha_i, \alpha_j > 0$. If $\alpha_i \geqslant \alpha_j$, then $\boldsymbol{\varphi}_i$ is higher than $\boldsymbol{\varphi}_j$ in the reversed hazard-rate order, and so $\omega_i \geqslant \omega_j$ by the theorem, regardless of the distributions of other agents.[15]

Let us now revisit our land-acquisition interpretation. The principal, an industrialist, wishes to buy a large plot of land whose ownership is dispersed across N individuals, with agent i owning share σ_i of the land. Each agent's valuation per unit of land is $\boldsymbol{\theta}_i$, and his utility is $\sigma_i \mathtt{m} - \sigma_i \boldsymbol{\theta}_i \mathtt{x}$. Because σ_i is a positive scalar multiplying $\mathtt{m} - \boldsymbol{\theta}_i \mathtt{x}$, it is strategically irrelevant. Therefore, if the agents' virtual cost distributions are ranked according to the reversed hazard-rate order, then a ranking of the weights follows. Land shares per se play no role in determining agents' optimal weights. For instance, if two agents have the same virtual cost distributions, then the optimal mechanism will weigh them equally however asymmetric their land shares are.

But could the amount of landholding be systematically related to the cost distribution? For example, consider two sellers with landholdings $\sigma_1 < \sigma_2$, and assume that the shares are sufficiently asymmetric. Then it is conceivable that the agent with a larger landholding may have uses of land that generate higher value (thus, a higher cost of trade) in per-unit terms. For example, an agent with a larger piece of land might install a manufacturing plant. The smaller landowner cannot do the same because of the associated fixed costs and minimum-size constraints. This difference in how they use their plots can lead to $\boldsymbol{\varphi}_2 \geqslant_{\text{rh}} \boldsymbol{\varphi}_1$; that is, the agent with a larger landholding may have higher productivity (and therefore cost of trade) per unit of land. As Theorem 2 says, the optimal mechanism assigns agent 2, the more productive agent, a higher weight.

Another compelling story could, however, apply to situations in which all the landowners have the same land use, say agriculture, and they differ only in the sizes of plots they own (in addition to idiosyncratic shocks). A negative relationship between the size of land and productivity is well documented (e.g., Banerjee et al., 2000; Berry, Cline et al., 1979). In fact, the magnitude of this difference in productivity is often sizable. As Banerjee et al. (2000) says:[16]

[15]As pointed out in Example 1, any $\theta_i \in [0,1]$ has $\varphi_i(\theta_i) = \frac{\alpha_i+1}{\alpha_i}\theta_i$, implying $\mathbb{P}[\boldsymbol{\varphi}_i \leqslant x] = \left(\frac{\alpha_i x}{\alpha_i+1}\right)^{\alpha_i}$ for $x \in \left[0, \frac{\alpha_i+1}{\alpha_i}\right]$—and analogously for j. Because $\frac{\alpha_i+1}{\alpha_i} \geqslant \frac{\alpha_j+1}{\alpha_j}$ and any x in the common support has

$$\frac{\mathbb{P}[\boldsymbol{\varphi}_i \leqslant x]}{\mathbb{P}[\boldsymbol{\varphi}_j \leqslant x]} \text{ proportional to } x^{\alpha_i - \alpha_j},$$

it follows that $\boldsymbol{\varphi}_i$ is larger than $\boldsymbol{\varphi}_j$ in the reversed hazard-rate order.

[16]One reason Banerjee et al. (2000) offers for this negative relationship is decreasing returns to scale arising from incentive costs. Smaller plots tend to be managed by families, while larger ones require significant external labor.

> In Punjab, Pakistan, productivity on the largest farms (as measured by value added per unit of land) is less than 40 percent that on the second smallest size group, while in Muda, Malaysia, productivity on the largest farms is just two-thirds that on the second smallest size farms.

In such contexts, in which the agents with smaller landholdings are more productive (in per-unit terms), we could have $\boldsymbol{\varphi}_1 \geqslant_{\mathrm{rh}} \boldsymbol{\varphi}_2$ and therefore $\omega_1 \geqslant \omega_2$ per Theorem 2. That is, the optimal mechanism would favor the agents with smaller landholdings.

In summary, a general qualitative feature emerges from the above two situations: the optimal mechanism favors the more productive agents, who are less ex ante inclined to part with their land. Given a systematic positive or negative relationship between agents' productivity and their landholdings, this observation further enables us to understand which agents the optimal mechanism favors.

IV. Posted-price mechanisms

In some mechanism design problems—for example, selling a single indivisible good to a single agent—the optimal mechanism is a take-it-or-leave-it posted-price mechanism (Myerson, 1981; Riley and Zeckhauser, 1983). Beyond the single-agent setting, there are environments in which such pricing mechanisms remain approximately optimal (Chawla et al., 2010; Chawla, Malec and Sivan, 2015; Hart and Nisan, 2017; Babaioff et al., 2020). Especially in our setting—in which any agent can unilaterally veto the mechanism and all the agents must pay a common price—a natural conjecture is that posted-price mechanisms remain optimal. The purpose of this section is to establish that this conjecture is false. Of course, before we can do so, we must first define a posted-price mechanism for our setting.

In the one-agent setting, the IC direct mechanisms that correspond to a posted price are those satisfying two properties. First, the transfer is directly proportional to the allocation probability. And second, the allocation probability is 1 for types above the price and 0 for those below it. The first condition—which we can interpret as a restriction that money never changes hands if the good is not sold and that the price at which trade occurs is constant when it does—generalizes immediately. But the second condition—which we can interpret as stating that the agent freely decides whether to execute a trade—does not immediately generalize to the multi-agent setting. Who decides whether trade occurs? Once the buyer announces a price for the good, a complex negotiation process could ensue between the agents deciding whether to sell. Might eventual trade outcomes arise from some mixed-strategy equilibrium of the resulting bargaining game between the agents?

In light of these difficulties, we define a collective posted price rather permissively, only incorporating the first of the two conditions mentioned in the previous paragraph. We also introduce a specific, interpretable pricing mechanism that will be important for our results.

Definition 4: *A mechanism (x, m) is a* ***collective posted-price mechanism*** *if some $p \in \mathbb{R}_+$ exists such that $m = px$. It is a* ***unanimous posted-price mechanism*** *if it is a collective posted-price mechanism with price p such that $x(\theta) = \mathbb{1}_{\theta_i \leqslant p \ \forall i \in N}$ for every $\theta \in \Theta$.*

One can envision several examples of collective posted-price mechanisms. For example, the buyer could set a price p and execute a sale if and only if all agents agree to the purchase—a unanimous posted price. Alternatively, the buyer could post a price and select an agent, or even a subset of agents, perhaps randomly, and execute the trade if all the agents in this chosen subset agree to the sale.

Although the space of all collective posted-price mechanisms is rather rich, the next result shows that arguably the simplest example of them is optimal.

Proposition 1 (Optimal posted price is unanimous)**:** *Some unanimous posted-price mechanism is optimal among IC and IR collective posted-price mechanisms.*

To show this result, we begin with an arbitrary collective posted-price mechanism, with a view to showing some unanimous posted price does better. If the price exceeds the benefit of trade b, then the mechanism is not profitable, and so a unanimous posted price slightly below b yields higher buyer profit. Now focus on the case of a price below b. Observe that IR implies an agent's interim allocation is zero whenever the agent's type is above the price. Therefore, trade has zero probability conditional on *any* agent's having a realized valuation above the price. Hence, a unanimous posted price (at the same price level) would generate profitable trade with a higher probability, and so it is more profitable.

Having characterized the optimal form of collective posted-price mechanism, we are poised to answer the question that motivated this subsection: when are collective posted-price mechanisms optimal? The result below establishes that, under a mild nondegeneracy condition, they never are.

Proposition 2 (Posted prices are suboptimal)**:** *If at least two $j \in N$ have $b < \bar{\theta}_j$, then no collective posted-price mechanism is optimal.*

To establish the above result, in light of Proposition 1, it suffices to show the optimal allocation mechanism is not a unanimous posted price. We show the two cannot coincide by examining their interim allocation rules for an agent who has positive weight in the optimal weighted allocation rule. His interim allocation rule is clearly a step function under a unanimous posted-price mechanism. Meanwhile, it cannot be one under the optimal mechanism: his interim probability of trade is nonconstant in his type because the allocation rule puts positive weight on his own virtual cost, and it is continuous in his type because it puts positive weight on the (atomlessly distributed) virtual cost of at least one other agent. So the two allocation rules cannot coincide.

Thus, optimal mechanisms incorporate sellers' private information, smoothly varying the terms of trade with an agent's reported type. Observe that such continuous incentives are reflected in the bidding game of Example 1, in which changing a bid leads to a change in the price conditional on trade.

V. Group versus single-agent mechanisms

In this section, we compare optimal mechanisms in our setting with optimal mechanisms for buying from a *single* agent. We use this comparison to highlight how optimal allocations are distorted and discuss welfare consequences of these distortions.

We start by defining the single-agent benchmark. In this benchmark, a single agent owns a good that has valuation $\mathbf{v} := \sigma \cdot \boldsymbol{\theta}$ for him, where $\sigma = (\sigma_1, \dots, \sigma_N)$ is a fixed vector of positive weights summing to 1, and the random vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N)$ has each $\boldsymbol{\theta}_i$ drawn independently from F_i. Using the terminology from our land acquisition application, the agent owns a plot of land that is divided into N parcels, possibly of different sizes. Land parcel i has size σ_i and selling it has per-unit cost $\boldsymbol{\theta}_i$ to the agent.[17] The agent privately knows $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$ (and hence privately knows $\mathbf{v}$). The buyer designs a (direct) mechanism in which the single seller reports her type $\boldsymbol{\theta}$, resulting in a probability of trade and a transfer the buyer pays him. We study mechanisms that maximize the buyer's profit subject to single-agent analogues of the IC and IR constraints. Let G denote the cumulative distribution function $\mathbf{v}$, and let g be the continuous and strictly positive density of $\mathbf{v}$. Although we do not make this assumption for our analysis, let us focus our discussion around the regular case in which the associated virtual cost $v + \frac{G(v)}{g(v)}$ is strictly increasing in v.[18]

[17] These costs might be independent (conditional on observables) if they represent productivity of different parcels of land, and any shocks that affect multiple parcels' productivity are observable to the buyer.

[18] We make this assumption here only to streamline the exposition. Regularity of the distribution of $\mathbf{v}$ would follow if we were to assume each $\frac{f_i}{F_i}$ is nonincreasing on $(\underline{\theta}_i, \bar{\theta}_i)$ for each $i \in N$. Indeed, in this case, the corresponding ratio for $\mathbf{v}_i = \sigma_i \boldsymbol{\theta}_i$ given by

$$v_i \mapsto \frac{f_i\left(\frac{1}{\sigma_i} v_i\right)}{\sigma_i F_i\left(\frac{1}{\sigma_i} v_i\right)}$$

is nonincreasing there too. Iteratively applying Corollary 3.3 from Barlow, Marshall and Proschan (1963) then tells us $\frac{g}{G}$ is nonincreasing there, so that the associated virtual cost is strictly increasing—that is, $\mathbf{v}$ has a regular distribution.

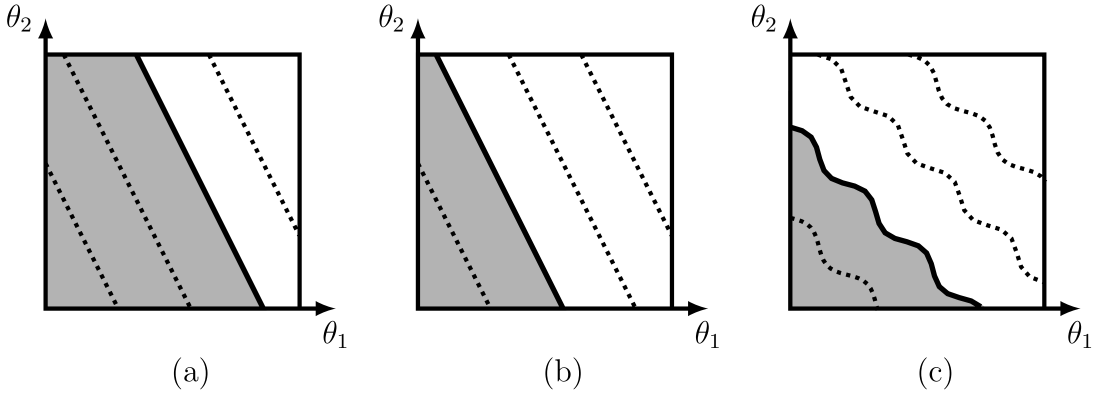

Figure 1: (a) In the efficient allocation, trade occurs below the iso cost curve given by $b = \sigma \cdot \boldsymbol{\theta}$. (b) In the optimal allocation for the single-agent benchmark, trade occurs below the iso-single-agent-virtual-cost curve given by $b = \sigma \cdot \boldsymbol{\theta} + \frac{G(\sigma \cdot \boldsymbol{\theta})}{g(\sigma \cdot \boldsymbol{\theta})}$. (c) In the optimal allocation for our group setting, trade occurs below the iso-group-virtual-cost curve given by $b = \omega \cdot \boldsymbol{\varphi}$.

Notice that, both in the single-agent benchmark and in our group setting, a mechanism stipulates a probability of trade as a function of the random variable $\boldsymbol{\theta}$, and that this random variable has the same distribution in both settings. Also, in both settings, the total monetary value of the good to the seller(s) is $\sigma \cdot \boldsymbol{\theta}$ and has the same distribution in both cases. From the perspective of the buyer, regardless of whether she interacts with a single agent (as described in the previous paragraph) or with the group (as in our main model), she is paying money to buy a good, and she has the same belief about how valuable the good is to the seller(s). Hence, the utilitarian-efficient allocation is the same in either setting: the good is efficiently traded whenever the benefit of doing so is greater than its cost,

$$b \geqslant \sigma \cdot \boldsymbol{\theta}.$$

For any c_0, let the **iso-cost curve for** c_0 be the set of all type profiles that have the same cost c_0, i.e., those $\theta \in \Theta$ that satisfy $c_0 = \sigma \cdot \theta$. Then, efficient trade occurs in the region of the type space that is below the iso-cost curve for b. Panel (a) of Figure 1 illustrates this region for the case of two agents ($N = 2$), in which iso-cost curves are straight lines with slope $-\frac{\sigma_1}{\sigma_2}$. Iso-cost curves that are further in the northeast direction correspond to larger cost levels. In what follows, we compare the efficiency of the buyer's optimal allocation rule across these two models.

In a buyer-optimal mechanism for the single-agent benchmark, the good is traded whenever the benefit exceeds its single-agent *virtual* cost,

$$b \geqslant \sigma \cdot \boldsymbol{\theta} + \frac{G(\sigma \cdot \boldsymbol{\theta})}{g(\sigma \cdot \boldsymbol{\theta})}.$$

For any c_0, let the **iso-single-agent-virtual-cost curve for** c_0 be the set of all

$\theta \in \Theta$ that have the same virtual cost c_0—that is, satisfying $c_0 = \sigma \cdot \theta + \frac{G(\sigma \cdot \theta)}{g(\sigma \cdot \theta)}$. Then trade occurs below the iso-single-agent-virtual-cost curve for b. This curve is shown for two agents in Panel (b) of Figure 1. Observe that every iso-single-agent-virtual-cost curve is also an iso-cost curve (associated with a lower cost level), so that the former curves are also straight lines with slope $-\frac{\sigma_1}{\sigma_2}$, and the virtual cost also increases as we move in the northeast direction. As is well known, optimal allocations for single-agent settings entail a downward distortion in trade: when $\sigma \cdot \boldsymbol{\theta} < b < \sigma \cdot \boldsymbol{\theta} + \frac{G(\sigma \cdot \boldsymbol{\theta})}{G(\sigma \cdot \boldsymbol{\theta})}$, trade occurs in the efficient allocation but not according to optimal allocations in the single-agent benchmark.

In buyer-optimal mechanisms in our group setting, trade occurs whenever the benefit exceeds the weighted virtual cost of all group members,

$$b \geqslant \omega \cdot \boldsymbol{\varphi} = \sum_i \omega_i \left[\boldsymbol{\theta}_i + \frac{F_i(\boldsymbol{\theta}_i)}{f_i(\boldsymbol{\theta}_i)} \right].$$

For any c_0, let the **iso-group-virtual-cost curve for** c_0 be the set of all type profiles that have the same weighted virtual cost c_0, i.e., those $\theta \in \Theta$ satisfying $c_0 = \sum_i \omega_i \left[\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)} \right]$. Then, in the optimal allocation rule for our group setting, trade occurs below the iso-group-virtual-cost curve for b. This allocation rule is illustrated for two agents in panel (c) of Figure 1. Notably, because the weights ω do not depend on the shares σ, iso-group-virtual-cost curves are unrelated to the shares (holding fixed the distribution of $\boldsymbol{\theta}$).

A comparison of the iso-cost curves with the iso-group-virtual-cost curves shows that an optimal allocation rule in our group setting differs from the efficient allocation for *three* reasons. To see this, let us convert the weighted virtual costs to (actual) costs in three steps, and study each conversion:

$$\sum_i \omega_i \left[\boldsymbol{\theta}_i + \frac{F_i(\boldsymbol{\theta}_i)}{f_i(\boldsymbol{\theta}_i)} \right] \rightsquigarrow \sum_i \sigma_i \left[\boldsymbol{\theta}_i + \frac{F_i(\boldsymbol{\theta}_i)}{f_i(\boldsymbol{\theta}_i)} \right] \rightsquigarrow \sum_i \sigma_i \left[\boldsymbol{\theta}_i + \frac{F_i(\sigma \cdot \boldsymbol{\theta})}{f_i(\sigma \cdot \boldsymbol{\theta})} \right] \rightsquigarrow \sum_i \sigma_i \boldsymbol{\theta}_i.$$

The first conversion highlights a *rotational* distortion. Whereas the iso-group-virtual-cost curves are unaffected by the shares σ, the iso-cost curves rotate as the shares change. With two agents, as σ_1 increases, each iso-cost curve rotates clockwise whereas iso-group-virtual-cost curves are unaltered. The second conversion highlights a *curvature* distortion. Unlike iso-cost curves, iso-group-virtual-cost curves might be non-linear because the inverse hazard rate functions F_i/f_i might be non-linear. The third conversion highlights the familiar *downward* distortion. The addition of the inverse hazard rate term elevates the iso-weighted-virtual-cost curves and leads to lower probability of trade. The optimal mechanism for the single-agent benchmark exhibits only the third distortion (with a different inverse hazard rate, G/g) and not the other two.

Two salient features emerge from examining how these three different distortions interact. First, as we will demonstrate, trade in the group setting may be *inefficiently high*. That is, optimal allocations in our main model may prescribe trade even when trading is inefficient. This type of inefficient trade cannot happen in the single-agent benchmark, which exhibits only the downward distortion. Second, an efficiency comparison between buying from a group and buying from a single agent is ambiguous in general. As we will show, a key determinant of the welfare ranking is how large a benefit the good yields for the buyer. Focusing on the natural case in which the per-unit costs $\{\boldsymbol{\theta}_i\}_i$ are identically distributed, we show optimal allocations in our group setting are more efficient than in the single-agent setting if b is large, and are less efficient when b is small. We elaborate more on these two observations below.

For the first observation, suppose each $i \in N$ has $\boldsymbol{\theta}_i \in [0, 1]$ following the power distribution $F_i(\theta_i) = \theta_i^\alpha$ for a power $\alpha > 0$. Recall that identically distributed $\{\boldsymbol{\theta}_i\}_i$ lead to $\omega = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$ being optimal in the group setting, and that $\varphi_i(\theta_i) = \frac{\alpha+1}{\alpha}\theta_i$ for any $\theta_i \in [0, 1]$. Given the latter linear form, iso-weighted-virtual-cost curves are linear, and so there is no curvature distortion in this example. The overall distortion depends on the interaction between rotational and downward distortions, and the rotational distortion might dominate for certain type profiles. In particular, compare the weighted virtual cost,

$$\omega \cdot \boldsymbol{\varphi} = \sum_i \frac{1}{N} \boldsymbol{\varphi}_i = \frac{1 + \alpha}{\alpha N} \sum_i \boldsymbol{\theta}_i,$$

to the (actual) cost,

$$\sum_i \sigma_i \boldsymbol{\theta}_i.$$

Suppose the share vector σ is asymmetric, so that some i has $\sigma_i > \frac{1}{N}$. Then, any large enough α admits a range of b for which

$$\frac{1 + \alpha}{\alpha N} < b < \sigma_i.$$

In this case, with positive probability—specifically, when $\boldsymbol{\theta}_i$ is high and $\{\boldsymbol{\theta}_j\}_{j \neq i}$ are low—trading is inefficient but still happens under the optimal allocation rule for the group setting. This example suggests that the irrelevance of land shares, σ, to the optimal mechanism may distort trade in favor of smaller landholders with low productivity.

Second, consider the efficiency of the allocation. Specializing to the case in which $\{\boldsymbol{\theta}_i\}_i$ are identically distributed, we provide an efficiency ranking between the group and single-agent settings for two cases: when the benefit to the buyer is very low, and when it is very high. First, when the benefit is low enough, we show our group setting generates a lower expected surplus than the single-agent benchmark. This surplus ranking holds in an ex-post sense—that is, the buyer's chosen mechanism for the single-agent setting stipulates trade whenever trade is efficient and happens in the group setting—if and only if the shares are sufficiently similar. On the other hand, when the benefit is large enough, then our group setting yields more surplus (in the stronger ex-post sense) than the single-agent one whatever the share vector. In particular, our results imply that the efficiency ranking between the group and single-agent settings will generally depend on the specific parameters of the model. We formally state and prove these results in the Online Appendix.

To provide some intuition, let us focus on the case in which the land shares are symmetric. In this case, because both weighted virtual costs and single-agent virtual costs are above actual costs, it follows that trading is efficient whenever the optimal allocation in the group or the single-agent setting prescribes it. Therefore, a surplus ranking will follow from showing one regime specifies trade in a bigger region than the other. When the actual cost realizations are extreme—either very high or very low—we can establish this ranking of trade regions. This is because, in these cases, we can rank the single-agent virtual costs against the weighted virtual costs. The case of high costs is simpler: Because the average cost can only be high if *all* sellers have a high cost, the density of the average cost must vanish at the tails, leading to an infinite single-agent virtual cost (whereas the weighted virtual cost is finite). The case of low costs requires a more detailed quantitative calculation, but we show in the appendix that single-agent virtual costs are indeed lower than weighted virtual costs when the average actual cost is low. The ranking follows: when the benefit to the buyer is very low, the single-agent optimal mechanism generates more surplus than the optimal mechanism in the group setting, while the reverse ranking holds when the benefit is high.

VI. Discussion

We now consider some variants of our main model and briefly discuss how our analysis can be extended in these directions. Any nontrivial formalism is deferred to the Online Appendix.

Dominant strategies. The notion of incentive compatibility we have employed so far is Bayesian incentive compatibility (BIC, which we have called IC throughout), which requires only that sellers' reports be best responses in expectation, given their own realized types. However, in our leading application—a group of sellers who collectively own a plot of land—one could envision scenarios without any private information inside the group. That is, the group members might know other members' costs, but the buyer does not.

Motivated by such situations, it is perhaps natural to consider more demanding incentive constraints. Specifically, we consider what happens when the buyer is constrained to offer a mechanism that is dominant-strategy incentive compatible (DIC). Whereas proportional transfers impose no constraints on what allocation rules can be implemented under Bayesian incentive constraints, we show they significantly constrain a buyer restricted to DIC mechanisms; that is, there is no counterpart to part (i) of Lemma 1 saying every ex-post monotone allocation rule is DIC-implementable by some transfer rule.[19] In particular, all deterministic DIC mechanisms take the form of a posted price (augmented by an upfront transfer) with trade occurring if and only if enough sellers approve the trade. Using this observation, we show that no optimal mechanism (i.e., those characterized by Theorem 1) is also DIC. The intuition is similar to that of Proposition 2: optimal mechanisms (putting weight on multiple agents) deliver smooth incentives to a single agent, whereas deterministic DIC mechanisms cannot.

Ex-post participation. With a view to respecting individual property rights, we have constrained our buyer to employ a mechanism that is individually rational for each seller—that is, such that every seller can keep his land rather than interacting with the mechanism. As with our other incentive constraints, we formulated IR in the interim sense, having each seller assess his participation decision in expectation over others' types. One may wish to consider a buyer constrained by a stronger form of property rights—namely, that any seller has the option to walk away from the mechanism even after all uncertainty has been resolved. For some examples, such an ex post IR constraint imposes no additional costs on the buyer.[20] For instance, in the equilibrium described in the bidding game of Example 1, no seller ever has an incentive to walk away, even after he learns the other seller's bid if the powers in the sellers' distributions multiply to at most one, $\alpha_1\alpha_2 \leqslant 1$. When does this property hold more generally? And can our analytical approach be applied to understand such ex post constraints?

To better understand the ex post IR constraint more generally, we first answer an implementability question: When can a given allocation rule, together with an expected transfer, be implemented in some IC and ex post IR mechanism? Using this characterization, we provide sufficient conditions on primitives under which

[19] This fact is reminiscent of other work showing BIC-DIC equivalence fails given financial constraints. Even though unidimensional private-values mechanism design settings with flexible transfers admit a strong form of this equivalence (Gershkov et al., 2013), a DIC constraint severely constrains implementable allocations when paired with ex post budget balance. See (Hagerty and Rogerson, 1987), for example.

[20] A similar equivalence arises in related work by Che (2002).

some buyer-optimal mechanism is also ex post IR. Applied to Example 1, these conditions say if $\alpha_1 = \alpha_2 \leqslant 1$, some optimal mechanism exists that is also ex post IR, consistent with our calculations for that example.

Pareto-optimal mechanisms. Our paper has focused on mechanisms that maximize the buyer's expected payoff. Although this objective is a natural benchmark, it assumes the buyer has extreme bargaining power relative to the seller group. More generally, one might wonder what mechanisms can arise naturally with different allocations of bargaining rights. Specifically, we study Pareto-optimal mechanisms—that is, IC and IR mechanisms for which there is no alternative IC and IR mechanism that delivers a weakly higher buyer profit, and a weakly higher agent i value for each agent i, with at least one of these $N + 1$ inequalities strict. Our land-acquisition application suggests another reason to care about the entire Pareto frontier. If the buyer is a government, state-owned enterprise, or other large stakeholder in the relevant community, then they may care about the welfare of the current landholders in addition to the purely financial consequences of trade. Understanding the range of optimal mechanisms all such stakeholders might wish to use amounts to understanding all Pareto optima of our space of IC and IR mechanisms.

Pareto-optimal mechanisms trade if and only if the buyer's benefit maximizes a weighted average of sellers' *virtual and actual* costs. Although this class of allocation rules is richer than the unique buyer optimum, it enjoys similar tractability and qualitative structure. For instance, Pareto-optimal mechanisms are deterministic and use weights that are fixed and do not depend on reports. The weight that applies to a seller's cost is exactly the Pareto weight of that seller, whereas the weight that applies to the virtual cost is identified endogenously and reflects the agent's influence over the outcomes. We also use our characterization to generalize the main result of Section IV, showing every Pareto-optimal mechanism entails complex pricing.

Our characterization of implementable allocation rules, along with the analytical approach we adopt in developing Theorem 1, proves useful in providing our characterization of Pareto-optimal mechanisms. A standard separation result enables us to represent Pareto optima as maximizers of weighted sums of the $N + 1$ individuals' objectives, and we can adapt our zero-sum game proof to this more general class of objectives.

Pre-market trade. Throughout, we have restricted attention to mechanisms in which agents are paid proportionally to their land shares. We mainly impose this structure as a fairness or institutional requirement that is natural in many applications. As formalized in the appendix, we point out another desirable property of such mechanisms for the case in which sellers' per-unit costs are identically distributed. We show that if the buyer uses these proportional transfer mechanisms, then sellers have no incentives to manipulate the outcome by trading their shares before interacting with the mechanism. We also show, by example, that this property could be violated if the buyer were not restricted to paying agents proportionally to their shares (as in Güth and Hellwig, 1986). Thus, in addition to being realistic in many settings, our assumption of collective transfers yields a desirable robustness property for buying mechanisms.

The result that the sellers have no incentives to trade shares is based on two observations. First, the optimal mechanism is independent of the shares. Second, in a mechanism that is independent of the shares, the sellers' incentives to trade shares disappear if agents are paid the same price per share. When discriminatory pricing is allowed, the buyer optimally treats sellers with different shares differently, opening the door to gaming by trading shares.

Beyond veto bargaining. An important feature of our environment is that any agent can unilaterally veto the mechanism. This feature, captured by the requirement that the mechanism be IR for all of the agents, is natural in settings with strong property rights. However, a more permissive bargaining arrangement may be more appropriate for modeling some contexts—for example, when eminent domain enables a government to forcibly acquire land from some individuals for public projects. As in some redevelopment projects, we could require that rather than unanimity, the terms of trade need to be approved by at least n agents for some given $n < N$. This flexibility raises new modeling questions concerning how exactly one determines whether a mechanism has sufficient approval.

In one approach, we could require this approval by n sellers determined ex ante—that is, independent of their type realizations. This formulation reduces nearly immediately to the analysis in our main model. Indeed, one need only replace the IR constraint (which we imposed for all N agents in our model) with a weaker assumption that at least n agents' IR constraints are satisfied. Because the buyer has no reason to condition on the types of agents facing no IR constraint, her problem reduces to an n-agent specification of our main model. The optimal mechanism allocates the good if and only if the benefit to the buyer exceeds a weighted sum of the chosen n agents' virtual costs. The buyer would then choose to tailor the mechanism to the n agents she finds most favorable to interact with ex ante—for instance (given Theorem 1), the n agents with the lowest virtual cost distributions if these distributions are first-order stochastically ranked.

Appendix

A. Proofs for main results

A.A. Proofs for Section II

We first reproduce the statement of Lemma 1.

Lemma: *Let x be some allocation rule.*

(i) Mechanism (x, m) is IC and IR for some transfer rule m if and only if x is interim monotone.

(ii) If some transfer rule m exists such that mechanism (x, m) is IC and IR, then a maximally profitable such mechanism exists, with resulting profit

$$\min_{i \in N} \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right].$$

Proof of Lemma 1. For each $i \in N$, let $X_i := X_i^x$, and define $M_i^* : \Theta_i \to \mathbb{R}$ by

$$M_i^*(\theta_i) := X_i(\theta_i)\theta_i + \int_{\theta_i}^{\bar{\theta}_i} X_i(\tilde{\theta}_i)\, \mathrm{d}\tilde{\theta}_i.$$

Given a transfer rule m, standard arguments (Myerson, 1981; Myerson and Satterthwaite, 1983) show that (x, m) is IC if and only if each $i \in N$ has X_i weakly decreasing and $M_i^m = M_i^* + \underline{U}_i$ for some constant $\underline{U}_i \in \mathbb{R}$; that such a mechanism is IR if and only if $\underline{U}_i \geqslant 0$ for each $i \in N$; and that $\mathbb{E}\left[M_i^*(\boldsymbol{\theta}_i)\right] = \mathbb{E}\left[X_i(\boldsymbol{\theta}_i)\boldsymbol{\varphi}_i\right]$. Given iterated expectations, the latter equation simplifies to $\mathbb{E}\left[M_i^*(\boldsymbol{\theta}_i)\right] = \mathbb{E}\left[x(\boldsymbol{\theta})\boldsymbol{\varphi}_i\right]$.

Using the above observations, let us prove the two parts of the lemma in turn.

Toward part (i), note the first paragraph says interim monotonicity is necessary for x to be IC implementable; and for sufficiency it suffices to show some transfer rule m^0 exists such that $M_i^{m^0} - M_i^*$ is constant for each $i \in N$ (since raising such a transfer rule by a large enough constant will ensure IR). The transfer rule m^0 given by $m^0(\theta) := \sum_{i \in N} M_i^*(\theta_i)$ has this property, and so part (i) follows.

Now, toward part (ii), suppose x is indeed implementable; say transfer rule m is such that (x, m) is IC and IR. Then each $i \in N$ admits $\underline{U}_i \geqslant 0$ such that $M_i^m = M_i^* + \underline{U}_i$. Hence, for any $i \in N$, we can write the expected transfer as

$$\mathbb{E}\left[m(\boldsymbol{\theta})\right] = \mathbb{E}\left[M_i^M(\boldsymbol{\theta}_i)\right] = \mathbb{E}\left[x(\boldsymbol{\theta})\boldsymbol{\varphi}_i\right] + \underline{U}_i,$$

so that the buyer's expected value can be written as

$$\mathbb{E}\left[bx(\boldsymbol{\theta}) - m(\boldsymbol{\theta})\right] = \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right] - \underline{U}_i.$$

Reducing the transfer rule by a constant will reduce each of $\{\underline{U}_i\}_i$ by the same constant, and so raise the buyer's expected value. The buyer therefore optimally sets $\min_{i \in N} \underline{U}_i = 0$. But in this case, we have $\mathbb{E}\left[bx(\boldsymbol{\theta}) - m(\boldsymbol{\theta})\right] \leqslant \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right]$ for every $i \in N$, with equality for some i. Said differently, we then have

$$\mathbb{E}\left[bx(\boldsymbol{\theta}) - m(\boldsymbol{\theta})\right] = \min_{i \in N} \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right],$$

delivering part (ii) □

The following notation will be convenient to us in making formal arguments.

Notation 1: *Let $\tilde{\mathcal{X}}$ denote the set of all allocation rules $\mathcal{X}$, modulo the F-almost everywhere equivalence relation, a subset of $L^\infty(\Theta, F)$. Each element of $\mathcal{X}$ corresponds to one of $\tilde{\mathcal{X}}$ in the obvious way.*

Consider now the relaxed buyer problem,

$$\max_{x \in \tilde{\mathcal{X}}} \left\{ \min_{i \in N} \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right] \right\}. \tag{RBP}$$

which is our buyer's problem without the interim-monotonicity constraint (and cast in $\tilde{\mathcal{X}}$). The following lemma characterizes solutions of this relaxed program.

Lemma 2: *A unique solution exists to program* (RBP)*. This solution is given by the ω-allocation rule, where $\omega \in \Delta N$ is any weight vector satisfying the two equivalent conditions* (i) *and* (ii) *in the statement of Theorem 1.*

Proof. Consider a two-player zero-sum game where the maximizer (Max) chooses $x \in \tilde{\mathcal{X}}$ and the minimizer (Min) chooses $\omega \in \Delta N$. The objective (that is, the payoff to Max) is

$$\mathcal{G}(x, \omega) := \mathbb{E}[x(\boldsymbol{\theta})(b - \omega \cdot \boldsymbol{\varphi})].$$

We will first argue that a Nash equilibrium exists for this zero-sum game; and that the Nash equilibria are exactly the pairs $(x^*, \omega^*) \in \tilde{\mathcal{X}} \times \Delta N$ for which x^* solves (RBP) and ω^* satisfies condition (i). Then we will argue that x_ω is Max's unique best response to any Min strategy ω; that Max has a unique Nash equilibrium strategy; and that condition (ii) is equivalent to being a Nash equilibrium strategy for Min. Establishing these facts will establish the lemma.

First, because $\tilde{\mathcal{X}}$ is weak*-compact (by Banach-Alaoglu) and convex, the space ΔN obviously is as well, and the objective is weak*-continuous in the strategy profile, it follows from Sion's minimax theorem that

$$\max_{x \in \tilde{\mathcal{X}}} \min_{\omega \in \Delta N} \mathcal{G}(x, \omega) = \min_{\omega \in \Delta N} \max_{x \in \tilde{\mathcal{X}}} \mathcal{G}(x, \omega),$$

where all extrema in the equation are attained by Berge's theorem. Then, because the auxiliary game is strictly competitive, Proposition 22.2 from Osborne and Rubinstein (1994) tells us some Nash equilibrium exists, and that the Nash equilibria are exactly the pairs $(x^*, \omega^*) \in \tilde{\mathcal{X}} \times \Delta N$ for which

$$\begin{aligned} x^* &\in \operatorname{argmax}_{x \in \tilde{\mathcal{X}}} \min_{\omega \in \Delta N} \mathcal{G}(x, \omega) \text{ and} \\ \omega^* &\in \operatorname{argmin}_{\omega \in \Delta N} \max_{x \in \tilde{\mathcal{X}}} \mathcal{G}(x, \omega). \end{aligned}$$

(In particular, the set of equilibria forms a product set.) Observe, though, that $\min_{\omega \in \Delta N} \mathcal{G}(x, \omega) = \min_{i \in N} \mathbb{E}\left[x(\boldsymbol{\theta})(b - \boldsymbol{\varphi}_i)\right]$ for each $x \in \tilde{\mathcal{X}}$ because $\mathcal{G}(x, \cdot)$ affine. Hence, x^* maximizes this quantity if and only if x^* solves (RBP). Moreover, $\max_{x \in \tilde{\mathcal{X}}} \mathcal{G}(x, \omega) = \mathbb{E}\left[\max_{\mathtt{x} \in [0,1]} (b - \omega \cdot \boldsymbol{\varphi})\mathtt{x}\right] = \mathbb{E}\left[(b - \omega \cdot \boldsymbol{\varphi})_+\right]$ for each $\omega \in \Delta N$, so that minimizing these expressions is equivalent—that is, the minimax strategies are exactly those satisfying condition (i). So we have established that some Nash equilibrium exists; and that the Nash equilibria are exactly the pairs $(x^*, \omega^*) \in \tilde{\mathcal{X}} \times \Delta N$ for which x^* solves (RBP) and ω^* satisfies condition (i).

It remains to show that x_ω is Max's unique best response to any Min strategy ω; that Max has a unique Nash equilibrium strategy; and that condition (ii) is equivalent to being a Nash equilibrium strategy for Min. Toward the first assertion, consider any $\omega \in \Delta N$. Because $\{\boldsymbol{\theta}_i\}_{i\in N}$ are atomless and independent and $\{\varphi_i\}_{i\in N}$ are all strictly increasing, it follows that $\mathbb{P}\{\omega \cdot \boldsymbol{\varphi} = b\} = 0$, so that the $\tilde{\mathcal{X}}$ element with representative x_ω is the unique $x \in \tilde{\mathcal{X}}$ such that

$$\mathbb{P}\left\{x(\boldsymbol{\theta}) \in \operatorname{argmax}_{\mathrm{x}\in[0,1]} [(b - \omega \cdot \boldsymbol{\varphi})\mathrm{x}]\right\} = 1.$$

Thus, the ω-allocation rule x_ω is Min's unique best response (in $\tilde{\mathcal{X}}$) to ω. From the product structure of the set of Nash equilibria, then, it follows that Max has a unique Nash equilibrium strategy x^*, which is then the unique solution to (RBP).

All that remains now is to show that condition (ii) is equivalent to being a Nash equilibrium strategy for Min. But because x_ω is the unique Max best response to $\omega \in \Delta N$, we know ω is a Nash equilibrium strategy if and only $\omega \in \operatorname{argmax}_{\tilde{\omega}\in\Delta N} \mathcal{G}(x_\omega, \tilde{\omega})$ or, equivalently (since $\mathcal{G}(x_\omega, \cdot)$ is affine) every $i \in \operatorname{supp}(\omega)$ belongs to $\operatorname{argmax}_{i\in N} \mathbb{E}\left[(b - \boldsymbol{\theta}_i)\, \mathbb{1}_{\omega\cdot\boldsymbol{\theta}\leqslant b}\right]$. Finally $b > \underline{\theta}_i$ for every $i \in N$, the event $\{\omega\cdot\boldsymbol{\theta} \leqslant b\}$ has strictly positive probability, so that the latter condition is equivalent to condition (i). The lemma follows. □

We now reproduce the statement of Theorem 1.

Theorem (Optimal allocation)**:**
A weighted allocation rule is essentially uniquely optimal. The unique optimal-weight vector ω is characterized by either of the following two equivalent conditions:

(i) $\omega \in \operatorname{argmin}_{\tilde{\omega}\in\Delta N} \mathbb{E}[(b - \tilde{\omega} \cdot \boldsymbol{\varphi})_+]$.

(ii) $\operatorname{supp}(\omega) \subseteq \operatorname{argmax}_{i\in N} \mathbb{E}\left[\boldsymbol{\varphi}_i \mid \omega \cdot \boldsymbol{\varphi} \leqslant b\right]$.

Moreover, if $b < \bar{\theta}_j$ for at least two $j \in N$, then every $i \in N$ has $\omega_i < 1$.

Proof of Theorem 1. First, by Lemma 1, an allocation rule is optimal if and only if it solves program (BP), so we focus on solutions to this program.

Now, Lemma 2 tells us that conditions (i) and (ii) in the theorem's statement are equivalent, that some $\omega \in \Delta N$ exists that satisfies those conditions, and that (the almost-sure equivalence class of) x_ω is uniquely optimal in (RBP). Because x_ω is interim monotone and solves a relaxation of (BP), it follows directly that x_ω solves (BP), and that every other solution x to (BP) has $x(\boldsymbol{\theta}) = x_\omega(\boldsymbol{\theta})$ almost surely.

Next, we establish $i \in N$ has $\omega_i < 1$ if some $j \in N\backslash\{i\}$ has $b < \bar{\theta}_j$. To see this fact, note that if $i \in N$ had $\omega_i = 1$, then $j \in N\backslash\{i\}$ would have

$$\mathbb{E}\left[\boldsymbol{\varphi}_i \mid \omega \cdot \boldsymbol{\varphi} \leqslant b\right] = \mathbb{E}\left[\boldsymbol{\varphi}_i \mid \boldsymbol{\varphi}_i \leqslant b\right] \leqslant b < \bar{\theta}_j = \mathbb{E}\left[\boldsymbol{\varphi}_j\right] = \mathbb{E}\left[\boldsymbol{\varphi}_j \mid \omega \cdot \boldsymbol{\varphi} \leqslant b\right],$$

in contradiction to condition (ii).

Finally, we turn to uniqueness of ω. Suppose $\tilde{\omega} \in \Delta N$ is such that $x_{\tilde{\omega}}$ is optimal, and so $x_{\tilde{\omega}}(\boldsymbol{\theta}) = x_\omega(\boldsymbol{\theta})$ almost surely; our aim is to show $\tilde{\omega} = \omega$. Toward establishing this equality, define $G := \prod_{i \in N} \left(b - \varphi_i(\bar{\theta}_i),\ b - \underline{\theta}_i\right)$, the interior of the support of $b\mathbb{1}_N - \boldsymbol{\varphi}$. Now, define the linear map $L : \mathbb{R}^N \to \mathbb{R}^2$ by letting $L(z) := (\omega \cdot z,\ \tilde{\omega} \cdot z)$ for each $z \in \mathbb{R}^N$. Let us now observe some properties of G and L. First, that $\omega \cdot \varphi(\underline{\theta}) < b < \omega \cdot \varphi(\bar{\theta})$ and $\tilde{\omega} \cdot \varphi(\underline{\theta}) < b < \tilde{\omega} \cdot \varphi(\bar{\theta})$ implies $L(G)$ is not a subset of $\mathbb{R}_+ \times \mathbb{R}$, of $\mathbb{R}_- \times \mathbb{R}$, of $\mathbb{R} \times \mathbb{R}_+$, or of $\mathbb{R} \times \mathbb{R}_-$. Second, that $\mathbb{P}\{x_{\tilde{\omega}}(\boldsymbol{\theta}) = x_\omega(\boldsymbol{\theta})\} = 1$ implies $L(G)$ is a subset of $\mathbb{R}^2_+ \cup \mathbb{R}^2_-$. Third, because L is linear and G is convex, the set $L(G)$ is convex. Combining these three observations tells us that $L(G)$ is contained in a single line through the origin. Because G is open and L is linear, then, $L(\mathbb{R}^N)$ is contained in the same line. Said differently, the rank of the linear map L is 1, so that vectors $\omega, \tilde{\omega} \in \mathbb{R}^N_+$ are proportional. Because $||\omega||_1 = 1 = ||\tilde{\omega}||_1$, it follows that $\omega = \tilde{\omega}$. □

Proof for Example 1. In what follows, we proceed in three steps. First, we show ω^* is the optimal weight vector, thus characterizing optimal allocation rules. Second, we name a specific (ex-post) transfer rule, and show that this transfer rule paired with our optimal allocation rule constitutes an optimal mechanism. Third, we show that the strategy profile we have named for the bidding game is an equilibrium that induces this optimal mechanisms.

We first prove that ω^* is the optimal vector of weights. To this end, let

$$\nu_i := 1 + \frac{1}{\alpha_i} = \frac{\alpha_i + 1}{\alpha_i}$$

for each agent i, and notice that $\varphi_i(\theta_i) = \nu_i \theta_i$ for every $\theta_i \in \Theta_i = [0, 1]$. For any $\omega \in \Delta N$ with $\min\{\omega_1\nu_1,\ \omega_2\nu_2\} \geqslant b$ (an interval of ω including ω^*), observe that

$$\mathbb{E}[(b - \omega \cdot \boldsymbol{\varphi})_+] = \int_0^{\frac{b}{\omega_1\nu_1}} \int_0^{\frac{b-\omega_1\nu_1\theta_1}{(1-\omega_1)\nu_2}} \left[b - \omega_1\nu_1\theta_1 - (1-\omega_1)\nu_2\theta_2\right] \alpha_1\theta_1^{\alpha_1 - 1}\alpha_2\theta_2^{\alpha_2-1}\,\mathrm{d}\theta_2\,\mathrm{d}\theta_1.$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\omega_1}\mathbb{E}\left\{[b - (\omega_1, 1-\omega_1)\cdot\boldsymbol{\varphi}]_+\right\} &= \int_0^{\frac{b}{\omega_1\nu_1}} \alpha_1\alpha_2 \int_0^{\frac{b-\omega_1\nu_1\theta_1}{(1-\omega_1)\nu_2}} (\nu_2\theta_2 - \nu_1\theta_1)\theta_1^{\alpha_1-1}\theta_2^{\alpha_2-1}\,\mathrm{d}\theta_2\,\mathrm{d}\theta_1 \\
&= \int_0^{\frac{b}{\omega_1\nu_1}} \alpha_1\alpha_2 \left\{ \nu_2\theta_1^{\alpha_1-1}\frac{\left[\frac{b-\omega_1\nu_1\theta_1}{(1-\omega_1)\nu_2}\right]^{\alpha_2+1}}{1+\alpha_2} - \nu_1\theta_1^{\alpha_1}\frac{\left[\frac{b-\omega_1\nu_1\theta_1}{(1-\omega_1)\nu_2}\right]^{\alpha_2}}{\alpha_2} \right\}\mathrm{d}\theta_1 \\
&= \int_0^{\frac{b}{\omega_1\nu_1}} \theta_1^{\alpha_1-1}\left[\tfrac{b-\omega_1\nu_1\theta_1}{(1-\omega_1)\nu_2}\right]^{\alpha_2}\left[\alpha_1\tfrac{b-\omega_1\nu_1\theta_1}{(1-\omega_1)\nu_2} - \alpha_2\tfrac{\omega_1^*\nu_1}{1-\omega_1^*}\theta_1\right]\mathrm{d}\theta_1.
\end{aligned}$$

Note now that if $\omega_1 = \omega_1^*$, the integrand is then equal to

$$\frac{\mathrm{d}}{\mathrm{d}\theta_1}\left\{\theta_1^{\alpha_1}\left[\tfrac{b-\omega_1\nu_1\theta_1}{(1-\omega_1)\nu_2}\right]^{\alpha_2+1}\right\},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\omega_1}\mathbb{E}\left\{[b-(\omega_1, 1-\omega_1)\cdot\boldsymbol{\varphi}]_+\right\} = \left(\tfrac{b}{\omega_1\nu_1}\right)^{\alpha_1} 0^{\alpha_2+1} - 0^{\alpha_1}\left[\tfrac{b}{(1-\omega_1)\nu_2}\right]^{\alpha_2+1} = 0.$$

Hence, ω^* solves the convex program from Theorem 1(i), meaning it is optimal.

Now, consider the mechanism (x, m) given by $x(\theta) := x_{\omega^*}(\theta)$ and

$$m(\theta) := x(\theta)[\kappa b + \beta_1\theta_1 + \beta_2\theta_2],$$

where

$$\begin{aligned} \kappa &= \tfrac{\alpha_1+\alpha_2}{(\alpha_1+1)(\alpha_2+1)}(1 - \alpha_1\alpha_2) \\ \beta_i &= \tfrac{\alpha_i+1}{\alpha_{-i}+1}\alpha_{-i} \text{ for } i \in N. \end{aligned}$$

Let us argue that (x, m) is an optimal mechanism. For each $i \in N$, define the interim allocation rule $X_i := X_i^x$, the interim transfer rule $M_i := M_i^m$, and the interim transfer rule M_i^* as defined in the proof of Lemma 1. If $M_i = M_i^*$ for both $i \in N$, then as explained in Lemma 1 proof, the mechanism (x, m) is IC and has binding IR for both agents, and is therefore best for the buyer among all IC and IR mechanisms with allocation rule x; because x is optimal, it will then follow that (x, m) is optimal. So we now turn to showing $M_i = M_i^*$ for both $i \in N$. To that end, let

$$\gamma_i := \omega_i^*\nu_i = \frac{\alpha_i + 1}{\alpha_1 + \alpha_2} > 0,$$

and note that $b \leqslant \gamma_i$ by hypothesis. Therefore, X_i is zero on $\left(\frac{b}{\gamma_i}, 1\right]$, so that any $\theta_i \in [0, 1]$ has

$$X_i(\theta_i) = \mathbb{P}\left[\gamma_{-i}\boldsymbol{\theta}_{-i} \leqslant b - \gamma_i\theta_i\right] = \left(\tfrac{b-\gamma_i\theta_i}{\gamma_{-i}}\right)_+^{\alpha_{-i}}$$

and

$$\begin{aligned} \int_{\theta_i}^1 X_i &= \mathbb{1}_{\theta_i \leqslant \frac{b}{\gamma_i}} \int_{\theta_i}^{\frac{b}{\gamma_i}} \left(\tfrac{b-\gamma_i\tilde{\theta}_i}{\gamma_{-i}}\right)^{\alpha_{-i}} \, \mathrm{d}\tilde{\theta}_i \\ &= \mathbb{1}_{\theta_i \leqslant \frac{b}{\gamma_i}} \int_{\theta_i}^{\frac{b}{\gamma_i}} \left(\tfrac{b-\gamma_i\tilde{\theta}_i}{\gamma_{-i}}\right)^{\alpha_{-i}} \, \mathrm{d}\tilde{\theta}_i \\ &= \tfrac{\gamma_{-i}}{\gamma_i}\mathbb{1}_{\theta_i \leqslant \frac{b}{\gamma_i}} \int_0^{\frac{b-\gamma_i\theta_i}{\gamma_{-i}}} y^{\alpha_{-i}} \, \mathrm{d}y \\ &= \tfrac{\alpha_{-i}+1}{\alpha_i+1}\mathbb{1}_{\theta_i \leqslant \frac{b}{\gamma_i}} \cdot \tfrac{1}{\alpha_{-i}+1}\left(\tfrac{b-\gamma_i\theta_i}{\gamma_{-i}}\right)^{\alpha_{-i}+1} \\ &= \tfrac{1}{\alpha_i+1}\left(\tfrac{b-\gamma_i\theta_i}{\gamma_{-i}}\right) X_i(\theta_i) \\ &= \left[\tfrac{1}{\alpha_i+1}\tfrac{1}{\gamma_{-i}}b - \tfrac{1}{\alpha_i+1}\tfrac{\gamma_i}{\gamma_{-i}}\theta_i\right] X_i(\theta_i) \\ &= \left[\tfrac{\alpha_1+\alpha_2}{(\alpha_1+1)(\alpha_2+1)}b - \tfrac{1}{\alpha_{-i}+1}\theta_i\right] X_i(\theta_i). \end{aligned}$$

Hence,

$$M_i^*(\theta_i) \;=\; X_i(\theta_i)\theta_i + \int_{\theta_i}^{\bar{\theta}_i} X_i = \left[\tfrac{\alpha_1+\alpha_2}{(\alpha_1+1)(\alpha_2+1)}b + \tfrac{\alpha_{-i}}{\alpha_{-i}+1}\theta_i\right] X_i(\theta_i).$$

Next, note that each $y \in [0,1]$ has

$$\mathbb{E}\left[\boldsymbol{\theta}_{-i}\mathbb{1}_{\boldsymbol{\theta}_{-i}\leqslant y}\right] = \int_0^y \theta_{-i}\left(\alpha_{-i}\theta_{-i}^{\alpha_{-i}-1}\right)\,\mathrm{d}\theta_{-i} = \tfrac{\alpha_{-i}}{\alpha_{-i}+1}y^{\alpha_{-i}+1}.$$

Therefore, each $i \in N$ and $\theta_i \in [0,1]$ has

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\theta}_{-i}x(\theta_i,\boldsymbol{\theta}_{-i})\right] &= \mathbb{E}\left[\boldsymbol{\theta}_{-i}\mathbb{1}_{\boldsymbol{\theta}_{-i}\leqslant\frac{b-\gamma_i\theta_i}{\gamma_{-i}}}\right] \\
&= \mathbb{1}_{\theta_i\leqslant\frac{b}{\gamma_i}}\cdot\tfrac{\alpha_{-i}}{\alpha_{-i}+1}\left(\tfrac{b-\gamma_i\theta_i}{\gamma_{-i}}\right)^{\alpha_{-i}+1} \\
&= \tfrac{\alpha_{-i}}{\alpha_{-i}+1}\left(\tfrac{b-\gamma_i\theta_i}{\gamma_{-i}}\right)X_i(\theta_i) \\
&= \alpha_{-i}\tfrac{\alpha_i+1}{\alpha_{-i}+1}\left[\tfrac{\alpha_1+\alpha_2}{(\alpha_1+1)(\alpha_2+1)}b-\tfrac{1}{\alpha_{-i}+1}\theta_i\right]X_i(\theta_i) \\
&= \tfrac{\alpha_{-i}}{(\alpha_{-i}+1)^2}\left[(\alpha_1+\alpha_2)b-(\alpha_i+1)\theta_i\right]X_i(\theta_i).
\end{aligned}$$

It follows that

$$\begin{aligned}
M(\theta_i) &= \mathbb{E}\left\{\left[\kappa b+\beta_i\theta_i+\beta_{-i}\boldsymbol{\theta}_{-i}\right]x(\theta_i,\boldsymbol{\theta}_{-i})\right\} \\
&= \left\{\kappa b+\beta_i\theta_i+\beta_{-i}\tfrac{\alpha_{-i}}{(\alpha_{-i}+1)^2}\left[(\alpha_1+\alpha_2)b-(\alpha_i+1)\theta_i\right]\right\}X_i(\theta_i) \\
&= \left\{\left[\kappa+\beta_{-i}\tfrac{\alpha_{-i}}{(\alpha_{-i}+1)^2}(\alpha_1+\alpha_2)\right]b+\left[\beta_i-\beta_{-i}\tfrac{\alpha_{-i}}{(\alpha_{-i}+1)^2}(\alpha_i+1)\right]\theta_i\right\}X_i(\theta_i) \\
&= \left\{\left[\kappa+\tfrac{\alpha_1\alpha_2}{(1+\alpha_1)(1+\alpha_2)}(\alpha_1+\alpha_2)\right]b+\left[\beta_i-\tfrac{\alpha_1\alpha_2}{(1+\alpha_1)(1+\alpha_2)}(\alpha_i+1)\right]\theta_i\right\}X_i(\theta_i) \\
&= \left[\tfrac{\alpha_1+\alpha_2}{(1+\alpha_1)(1+\alpha_2)}b+\left(\beta_i-\tfrac{\alpha_1\alpha_2}{1+\alpha_{-i}}\right)\theta_i\right]X_i(\theta_i) \\
&= \left[\tfrac{\alpha_1+\alpha_2}{(1+\alpha_1)(1+\alpha_2)}b+\tfrac{\alpha_{-i}}{1+\alpha_{-i}}\theta_i\right]X_i(\theta_i) \\
&= M_i^*(\theta_i).
\end{aligned}$$

Hence, the given mechanism (x,m) is optimal.

Finally, let us turn to the bidding game in which each agent i can submit any bid $s_i \geqslant 0$; trade occurs if and only if

$$\tau_1 s_1+\tau_2 s_2 \leqslant b,$$

where

$$\tau_i := \frac{\alpha_{-i}+1}{(\alpha_1+\alpha_2)\alpha_{-i}} \text{ for } i \in N;$$

and the price (paid if and only if trade occurs) is $p = \kappa b + s_1 + s_2$. Let us consider the strategy profile in which each type θ_i of each agent i bids $\beta_i\theta_i$. We will argue that this strategy profile constitutes a Bayes Nash equilibrium and that it generates allocation rule x and transfer rule m; optimality will then follow from optimality of the mechanism (x,m). First, any type profile $\theta \in \Theta$ has

$$\tau_i\beta_i\theta_i = \omega_i^*\nu_i\theta_i = \omega_i^*\varphi_i(\theta_i)\ \forall i \in N \implies \tau_1\beta_1\theta_1+\tau_2\beta_2\theta_2 = \omega^*\cdot\varphi(\theta).$$

Because each agent i bids $s_i = \beta_i\theta_i$, it follows that trade occurs if and only if $\omega^* \cdot \varphi(\theta) \leqslant b$—that is, the induced allocation rule is exactly x. Second, if trade happens at type profile θ, the price paid under this strategy profile is

$$p = \kappa b + s_1 + s_2 = \kappa b + \beta_1\theta_1 + \beta_2\theta_2.$$

Thus, the induced transfer rule is exactly m. All that remains then is to check that the described bidding rule is an equilibrium. To that end, consider any type $\theta_i \in [0,1]$ of any agent i; we want to show $\beta_i\theta_i$ yields a weakly higher expected payoff for this type than any other $s_i \geqslant 0$. Because the mechanism (x,m) is IC, we know that $\theta_i \in \operatorname{argmax}_{\tilde{\theta}_i\in[0,1]} \mathbb{E}\left[m(\tilde{\theta}_i, \boldsymbol{\theta}_{-i}) - \theta_i x(\tilde{\theta}_i, \boldsymbol{\theta}_{-i})\right]$. But then, because the bidding game and strategy profile induce x and m, it follows that θ_i has no profitable deviation in $\left\{\beta_i\tilde{\theta}_i\right\}_{\tilde{\theta}_i\in[0,1]} = [0, \beta_i]$. Meanwhile, every bid $s_i \geqslant \beta_i$ has

$$\tau_i s_i \geqslant \tau_i\beta_i = \gamma_i \geqslant b,$$

and so (because agent $-i$ has a strictly positive bid almost surely) leads to a zero probability of trade. Thus, all bids $s_i \geqslant \beta_i$ are payoff-equivalent for i, and so do not constitute profitable deviations because bid β_i does not. Hence, the given strategy profile is an equilibrium, as required. □

A.B. Proofs for Section III

The following lemma provides a sufficient condition to be able to weakly rank agents' weights in the optimal mechanism, and further provides a quantitative sufficient conditions for one agent's weight to be substantially higher than another's.

Lemma 3: *Suppose constants $\alpha \in (0,1]$ and $\beta \geqslant (1-\alpha)\frac{(\bar{\theta}_i+\alpha\bar{\theta}_j)}{(1+\alpha)}$ are such that*

$$\boldsymbol{\varphi}_i \geqslant_{\text{rh}} \alpha\boldsymbol{\varphi}_j + \beta.$$

Then, the optimal weight vector ω satisfies $\alpha\omega_i \geqslant \omega_j$.

Proof. Suppose $\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j, \alpha, \beta$ satisfy the given hypotheses, and let $\omega \in \Delta N$ have $\alpha\omega_i < \omega_j$ (which in particular implies $\omega_j > 0$). To establish the lemma, we need to show that ω is not optimal. To do so, we construct $\tilde{\omega}_i, \tilde{\omega}_j \geqslant 0$ such that $\tilde{\omega}_i + \tilde{\omega}_j = \omega_i + \omega_j$ and $(\tilde{\omega}_i, \tilde{\omega}_j) \neq (\omega_i, \omega_j)$, with $\tilde{\omega}_i\boldsymbol{\varphi}_i + \tilde{\omega}_j\boldsymbol{\varphi}_j \geqslant_{\text{icv}} \omega_i\boldsymbol{\varphi}_i + \omega_j\boldsymbol{\varphi}_j$, where $\geqslant_{\text{icv}}$ is the increasing concave order (a.k.a second-order stochastic dominance). Because $h : \mathbb{R} \to \mathbb{R}$ given by $h(z) := -\mathbb{E}[(b - z - \sum_{k\in N\setminus\{i,j\}} \omega_k\boldsymbol{\varphi}_k)_+]$ is (weakly) increasing and concave, finding such $\tilde{\omega}_i, \tilde{\omega}_j$ would show that ω is not the unique minimizer of $\hat{\omega} \mapsto \mathbb{E}[(b-\hat{\omega}\cdot\boldsymbol{\varphi})_+]$—the objective in condition (i) of Theorem 1—and so is not optimal.

Now, define $\gamma := \frac{\alpha(\omega_i+\omega_j)}{\alpha^2\omega_i+\omega_j} > 0$, and let $\tilde{\omega}_i := \gamma\frac{1}{\alpha}\omega_j$ and $\tilde{\omega}_j := \gamma\alpha\omega_i$. By construction, $\tilde{\omega}_i + \tilde{\omega}_j = \omega_i + \omega_j$. Moreover, $(\tilde{\omega}_i, \tilde{\omega}_j) \neq (\omega_i, \omega_j)$—obviously if $\omega_i = 0 < \tilde{\omega}_i$, and otherwise because $\frac{\tilde{\omega}_j}{\tilde{\omega}_i} = \alpha\frac{\alpha\omega_i}{\omega_j} < \alpha < \frac{\omega_j}{\omega_i}$. It thus remains to show that

$\tilde{\omega}_i \boldsymbol{\varphi}_i + \tilde{\omega}_j \boldsymbol{\varphi}_j \succcurlyeq_{\text{icv}} \omega_i \boldsymbol{\varphi}_i + \omega_j \boldsymbol{\varphi}_j$. To that end, first observe that

$$\begin{aligned}
\tilde{\omega}_i \boldsymbol{\varphi}_i + \tilde{\omega}_j \boldsymbol{\varphi}_j &= \gamma \tfrac{\omega_j}{\alpha} \boldsymbol{\varphi}_i + \gamma \alpha \omega_i \boldsymbol{\varphi}_j \\
&= \gamma \tfrac{\omega_j}{\alpha} \boldsymbol{\varphi}_i + \gamma \omega_i (\alpha \boldsymbol{\varphi}_j + \beta) - \gamma \omega_i \beta \\
&\succcurlyeq_{\text{icv}} \gamma \omega_i \boldsymbol{\varphi}_i + \gamma \tfrac{\omega_j}{\alpha} (\alpha \boldsymbol{\varphi}_j + \beta) - \gamma \omega_i \beta \\
&= \gamma (\omega_i \boldsymbol{\varphi}_i + \omega_j \boldsymbol{\varphi}_j) + \gamma \left(\tfrac{\omega_j}{\alpha} - \omega_i \right) \beta \\
&= \gamma \left[\omega_i (\boldsymbol{\varphi}_i - \bar{\theta}_i) + \omega_j (\boldsymbol{\varphi}_j - \bar{\theta}_j) \right] + \gamma \left[\left(\tfrac{\omega_j}{\alpha} - \omega_i \right) \beta + \omega_i \bar{\theta}_i + \omega_j \bar{\theta}_j \right],
\end{aligned}$$

where the inequality comes from Theorem 4.A.37 of Shaked and Shanthikumar (2007). Next, we establish that

$$\begin{aligned}
&\gamma \left[\omega_i (\boldsymbol{\varphi}_i - \bar{\theta}_i) + \omega_j (\boldsymbol{\varphi}_j - \bar{\theta}_j) \right] + \gamma \left[\left(\tfrac{\omega_j}{\alpha} - \omega_i \right) \beta + \omega_i \bar{\theta}_i + \omega_j \bar{\theta}_j \right] \\
\succcurlyeq_{\text{icv}} \; &\omega_i (\boldsymbol{\varphi}_i - \bar{\theta}_i) + \omega_j (\boldsymbol{\varphi}_j - \bar{\theta}_j) + \gamma \left[\left(\tfrac{\omega_j}{\alpha} - \omega_i \right) \beta + \omega_i \bar{\theta}_i + \omega_j \bar{\theta}_j \right]
\end{aligned}$$

To do so, observe that $\gamma = 1 - \frac{(1-\alpha)}{(\alpha^2 \omega_i + \omega_j)} (\omega_j - \alpha \omega_i) \leqslant 1$ and $\mathbf{z} := \omega_i (\boldsymbol{\varphi}_i - \bar{\theta}_i) + \omega_j (\boldsymbol{\varphi}_j - \bar{\theta}_j)$ has zero mean. Because a constant shift obviously preserves $\succcurlyeq_{\text{icv}}$, we need only observe $\gamma \mathbf{z} \succcurlyeq_{\text{icv}} \mathbf{z}$, which follows directly from Jensen's inequality.[21]

Therefore,

$$\begin{aligned}
\tilde{\omega}_i \boldsymbol{\varphi}_i + \tilde{\omega}_j \boldsymbol{\varphi}_j &\succcurlyeq_{\text{icv}} \gamma \left[\omega_i (\boldsymbol{\varphi}_i - \bar{\theta}_i) + \omega_j (\boldsymbol{\varphi}_j - \bar{\theta}_j) \right] + \gamma \left[\left(\tfrac{\omega_j}{\alpha} - \omega_i \right) \beta + \omega_i \bar{\theta}_i + \omega_j \bar{\theta}_j \right] \\
&\succcurlyeq_{\text{icv}} \omega_i (\boldsymbol{\varphi}_i - \bar{\theta}_i) + \omega_j (\boldsymbol{\varphi}_j - \bar{\theta}_j) + \gamma \left[\left(\tfrac{\omega_j}{\alpha} - \omega_i \right) \beta + \omega_i \bar{\theta}_i + \omega_j \bar{\theta}_j \right] \\
&= \omega_i \boldsymbol{\varphi}_i + \omega_j \boldsymbol{\varphi}_j + \gamma \left(\tfrac{\omega_j}{\alpha} - \omega_i \right) \beta - (1 - \gamma) \left(\omega_i \bar{\theta}_i + \omega_j \bar{\theta}_j \right),
\end{aligned}$$

Because $\beta \geqslant (1-\alpha) \frac{\bar{\theta}_i + \alpha \bar{\theta}_j}{1+\alpha}$, it will therefore follow that $\tilde{\omega}_i \boldsymbol{\varphi}_i + \tilde{\omega}_j \boldsymbol{\varphi}_j \succcurlyeq_{\text{icv}} \omega_i \boldsymbol{\varphi}_i + \omega_j \boldsymbol{\varphi}_j$ if we establish that

$$\lambda := \gamma \left(\tfrac{\omega_j}{\alpha} - \omega_i \right) (1 - \alpha) \tfrac{\bar{\theta}_i + \alpha \bar{\theta}_j}{1+\alpha} - (1 - \gamma) \left(\omega_i \bar{\theta}_i + \omega_j \bar{\theta}_j \right)$$

is nonnegative. And indeed, $\lambda = \frac{(1-\alpha)(\alpha \omega_i - \omega_j)^2}{(1+\alpha)(\alpha^2 \omega_i + \omega_j)} (\bar{\theta}_i - \bar{\theta}_j)$, so the lemma will follow as long as we have $\bar{\theta}_i \geqslant \bar{\theta}_j$. For this ranking, note Theorem 1.B.42 of Shaked and Shanthikumar (2007) implies $\mathbb{E}[\boldsymbol{\varphi}_i] \geqslant \mathbb{E}\left[\alpha \boldsymbol{\varphi}_j + \beta\right]$, i.e.,

$$\bar{\theta}_i \geqslant \alpha \bar{\theta}_j + \beta \geqslant \alpha \bar{\theta}_j + (1 - \alpha) \tfrac{\bar{\theta}_i + \alpha \bar{\theta}_j}{1+\alpha} = \bar{\theta}_i - \tfrac{2\alpha}{1+\alpha} (\bar{\theta}_i - \bar{\theta}_j).$$

Hence, $\bar{\theta}_i \geqslant \bar{\theta}_j$, as required. □

The following lemma sharpens the previous one by showing the weight ranking result often holds strictly. Whereas the previous lemma's proof uses the characterization of optimal weights as a minimax strategy, the following one uses the characterization as Minimizer's best response.

Lemma 4: *Suppose constants $\alpha \in (0, 1]$ and $\beta \geqslant (1 - \alpha) \frac{\bar{\theta}_i + \alpha \bar{\theta}_j}{1+\alpha}$ are such that*

$$\boldsymbol{\varphi}_i \succcurlyeq_{\text{rh}} \alpha \boldsymbol{\varphi}_j + \beta$$

and $\beta > 0$. Then, the optimal weight vector ω cannot satisfy $\alpha \omega_i = \omega_j > 0$.

[21] For $\eta : \mathbb{R} \to \mathbb{R}$ concave, $\mathbb{E}\eta(\gamma \mathbf{z}) \geqslant \mathbb{E}\left[\gamma \eta(\mathbf{z}) + (1 - \gamma) \eta(0)\right] = \gamma \mathbb{E}\eta(\mathbf{z}) + (1 - \gamma) \eta(\mathbb{E}\mathbf{z}) \geqslant \mathbb{E}\eta(\mathbf{z})$.

Proof. Consider any $\omega \in \Delta N$ with $\alpha\omega_i = \omega_j > 0$, with a view to showing it cannot be optimal. Defining the random variables $\tilde{\boldsymbol{\varphi}}_j := \alpha\boldsymbol{\varphi}_j + \beta$ and $\mathbf{y} := \frac{1}{\omega_i}\left(b - \sum_{k\in N\setminus\{i,j\}} \omega_k \boldsymbol{\varphi}_k\right) + \beta$, observe that $x_\omega(\boldsymbol{\theta}) = \mathbb{1}_{\boldsymbol{\varphi}_i + \tilde{\boldsymbol{\varphi}}_j \leqslant \mathbf{y}}$. Meanwhile, the random variables $\boldsymbol{\varphi}_i, \tilde{\boldsymbol{\varphi}}_j, \mathbf{y}$ are independent of $\mathbf{y}$ and $\boldsymbol{\varphi}_i \geqslant_{\text{rh}} \tilde{\boldsymbol{\varphi}}_j$.

Now, let us observe that $\mathbb{E}\left[\boldsymbol{\varphi}_i \mid \omega\cdot\boldsymbol{\varphi} \leqslant b\right] \geqslant \mathbb{E}\left[\tilde{\boldsymbol{\varphi}}_j \mid \omega\cdot\boldsymbol{\varphi} \leqslant b\right]$. Indeed, this inequality is equivalent to showing $\mathbb{E}\eta(\tilde{\boldsymbol{\varphi}}_j, \boldsymbol{\varphi}_i) \geqslant 0$, where $\eta : \mathbb{R}^2 \to \mathbb{R}$ is given by $\eta(s,t) := (t-s)\mathbb{E}\left[\mathbb{1}_{s+t\leqslant \mathbf{y}}\right]$. Because $\eta(s,t) + \eta(t,s) = 0$ for every $s,t \in \mathbb{R}$ and η is nonincreasing in its first argument (as a product of two nonnegative nonincreasing functions) on $\{(s,t) \in \mathbb{R}^2 : \ s \leqslant t\}$, the inequality follows directly from Theorem 1.B.48 of Shaked and Shanthikumar (2007).

Hence, ω satisfies

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{\varphi}_i \mid \omega\cdot\boldsymbol{\varphi} \leqslant b\right] &\geqslant \mathbb{E}\left[\tilde{\boldsymbol{\varphi}}_j \mid \omega\cdot\boldsymbol{\varphi} \leqslant b\right] \\ &= \alpha\mathbb{E}\left[\boldsymbol{\varphi}_j \mid \omega\cdot\boldsymbol{\varphi} \leqslant b\right] + \beta.\end{aligned}$$

Assume now, for a contradiction, that ω is optimal. In this case, Theorem 1(ii) yields

$$\mathbb{E}\left[\boldsymbol{\varphi}_i \mid \omega\cdot\boldsymbol{\varphi} \leqslant b\right] = \mathbb{E}\left[\boldsymbol{\varphi}_j \mid \omega\cdot\boldsymbol{\varphi} \leqslant b\right] =: \hat{\theta}.$$

Because the (interior-probability) event that $\omega\cdot\boldsymbol{\varphi} \leqslant b$ is the event that the bounded random variable $\boldsymbol{\varphi}_i$ [resp. $\boldsymbol{\varphi}_j$] lies below some random variable independent of it, it follows that $\hat{\theta} < \mathbb{E}[\boldsymbol{\varphi}_i] = \bar{\theta}_i$ [resp. $\hat{\theta} < \bar{\theta}_j$]. Therefore, $\frac{\bar{\theta}_i + \alpha\bar{\theta}_j}{1+\alpha} > \hat{\theta}$, implying $\beta > (1-\alpha)\hat{\theta}$. Hence, $\hat{\theta} \geqslant \alpha\hat{\theta} + \beta > \hat{\theta}$, a contradiction. □

We now reproduce the statement of Theorem 2.

Theorem (Ranking allocation weights)**:** *If $\boldsymbol{\varphi}_i \geqslant_{\text{rh}} \boldsymbol{\varphi}_j + \beta$ for some $\beta \geqslant 0$, then the optimal vector of allocation weights ω satisfies $\omega_i \geqslant \omega_j$. Moreover, $\omega_i > \omega_j$ whenever $\beta > 0$ and $\omega_j > 0$.*

Proof of Theorem 2. The first statement is exactly Lemma 3, specialized to the case of $\alpha = 1$. Given this result, the second statement corresponds exactly to Lemma 4, specialized to the case of $\alpha = 1$. □

A.C. Proofs for Section IV

We now reproduce the statement of Proposition 1.

Proposition (Optimal posted price is unanimous)**:** *Some unanimous posted-price mechanism is optimal among IC and IR collective posted-price mechanisms.*

Proof of Proposition 1. Consider an arbitrary collective posted-price mechanism (x,m) with price p. Let us show a unanimous posted price performs better.[22]

If $p \geqslant b$, then the profit associated with the mechanism is always nonpositive, and so a unanimous posted price with price in $(\max_{i\in N} \underline{\theta}_i, b)$ is more profitable.

[22]Our proof establishes any IC and IR collective posted price that is not almost surely identical to a unanimous one is *strictly* worse than some unanimous posted price.

Now, suppose $p < b$. For any agent $i \in N$ and $\theta_i \in (p, \bar{\theta}_i]$, IR implies $X_i^x(\theta_i) = 0$—and so $x(\theta_i, \boldsymbol{\theta}_{-i})$ must be zero almost surely. It follows that $x(\boldsymbol{\theta}) \leqslant x^U(\boldsymbol{\theta})$ almost surely, where x^U is the allocation rule

$$x^U(\theta) := \mathbb{1}_{\theta_j \leqslant p \ \forall j \in N}$$

associated with a unanimous posted price of p. Hence, $(b-p)\mathbb{E}[x(\boldsymbol{\theta})] \leqslant (b-p)\mathbb{E}[x^U(\boldsymbol{\theta})]$—strictly so unless $x(\boldsymbol{\theta}) = x^U(\boldsymbol{\theta})$ almost surely. Therefore, the unanimous posted-price mechanism (x^U, px^U) yields a higher profit.

Having shown every collective posted price is outperformed by some unanimous posted price, it remains to note that an optimal posted price exists. Any posted price outside of $(\max_{i \in N} \underline{\theta}_i, b)$ yields a nonpositive profit, whereas unanimous posted prices in this interval yield strictly positive profit. It thus suffices to show the buyer has some preferred price in $[\max_{i \in N} \underline{\theta}_i, b]$—which follows from compactness of this interval and continuity of the objective $p \mapsto (b-p)\prod_{i \in N} F_i(p)$. □

Lemma 5: *If x is an optimal allocation rule, then $X_i^x(\cdot)$ is continuous on $(\underline{\theta}_i, \bar{\theta}_i)$ for every $i \in N$ with $\omega_i < 1$, and is nonconstant on $(\underline{\theta}_i, \bar{\theta}_i)$ if the optimal weights ω have $\omega_i > 0$.*

Proof. Let $\omega \in \Delta N$ be optimal, and let $X_i := X_i^{x_\omega}$ for each $i \in N$. Essential uniqueness of the optimal allocation rule (assured by Theorem 1) means it suffices to show $X_i(\cdot)$ is continuous for every $i \in N$, and is nonconstant on $(\underline{\theta}_i, \bar{\theta}_i)$ if the optimal weights ω have $\omega_i > 0$.[23]

First, let us see any given $i \in \operatorname{supp}(\omega)$ is nonconstant. Indeed, a nonempty open neighborhood in Θ_{-i} exists such that $\omega \cdot (\underline{\theta}_i, \theta_{-i}) < b < \omega \cdot \varphi(\bar{\theta}_i, \theta_{-i})$ for any θ_{-i} in this neighborhood.[24] Because $\boldsymbol{\theta}_{-i}$ has full support and x is decreasing, it follows that $\lim_{\theta_i \searrow \underline{\theta}_i} X_i(\theta_i) < \lim_{\theta_i \nearrow \bar{\theta}_i} X_i(\theta_i)$. Hence, X_i is not constant on $(\underline{\theta}_i, \bar{\theta}_i)$.

Next, let us show that any $i \in N$ with $\omega_i < 1$ has X_i continuous. For each $\theta_i \in \Theta_i$, the interim probability of trade is given by

$$X_i^x(\theta_i) = \mathbb{P}\left\{b - \sum\nolimits_{j \in N \setminus \{i\}} \omega_j \varphi_j(\boldsymbol{\theta}_j) \leqslant \omega_i \varphi_i(\theta_i)\right\}.$$

Recall that $\{\boldsymbol{\theta}_j\}_{j \in N}$ are independent and atomlessly distributed, ω_{-i} is nonzero, and $\varphi_i(\cdot)$ is continuous. It follows that the random variable on the left side of the above inequality is atomlessly distributed, while the quantity on the right side varies continuously with θ_i. Hence, X_i^x is continuous, as desired. □

We now reproduce the statement of Proposition 2.

Proposition (Posted prices are suboptimal): *If at least two $j \in N$ have $b < \bar{\theta}_j$, then no collective posted-price mechanism is optimal.*

[23] Essential uniqueness implies $X_i^x(\boldsymbol{\theta}_i) = X_i(\boldsymbol{\theta}_i)$ almost surely. Because $\boldsymbol{\theta}_i$ has convex support and X_i^x is monotone, it then follows (after establishing continuity of X_i) that the two functions are identical on $(\underline{\theta}_i, \bar{\theta}_i)$.

[24] Indeed, these inequalities hold whenever all of $\{\varphi_j(\theta_j)\}_{j \in N \setminus \{i\}}$ are within ϵ of b, where $\epsilon > 0$ is smaller than $\omega_i \min\{b - \underline{\theta}_i,\ \varphi_i(\bar{\theta}_i) - b\}$. This condition describes an open neighborhood because $\{\varphi_j\}_{j \in N}$ are continuous.

Proof of Proposition 2. Given Proposition 1, we need only show the unanimous posted-price mechanism is not an optimal mechanism for any price. Let ω denote the optimal weight vector assured by Theorem 1, fix some $i \in N$ such that $\omega_i > 0$, and let X_i denote i's interim allocation rule induced by a unanimous posted-price mechanism. By iterated expectations, constants p and $\bar{\mathtt{x}}$ exist such that every $\theta_i \in \Theta_i$ has $X_i(\theta_i) = \bar{\mathtt{x}}\mathbb{1}_{\theta_i \leqslant p}$. The function X_i therefore cannot be both continuous and nonconstant on $(\underline{\theta}_i, \bar{\theta}_i)$—it is discontinuous there if $\underline{\theta}_i < p < \bar{\theta}_i$ and $\bar{\mathtt{x}} \neq 0$, and is constant there otherwise. Given that the last assertion of Theorem 1 tells us $\omega_i < 1$, Lemma 5 thus delivers the proposition. □

References

Ausubel, Lawrence M, Peter Cramton, Marek Pycia, Marzena Rostek, and Marek Weretka. 2014. "Demand reduction and inefficiency in multi-unit auctions." *The Review of Economic Studies*, 81(4): 1366–1400. 5

Azrieli, Yaron, and Semin Kim. 2014. "Pareto efficiency and weighted majority rules." *International Economic Review*, 55(4): 1067–1088. 6

Babaioff, Moshe, Nicole Immorlica, Brendan Lucier, and S Matthew Weinberg. 2020. "A simple and approximately optimal mechanism for an additive buyer." *Journal of the ACM (JACM)*, 67(4): 1–40. 18

Banerjee, Abhijit V, et al. 2000. "Prospects and strategies for land reform." 253–84, World Bank. 4, 17

Barlow, Richard E, Albert W Marshall, and Frank Proschan. 1963. "Properties of probability distributions with monotone hazard rate." *The Annals of Mathematical Statistics*, 375–389. 20

Bergstrom, Theodore C. 1978. "Cournot equilibrium in factor markets." 6

Berry, R Albert, William R Cline, et al. 1979. *Agrarian structure and productivity in developing countries: a study prepared for the International Labour Office within the framework of the World Employment Programme.* Johns Hopkins Univ. Press. 17

Börgers, Tilman, and Peter Norman. 2009. "A note on budget balance under interim participation constraints: the case of independent types." *Economic Theory*, 39(3): 477–489. 11

Cai, Yang, Constantinos Daskalakis, and S Matthew Weinberg. 2013. "Understanding incentives: Mechanism design becomes algorithm design." 618–627, IEEE. 6

Chand, Ramesh, PA Lakshmi Prasanna, and Aruna Singh. 2011. "Farm size and productivity: Understanding the strengths of smallholders and improving their livelihoods." *Economic and Political Weekly*, 5–11. 1

Chawla, Shuchi, David Malec, and Balasubramanian Sivan. 2015. "The power of randomness in bayesian optimal mechanism design." *Games and Economic Behavior*, 91: 297–317. 18

Chawla, Shuchi, Jason D Hartline, David L Malec, and Balasubramanian Sivan. 2010. "Multi-parameter mechanism design and sequential posted pricing." 311–320. 18

Che, Yeon-Koo. 2002. "The economics of collective negotiation in pretrial bargaining." *International Economic Review*, 549–575. 6, 25

Che, Yeon-Koo, and Jinwoo Kim. 2006. "Robustly collusion-proof implementation." *Econometrica*, 74(4): 1063–1107. 11

Cramton, Peter, Robert Gibbons, and Paul Klemperer. 1987. "Dissolving a partnership efficiently." *Econometrica: Journal of the Econometric Society*, 615–632. 5

d'Aspremont, Claude, and Louis-André Gérard-Varet. 1979. "Incentives and incomplete information." *Journal of Public economics*, 11(1): 25–45. 5

d'Aspremont, Claude, Jacques Crémer, and Louis-André Gérard-Varet. 2004. "Balanced bayesian mechanisms." *Journal of Economic Theory*, 115(2): 385–396. 11

Ekmekci, Mehmet, Nenad Kos, and Rakesh Vohra. 2016. "Just enough or all: Selling a firm." *American Economic Journal: Microeconomics*, 8(3): 223–256. 5

Friedman, Milton. 1960. *A program for monetary stability.* Ravenio Books. 5

Gandhi, Sahil, Vaidehi Tandel, Alexander Tabarrok, and Shamika Ravi. 2021. "Too slow for the urban march: Litigations and the real estate market in Mumbai, India." *Journal of Urban Economics*, 123: 103330. 1

Gershkov, Alex, Benny Moldovanu, and Xianwen Shi. 2017. "Optimal voting rules." *The Review of Economic Studies*, 84(2): 688–717. 6

Gershkov, Alex, Jacob K Goeree, Alexey Kushnir, Benny Moldovanu, and Xianwen Shi. 2013. "On the equivalence of Bayesian and dominant strategy implementation." *Econometrica*, 81(1): 197–220. 25

Gopalan, Parikshit, Noam Nisan, and Tim Roughgarden. 2018. "Public projects, Boolean functions, and the borders of Border's theorem." *ACM Transactions on Economics and Computation (TEAC)*, 6(3-4): 1–21. 10

Grossman, Sanford J, and Oliver D Hart. 1980. "Takeover bids, the free-rider problem, and the theory of the corporation." *The Bell Journal of Economics*, 42–64. 6

Güth, Werner, and Martin Hellwig. 1986. "The private supply of a public good." *Journal of Economics*, 46(1): 121–159. 5, 27

Hagerty, Kathleen M, and William P Rogerson. 1987. "Robust trading mechanisms." *Journal of Economic Theory*, 42(1): 94–107. 25

Hart, Sergiu, and Noam Nisan. 2017. "Approximate revenue maximization with multiple items." *Journal of Economic Theory*, 172: 313–347. 18

Heller, Michael A. 1998. "The tragedy of the anticommons: property in the transition from Marx to markets." *Harvard law review*, 621–688. 1

Kirkegaard, René. 2012. "A mechanism design approach to ranking asymmetric auctions." *Econometrica*, 80(5): 2349–2364. 16

Krishna, Vijay, and John Morgan. 2015. "Majority rule and utilitarian welfare." *American Economic Journal: Microeconomics*, 7(4): 339–75. 6

Kuran, Timur. 2004. "Why the Middle East is economically underdeveloped: historical mechanisms of institutional stagnation." *Journal of economic perspectives*, 18(3): 71–90. 2

Mailath, George J, and Andrew Postlewaite. 1990. "Asymmetric information bargaining problems with many agents." *The Review of Economic Studies*, 57(3): 351–367. 5

Makowski, Louis, and Claudio Mezzetti. 1994. "Bayesian and weakly robust first best mechanisms: characterizations." *Journal of Economic Theory*, 64(2): 500–519. 11

Malvey, Paul F, and Christine M Archibald. 1998. "Uniform-price auctions: Update of the Treasury experience." *US Treasury.* 5

Manjunatha, AV, Asif Reza Anik, Stijn Speelman, and EA Nuppenau. 2013. "Impact of land fragmentation, farm size, land ownership and crop diversity on profit and efficiency of irrigated farms in India." *Land use policy*, 31: 397–405. 1

Maskin, Eric, and John Riley. 2000. "Asymmetric auctions." *The review of economic studies*, 67(3): 413–438. 16

Myerson, Roger B. 1981. "Optimal auction design." *Mathematics of operations research*, 6(1): 58–73. 3, 10, 18, 28

Myerson, Roger B, and Mark A Satterthwaite. 1983. "Efficient mechanisms for bilateral trading." *Journal of economic theory*, 29(2): 265–281. 10, 28

Oliveros, Santiago, and Matias Iaryczower. 2022. "Collective Hold-Up." *Theoretical Economics.* 6

Osborne, Martin J, and Ariel Rubinstein. 1994. *A course in game theory.* MIT press. 29

Rae, Douglas W. 1969. "Decision-rules and individual values in constitutional choice." *The American Political Science Review*, 63(1): 40–56. 6

Riley, John, and Richard Zeckhauser. 1983. "Optimal selling strategies: When to haggle, when to hold firm." *The Quarterly Journal of Economics*, 98(2): 267–289. 18

Rob, Rafael. 1989. "Pollution claim settlements under private information." *Journal of Economic Theory*, 47(2): 307–333. 5

Schmitz, Patrick W, and Thomas Tröger. 2012. "The (sub-) optimality of the majority rule." *Games and Economic Behavior*, 74(2): 651–665. 6

Shaked, Moshe, and J George Shanthikumar. 2007. *Stochastic orders.* Springer Science & Business Media. 35, 36

Sood, Aradhya. 2020. "Land Market Frictions in Developing Countries: Evidence from Manufacturing Firms in India." 1