Dynamics Analysis of Discrete-time Sliding Mode Controller for the Nonlinear Systems

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***Abstract*—This brief research proposes a discrete-time sliding mode controller and studies its dynamic characteristics. First, this research develops a discrete-time sliding mode reaching law by redefining the change rate of disturbance through difference function, estimating disturbances using disturbance estimation techniques and compensating them into the reaching law, and further designing a discrete-time sliding mode controller. Second, through theoretical analysis, the width of quasi-sliding mode domain(QSMD), and convergence steps are obtained, and the bounded convergence property of system state variables is proven. Lastly, the correctness of the theoretical analysis and effectiveness of the controller are verified through simulations. Simulation results demonstrate that the proposed controller can ensure that the switching function reaches QSMD within a finite time and continue to move within QSMD, and the system state will eventually converge.**

***Index Terms*—Discrete-time sliding mode control, sliding mode reaching law, quasi-sliding mode domain, nonlinear System.**

I. Introduction

Sliding mode control, as a type of variable structure control strategy, has attracted widespread attention and research in the academic and applied fields owing to its powerful robustness, rapid response capability, and excellent disturbance rejection ability[1], [2], [3], [4], [5]. Sliding mode control methods include continuous-time sliding mode control (CSMC) and discrete-time sliding mode control (DSMC). The main advantage of DSMC, compared with CSMC, lies in its finite switching frequency[6], making it more suitable for application in industrial computer systems.

At presents, a series of DSMC has been proposed in [7], [8], [9]. Among the existing DSMC, the discrete-time reaching law was initially proposed by Gao et al.[10], [11], which can ensure that the system reaches the desired sliding manifold as expected. However, the sliding mode switching function cannot reach the sliding manifold but generates chattering motion nearby, thereby forming a quasi-sliding mode domain(QSMD), which compromises system performance and even result in instability. Therefore, reducing the width of QSMD remains a major challenge for DSMC. Many DSMC has been developed to reduce the width of a system's QSMD[12], [13], [14]. Noted that the width of QSMD is determined by the change rate of system disturbances[15].Qu et al.[14] proposed a discrete-time sliding mode controller with disturbance compensation, which decreases the width of QSMD to the $O(T^2)$ order of the disturbances. Ma et al.[15] proposed a discrete-time reaching law based on difference operators, which uses modified continuous terms instead of sign functions, effectively reducing the QSMD width. However, the majority of the current relevant studies have been based on assumptions of known upper or lower bounds of disturbances, which are difficult to obtain in practical systems. Furthermore, reducing the QSMD width remains a critical factor in improving the DSMC performance. Therefore, DSMC still deserve further research.

This study, which was inspired by the preceding problems, presents the design of a discrete-time sliding mode controller and analyzes its dynamic characteristics. First, a discrete-time reaching law is proposed, and a discrete-time sliding mode controller is designed. The change rate of system disturbances is redefined using a difference function. Moreover, disturbance estimation techniques are used to estimate and compensate for system disturbances into the reaching law, without requiring prior information on disturbances. Second, the QSMD width is analyzed, and the number of steps required to converge to QSMD is calculated. In additional, the state variables of the system will eventually converge within a bounded neighborhood of the origin. Lastly, numerical simulation results demonstrate the effectiveness of the proposed controller.

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II. System Description

Consider the following continuous-time nonlinear single-input single-output system:

$$\dot{x} = Ax + Bu + D \tag{1}$$

where $x \in R^{n\times 1}$ represents the system state variables, $A \in R^{n\times n}$ and $B \in R^{n\times n}$ are constant matrices, $u \in R^{n\times 1}$ represents the system control input, and $D \in R^{l\times 1}, l \le n$ is the disturbance. Consider the following assumption 1:

1) The pair (A, B) is completely controllable;
2) System disturbance D is smooth function and bounded.

Assuming system sampling period is T, when the input signal of the controller is passed through a zero-order holder(ZOH) to the system(i.e., within the sampling time $[nT,(n+1)T]$), denoted as $u(t)=u(n)$, the system function can be rewritten as follows:

$$x_{k+1}=\Phi x_k+\Gamma u_k+d_k \tag{2}$$

where $\Phi=e^{AT}$, $\Gamma=\int_0^T e^{A\tau}d\tau B$, $d_k=\int_0^T e^{A\tau}D((k+1)T-\tau)d\tau$.

$Assumption 2$: According to Eq.(2), d_k is of the $O(T)$ order with respect to T. Thus, $d_k=O(T)$, $d_k-d_{k-1}=O(T^2)$ and $d_k-2d_{k-1}+d_{k-2}=O(T^3)$.

The switching function is chosen as follows:

$$s_k=Cx_k \tag{3}$$

where $C\in R^{1\times n}$ and satisfies $C\Gamma\neq 0$. The control objective of this paper is to design a discrete sliding mode controller to achieve the desired dynamics when traveling along the $s=\{x_k\,|Cx_k=0\}$

The quasi-sliding mode(QSM) and the reaching condition are defined as follows:

$Definition 1$: System (2) is considered to enter into a QSM into the Δ vicinity of the switching function if the state of the system satisfies $\forall k>k^*>0, |s_k|\leq\Delta$. The specific space domain where QSM occurs is called QSMD, where the value of $\Delta>0$ is called the QSMD width.

$Definition 2$: The system satisfies the quasi-sliding mode reaching condition of the sliding mode manifold in Δ as follows: if $s_k>\Delta$, then $-\Delta\leq s_{k+1}<s_k$; if $s_k<-\Delta$, then $s_k<s_{k+1}\leq\Delta$; and if $|s_k|\leq\Delta$, then $|s_{k+1}|\leq\Delta$.

III. Design of the Controller with the Proposed Reaching Law

This section proposes a discrete-time sliding mode reaching law. Moreover, a discrete-time sliding mode controller is further designed to achieve the control object.

A. Reaching Law

This study proposes a discrete-time reaching law is proposed, and the specific form is as follows:

$$\begin{aligned} s_{k+1}= \quad & (1-qT)s_k-\tfrac{\varepsilon T|s_k|^{1/2}}{\lambda+\gamma e^{-\sigma|s_k|}}\operatorname{sgn}(s_k) \\ & +(1-z^{-1})(Cd_k-Cd_{k-1}) \end{aligned} \tag{4}$$

where q and ε are positive constant; λ, γ, and σ are positive gain coefficients, and z^{-1} is the unit delay operator.

$Remark 1$: The difference equation in the reaching law(4) is used to redefine the change rate of the disturbance. Through basic algebraic operations, the reaching law can be simplified as follows:

$$\begin{aligned} s_{k+1}= \quad & (1-qT)s_k-\tfrac{\varepsilon T|s_k|^{1/2}}{\lambda+\gamma e^{-\sigma|s_k|}}\operatorname{sgn}(s_k) \\ & +(Cd_k-2Cd_{k-1}+Cd_{k-2}) \end{aligned} \tag{5}$$

$Assumption 3$: The change rate of disturbance d_k in this study is bounded and satisfies the following formula:

$$|\delta_k|=|C(d_k-2d_{k-1}+d_{k-2})|\leq\delta \tag{6}$$

$Remark 2$: Given that $d(k)=O(T)$, considering Eq.(6) and assumption 1, we can obtain that the upper bound of δ in the proposed reaching law $\delta(k)\leq\delta=O(T^3)$ is effectively reduced compared with $d(k)=O(T)$ and $d(k)-d(k-1)=O(T^2)$][13], [14]. Given that the QSMD width is related to the change rate of disturbance, smaller change rate of d_k will yield a smaller QSMD width.

This research uses the perturbation estimation technique to estimate the disturbance and integrate it into the reaching law design. The estimation equation is as follows:

$$d_{k-1}=x_k-\Phi x_{k-1}-\Gamma u_{k-1} \tag{7}$$

B. Sliding Mode Controller

By substituting the system dynamics Eq.(2) into the switching function(3), the following equation can be obtained:

$$\begin{aligned} s_{k+1}=&Cx_{k+1} \\ =&C\Phi x_k+C\Gamma u_k+Cd_k \end{aligned} \tag{8}$$

Given that $C\Gamma\neq 0$, integrated the Eqs.(4) and (11), the form of controller u_k designed as follows:

$$u_k=-(C\Gamma)^{-1}\left\{\begin{array}{l} C\Phi x_k-(1-qT)s_k+\frac{\varepsilon T|s_k|^{1/2}}{\lambda+\gamma e^{-\sigma|s_k|}}\operatorname{sgn}(s_k) \\ +2Cd_{k-1}-Cd_{k-2} \end{array}\right\} \tag{9}$$

$Remark 3$: Note that although control law(9) depends on the disturbance information, Eq.(4) and (9) show that the information of disturbance can be estimated by using the difference function and perturbation estimation technique. Therefore, all state variable in the controller are known and can be directly applied.

IV. Dynamic Analysis of System

Three theorems have been proven to analyze the main dynamic characteristics of the controller: QSMD width, convergence steps and convergence.

Theorem 1: For system(1) satisfying assumption 2, if the upper bound δ of the disturbance satisfies $\delta<\frac{q\varepsilon^2}{4\lambda^2(1-qT)}T^3$, then the switching function s_k converges to the QSMD as follows:

$$\Delta=\frac{(\varepsilon T)^2}{4\lambda^2(1-qT)^2}+\delta \tag{10}$$

from any initial position under the action of contoller(9). In addition, once s_k enters QSMD, it will always move in the zone.

Proof: The following two cases are considered:

$Case 1$: If the initial position $s_k>\Delta$, in order to make s_k convergent from any positive initial time to QSMD(i.e., $s_{k+1}-s_k<0$), then

$$\begin{aligned} s_{k+1}-s_k&=-qTs_k-\frac{\varepsilon Ts_k^{1/2}}{\lambda+\gamma e^{-\sigma s_k}}+\delta_k \\ &\leq-qTs_k-\frac{\varepsilon Ts_k^{1/2}}{\lambda+\gamma}+\delta<0 \end{aligned} \tag{11}$$

According to Eq.(11), if $\sqrt{s_k}\geq\frac{\sqrt{\left(\frac{\varepsilon T}{\lambda+\gamma}\right)^2+4qT\delta}-\frac{\varepsilon T}{\lambda+\gamma}}{2qT}=\Omega_1$ is satisfied, then $s_{k+1}-s_k<0$. However, proving $\Omega_1^2<\Delta$ is not difficult owing to $\delta < \frac{q\varepsilon^2}{4\lambda^2(1-qT)}T^3$. Thus, the switching function s_k will converge to QSMD from any positive initial position.

If $s_k \in [0, \Delta]$, then

$$\begin{aligned} s_{k+1} &= (1-qT)s_k - \tfrac{\varepsilon T s_k^{1/2}}{\lambda+\gamma e^{-\sigma s_k}} + \delta_k \\ &\geq (1-qT)s_k - \tfrac{\varepsilon T s_k^{1/2}}{\lambda} - \delta \\ &= -\tfrac{(\varepsilon T)^2}{4\lambda^2(1-qT)} - \delta\Big|_{s_k=\frac{(\varepsilon T)^2}{4\lambda^2(1-qT)^2}} \geq -\Delta \end{aligned} \tag{12}$$

Definitions 1 and 2, indicate that when s_k starts from any positive initial position, it will eventually converge to and always move in QSMD.

$Case2$: If the initial position $s_k < -\Delta$, in order to make s_k convergent from any negative initial time to QSMD(i.e., $s_{k+1} - s_k > 0$), then

$$\begin{aligned} s_{k+1} - s_k &= -qTs_k + \frac{\varepsilon T(-s_k)^{1/2}}{\lambda+\gamma e^{\sigma s_k}} + \delta_k \\ &\geq -qTs_k + \frac{\varepsilon T(-s_k)^{1/2}}{\lambda+\gamma} - \delta > 0 \end{aligned} \tag{13}$$

If Eq.(13) is satisfied, then $\sqrt{-s_k} > \frac{-\frac{\varepsilon T}{\lambda+\gamma}+\sqrt{\left(\frac{\varepsilon T}{\lambda+\gamma}\right)^2+4qT\delta}}{2qT} = \Omega_2$. Similar with Case 1, $-\Omega_2^2 > -\Delta$ is obtained using simple calculation. Hence, s_k will converge to QSMD from any negative initial position.

If $s_k \in [-\Delta, 0]$, then

$$\begin{aligned} s_{k+1} &= (1-qT)s_k + \tfrac{\varepsilon T(-s_k)^{1/2}}{\lambda+\gamma e^{\sigma s_k}} + \delta_k \\ &\leq -(1-qT)(-s_k) + \tfrac{\varepsilon T(-s_k)^{1/2}}{\lambda} + \delta \\ &= \tfrac{(\varepsilon T)^2}{4\lambda^2(1-qT)} + \delta\Big|_{s_k=-\frac{(\varepsilon T)^2}{4\lambda^2(1-qT)^2}} \leq \Delta \end{aligned} \tag{14}$$

Similar to Case1, s_k starts from any negative initial position and gradually reaches the QSMD without escaping.

In summary, if the initial state of the system is distant from the sliding mode manifold, then the system will gradually move towards the vicinity of the sliding mode manifold, until it reaches QSMD. In addition, when the system reaches QSMD, the sliding mode manifold will move in QSMD instead of escaping.

Theorem 2: For system(2) satisfying assumption 2, driven by controller(9), the trajectory of the switch function s_k starts from any initial position and reaches QSMD within a finite number of steps η^*, which is expressed as follows:

$$\eta^* = \lfloor n \rfloor + 1 = \left\lfloor \log_{1-qT}\left[\frac{\varepsilon T\varphi - (\lambda+\gamma)\delta}{qT(\lambda+\gamma)\left|s_0\right| + \varepsilon T\varphi - (\lambda+\gamma)\delta}\right]\right\rfloor + 1 \tag{15}$$

where $\lfloor\rfloor$ is the integer down function.

Proof: Under the controller action, the sliding mode switching function can move gradually from the initial state to QSMD, assuming that s_{η^*} does not change the sign at step η^*. Therefore, it will be divided into the following two cases:

$Case1$: If $s_0 > 0$, then,

$$\begin{aligned} s_0 &= s_0 \\ s_1 &= (1-qT)s_0 - \tfrac{\varepsilon T}{\lambda+\gamma e^{-\sigma s_0}}s_0^{1/2} + \delta_0 \\ &\leq (1-qT)s_0 - \tfrac{\varepsilon T}{\lambda+\gamma}s_0^{1/2} + \delta_0 \\ s_2 &= (1-qT)s_1 - \tfrac{\varepsilon T}{\lambda+\gamma e^{-\sigma s_1}}s_1^{1/2} + \delta_1 \\ &\leq (1-qT)^2 s_0 - (1-qT)\tfrac{\varepsilon T}{\lambda+\gamma}s_0^{1/2} + (1-qT)\delta_0 - \tfrac{\varepsilon T}{\lambda+\gamma}s_0^{1/2} + \delta_1 \\ &\vdots \\ s_n &\leq (1-qT)^n s_0 - \sum_{i=0}^{n-1}(1-qT)^{n-1-i}\left(\tfrac{\varepsilon T}{\lambda+\gamma}s_i^{1/2} - \delta\right) \end{aligned} \tag{16}$$

Suppose there is a positive constant $\varphi > \frac{(\lambda+\gamma)\varepsilon qT^2}{4\lambda^2(1-qT)} + \frac{1}{\varepsilon}$ that makes the following equation true

$$\sum_{i=0}^{n-1}(1-qT)^{n-1-i}\left(\frac{\varepsilon T}{\lambda+\gamma}s_i^{1/2} - \delta\right) = \sum_{i=0}^{n-1}(1-qT)^{n-1-i}\left(\frac{\varepsilon T}{\lambda+\gamma}\varphi - \delta\right) \tag{17}$$

Therefore, the Eq.(16) can be rewritten as follows:

$$\begin{aligned} s_n &\leq (1-qT)^n s_0 - \sum_{i=0}^{n-1}(1-qT)^{n-1-i}\left(\frac{\varepsilon T}{\lambda+\gamma}\varphi - \delta\right) \\ &= (1-qT)^n s_0 - \frac{1-(1-qT)^n}{qT}\left(\frac{\varepsilon T}{\lambda+\gamma}\varphi - \delta\right) \end{aligned} \tag{18}$$

Suppose that at step n, the switching function s_k reaches QSMD. Therefore, assuming $s_n > 0, s_{n+1} < 0$. To facilitate calculation that makes

$$(1-qT)^n s_0 - \frac{1-(1-qT)^n}{qT}\left(\frac{\varepsilon T}{\lambda+\gamma}\varphi - \delta\right) = 0 \tag{19}$$

Thus,

$$n = \log_{1-qT}\left[\frac{\varepsilon T\varphi - (\lambda+\gamma)\delta}{qT(\lambda+\gamma)s_0 + \varepsilon T\varphi - (\lambda+\gamma)\delta}\right] \tag{20}$$

$Case2$: If $s_0 < 0$, then,

$$\begin{aligned} s_0 &= s_0 \\ s_1 &= (1-qT)s_0 + \tfrac{\varepsilon T}{\lambda+\gamma e^{-\sigma s_0}}(-s_0)^{1/2} + \delta_0 \\ &\geq (1-qT)s_0 + \tfrac{\varepsilon T}{\lambda+\gamma}(-s_0)^{1/2} + \delta_0 \\ s_2 &= (1-qT)s_1 + \tfrac{\varepsilon T}{\lambda+\gamma e^{-\sigma s_1}}(-s_1)^{1/2} + \delta_1 \\ &\geq (1-qT)^2 s_0 + (1-qT)\tfrac{\varepsilon T}{\lambda+\gamma}(-s_1)^{1/2} + (1-qT)\delta_0 + \tfrac{\varepsilon T}{\lambda+\gamma}s_0^{1/2} + \delta_1 \\ &\vdots \\ s_n &\geq (1-qT)^n s_0 + \sum_{i=0}^{n-1}(1-qT)^{n-1-i}\left(\tfrac{\varepsilon T}{\lambda+\gamma}(-s_i)^{1/2} - \delta\right) \end{aligned} \tag{21}$$

Similar with Eq.(16)-(18) in Case 1, Eq.(21) can be rewritten as follows:

$$\begin{aligned} s_n &\geq (1-qT)^n s_0 + \sum_{i=0}^{n-1}(1-qT)^{n-1-i}\left(\frac{\varepsilon T}{\lambda+\gamma}\varphi - \delta\right) \\ &= (1-qT)^n s_0 + \frac{1-(1-qT)^n}{qT}\left(\frac{\varepsilon T}{\lambda+\gamma}\varphi - \delta\right) \end{aligned} \tag{22}$$

Suppose that at step n, the switching function s_k reaches the QSMD. Therefore, assuming $s_n < 0, s_{n+1} > 0$. In order to facilitate calculation that makes

$$(1-qT)^n s_0 + \frac{1-(1-qT)^n}{qT}\left(\frac{\varepsilon T}{\lambda+\gamma}\varphi - \delta\right) = 0 \tag{23}$$

Thus,

$$n = \log_{1-qT}\left[\frac{\varepsilon T\varphi - (\lambda+\gamma)\delta}{-qT(\lambda+\gamma)s_0 + \varepsilon T\varphi - (\lambda+\gamma)\delta}\right] \tag{24}$$

Combining the two types of cases, whether or not switch function s_k starts from any initial position, it can reach QSMD within a finite number of steps η^*, the expression of η^* is as follows:

$$\eta^* = \lfloor n \rfloor + 1 = \left\lfloor \log_{1-qT}\left[\frac{\varepsilon T\varphi - (\lambda+\gamma)\delta}{qT(\lambda+\gamma)|s_0| + \varepsilon T\varphi - (\lambda+\gamma)\delta}\right]\right\rfloor + 1 \tag{25}$$

Through the preceding analysis and under the action of controller(9), switching function s_k will enter and remain in QSMD. Therefore, the system state will converge to the bounded neighborhood of zero under the action of the discrete sliding mode controller rather than reaching zero.

Theorem 3: Suppose that by adjusting the vector matrix C in the sliding mode switch function s_k to satisfy $\|N\| < 1$, where $N = \left[(I - \Gamma(C\Gamma)^{-1})C\right]\Phi$, and the disturbance satisfied $\|d_k\| < \Omega$, then the trajectory of the state x_k of the sliding mode control system will converge to the interval as follows:

$$\|x_\infty\| \le \varpi(1 - \|N\|)^{-1} \tag{26}$$

where $\varpi = \left\|\Gamma(C\Gamma)^{-1}\right\|\left[(3\|C\|+1)\Omega + (1-qT)\Delta + \frac{\varepsilon T}{\lambda+\gamma}\sqrt{\Delta}\right]$.

Proof: Substituting the controller(9) into the system(2), we can obtain:

$$x_{k+1} = Nx_k + M_k + Y_k \tag{27}$$

where $M_k = \Gamma(C\Gamma)^{-1}\{(1-qT)s_k - \frac{\varepsilon T|s_k|^{1/2}}{\lambda+\gamma}\mathrm{sgn}(s_k)\}$, $Y_k = d_k - \Gamma(C\Gamma)^{-1}(2Cd_{k-1} - Cd_{k-2})$.

Given that d_k is bounded by $\|d_k\| < \Omega$. Therefore:

$$\begin{aligned}\|Y_k\| &\le d_k + \Gamma(C\Gamma)^{-1}(2Cd_{k-1} + Cd_{k-2}) \\ &\le \Omega + \left\|\Gamma(C\Gamma)^{-1}\right\| \|C\| (\|2d_{k-1}\| + \|d_{k-2}\|) \\ &\le (1 + 3\left\|\Gamma(C\Gamma)^{-1}\right\| \|C\|)\Omega\end{aligned} \tag{28}$$

Once the switch function s_k enters QSMD, then

$$\begin{aligned}\|M_k\| &\le \left\|\Gamma(C\Gamma)^{-1}\right\| [(1-qT)\|s_k\| + \frac{\varepsilon T}{\lambda+\gamma}\|s_k\|^{1/2}] \\ &\le \left\|\Gamma(C\Gamma)^{-1}\right\| [(1-qT)\Delta + \frac{\varepsilon T}{\lambda+\gamma}\sqrt{\Delta}]\end{aligned} \tag{29}$$

On the bases of the preceding results, Eq.(29) is further developed as follows:

$$\begin{aligned}\|x_{k+1}\| &= \|Nx_k + M_k + Y_k\| \\ &\le \|A\|\|x_k\| + \|M_k\| + \|Y_k\| \\ &\le \|A\|\|x_k\| + \varpi\end{aligned} \tag{30}$$

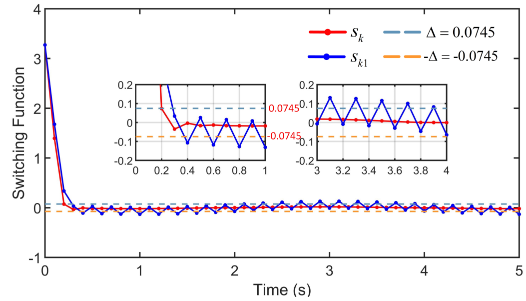

Fig. 1. Trajectory of the switching function in Case 1.

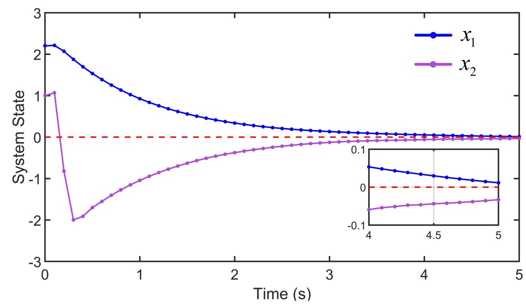

Fig. 2. Trajectories of system states in Case 1.

By iteratively calculating the state after l time constants of the state vector x_k, the inequality is as follows

$$\|x_{k+l}\| \le \|N\|^l \|x_k\| + \varpi \sum_{i=0}^{l-1} \|N\|^{l-1-i} \tag{31}$$

Therefore, if the vector $\|N\| < 1$ is true, then $\|N\|^\infty \to 0$, inequality $\|x_\infty\| \le \varpi(1 - \|N\|)^{-1}$, then the state trajectories of the system will eventually be bounded.

V. Numerical Examples

Considers discrete-time nonlinear system(2) with following parameters[16]:

$$\Phi = \begin{bmatrix} 1.02351 & 0.09139 \\ 0.45696 & 0.84073 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.00470 \\ 0.09139 \end{bmatrix}$$
$$d_k = \begin{bmatrix} 0.10080 & 0.00470 \\ 0.02351 & 0.09139 \end{bmatrix}\begin{bmatrix} 0 \\ 1 + 2.2\cos(0.5\pi KT) \end{bmatrix} \tag{32}$$

to implement numerical simulation. The initial system state is set as $x_0 = \begin{bmatrix} 2.1 & 1 \end{bmatrix}^T$, and the gain coefficient of the switching function is $C = \begin{bmatrix} 1, & 1 \end{bmatrix}$. This study compares the reaching law $s_{k1+1} = (1-qT)s_{k1} - \varepsilon T\mathrm{sgn}(s_{k1}) + (Cd_k - 2Cd_{k-1} + Cd_{k-2})$, and conducts a simulation under different periods to verify the effectiveness of the proposed controller.

1)Case 1: In this case, the sampling period $T = 0.1s$, and the parameters of the discrete sliding mode reaching law are selected as follows: $q = 5.5$, $\varepsilon = 1.1$, $\lambda = 0.5$, $\gamma = 0.1$, and $\sigma = 0.5$. Theorem 2 indicates that the QSMD width is $\Delta_{Caculate} = 0.0745$, and the steps η^* from the initial state to QSMD is $\eta^* = \lfloor 2.7 \rfloor + 1 = 3$. The change of switching function and system state is shown in Figs. 1 and

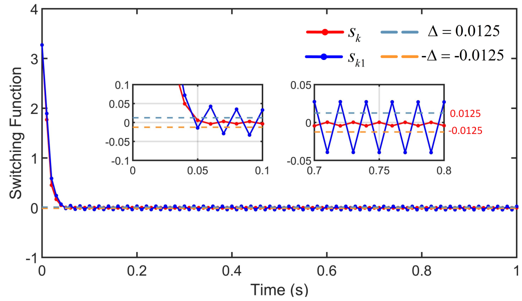

Fig. 3. Trajectory of the switching function in Case 2.

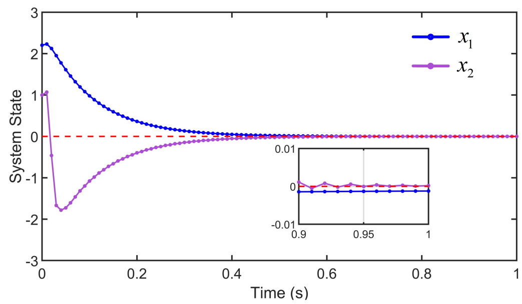

Fig. 4. Trajectories of system states in Case 2.

2, respectively. As shown in Fig.1, the switch function has been moving in QSMD since it first entered. In addition, the width of the QSMD under the proposed reaching law is $\Delta = 0.0701$, and s_{k1} is $\Delta_1 = 0.138$, showing the effectiveness of the proposed reaching law in reducing the QSMD width. By comparing the data in Table 1, note that the calculated values of theorems 1 and 2 are slightly higher than the actual values, thereby verifying the correctness of the theoretical analysis.

2)Case 2: In this case, the sampling period is set as $T = 0.01s$, and the parameters of the reaching law are selected as follows: $q = 50$, $\varepsilon = 5$, $\lambda = 0.5$, $\gamma = 0.1$, and $\sigma = 0.5$. In this case, the QSMD width is $\Delta_{Caculate} = 0.0125$ and convergent steps $\eta^* = \lfloor 5.69 \rfloor + 1 = 6$. The numerical simulation results are shown in Figs. 3 and 4. As shown in Fig.3, the QSMD width for the proposed switching function is $\Delta = 0.005$. In Addition, the QSMD width for s_{k1} is $\Delta_1 = 0.04$ when the convergence speed is the same. Similar with case 1, the calculated convergence steps are greater than the simulation data, which can further prove the rationality of the theoretical analysis. In addition, the state quantity of the system does not eventually converge to zero, but gradually converges to a bounded neighborhood of zero. This result, is consistent with the analysis of theorem 3.

VI. Conclusion

This research develops a discrete-time sliding mode controller and analyzes the dynamic characteristics. A discrete sliding mode reaching law is proposed and discrete sliding mode controller is further designed. The QSMD width is obtained, and the convergent steps required for the system switching variable is analyzed. In addition, the convergence of the system state variables under the action of the controller is proven theoretically. The simulation results show that the proposed discrete sliding mode controller can reach QSMD in a finite time and continue moving in QSMD. In future studies, we will further design a discrete-time sliding mode controller to suppress the QSMD width and improve the convergence rate of the system.

TABLE I
Simulation Result

		Calculated	s_k	s_{k1}
Case 1	QSMD	0.0745	0.0701	0.138
	Step	3	2	3
Case 2	QSMD	0.0125	0.005	0.04
	Step	6	5	5

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