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Stability Criteria for Time-Delay Systems from an Insightful Perspective on the Characteristic Equation

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***Abstract*—This note provides a delay-dependent and a necessary and sufficient delay-independent stability criterion for linear autonomous continuous-time systems with a discrete delay. We take a simple perspective on the two-variable formulation of the characteristic equation, which leads to the following advantageous but not widespread delay-independent criterion: the sum of the coefficient matrix of the delay-free term and the elementwise unitarily rotated matrix of the delay term must remain Hurwitz for all rotation angles. A graphical test for the latter is shown to require no more than three lines of code. Concerning delay-independent stability, our main contribution is to extend this sufficient criterion to a necessary and sufficient one. Concerning delay-dependent stability, the focus is on the critical delay that bounds the initial delay interval of stability. We formulate a constrained minimization problem that gives the exact value of this critical delay. The taken perspective is especially insightful in terms of how the coefficient matrices may look like, and, for scalar systems, the delay-dependent stability chart becomes obvious at a first glance. The presented criteria are complementary to the well-known frequency-sweeping test, which results from another of three possible perspectives on the two-variable criterion. Besides of a unified treatment of these different perspectives, the note also discusses corollaries of the one taken, which include Mori's famous criterion.**

***Index Terms*—Asymptotic stability, delay margin, delay systems, delay-independent stability, spectral abscissa**

I. Introduction

Once measurements, network communication, data processing, or actuator reactions are no longer assumed to take place instantaneously, a delayed term occurs in the system equations. As a consequence, the well-known stability theory of differential equations is no longer applicable. Fortunately, the principle of linearized stability still holds for hyperbolic equilibria in nonlinear systems [1]. That is why we focus on linear equations

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) \tag{1}$$

$A_0,\ A_1 \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, where $h \geq 0$ describes the discrete delay. The present note is concerned with exponential stability (ES) (equivalently, asymptotic stability [2]) of the zero equilibrium in (1). We are interested in delay-independent ES, i.e., exponential stability for all delays $h \geq 0$, and alternatively, in delay-dependent ES with exponential stability for sufficiently small delays $h \in [0, h_c)$. For each explicit value of the delay h, the characteristic equation

$$\det(sI_n - A_0 - \mathrm{e}^{-sh} A_1) = 0 \tag{2}$$

has generically an infinite number of roots $s \in \mathbb{C}$. See, e.g., [3] for numerical root finding. It is well known that the equilibrium of (1) is exponentially stable (ES) for an explicitly given delay h if and only if all these roots s have negative real parts [2]. Proving the latter for all positive delays or for a delay interval is no trivial task.

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In the last decades, various analytical stability criteria have been established, see, e.g., [4]–[8] and references therein. Some criteria [9]–[16] are based on the characteristic quasipolynomial (2) or the corresponding bivariate polynomial that is defined in the so-called two-variable criterion [17]. However, due to the evaluation of the determinant in (2), interpretability in terms of the given matrices A_0 and A_1 is lost in general. Other criteria [18]–[20] are based on eigenvalues of matrix pencils with block matrices containing Kronecker products of A_0 and A_1. These are numerically appealing but also do not make the influence of A_0 and A_1 visible. Frequency sweeping tests involve matrix inverses or generalized eigenvalues [21], [22]. Criteria based on LMIs that result from Lyapunov-Krasovskii functionals, Lyapunov-Razumikhin functions, or robustness theory, are by no means more descriptive [2], [7], [8], [23]. In contrast, Mori's famous criterion [24] provides some insights: A_0 should be Hurwitz and A_1, in some sense, not too large, no matter which sign. However, Mori's criterion is a conservative result for delay-independent stability. Similarly insightful but less conservative is

$$\max_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) < 0, \tag{3}$$

where $\alpha(M) \overset{\mathrm{def}}{=} \max_{k\in\{1,\ldots,n\}} \operatorname{Re}\lambda_k(M)$ denotes the spectral abscissa of $M \in \mathbb{C}^{n\times n}$. Criterion (3), which seems to be little known in this form, is a direct consequence of results by Datko [25, Thm. 1.3] and Kamen [26]. Firstly, it offers a descriptive interpretation: the sum of A_0 and the elementwise unitarily rotated matrix A_1 must remain Hurwitz. Secondly, a numerical test is simple: even for large systems, the spectral abscissa of $M(\varphi) := A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1$ can easily be computed and plotted over the bounded interval $\varphi \in [0, \pi]$. If this scalar continuous function $\varphi \mapsto \alpha(M(\varphi))$ remains smaller than zero, delay-independent ES is proven. Still, although the gap to necessity is small, (3) is only sufficient and not necessary for delay-independent stability.

The present note aims to extend (3) to a necessary and sufficient criterion and, if stability only holds for sufficiently small delays $h \in [0, h_c)$, to provide the bound h_c. The focus is on the simplicity and interpretability of the gained results.

Starting point of our considerations is a fundamental stability result due to Hale et al. [27]. It refers to a two-variable formulation [17] of the characteristic equation (2), which, conceived as a bivariate polynomial, also gave rise to [9]–[16] mentioned above. In contrast to the latter, we introduce a framework of three possible perspectives ($\mathrm{P_S}$), ($\mathrm{P_Z}$), ($\mathrm{P_{SZ}}$) on the two-variable formulation in terms of classical eigenvalue problems. Inequality (3) exploits perspective ($\mathrm{P_S}$), which is related to [18], [25], [28]–[30]. The well-known frequency-sweeping approaches [25, Thm. 1.3], [21], [22], [31] can be recognized as a consequence of perspective ($\mathrm{P_Z}$) instead. Frequency-sweeping is not only associated to a necessary and sufficient delay-independent criterion but also to a formula for the critical delay h_c. Thus, the present note aims to describe a completely analogous theory based on the simpler perspective ($\mathrm{P_S}$). The interrelations will be pointed out clearly.

First, we formulate a constrained minimization problem for the critical delay h_c and provide some meaningful examples. Then we give the necessary and sufficient criterion for delay-independent ES.

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We show that the required extension of (3) only has to incorporate special cases of $\max_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1) = 0$. To this end, we have to prove the following non-trivial consequence of delay-independent ES: eigenvalues of $M(\varphi) := A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1$, as φ increases, cannot move between the left and right complex half-plane only by tunneling through the origin.

Structure. The note is organized as follows. Section II provides the framework of possible perspectives on the two-variable criterion. Section III and IV present the formula for the first delay interval of ES and the criterion for delay-independent ES. Finally, we derive some corollaries including Mori's criterion.

Notation. Eigenvalues of $M \in \mathbb{C}^{n\times n}$ are denoted by $\lambda_k(M)$, $k \in \{1,\dots,n\}$, with arbitrary ordering. Similarly, $\lambda_k(M,E) \in \sigma(M,E) = \{\lambda \in \mathbb{C} : \det(\lambda E - M) = 0\}$ for some $E \in \mathbb{C}^{n\times n}$ refers to eigenvalues of a matrix pencil (M,E). The spectral radius of $M \in \mathbb{C}^{n\times n}$ is $\rho(M) = \max_k |\lambda_k(M)|$ and the spectral abscissa $\alpha(M) = \max_k \operatorname{Re}\lambda_k(M)$. If $\alpha(M) < 0$, the matrix M is said to be Hurwitz. $\mathbb{C} = \mathbb{C}^- \cup \overline{\mathbb{C}^+}$ refers to the open left and closed right complex half-plane. We write $\overline{z}$ and $\overline{M}$ for the complex conjugate of $z \in \mathbb{C}$, $M \in \mathbb{C}^{n\times n}$ and $M^H = \overline{M}^\top$. The identity matrix in $\mathbb{C}^{n\times n}$ is I_n and the zero matrix $0_{n\times n}$. The notation $\|\cdot\|_\nu$ is a norm in $\mathbb{C}^n \ni x$, i.e., $\|x\|_\nu$, or its induced matrix norm $\|M\|_\nu$, $M \in \mathbb{C}^{n\times n}$.

II. Possible Perspectives

Subsequent considerations are based on a reformulation [17] of the characteristic equation (2) with two variables s and $z := \mathrm{e}^{-sh}$

$$\det(sI_n - A_0 - zA_1) = 0. \tag{4}$$

Before discussing possible perspectives on (4), we will introduce various forbidden sets $\mathcal{S}_{\overline{\mathbb{C}^+}}, \mathcal{S}_{\mathrm{i}\mathbb{R}}, \mathcal{S}_0, \mathcal{S}_{\mathrm{str}}$ for s and z to describe some required preliminaries in a uniform manner.

Delay-independent ES holds if and only if for all $h \geq 0$ no root s of (2) occurs in $\overline{\mathbb{C}^+}$, and thus if and only if $\nexists (s,z) \in \mathcal{S}_{\overline{\mathbb{C}^+}} : (4)$, with the forbidden set

$$\mathcal{S}_{\overline{\mathbb{C}^+}} := \{(s,z) \in \mathbb{C}^2 \colon s \in \overline{\mathbb{C}^+}, z = \mathrm{e}^{-sh}, h \in \mathbb{R}_{\geq 0}\}. \tag{5}$$

Assume for a given $h \geq 0$ there is indeed no root s in $\overline{\mathbb{C}^+}$. If h is varied, then the occurrence of a root in $\mathbb{C}^+$ must be preceded by a crossing of the imaginary axis, i.e., $s = \mathrm{i}\omega, \omega \in \mathbb{R}$, at some $h = h_c$ (continuous dependence of the real part of the rightmost root on changes in h is proven in [32]). Hence, already a forbidden set

$$\mathcal{S}_{\mathrm{i}\mathbb{R}} := \{(s,z) \in \mathbb{C}^2 \colon s = \mathrm{i}\omega, z = \mathrm{e}^{-\mathrm{i}\omega h_c}, \omega \in \mathbb{R}, h_c \in \mathbb{R}_{\geq 0}\} \tag{6}$$

is decisive: The equilibrium is delay-independently ES if and only if $\nexists (s,z) \in \mathcal{S}_{\mathrm{i}\mathbb{R}} : (4)$, provided ES is proven for an arbitrary explicit delay, e.g., for zero delay. For zero delay, (1) simplifies to

$$\dot{x}(t) = (A_0 + A_1)x(t) \tag{7}$$

and thus $A_0 + A_1$ is required to be Hurwitz. Furthermore, the range of ω considered in (6) can be restricted to $\omega \in \mathbb{R}\setminus\{0\}$ since

a) $s = \mathrm{i}\omega = 0$ with $h_c < \infty$ (and thus $z = 1$) would be a root for all $h \geq 0$ in (2) and contradicts $A_0 + A_1$ Hurwitz;
b) $s \to \mathrm{i}\omega = 0$ as $h \to \infty$ is beyond the definition of delay-independent ES which considers only finite delays.

That is why a new parameter $\varphi \in \mathbb{R}$ can be introduced that substitutes the free parameter $h_c \geq 0$ by

$$h_c = -\frac{\varphi}{\omega}, \quad \omega \neq 0, \tag{8}$$

which will be decisive in Section III. Hence, the variable z in (6),

$$z = \mathrm{e}^{-\mathrm{i}\omega h_c} = \mathrm{e}^{\mathrm{i}\varphi}, \quad \varphi \in \mathbb{R}, \tag{9}$$

becomes an arbitrary complex number with $|z| = 1$. This decoupling of s and z leads to the following two-variable criterion.

Theorem 2.1 (Hale, Infante and Tsen [27, Thm. 2.4]): The equilibrium of (1) is delay-independently ES if and only if $A_0 + A_1$ is Hurwitz and $\nexists (s,z) \in \mathcal{S}_0 : (4)$ with

$$\mathcal{S}_0 := \{(s,z) \in \mathbb{C}^2 \colon s = \mathrm{i}\omega, |z| = 1, \omega \in \mathbb{R}\setminus\{0\}\}. \tag{10}$$

◀

In contrast to the classical two-variable criterion by Kamen [17], $s = \mathrm{i}\omega = 0$ is not part of the forbidden set $\mathcal{S}_0$. Systems that fulfill $A_0 + A_1$ Hurwitz and even $\nexists (s,z) \in \mathcal{S}_{\mathrm{str}} : (4)$ with

$$\mathcal{S}_{\mathrm{str}} := \{(s,z) \in \mathbb{C}^2 \colon s = \mathrm{i}\omega, |z| = 1, \omega \in \mathbb{R}\}, \tag{11}$$

where $s = \mathrm{i}\omega = 0$ is included, are referred to as *strongly* delay-independently ES [33].

Example 2.1 (Non-strong delay-independent ES): The zero equilibrium of $\dot{x}(t) = -x(t) - x(t-h)$ is, despite of being delay-independently ES, not strongly delay-independently ES. The decisive element turns out to be $(s_0, z_0) = (0, -1)$, which satisfies (4) with $s + 1 + z = 0$. It hampers strong delay-independent ES by $(s_0, z_0) \in \mathcal{S}_{\mathrm{str}}$, while not belonging to the forbidden set $\mathcal{S}_0$ for delay-independent ES. Indeed, although $s_0 = 0$ is an element of the imaginary axis and $|z_0| = 1$, it holds $(s_0, z_0) \notin \mathcal{S}_{\mathrm{i}\mathbb{R}}$ in (6) since there is no finite h_c such that $\mathrm{e}^{-\mathrm{i}\omega h_c} = z_0$ when $\omega = 0$. ◀

In order to determine whether any forbidden $(s,z) \in \mathcal{S}_0$ satisfies (4), several perspectives on this two-variable formulation in terms of classical eigenvalue problems are appropriate.

($\mathrm{P_S}$) For any given z, the variable s in (4) can be seen as an eigenvalue of the matrix $A_0 + zA_1$, cf. [28]. In $\mathcal{S}_0$, or whenever (9) is used, values $z = \mathrm{e}^{\mathrm{i}\varphi}$ are of interest. Hence, we define

$$s_k(\varphi) := \lambda_k(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1), \quad \varphi \in \mathbb{R}, \quad k \in \{1,\dots,n\}. \tag{12}$$

Thm. 2.1 requires that no $(s_k(\varphi), \mathrm{e}^{\mathrm{i}\varphi})$ belongs to $\mathcal{S}_0$, i.e., $s_k(\varphi)$ must satisfy

$$\nexists (\varphi,k) : \operatorname{Re}(s_k(\varphi)) = 0 \text{ with } \operatorname{Im}(s_k(\varphi)) \neq 0. \tag{13}$$

($\mathrm{P_Z}$) For any given s, the variable z in (4) can be seen as an eigenvalue of the matrix pencil $(sI_n - A_0,\ A_1)$, cf. [22]. In $\mathcal{S}_0$, values $s = \mathrm{i}\omega \neq 0$ are of interest. Hence, we define[1]

$$z_k(\omega) := \lambda_k(\mathrm{i}\omega I_n - A_0,\ A_1), \quad \omega \in \mathbb{R}\setminus\{0\}, \quad k \in \{1,\dots,n\}. \tag{14}$$

Thm. 2.1 requires $(\mathrm{i}\omega, z_k(\omega)) \notin \mathcal{S}_0$, i.e., $z_k(\omega)$ must satisfy

$$\nexists (\omega,k) : |z_k(\omega)| = 1. \tag{15}$$

Alternatively, $\frac{1}{z}$ is seen as an eigenvalue of the dual pencil $(A_1,\ sI_n - A_0)$. If A_0 is Hurwitz, $sI_n - A_0$ in the latter is invertible for $s = \mathrm{i}\omega$, and thus (14) also results from

$$\frac{1}{z_k(\omega)} = \lambda_k\Big((\mathrm{i}\omega I_n - A_0)^{-1}A_1\Big) \tag{16}$$

(provided k in (14) and (16) is chosen correspondingly).

($\mathrm{P_{SZ}}$) Both s and z can be seen as eigenvalues, cf. [34]. To this end,

$$(sI_n - A_0 - zA_1)v = 0_{n\times 1}, \quad s,z \in \mathbb{C}, v \in \mathbb{C}^n \tag{17}$$

must be complemented by a second equation, incorporating that $\overline{s} = \overline{\mathrm{i}\omega} = -s$ and $\overline{z} = \overline{\mathrm{e}^{-\mathrm{i}\varphi}} = \frac{1}{z}$ are characteristic properties in $\mathcal{S}_0$, cf. [12], [14]. The conjugate complex of (17)

[1] Some of the n eigenvalues can be infinite. Since their number n_∞ (algebraic multiplicity) equals the dimension of the nilpotent matrix N in Weierstrass' canonical form with $\mathrm{rk}(N) \geq 0$, it holds $n_\infty \geq n - \mathrm{rk}(A_1)$.

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with $w := \overline{v}$ serves this purpose. A quadratic two-parameter eigenvalue problem in s and z emerges [34]

$$(A_0 - sI_n + zA_1)v = 0_{n\times 1} \tag{18a}$$
$$(A_1 + zA_0 + szI_n)w = 0_{n\times 1} \tag{18b}$$

with solution tuples (s_k, z_k), $k \in \{1, \dots, 2n^2\}$. Thm. 2.1 requires that $(s_k, z_k) \notin \mathcal{S}_0$, i.e., s_k and z_k must satisfy

$$\nexists k : \mathrm{Re}(s_k) = 0, \mathrm{Im}(s_k) \neq 0 \text{ and } |z_k| = 1. \tag{19}$$

If elements (s, z) in the forbidden set $\mathcal{S}_0$ have been identified, no matter by which perspective, stability is only delay-dependent. Then, based on the corresponding values of φ in $z = \mathrm{e}^{\mathrm{i}\varphi}$ and ω in $s = \mathrm{i}\omega$, critical delays at which roots on the imaginary axis occur can be concluded from (8), cf. [14], [18], [22]. Provided ES holds for the system with zero delay (7), the initial exponential stability gets lost at the smallest of these critical delays, which thus bounds the exponentially stable initial delay interval $[0, h_c)$.

III. Delay-Dependent Stability

The main result of this section, Thm. 3.2, gives a constrained minimization problem based on $(\mathrm{P_S})$ for the first critical delay h_c. The result is complementary to the $(\mathrm{P_Z})$-based frequency sweeping approach [4, Thm. 2.2], which can be reformulated into a constrained optimization form as follows. Note that (20) describes the minimum[2] positive value of (8) for (ω, k)-pairs in $(\mathrm{P_Z})$ that hamper (15).

Theorem 3.1 (Delay interval of ES by $(\mathrm{P_Z})$, cf. [4, Thm. 2.2]): Based on (14) define

$$h_c := \inf_{(\omega,k)\in(0,\infty)\times\{1,\dots,n\}} \left(-\frac{\arg^- z_k(\omega)}{\omega} \right) \quad \text{subject to} \quad |z_k(\omega)| = 1 \tag{20}$$

with $\arg^- z := \varphi \in (-2\pi, 0]$ such that $z = |z|\mathrm{e}^{\mathrm{i}\varphi}$. If $A_0 + A_1$ is Hurwitz and $h_c < \infty$, then the equilibrium of (1) is exponentially stable for $h \in [0, h_c)$ and not exponentially stable at $h = h_c$. ◀

Similarly, we will describe the minimum positive value of (8) for (φ, k)-pairs in $(\mathrm{P_S})$ that hamper (13).

A disadvantage of (20) is that $z_k(\omega)$ is considered on an unbounded set $\omega \in (0, \infty)$. In contrast, for perspective $(\mathrm{P_S})$, the evaluation of eigenvalues $s_k(\varphi)$, (12), can be restricted to the bounded set $\varphi \in [0, \pi]$. Values of $s_k(\varphi)$ on the whole domain $\varphi \in (-\infty, \infty)$ are still needed to gain all critical delays from (8), but they can be reconstructed from those on $[0, \pi]$ due to symmetry.

Note that the objective function in (22) below is nothing more than

$$h(\varphi, k) := \begin{cases} -\frac{\varphi}{\mathrm{Im}\, s_k(\varphi)}, & \text{if } \mathrm{Im}\, s_k(\varphi) < 0 \\ \infty, & \text{if } \mathrm{Im}\, s_k(\varphi) = 0 \\ \frac{2\pi - \varphi}{\mathrm{Im}\, s_k(\varphi)}, & \text{if } \mathrm{Im}\, s_k(\varphi) > 0, \end{cases} \tag{21}$$

which we write for the sake of compactness with the modulo operation, i.e., $(\pm\varphi \mod 2\pi) \in [0, 2\pi)$.

Theorem 3.2 (Delay interval of ES): Based on the eigenvalues $s_k(\varphi) := \lambda_k(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)$ define

$$h_c := \inf_{(\varphi,k)\in(0,\pi]\times\{1,\dots,n\}} \frac{\mathrm{sgn}\big(-\mathrm{Im}\, s_k(\varphi)\big)\varphi \mod 2\pi}{|\mathrm{Im}\, s_k(\varphi)|} \quad \text{subject to} \quad \mathrm{Re}\, s_k(\varphi) = 0 \tag{22}$$

with $\inf \emptyset = \infty$ and $\frac{\mathrm{sgn}(0)\varphi \mod 2\pi}{0} := \infty$.
If $A_0 + A_1$ is Hurwitz and $h_c < \infty$, then the equilibrium of (1) is exponentially stable for $h \in [0, h_c)$ and not exponentially stable at $h = h_c$. The equilibrium of (1) is delay-independently exponentially stable if and only if $A_0 + A_1$ is Hurwitz and $h_c = \infty$. ◀

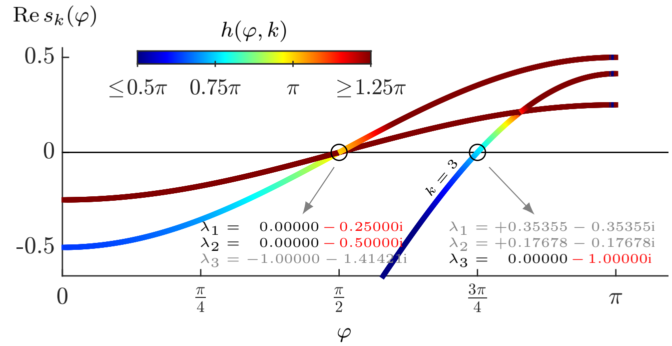

Fig. 1: Example 3.1. Zeros of $\mathrm{Re}\, s_k(\varphi) = \mathrm{Re}\, \lambda_k(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)$, $k \in \{1, 2, 3\}$, form the constraint set in Thm. 3.2 (circles). At these (φ, k), the smallest $h(\varphi, k) := \frac{\mathrm{sgn}\left(-\mathrm{Im}\, s_k(\varphi)\right)\varphi \mod 2\pi}{|\mathrm{Im}\, s_k(\varphi)|}$ is $h_c = h(\frac{3\pi}{4}, 3) = \frac{3}{4}\pi$. Thus, stability holds for delays $h \in [0, \frac{3}{4}\pi)$.

Proof: Due to (7), $A_0 + A_1$ being Hurwitz is necessary. As defined in (8), critical delays can be expressed by $h_c = -\frac{\varphi}{\omega}$, where $\omega = \mathrm{Im}\, s_k(\varphi)$ in perspective $(\mathrm{P_S})$. According to Section II, critical delays occur for any $(\varphi, k) \in \mathbb{R} \times \{1, \dots, n\}$ with $\mathrm{Re}\, s_k(\varphi) = 0$ and $\mathrm{Im}\, s_k(\varphi) \neq 0$. Hence, the minimum positive critical delay is

$$\begin{aligned} h_c = \inf h_0(\varphi, k) \quad & \text{with } h_0(\varphi, k) := -\frac{\varphi}{\mathrm{Im}\, s_k(\varphi)} \\ \text{subject to} \quad & \mathrm{Re}\, s_k(\varphi) = 0 \text{ and } \mathrm{Im}\, s_k(\varphi) \neq 0 \\ & h_0(\varphi, k) > 0 \\ & \varphi \in (-\infty, \infty) \text{ and } k \in \{1, \dots, n\}. \end{aligned}$$

The requirement $\mathrm{Im}\, s_k(\varphi) \neq 0$ can be dropped since the corresponding objective function value, which is ∞ by definition, can only be optimal if there is no other element in the constraint set, while an empty constraint set yields the same result $h_c = \inf \emptyset = \infty$. We have to show that only $\varphi \in (0, \pi]$ instead of $\varphi \in (-\infty, \infty)$ is relevant. Only $\varphi \geq 0$ must be considered because

$$s_k(-\varphi) = \lambda_k\Big(\overline{A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1}\Big) = \overline{\lambda_{\tilde{k}}(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)} = \overline{s_{\tilde{k}}(\varphi)} \tag{23}$$

holds for some $\tilde{k} \in \{1, \dots, n\}$, and thus $h_0(-\varphi, k) = h_0(\varphi, \tilde{k})$. Furthermore, only $\varphi \in [0, 2\pi)$ can lead to an optimum since $s_k(\varphi + 2l\pi) = s_k(\varphi)$, $l \in \mathbb{Z}$, implies $h_0(\varphi + 2l\pi, k) > h_0(\varphi, k)$. Additionally, $\varphi \neq 0$ since $A_0 + A_1$ Hurwitz implies $\mathrm{Re}\, s_k(0) \neq 0$. Hence, in a first step, only $\varphi \in (0, 2\pi)$ is relevant. Positivity $h_0(\varphi, k) > 0$ is achieved iff $\mathrm{Im}\, s_k(\varphi) < 0$, which gives the first case in (21) and allows to drop $h_0(\varphi, k) > 0$ from the constraints. In a second step, the domain can be restricted to $(0, \pi]$ by considering for any $\varphi \in [\pi, 2\pi)$ the corresponding $\tilde{\varphi} \in (0, \pi]$ with $\varphi = 2\pi - \tilde{\varphi}$. Since $s_k(\varphi) = \overline{s_{\tilde{k}}(\tilde{\varphi})}$, it holds $h_0(\varphi, k) = -\frac{\varphi}{\mathrm{Im}\, s_k(\varphi)} = -\frac{2\pi - \tilde{\varphi}}{(-\mathrm{Im}\, s_{\tilde{k}}(\tilde{\varphi}))}$, which gives the third case in (21) for $\mathrm{Im}\, s_{\tilde{k}}(\tilde{\varphi}) > 0$. ■

The constraint set in (22)

$$C := \{(\varphi, k) : \mathrm{Re}(s_k(\varphi)) = 0\} \tag{24}$$

contains zeros of the real parts from all n eigenvalue functions $\varphi \mapsto \mathrm{Re}\, s_k(\varphi)$, $k \in \{1 \dots, n\}$. The next example demonstrates that a restriction to zeros of the spectral abscissa function $\varphi \mapsto \max_k \mathrm{Re}\, s_k(\varphi)$ is indeed not possible. Moreover, Fig. 1 provides a simple graphical evaluation of Thm. 3.2.

[2] In (20) and (22), the infimum is only required to cope with an empty constraint set and can otherwise be replaced by a minimum operator.

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Example 3.1 (Relevance of all eigenvalues in Thm. 3.2): Consider

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix} x(t-h).$$

The eigenvalues required in Thm. 3.2 are $s_1(\varphi) = -\frac{1}{4}\mathrm{e}^{\mathrm{i}\varphi}$, $s_2(\varphi) = -\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi}$, $s_3(\varphi) = -1-\sqrt{2}\mathrm{e}^{\mathrm{i}\varphi}$. Fig. 1 visualizes that the constraint set consists of three elements $C = \{(\frac{\pi}{2},1),(\frac{\pi}{2},2),(\frac{3\pi}{4},3)\}$. At these (φ,k)-pairs, the objective function of (22), cf. color bar, takes the values $\{2\pi,\pi,\frac{3}{4}\pi\} =: H$. Hence, $h_c = \min H = \frac{3}{4}\pi$. ◀

Remark 3.1 (Numerical determination of φ): The graphical evaluation of the optimization problem already provides a rough result for the critical delay with little effort. To evaluate the optimum (22) precisely, the values of φ in (24) can, e.g., be determined as minima of $\varphi \mapsto \min_{k\in\{1,\dots,n\}} |\operatorname{Re} s_k(\varphi)|$. To this end, local minima from a pointwise evaluation can be refined by `fmindnd` in MATLAB with tightened tolerances. Alternatively, the values of φ or $2\pi-\varphi$ are derived by perspective ($\mathrm{P_{SZ}}$) as arguments of $z = \mathrm{e}^{\mathrm{i}\varphi}$ if a routine like `quad_twopareig` [35], [36] is available to compute (s_k, z_k). The proof of [18, Thm. 3.1] reveals that another way is to search for valid values of z in the spectrum of a $2n^2$-dimensional matrix pencil $\sigma\left(\begin{bmatrix} 0_{n^2\times n^2} & I_{n^2} \\ -I_n\otimes A_1^\top & -A_0\oplus A_0^\top \end{bmatrix}, \begin{bmatrix} I_{n^2} & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & A_1\otimes I_n \end{bmatrix}\right)$, where $\otimes,\oplus$ denote Kronecker product and sum. ◀

The system in the following example is a frequently used benchmark in the context of LMI-based criteria [37], [38].

Example 3.2 (Analytically determined h_c): The equilibrium of

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h)$$

is ES for $h < h_c := \frac{\arccos(-0.9)}{\sqrt{1-0.9^2}} \approx 6.17258$ and not ES at $h = h_c$.

Reasoning: $A_0 + A_1$ is Hurwitz as required in Thm. 3.2. The eigenvalues $s_k(\varphi) := \lambda_k(M(\varphi))$ of the triangular matrix $M(\varphi) := A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1$ are obvious. Since $\operatorname{Re} s_2(\varphi) = -2-\cos\varphi \neq 0$ and $\operatorname{Re} s_1(\varphi) = -0.9-\cos\varphi$, the constraint set (24) consists of the single point $(\varphi_c,k_c) := (\arccos(-0.9),1)$ with $s_{k_c}(\varphi_c) = -\mathrm{i}\sin\varphi_c = -\mathrm{i}\sqrt{1-0.9^2}$. Since $\operatorname{Im}(s_{k_c}(\varphi_c)) < 0$, Thm. 3.2 yields $h_c = \frac{\varphi_c}{|\operatorname{Im} s_{k_c}(\varphi_c)|} = \frac{\arccos(-0.9)}{\sqrt{1-0.9^2}}$. ◀

In fact, the triangular structure of Example 3.2 allows the stability analysis to be reduced to an analysis of scalar systems (so-called Hayes equations [39]). The following example shows that perspective ($\mathrm{P_S}$) is particularly insightful in this case. The results are well known [2, p. 135 / Fig. 5.1], but the usual derivations are quite laborious.

Example 3.3 (The scalar case): Consider (1) with $n = 1$, i.e., $\dot{x}(t) = a_0x(t) + a_1x(t-h)$, $a_0,a_1 \in \mathbb{R}$. The Hurwitz condition[3]

$$a_0 + a_1 < 0 \tag{25}$$

forms the non-red open region in Fig. 2. Thm. 3.2 only depends on $s_1(\varphi) = a_0+\mathrm{e}^{\mathrm{i}\varphi}a_1$ for $\varphi\in[0,\pi]$. In the complex plane the latter can be visualized as a 180° rotation around a_0 from a_0+a_1 to a_0-a_1, cf. the grey semicircle in Fig. 2 with $(a_0,a_1) = (-3,0.5)$. Depending on the end point a_0-a_1, three cases have to be distinguished:
a) For (25) combined with $a_0-a_1 < 0$, i.e., $a_0+|a_1| < 0$, the constraint set in (22) is empty, and thus $h_c = \infty$ (dark-green triangle in the stability chart, strong delay-independent ES).
b) For (25) combined with $a_0-a_1 = 0$, i.e., $a_0 = a_1 < 0$, the denominator of the objective function in (22) is zero, and thus

[3] ES cannot be gained by increasing the delay in the scalar system [40]. Hence, $a_0+a_1<0$, which ensures that ES holds for $h=0$, is not only necessary for ES $\forall h\geq 0$ or ES $\forall h\in[0,h_c)$, but even necessary for ES at some $h\in\mathbb{R}_{\geq 0}$.

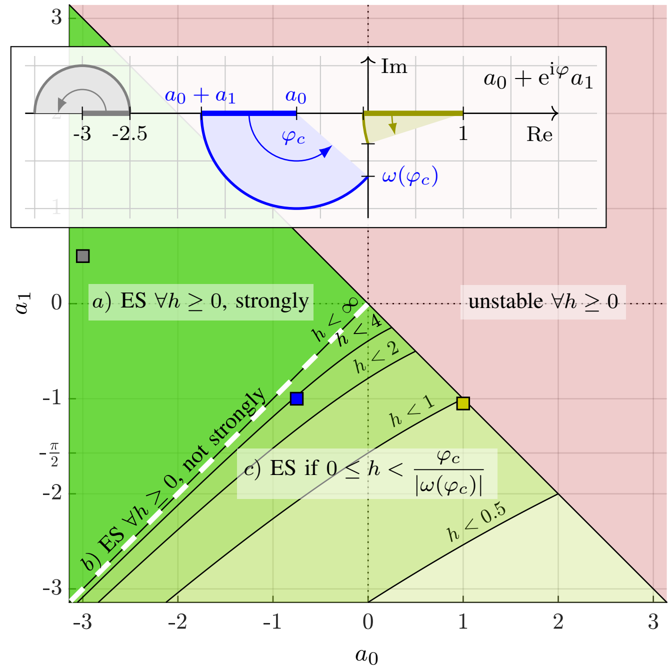

Fig. 2: Example 3.3. Stability chart for scalar systems. The white box above gives for highlighted points an evaluation of $\varphi\mapsto a_0+\mathrm{e}^{\mathrm{i}\varphi}a_1$ in the complex plane. The grey point $(a_0,a_1) = (-3,0.5)$ provides delay-independent ES, whereas $(-0.75,-1)$ in blue gives ES for $h<3.66$ and $(1,-1.05)$ in ochre gives ES for $h<0.97$.

$h_c = \infty$ (white dashed line in the stability chart, non-strong delay-independent ES).
c) For (25) combined with $a_0-a_1>0$, consider in Fig. 2 the blue and ochre arcs. Thm. 3.2 yields $h_c = \frac{\varphi_c}{|\omega(\varphi_c)|}$ as the quotient of the rotation angle $\varphi_c\in(0,\pi)$ at which the imaginary axis is met and the corresponding section of the imaginary axis $|\omega(\varphi_c)|$. That is

$$\begin{aligned} h_c &= \frac{\varphi_c}{|a_1|\sin\varphi_c} \quad \text{with } \varphi_c = \arccos(-\tfrac{a_0}{a_1}) \\ &= \frac{\arccos(-\frac{a_0}{a_1})}{\sqrt{a_1^2-a_0^2}}. \end{aligned} \tag{26}$$

**) If (25) is satisfied, ES holds at least for sufficiently small delays (differently shaded green areas in Fig. 2). Given a fixed delay $h>0$, the exponentially stable region in the (a_0,a_1)-parameter plane is derived by solving $h<h_c$ in (26) for $|a_1| = -a_1 < \frac{\varphi_c}{h\sin\varphi_c}$ and $a_0 = -a_1\cos(\varphi_c) < \frac{\varphi_c}{h}\cot(\varphi_c)$. Its boundary $\{[a_0,a_1]^\top = \frac{\varphi_c}{h\sin\varphi_c}[\cos\varphi_c,-1]^\top : \varphi_c\in(0,\pi)\}$ for $h=h_c\in\{0.5,1,2,4\}$ is shown as boundary of the $(h<h_c)$ regions in Fig. 2.
***)* Provided $a_0<0$, the critical delay is always larger than the critical delay with $a_0=0$ since the blue arc reveals that a larger angle $\varphi_c>\frac{\pi}{2}$ is combined with a smaller $|\omega(\varphi_c)|<|a_1|$. Hence, $h_c>\frac{\pi}{2|a_1|}$ if $a_0<0$ and $a_1<0$. ◀

IV. Delay-Independent Stability

The main result in this section, stated in Thm. 4.2, provides a necessary and sufficient delay-independent stability criterion based on ($\mathrm{P_S}$). It aims to be complementary to the following well-known ($\mathrm{P_Z}$)-based criterion, where (27) ensures that $z_k(\omega)$ in (16) satisfies (15).

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Theorem 4.1 (Frequency sweeping [4, Thm. 2.1]): Delay-independent ES holds iff A_0 and $A_0 + A_1$ are Hurwitz and

$$\forall \omega > 0: \quad \rho\big((\mathrm{i}\omega I_n - A_0)^{-1} A_1\big) < 1. \tag{27}$$

◀

A graphical evaluation of the spectral radius (27) over $\omega \in (0, \infty)$ is proposed in [4], [21]. Similarly, we are concerned with an evaluation of the spectral abscissa

$$\alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) = \max_{k\in\{1,\ldots,n\}} \operatorname{Re} s_k(\varphi) \tag{28}$$

over $\varphi \in [0, \pi]$ to ensure that $s_k(\varphi)$ in (12) satisfies (13).

According to Thm. 3.2, there are two scenarios that lead to delay-independent ES:
(i) $A_0 + A_1$ is Hurwitz and the constraint set of (22) is empty, i.e., for all $\varphi \in [0, \pi]$ no eigenvalue $s_k(\varphi)$ occurs on the imaginary axis;
(ii) $A_0 + A_1$ is Hurwitz and there is no other denominator in (22) than zero, i.e., eigenvalues on the imaginary axis occur for some (φ, k), but these are exclusively located at the origin.

Case (i) describes strong delay-independent ES (11). It will be addressed by $\alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) < 0$ for all $\varphi \in [0, \pi]$, cf. (3). Case (ii) is the special case of non-strong delay-independent ES. Obviously, it can occur with $\max_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) = 0$, which becomes visible in Example 3.3 b). Whether case (ii) can also be accompanied by $\max_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) > 0$ is not that obvious. Starting in $\mathbb{C}^-$ for $\varphi = 0$, the n eigenvalues $s_k(\varphi)$, $k \in \{1, \ldots, n\}$, move continuously in the complex plane as φ increases. Case (i) bans eigenvalues from the imaginary axis, and thus they cannot reach the right half-plane. However, case (ii) describes a gap in the imaginary axis: the occurrence of eigenvalues at the origin does not hamper delay-independent ES. Thus, the question arises whether eigenvalues can move from the left half-plane to the right half-plane only by tunneling through the origin. This question is motivated further in the following remark.

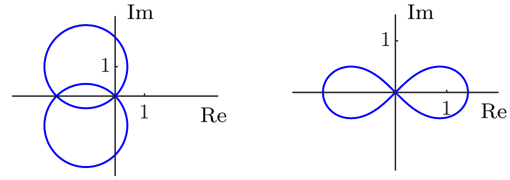
Remark 4.1 (Crossing of the origin): In contrast to the roots[4] of (2), the eigenvalues $s_k(\varphi)$ in (12) can move from $\mathbb{C}^-$ through the origin to $\mathbb{C}^+$ as φ increases. Such an example is provided in Fig. 3a, which shows for $A_0 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, $A_1 = \sqrt{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ the union of eigenvalue paths[5] $\{s_k(\varphi) : \varphi \in \mathbb{R}, k \in \{1, 2\}\}$. However, since the movement back to $\mathbb{C}^-$ in this example is not through the origin, $h_c = \infty$ does not result as a minimum in Thm. 3.2. In Fig. 3b there is indeed no other crossing point of the imaginary axis than the origin. Thus, Thm. 3.2 would lead to the conclusion of delay-independent ES, provided $A_0 + A_1$ was Hurwitz. Fig. 3b can, e.g., be realized by $A_0 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ or by $A_0 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, but both are not fulfilling $A_0 + A_1$ Hurwitz, although the latter example with $\alpha(A_0 + A_1) = 0$ is very close. ◀

The proof of Thm. 4.2 will show that case (ii) indeed cannot be accompanied by zero crossings in the manner of Fig. 3b. Thus, if there is a $\varphi \in [0, \pi]$ with $\alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) > 0$, non-delay-independent ES is proven immediately. To this end, we need the following lemma.

Lemma 4.1 ([27, Corollary 2.7]): If Thm. 2.1 holds, then A_0 is Hurwitz. ◀

[4]Because of case a) in Section II, roots s of the characteristic quasi-polynomial (2) cannot move through the origin in the complex plane as h increases. Note that the only relation between $M(\varphi) = A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1$ and the delay equation (1) is that non-zero purely imaginary eigenvalues $s_k(\varphi) = \mathrm{i}\omega \neq 0$ of $M(\varphi)$ at some φ coincide with non-zero purely imaginary roots s of (2) at some h.

[5]An evaluation of the union of eigenvalue paths is proposed in [29], where, however, zero crossings in the manner of Fig. 3b are not taken into account.



(a) (b)

Fig. 3: Remark 4.1. Eigenvalues $s_k(\varphi)$ can cross the origin.

Theorem 4.2 (Delay-independent ES): The zero equilibrium of (1) is delay-independently exponentially stable if and only if either

$$(i) \quad \max_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) < 0 \tag{29}$$

$$\text{or} \quad (ii) \quad \alpha(A_0 + A_1) < 0, \quad \max_{\varphi\in(0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) = 0,$$
$$\text{and} \quad \nexists(\varphi_c, k) : \operatorname{Re} s_k(\varphi_c) = 0,\ \operatorname{Im} s_k(\varphi_c) \neq 0,$$

where $\varphi_c \in \operatorname{argmax}_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1)$ and $s_k(\varphi) := \lambda_k(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1)$, $k \in \{1, \ldots, n\}$. It is strongly delay-independently ES, cf. (11), if and only if (i) holds. ◀

Proof: According to Thm. 3.2, $h_c = \infty$ occurs iff either

$$(i') \ A_0 + A_1 \text{ is Hurwitz and } \ \nexists(\varphi, k) : \operatorname{Re} s_k(\varphi) = 0 \quad \text{or}$$
$$(ii') \ A_0 + A_1 \text{ is Hurwitz, } \exists(\varphi, k) : s_k(\varphi) = 0,$$
$$\text{and } \nexists(\varphi, k) : s_k(\varphi) = \mathrm{i}\omega, \omega \in \mathbb{R} \setminus \{0\}.$$

In the following, we will infer sufficiency and necessity of (i) and (ii) in Thm. 4.2 for (i') and (ii'). Unless otherwise stated, consider $\varphi \in [0, \pi]$, $k \in \{1, \ldots, n\}$, $s \in \mathbb{C}$, $\omega \in \mathbb{R}$, $M(\varphi) := A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1$.
$(i) \Rightarrow (i')$: For $\varphi = 0$, (i) yields $\alpha(M(0)) = \alpha(A_0 + A_1) < 0$, i.e., $A_0 + A_1$ must be Hurwitz. Furthermore, since (i) requires $\max_{\varphi\in[0,\pi]} \alpha(M(\varphi)) = \max_{\varphi\in[0,\pi]} \max_{k\in\{1,\ldots,n\}} \operatorname{Re} s_k(\varphi) < 0$, it holds that $\nexists(\varphi, k) : \operatorname{Re} s_k(\varphi) = 0$.
$(i') \Rightarrow (i)$: Continuity of $\varphi \mapsto \alpha(M(\varphi))$ and $\alpha(A_0 + A_1) = \alpha(M(0)) < 0$, while $\nexists\varphi : \alpha(M(\varphi)) = 0$, implies $\alpha(M(\varphi)) < 0, \forall\varphi$.
$(ii) \Leftrightarrow (ii')$: Both, (ii) and (ii') consist of three requirements. For the first one ($A_0 + A_1$ Hurwitz) as well as the third one (nonexistence of non-zero purely imaginary roots) equivalence between (ii) and (ii') is obvious. The second combined with the third one in (ii) implies $\exists(\varphi, k) : s_k(\varphi) = 0$ in (ii'). It remains to show that (ii') implies $\max_{\varphi\in(0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi} A_1) = 0$ by proving that (ii') implies

$$\nexists(\varphi, k) : s_k(\varphi) \in \mathbb{C}^+, \tag{30}$$

cf. the discussion above. To this end, define $s \mapsto \tilde{z}_k(s)$ with $k \in \{1, \ldots, n\}$ as a mapping of $s \in \mathbb{C}$ to a corresponding z in (4), i.e.,

$$\det(sI_n - A_0 - \tilde{z}_k(s)\, A_1) = 0. \tag{31}$$

As in (16), and because of Lemma 4.1, $\tilde{z}_k(s)$ can be determined by

$$\frac{1}{\tilde{z}_k(s)} = \lambda_k\big((sI_n - A_0)^{-1} A_1\big) \text{ for } s \in \overline{\mathbb{C}^+}. \tag{32}$$

Consider at each s the largest absolute value of (32)

$$\frac{1}{\min\limits_{k\in\{1,\ldots,n\}} |\tilde{z}_k(s)|} = \rho\big((sI_n - A_0)^{-1} A_1\big). \tag{33}$$

Due to Lemma 4.1, $s \mapsto \tilde{N}(s) := (sI_n - A_0)^{-1} A_1$ is holomorphic on $\mathbb{C}^+$. For holomorphic $\tilde{N}(s)$ the spectral radius $s \mapsto \rho(\tilde{N}(s)) = \max_{k\in\{1,\ldots,n\}} |\lambda_k(\tilde{N}(s))|$ is a subharmonic function

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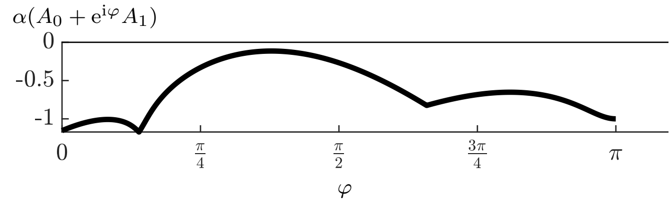

Fig. 4: Example 4.1. Since $\alpha(A_0+\mathrm{e}^{\mathrm{i}\varphi}A_1)$ takes only negative values on $\varphi \in [0,\pi]$, strong delay-independent ES holds.

[41]. Since the domain $\mathbb{C}^+$ is unbounded, the maximum principle for subharmonic functions only applies if the function is bounded above [42, Thm. A.2.28], which, however, is true because $\lim_{s\to\infty}\rho(\tilde{N}(s)) = 0$. Thus, for $s \in \overline{\mathbb{C}^+}$ the maximum of (33) is attained on the imaginary axis $\partial\overline{\mathbb{C}^+} = \mathrm{i}\mathbb{R}$, and, consequently, the minimum of $s \mapsto \min_{k\in\{1,\dots,n\}} |\tilde{z}_k(s)|$ as well. Hence, unless $s \mapsto \min_{k\in\{1,\dots,n\}} |\tilde{z}_k(s)|$ is constant on $\mathbb{C}^+$, the strict inequality

$$\forall (s,k) \text{ with } s\in\mathbb{C}^+ : |\tilde{z}_k(s)| > \min_{\omega\in\mathbb{R}} \min_{k\in\{1,\dots,n\}} |\tilde{z}_k(\mathrm{i}\omega)| \tag{34}$$

holds. Consider the right hand side of (34). On the one hand, because of (33) with $\rho(\tilde{N}(\mathrm{i}\omega)) \to 0$ as $\omega \to \pm\infty$, it holds

$$\min_{k\in\{1,\dots,n\}} |\tilde{z}_k(\mathrm{i}\omega)| \to \infty \quad \text{as } \omega \to \pm\infty. \tag{35}$$

On the other hand, the second requirement in (ii') yields

$$\begin{aligned} &\exists (\varphi,k) : s_k(\varphi) = 0 \\ \overset{(12)}{\Leftrightarrow}\;\; &\exists \varphi : \det(-A_0 - \mathrm{e}^{\mathrm{i}\varphi}A_1) = 0 \\ \overset{(31)}{\Leftrightarrow}\;\; &\exists k : |\tilde{z}_k(0)| = 1 \end{aligned} \tag{36}$$

for the point $\omega = 0$. Consequently, by continuity, $s \mapsto \min_{k\in\{1,\dots,n\}} |\tilde{z}_k(s)|$ is indeed non-constant and (34) applies. Furthermore, the third requirement in (ii') gives

$$\begin{aligned} &\nexists (\varphi,k) : s_k(\varphi) = \mathrm{i}\omega, \quad \omega \neq 0 \\ \overset{(12)}{\Leftrightarrow}\;\; &\nexists \varphi : \det(\mathrm{i}\omega I_n - A_0 - \mathrm{e}^{\mathrm{i}\varphi}A_1) = 0, \quad \omega\neq 0 \\ \overset{(31)}{\Leftrightarrow}\;\; &\nexists k : |\tilde{z}_k(\mathrm{i}\omega)| = 1 \quad \text{for all } \omega \neq 0. \end{aligned} \tag{37}$$

Continuity of $\omega \mapsto \tilde{z}_k(\mathrm{i}\omega)$ combined with the results for $\omega = 0$, $\omega \in \mathbb{R}\setminus\{0\}$, and $\omega \to \pm\infty$, which are obtained in (36), (37), and (35), leads (similar to [27, Lemma 2.5]) to

$$\min_{\omega\in\mathbb{R}} \min_{k\in\{1,\dots,n\}} |\tilde{z}_k(\mathrm{i}\omega)| = 1$$

for the right hand side in (34). Thus, (34) implies

$$\begin{aligned} &\nexists (s,k) : |\tilde{z}_k(s)| = 1 \text{ with } s\in\mathbb{C}^+ \\ \overset{(31)}{\Leftrightarrow}\;\; &\nexists s \in \mathbb{C}^+ : \det(sI_n - A_0 - \mathrm{e}^{\mathrm{i}\varphi}A_1) = 0,\ \varphi\in(-\pi,\pi] \\ \overset{(12),(23)}{\Leftrightarrow}\;\; &\nexists (\varphi,k) : s_k(\varphi) \in \mathbb{C}^+, \end{aligned}$$

which completes the proof of (30). ■

A main advantage of Thm. 4.2 (i) is its simple implementation.

Example 4.1 (Numerical evaluation of Thm. 4.2 (i)): Thm. 4.2(i) only requires an evaluation of the function $\varphi \mapsto \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)$ over $\varphi \in [0,\pi]$. In MATLAB or GNU Octave, the following lines serve this purpose for an exemplary step size of φ, provided the system matrices have been assigned to `A0` and `A1`.

```
P=0:1e-3:pi;
ALPHA=arrayfun(@(x) ...
   max(real(eig(A0+exp(1i*x)*A1))), P);
plot(P,ALPHA);
```

Exclusively negative values indicate strong delay-independent ES. For $A_0 = \left[\begin{smallmatrix} -1 & 2 & 1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{smallmatrix}\right]$ and $A_1 = \left[\begin{smallmatrix} -1 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & 2 & 2 \end{smallmatrix}\right]$ the result is shown in Fig. 4. Thus, the equilibrium of $\dot{x}(t) = A_0x(t) + A_1x(t-h)$ is exponentially stable for any delay $h \geq 0$. ◀

In the following Examples 4.2a and 4.2b, the maximum of the spectral abscissa function $\varphi \mapsto \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)$ is zero. Hence, part (ii) of Thm. 4.2 must be considered and the eigenvalues $s_k(\varphi_c), k \in \{1,\dots,n\}$, at the maximizers φ_c become decisive. Examples 4.2a and 4.2b lead to identical spectral abscissa functions $\varphi \mapsto \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)$, shown as a black line in Fig. 5. Nevertheless, Example 4.2a is delay-independently ES, whereas 4.2b, with values of $s_k(\varphi_c)$ explicitly listed in Fig. 5, is only delay-dependently ES.

Example 4.2 (Zero as maximum of $\varphi \mapsto \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)$): Consider (1) with $x(t) \in \mathbb{R}^{2p}$, $p \in \mathbb{N}_{>0}$, where the coefficients are given by the block diagonal matrices

$$\begin{aligned} A_0 &= \operatorname{blkdiag}\Big(\left[\begin{smallmatrix} -1 & -\beta \\ \beta & -1 \end{smallmatrix}\right], -I_2, \dots, -I_2 \Big) \\ A_1 &= \operatorname{blkdiag}\big(Q(\vartheta_1),\dots,Q(\vartheta_p)\big), \end{aligned}$$

with $Q(\vartheta) := \left[\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right]$ and $\vartheta_j \in (-\pi,\pi]\setminus\{0\}$, $j \in \{1,\dots,p\}$.

a) With $\beta = 0$, i.e., $A_0 = -I_{2p}$, delay-independent ES holds.

b) With $\beta = 1$, the zero equilibrium is ES for $h < h_c$ and not ES for $h = h_c$,

$$h_c := \begin{cases} -\vartheta_1, & \text{if } \vartheta_1 < 0 \\ 2\pi - \vartheta_1, & \text{if } \vartheta_1 > 0. \end{cases}$$

Reasoning: The $2p$ eigenvalues of $M(\varphi) = A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1$ are

$$\begin{aligned} s_{1,2}(\varphi) &= -1 \pm \mathrm{i}(-\beta) + \mathrm{e}^{\mathrm{i}(\varphi\pm\vartheta_1)} \\ &= -1 + \cos(\varphi_1 \pm \vartheta_1) + \mathrm{i}\big(\mp\beta + \sin(\varphi\pm\vartheta_1)\big) \\ s_k(\varphi) &= -1 + \mathrm{e}^{\mathrm{i}(\varphi\pm\vartheta_j)}, \end{aligned}$$

$k := 2j - \frac{1}{2}(1\pm 1)$, $j \in \{2,\dots,p\}$, and their largest real part is

$$\alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1) = -1 + \max_{j\in\{1,\dots,p\}} \cos(\varphi\pm\vartheta_j)$$

with $\max_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1) = 0$. This maximum on $[0,\pi]$ is attained at

$$\varphi_c \in \Phi := \{|\vartheta_1|,\dots,|\vartheta_p|\}.$$

a) If $\beta = 0$, then at any $\varphi_c \in \Phi$ no non-zero purely imaginary eigenvalues exist since $\operatorname{Im}(s_k(\varphi_c)) = \sin(\varphi_c \pm \vartheta_j) = 0$ when $\cos(\varphi_c \pm \vartheta_j) = 1$. Hence, case (ii) in Thm. 4.2 applies.

b) In contrast, if $\beta = 1$, there is a non-zero purely imaginary eigenvalue at $\varphi_c = |\vartheta_1|$, namely $s_1(-\vartheta_1) = -\mathrm{i}$ if $\vartheta_1 < 0$, cf. Fig. 5, or $s_2(\vartheta_1) = \mathrm{i}$ if $\vartheta_1 > 0$. Hence, Thm. 4.2 (ii) does not hold. Instead, Thm. 3.2 provides the delay interval of ES. Since $\operatorname{Im} s_1(|\vartheta_1|) = -1 < 0$ for $\vartheta_1 < 0$, the modulo operation in (22) is without effect for $\vartheta_1 < 0$, while for $\vartheta_1 > 0$ the numerator of h_c becomes $2\pi - \varphi = 2\pi - |\vartheta_1|$. ◀

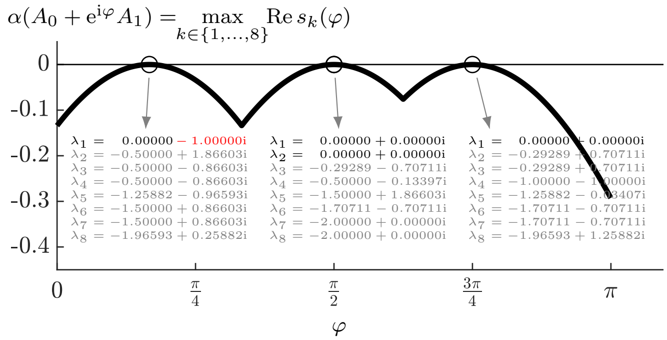

Fig. 5: Example 4.2b with $x(t) \in \mathbb{R}^8$ and $\vartheta_1 = -\pi/6, \vartheta_2 = \vartheta_3 = \pi/2, \vartheta_4 = 3/4\,\pi$. Since a non-zero purely imaginary eigenvalue $s_k(\varphi) = \mathrm{i}\omega \neq 0$ occurs (red), Thm. 4.2 (ii) does not apply. Instead, Thm. 3.2 yields $h_c = \pi/6$.

V. Corollaries

Considerations so far are based on the spectral abscissa. The logarithmic norm is related to the spectral abscissa, but it exhibits advantageous properties allowing further simplifications. Based on these, some known stability criteria can be inferred directly from Thm. 4.2 without elaborate proofs.

The spectral abscissa $\alpha(M)$ of a matrix $M \in \mathbb{C}^{n\times n}$ can be approached as close as desired by a logarithmic norm of M

$$\mu_\nu(M) \overset{\text{def}}{=} \lim_{h\to 0+} \frac{\|I_n + hM\|_\nu - 1}{h}, \tag{38}$$

provided the involved matrix norm $\|\cdot\|_\nu$ is chosen in an optimal way depending on M. To be more precise, $\alpha(M) = \inf_\nu \mu_\nu(M)$ [43]. Thus, Thm. 4.2(i) can equivalently be expressed in this manner. Usually, however, a matrix norm is chosen a priori. For common norms, (38) simplifies to well-known formulas [44, p. 33], e.g., the logarithmic norm w.r.t. the spectral norm $\|\cdot\|_2$ equals the maximum eigenvalue of the Hermitian part of M

$$\mu_2(M) = \lambda_{\max}\big(\tfrac{1}{2}(M + M^H)\big). \tag{39}$$

In any case, inequality (40a) holds.

Lemma 5.1 (Properties of $\mu_\nu(\cdot)$ [44]): Let $M, N \in \mathbb{C}^{n\times n}$. Then

$$\alpha(M) \leq \mu_\nu(M), \tag{40a}$$
$$\mu_\nu(M+N) \leq \mu_\nu(M) + \mu_\nu(N), \tag{40b}$$
$$\mu_\nu(M) \leq \|M\|_\nu. \tag{40c}$$
$$\mu_\nu(\overline{M}) = \mu_\nu(M), \tag{40d}$$
$$\mu_\nu(M) = \sup_{\|x\|_\nu=1} \operatorname{Re}[Mx, x]_\nu \quad \text{[45, Lem. 12]}, \tag{40e}$$

where $[\cdot,\cdot]_\nu$ is a semi-inner product with $[x,x]_\nu = \|x\|_\nu^2$. ◀

Consider the following expressions and obvious relations

$$\begin{aligned}\max_{\varphi\in[0,\pi]} \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1) &\overset{(40a)}{\leq} \max_{\varphi\in[0,\pi]} \mu_\nu(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1) =: \mathcal{B}_\mu \\ &\overset{(40b)}{\leq} \mu_\nu(A_0) + \max_{\varphi\in[0,\pi]} \mu_\nu(\mathrm{e}^{\mathrm{i}\varphi}A_1) =: \tilde{\mathcal{B}}_r \\ &\overset{(40c)}{\leq} \mu_\nu(A_0) + \|A_1\|_\nu =: \mathcal{B}_M,\end{aligned} \tag{41}$$

as well as

$$\mu_\nu(A_0) + r_{\mathrm{num},\nu}(A_1) =: \mathcal{B}_r,$$

with the numerical radius $r_{\mathrm{num},\nu}(A) = \sup_{\|x\|_\nu=1} |[Ax, x]_\nu|$.

Lemma 5.2: The equality $\tilde{\mathcal{B}}_r = \mathcal{B}_r$ holds. ◀

Proof:

$$\begin{aligned}\max_{\varphi\in[0,\pi]} \mu_\nu(\mathrm{e}^{\mathrm{i}\varphi}A_1) &\overset{(40d,e)}{=} \max_{\varphi\in(-\pi,\pi]} \sup_{\|x\|_\nu=1} \operatorname{Re}(\mathrm{e}^{\mathrm{i}\varphi}[A_1x, x]_\nu) \\ &= \sup_{\|x\|_\nu=1} \big|[A_1x, x]_\nu\big| \overset{\text{def}}{=} r_{\mathrm{num},\nu}(A_1).\end{aligned}$$

■

Based on the relations of $\mathcal{B}_M, \mathcal{B}_\mu$, and $\tilde{\mathcal{B}}_r = \mathcal{B}_r$ in (41), we can summarize three immediate corollaries of Thm. 4.2.

Corollary 5.1: If for some norm $\|\cdot\|_\nu$ one of the following inequalities holds, which are ordered by decreasing conservativity,

(I) $\mathcal{B}_M < 0$, (Mori's criterion [24, Thm. 1]),

(II) $\mathcal{B}_r < 0$, (cf. [46, Thm. 2.2]),

(III) $\mathcal{B}_\mu < 0$, (cf. [47, Thm. 1]),

then the equilibrium of (1) is strongly delay-independently ES. ◀

VI. Conclusion

The note introduces a framework of three possible perspectives on the two-variable formulation of the characteristic equation. Based on this framework, we formulate consequent analogues to the well-known delay-dependent and delay-independent frequency-sweeping criteria. The derived criteria focus on eigenvalues of the matrix $M(\varphi) = A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1$. Contrary to the frequency sweeping approach, no generalized eigenvalues or matrix inverses are needed and eigenvalues of $M(\varphi)$ must only be evaluated on the bounded domain $\varphi \in [0, \pi]$.

Exclusively negative values of the spectral abscissa function $\varphi \mapsto \alpha(A_0 + \mathrm{e}^{\mathrm{i}\varphi}A_1)$ indicate delay-independent stability – a test that is shown to require no more than three lines of code. The present note proves that, if positive values of the function occur, delay-independent exponential stability can be excluded immediately. Moreover, the ambiguous case of a zero maximum is discussed in a meaningful example.

If stability is only delay-dependent, the occurring zeros of the spectral abscissa function become part of a constraint set in a proposed minimization problem for the first critical delay. Although these zeros will be decisive in most cases, a counterexample shows that zeros of the remaining eigenvalue real parts cannot be ignored in the constraint set. A rough graphical evaluation of the optimization problem requires only few lines of code more. Furthermore, we demonstrate that, in simple cases, even analytical results for the first critical delay can be achieved. In fact, the delay-dependent stability chart for scalar systems becomes obvious at a first glance. Additionally, the formulation as a constrained optimization problem is particularly insightful when generalized to multiple incommensurate delays, multiple commensurate delays, and perturbations thereof, which, however, is beyond the scope of the present note.

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