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Reference Tracking and Profit Optimization of a Power Plant

Martin Kragelund, John Leth, and Rafał Wisniewski

***Abstract*— In this paper we discuss two different methods for implementing reference tracking in a profit optimization problem of a power plant. It is shown that tracking included as a side constraint results in an significant tracking error only when the reference gradient is large. When tracking is included in the cost function, as a quadratic term, the reference is tracked with a small accumulated error. Finally, the two methods are compared both in terms of tracking performance and computational burden.**

I. Introduction

Traditional thermal power plants, i.e., coal, gas, or oil fired power plants, have been studied in details [1]. In brief, a thermal power plant functions by burning a fuel in a boiler which evaporates water to steam under high pressure. The stream then drives a turbine generating electrical power which is delivered to the electrical grid.

A thermal power plant is modeled by first principle in [2], where the considered fuel is coal dust which is fed by four coal mills grinding the raw coal. The detailed model in [2] was used to establish an observer for the flow of coal into the boiler to improve the control of the coal mills. Simpler models for system control are presented in [3], where the different methods for changing the output from the complete portfolio of DONG Energy in Denmark are described. The means of changing the output is denoted an effectuator in [3], and the models of typical effectuators in a power plant are derived. An example of an effectuator is the boiler load in a thermal power plant which can be modeled as a 3rd order system.

In production economics the possible outputs from a production unit or "firm" are identified and called the production set [4, Chapter 5]. The production units are seen as black boxes which are capable of transforming some goods (input) to other goods (output). Some assumptions are often made about the production set e.g. *No free lunch* and *Free disposal*, i.e. the production set, Y, cannot contain $\mathbb{R}^l_+$ as this would yield production of some quantity without consumption and the company can absorb any additional input without reducing the output. In [4, Chapter 5] it is concluded that the objective of a company is to maximize its profit, which at first seems reasonable. However, it is possible to imagine companies which have the objective of maximizing sales revenue or the size of the company, but if the company is owned by the consumers in a market they will agree that profit maximization is preferable regardless of their own preference function.

The electricity market place for Nordic Countries is called Nord Pool. Here the price of electricity, as known by the average electricity consumer, is negotiated. Furthermore Nord Pool regulates related to the quality of the power deliverance. These quantities are traded on the hourly spot market, elspot. The transmission system operator maintains the energy balance. In other words it takes care of the situation when a power plant delivers too much or too little electricity to the grid than agreed. To ensure that sufficient reserve capacity is available the transmission system operator pays two prices, an up price and a down price, i.e. price for producing more or less electricity than previously agreed.

The data from Nord Pool has been used before to schedule the usage of hydro power plant in Norway such that the production plan commitment of the current day is fulfilled while maximizing the profit of the hydro plant [6].

This work focus on two different methods for including reference tracking into the design of an optimal profit strategy for a power plant, using coal, gas and oil, under the consideration of two business objectives, efficiency and controllability.

A. Outline

In Section II the plant dynamics, business objectives and profit function are described. In Section III the continuous optimization problem is formulated without reference tracking. For this purpose a discrete formulation of the problem is derived. In Section IV and Section V the reference tracking is included into the optimization problem as a side constraint and as a quadratic term in cost function, respectively. Section VI contain a discussion of the methods for implementing the tracking.

II. Plant Model

In this section a model of the power plant considered in this work is presented. The plant is capable of using three different fuel systems; coal, gas, and oil. For further details about the presented models and quantities the reader is referred to [7]–[10].

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A. Plant Dynamics

The fuel flow, $\boldsymbol{x}(t)$ $[kg/s]$, into the power plant is governed by third order differential equations (these equations also include the power plant dynamics). The control signal to the valves controlling these flows is denoted $\boldsymbol{u} = (u_c, u_g, u_o) \in U,\ U = \left\{ \boldsymbol{v} \in \mathbb{R}^3_+ \mid 0 \leq \boldsymbol{v}^T \boldsymbol{e}_u \leq 400 \right\}$, where $\boldsymbol{e}_u = (10.77, 18.87, 15.77)$ $[kg/s]$, and the dynamics is given by

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= \boldsymbol{A}\boldsymbol{z}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{x}(t) &= \boldsymbol{C}\boldsymbol{z}(t), \end{aligned} \tag{1}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_c & \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x3} \\ \boldsymbol{0}_{3x3} & \boldsymbol{A}_g & \boldsymbol{0}_{3x3} \\ \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x3} & \boldsymbol{A}_o \end{bmatrix}, \quad \boldsymbol{A}_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ h_{i_1} & h_{i_2} & h_{i_3} \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_c & \boldsymbol{0}_{3x1} & \boldsymbol{0}_{3x1} \\ \boldsymbol{0}_{3x1} & \boldsymbol{B}_g & \boldsymbol{0}_{3x1} \\ \boldsymbol{0}_{3x1} & \boldsymbol{0}_{3x1} & \boldsymbol{B}_o \end{bmatrix}, \quad \boldsymbol{B}_i = \begin{bmatrix} 0 \\ 0 \\ h_{i_0} \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{0}_{1x3} & \boldsymbol{0}_{1x3} \\ \boldsymbol{0}_{1x3} & \boldsymbol{C}_1 & \boldsymbol{0}_{1x3} \\ \boldsymbol{0}_{1x3} & \boldsymbol{0}_{1x3} & \boldsymbol{C}_1 \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

and $h_{i_j}, i \in \mathcal{I}$, are constants describing the dynamics of the three fuel systems which are obtained from transfer functions of the form

$$H_i(s) = \frac{1}{(\tau_i s + 1)^3},$$

where $\tau_i, i \in \mathcal{I}$, is 90, 60, and 70, respectively. The three fuel systems may have some shared dynamics but to simplify the model in this work the systems are assumed decoupled.

Functions describing the two business objectives are derived in the following.

B. Efficiency

The efficiency objective, $y_e = y_e(\boldsymbol{z})$, deals with how much electricity is produced from a certain amount of fuel. Three affine functions describing the contribution of the individual fuels to the efficiency objective have been established using measurement data from two Danish power plants and can be expressed as

$$\tilde{\boldsymbol{y}}_e(\boldsymbol{z}) = \tilde{\boldsymbol{Q}}\boldsymbol{z} + \boldsymbol{b}, \tag{2}$$

where

$$\begin{aligned} \tilde{\boldsymbol{Q}} &= \operatorname{diag}(\boldsymbol{e}_x)\boldsymbol{C}, \quad \boldsymbol{e}_x = (10.77, 18.87, 15.77), \\ \boldsymbol{b} &= (-1.76, 1.85, -0.37), \end{aligned}$$

and $\boldsymbol{C}$ defined in (1). The values of $\boldsymbol{e}_x$ and $\boldsymbol{b}$ have been established using measurement data and are measured in $[MJ/kg]$ and $[MW]$ respectively. The energy used for preprocessing the individual fuels is expressed by the b_i's, and the e_{x_i}'s are conversion factors which are a combination of the boiler efficiency and energy storage in the different fuels.

The total amount of efficiency is described by the function

$$Z \to Y_1;\ \boldsymbol{z} \mapsto y_e(\boldsymbol{z}) = \boldsymbol{\gamma}^T \tilde{\boldsymbol{y}}_e(\boldsymbol{z}),$$

where

$$\boldsymbol{\gamma} = (1, 1, 1).$$

C. Controllability

The controllability objective, $y_c = y_c(\boldsymbol{z})$, deals with a measure of how fast the production of electricity can be changed. Allowed changes in the production is limited to a certain gradient depending on the current efficiency. The reason for this limit is a compliance to maximum temperature gradients in the boiler (these have not been explicitly modelled and are therefore indirectly considered by limiting the allowed changes). When using coal it is allowed to change production with 0.133 $[MW/s]$ when running the plant at low and high production and 0.267 $[MW/s]$ in the middle range from 200 $[MW]$ to 360 $[MW]$. When using oil or gas the values are 0.133 $[MW/s]$ and 0.534 $[MW/s]$. If a mixture of the three fuels are used it is assumed that the allowed change is a linear combination of the allowed change of the individual fuels. The controllability objective is, therefore, modelled as

$$Z \to Y_2;\ \boldsymbol{z} \mapsto y_c(\boldsymbol{z}) = \begin{cases} 0.133 & y_e(\boldsymbol{z}) \in S_1 \\ \frac{\boldsymbol{\xi}^T \tilde{\boldsymbol{y}}_e(\boldsymbol{z})}{y_e(\boldsymbol{z})} & y_e(\boldsymbol{z}) \in S_2 \\ 0.133 & y_e(\boldsymbol{z}) \in S_3, \end{cases} \tag{3}$$

where

$$\begin{aligned} &\boldsymbol{\xi} = (0.267, 0.534, 0.534), \quad S_1 = \{s \in \mathbb{R} | 0 \leq s \leq 200\}, \\ &S_2 = \{s \in \mathbb{R} | 200 < s < 360\}, \text{ and} \\ &S_3 = \{s \in \mathbb{R} | 360 \leq s \leq 400\}. \end{aligned}$$

D. Prices

The cost of using the fuel, revenue from production of output, and the profit of operating the power plant can now be determined. The above constructions yields a product (or output) function, y_P, of the system given by

$$y_P : Z \to Y;\ \boldsymbol{z} \mapsto (y_e(\boldsymbol{z}), y_c(\boldsymbol{z})).$$

The growth of cost and growth of revenue for the system are defined by the following functions (both with units in [DKK/s])

$$\begin{aligned} &g_C : Z \to \mathbb{R};\ \boldsymbol{z} \mapsto \boldsymbol{z}^T \boldsymbol{C}^T \boldsymbol{p}_C, \\ &g_R : Y \times \mathbb{R}_+ \to \mathbb{R};\ (\boldsymbol{y}, t) \mapsto \boldsymbol{y}^T \boldsymbol{p}_R(t), \quad \boldsymbol{p}_R(t) > 0, \end{aligned}$$

where $\boldsymbol{p}_C = (1.20, 3.74, 6.00)$ is the price of coal, gas, and oil respectively and

$$\boldsymbol{p}_R(t) = (p_{R1}(t), p_{R2}(t))$$

the price of the efficiency and controllability respectively.[1]

The growth of profit is hence defined by

$$Z \times Y \times \mathbb{R}_+ \to \mathbb{R};\ (\boldsymbol{z}, \boldsymbol{y}, t) \mapsto g_R(\boldsymbol{y}, t) - g_C(\boldsymbol{z}),$$

which for the system yields the function

$$g_P : Z \times \mathbb{R}_+ \to \mathbb{R};\ (\boldsymbol{z}, t) \mapsto g_R(\boldsymbol{y}_P(\boldsymbol{z}), t) - g_C(\boldsymbol{z}).$$

Therefore, the profit is given by

$$P : \mathbb{R}_+ \to \mathbb{R};\ t \mapsto \int_0^t g_P(\boldsymbol{z}(\tau), \tau) d\tau. \tag{4}$$

III. Problem Formulation

Using the above it is now possible to formulated the following optimization problem

$$\begin{array}{ll} \max\limits_{u \in U} & P(T) = \int_0^T g_P(\boldsymbol{z}, t) dt \\ \text{subject to} & \dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}, \end{array} \tag{5}$$

and with the additional requirement that, $y_e(\boldsymbol{z}(t))$ should track a predefined reference signal, $y_r(t)$.

For computational reasons the optimization problem above will be simplified by introduction two approximations. One which assumes good reference tracking and one which deals with condition for discretization of (5).

The growth of profit function, g_P, can when $y_e \approx y_r$ be approximated by

$$g_p(\boldsymbol{z}, t) = \boldsymbol{\Theta}(t)\boldsymbol{z} + \tilde{\varphi}(t), \tag{6}$$

where

$$\begin{aligned} \boldsymbol{\Theta}(t) &= p_{R1}(t)\boldsymbol{\gamma}^T\boldsymbol{Q} - p_{R2}(t)\boldsymbol{p}_C^T\boldsymbol{C} + \boldsymbol{\vartheta}(t), \\ \tilde{\varphi}(t) &= p_{R1}(t)\boldsymbol{\gamma}^T\boldsymbol{b} + p_{R2}(t)\zeta(t), \end{aligned}$$

and $\boldsymbol{\vartheta}(t)$ and $\zeta(t)$ makes up for the switching function in the original formulation of the controllability, i.e.,

$$\boldsymbol{\vartheta}(t) = \begin{cases} 0 & y_r(t) \in S_1 \\ \frac{\boldsymbol{\xi}^T\boldsymbol{Q}}{y_r(t)} & y_r(t) \in S_2 \\ 0 & y_r(t) \in S_3, \end{cases}$$

$$\zeta(t) = \begin{cases} 0.133 & y_r(t) \in S_1 \\ \frac{\boldsymbol{\xi}^T\boldsymbol{b}}{y_r(t)} & y_r(t) \in S_2 \\ 0.133 & y_r(t) \in S_3, \end{cases}$$

Hence the assumption $y_e \approx y_r$ enables us to consider the growth of profit (4) as a affine function of the state as is (6). Note that the assumption also implies that the switching condition $y_r(t) \in S_i$ in the expression for $\boldsymbol{\vartheta}$ and ξ are time dependent, this switching condition would be state dependent otherwise.

The time period T is divided into N equally sized time units, h, i.e., $T = Nh$. It is assumed that $\boldsymbol{\Theta}(t)$, $\varphi(t)$, $\boldsymbol{\psi}(t)$, $y_r(t)$ can be approximated by piecewise constant functions for each time step, i.e.,

$$\begin{aligned} \boldsymbol{\Theta}(t) &= \boldsymbol{\Theta}_k, \quad kh < t < (k+1)h, \\ \tilde{\varphi}(t) &= \tilde{\varphi}_k, \quad kh < t < (k+1)h, \\ y_r(t) &= y_{r_k}, \quad kh < t < (k+1)h. \end{aligned}$$

Furthermore, the control will be assumed piecewise constant as customary when digital to analogue conversion is performed using sample-hold circuits.

Using a fact from [11] the continuous time state $\boldsymbol{z}(t)$ in the dynamical system in (5) can be described by

$$\begin{aligned} \boldsymbol{z}(t) &= e^{\boldsymbol{A}t}\boldsymbol{z}_0 + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{B}\boldsymbol{u}_0(s) ds \\ &= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \exp\left\{ \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} t \right\} \begin{bmatrix} \boldsymbol{z}_0 \\ \boldsymbol{u}_0 \end{bmatrix}, \end{aligned} \tag{7}$$

where $\boldsymbol{I}$ is an identity matrix with appropriate dimension. Using (7) it is possible to derive the following formula which is used during the discretization of the cost and constraint

$$\begin{aligned} \int_0^h e^{\boldsymbol{A}t} dt &= e^{\boldsymbol{A}h} \int_0^h e^{-\boldsymbol{A}(h-t)} dt \\ &= e^{\boldsymbol{A}h} \left(e^{-\boldsymbol{A}h} \cdot 0 + \int_0^h e^{-\boldsymbol{A}(h-t)} \boldsymbol{I} dt \right) \\ &= e^{\boldsymbol{A}h} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \exp\left\{ \begin{bmatrix} -\boldsymbol{A} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} h \right\} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}. \end{aligned} \tag{8}$$

The objective function, $P(T)$, in the optimization problem in (5) is converted to discrete time by using the above, i.e.,

$$\begin{aligned} P(T) &= \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} \left(\boldsymbol{\Theta}(t)\boldsymbol{z}(t) + \tilde{\varphi}(t) \right) dt \\ &= \sum_{k=0}^{N-1} \boldsymbol{\Theta}_k \int_0^h \left(e^{\boldsymbol{A}t}\boldsymbol{z}_k + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{B} ds \boldsymbol{u}_k \right) dt + h\tilde{\varphi}_k \\ &= \sum_{k=0}^{N-1} \boldsymbol{\Theta}_k \int_0^h \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\tilde{\boldsymbol{A}}t} \begin{bmatrix} \boldsymbol{z}_k \\ \boldsymbol{u}_k \end{bmatrix} dt + h\tilde{\varphi}_k \\ &= \sum_{k=0}^{N-1} \boldsymbol{\Theta}_k \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\tilde{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\hat{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_k \\ \boldsymbol{u}_k \end{bmatrix} + h\tilde{\varphi}_k, \end{aligned}$$

where

$$\hat{\boldsymbol{A}} = \begin{bmatrix} -\tilde{\boldsymbol{A}} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

With the reference tracking disregarded and the growth of profit function as in (6), the optimization problem (5) can be reformulated as

$$\begin{array}{ll} \max\limits_{u_k \in U} & \sum\limits_{k=0}^{N-1} \boldsymbol{C}_k\boldsymbol{z}_k + \boldsymbol{D}_k\boldsymbol{u}_k + \boldsymbol{E}_k \\ \text{subject to} & \boldsymbol{z}_{k+1} = \boldsymbol{\Phi}\boldsymbol{z}_k + \boldsymbol{\Gamma}\boldsymbol{u}_k, \end{array} \tag{9}$$

where

$$C_k = \boldsymbol{\Theta}_k \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\tilde{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\hat{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix},$$

$$D_k = \boldsymbol{\Theta}_k \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\tilde{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\hat{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix},$$

$$E_k = h\varphi_k,\ \boldsymbol{\Phi} = e^{\boldsymbol{A}(t_{k+1}-t_k)}, \text{ and } \boldsymbol{\Gamma} = \int_0^{t_{k+1}-t_k} e^{\boldsymbol{A}s} ds \boldsymbol{B}.$$

[1] The prices used in this work corresponds to the market prices the 29th of June, 2008 and has been established using internal DONG Energy documents and the archive of power price at www.nordpool.dk, which is a marketplace for trading power contracts.

When considering the reference tracking different approaches can be used to formulate them. In this work two different methods are considered - briefly these are:

Quadratic: In this approach the tracking constraint is included in the profit function as a norm of the difference between the efficiency and the reference and thus penalizing deviations.

Side Constraint: In this approach the tracking is formulated as a constraint in the optimization such that the reference is followed within a reference band. This is implemented as additional side constraints to problem (9).

IV. Side Constraint

To include the reference tracking in problem (5) we introduce in this section a reference band with time dependent width, $\alpha(t)$, i.e., $\alpha(t)$ is the normed error at time t. In continuous time the reference band can be formulated as

$$h(\boldsymbol{z}(t), t) \geq 0, \tag{10}$$

where

$$h(\boldsymbol{z}(t), t) = \boldsymbol{\Upsilon}\boldsymbol{z}(t) + \boldsymbol{\psi}(t), \tag{11}$$

with

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \gamma^T \tilde{\boldsymbol{Q}} \\ -\gamma^T \tilde{\boldsymbol{Q}} \end{bmatrix},$$
$$\boldsymbol{\psi}(t) = \begin{bmatrix} \boldsymbol{\gamma}^T \boldsymbol{b} - y_r(t) + \alpha \\ -\boldsymbol{\gamma}^T \boldsymbol{b} + y_r(t) + \alpha \end{bmatrix}.$$

By direct calculation the discrete time approximation then yields

$$\boldsymbol{\Psi}_l \boldsymbol{z}_k + \boldsymbol{\Pi}_l \boldsymbol{u}_k + \boldsymbol{\Omega}_{k,l} \geq 0$$

where for $l = 0, 1, 2, ..., L$

$$\boldsymbol{\Psi}_l = \boldsymbol{\Upsilon} e^{\boldsymbol{A}\frac{l-1}{L}h},$$
$$\boldsymbol{\Pi}_l = \boldsymbol{\Upsilon} \int_0^{\frac{l-1}{L}h} e^{\boldsymbol{A}(\frac{l-1}{L}h - s)} \boldsymbol{B} ds,$$
$$\boldsymbol{\Omega}_{k,l} = \boldsymbol{\psi}(\tfrac{l-1}{L}h + kh).$$

Note that the constraint is guaranteed to be satisfied L times between each sampling of the system in (9).

Hence the optimization problem (9) together with tracking constraint can be formulated as

$$\max_{\substack{u \in U \\ \alpha \geq 0}} \sum_{k=0}^{N-1} (\boldsymbol{C}_k \boldsymbol{z}_k + \boldsymbol{D}_k \boldsymbol{u}_k + \boldsymbol{E}_k - W_k \alpha_k)$$
$$\text{subject to} \quad \boldsymbol{z}_{k+1} = \boldsymbol{\Phi}\boldsymbol{z}_k + \boldsymbol{\Gamma}\boldsymbol{u}_k,$$
$$\boldsymbol{\Psi}_l \boldsymbol{z}_k + \boldsymbol{\Pi}_l \boldsymbol{u}_k + \boldsymbol{\Omega}_{k,l} \geq 0.$$

Note that the tracking width α_k is included in the optimization problem, i.e., the tracking error is minimized as well.

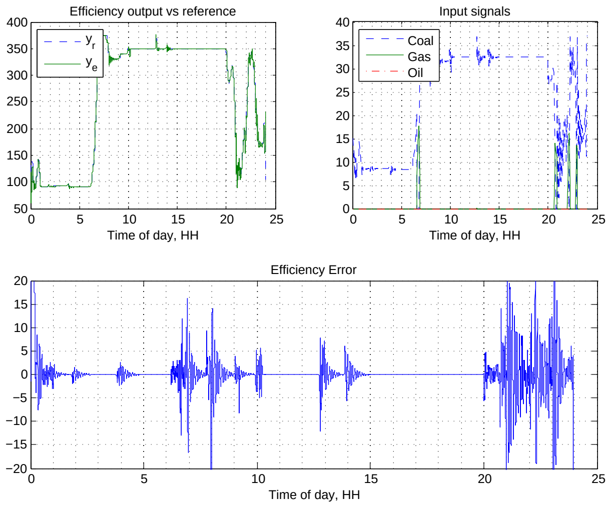

Fig. 1. Graphs of the efficiency output, input usage, and tracking error for the optimization problem with reference band tracking, $L = 1$.

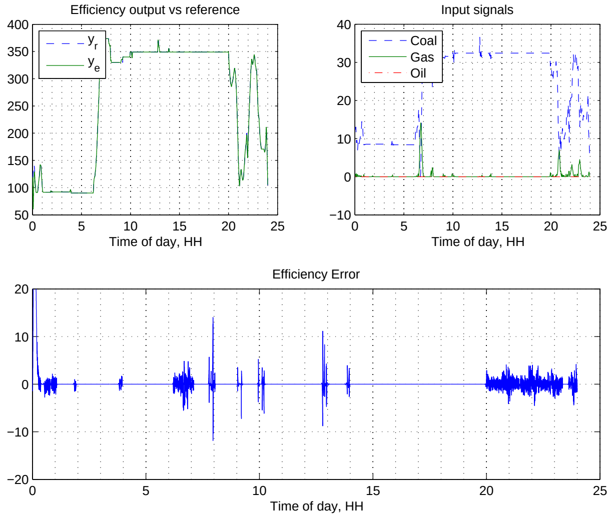

Fig. 2. Graphs of the efficiency output, input usage, and tracking error for the optimization problem with reference band tracking, $L = 5$.

The optimization problem above has been solved for $L = 1$ and $L = 5$, the results are depicted in Figure 2 and Figure 1. As seen in these figures, reference is tracked well with an significant error only present at times with large gradients in the reference signal. Furthermore, the tracking of the reference is considerably better when $L = 5$ as both the intensity and the value of the efficiency error is smaller.

V. Quadratic

In this section we include the reference tracking as a cost on the deviation from the reference. This is formulated as

$$Q(T) = \int_0^T -\beta_q \left\| \boldsymbol{\gamma}^T \boldsymbol{Q} \boldsymbol{z}(t) - y_r(t) \right\|^2 dt, \tag{12}$$

where $\|\cdot\|$ is the Euclidean norm. The tracking is included in the objective function as

$$P(T) = \int_0^T g_p(\boldsymbol{z},t) - \beta_q \left\|\boldsymbol{\gamma}^T\boldsymbol{Q}\boldsymbol{z}(t) - y_r(t)\right\|^2 dt$$
$$= \int_0^T \left(\underbrace{-\boldsymbol{z}(t)^T\boldsymbol{Q}\boldsymbol{z}(t)}_{P_2(T)} + \underbrace{2\boldsymbol{q}(t)^T\boldsymbol{z}(t) + \varphi(t)}_{P_1(T)}\right) dt, \tag{13}$$

with

$$\begin{aligned}\boldsymbol{Q} &= \beta_q\tilde{\boldsymbol{Q}}^T\boldsymbol{\gamma}\boldsymbol{\gamma}^T\tilde{\boldsymbol{Q}}\\ \boldsymbol{q}(t)^T &= \frac{1}{2}\boldsymbol{\Theta}(t) + \beta_q y_r(t)\boldsymbol{\gamma}^T\tilde{\boldsymbol{Q}}\\ \varphi(t) &= \tilde{\varphi}(t) - \beta_q y_r(t)^2.\end{aligned}$$

As our maximization problem is formulated in discrete time we need to discretize (13). This is done in the sequel by apply (7) and (8).

$$\begin{aligned}P_1(T) &= \int_0^T \left(2\boldsymbol{q}(t)^T\boldsymbol{z}(t) + \varphi(t)\right) dt\\ &= \sum_{k=0}^{N-1} 2\boldsymbol{q}(t)^T \int_0^h \left(e^{\boldsymbol{A}t}\boldsymbol{z}_k + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{B}ds\boldsymbol{u}_k\right) dt\\ &\quad + h\sum_{k=0}^{N-1}\varphi_k\\ &= \sum_{k=0}^{N-1} 2\boldsymbol{q}(t)^T \int_0^h \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} e^{\tilde{\boldsymbol{A}}t}\begin{bmatrix}\boldsymbol{z}_k\\ \boldsymbol{u}_k\end{bmatrix} dt + h\sum_{k=0}^{N-1}\varphi_k\\ &= \sum_{k=0}^{N-1} 2\boldsymbol{q}(t)^T e^{\tilde{\boldsymbol{A}}h}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} e^{\hat{\boldsymbol{A}}h}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}_k\\ \boldsymbol{u}_k\end{bmatrix} + h\sum_{k=0}^{N-1}\varphi_k\\ &= \sum_{k=0}^{N-1}\left(\boldsymbol{M}_z\boldsymbol{z}_k + \boldsymbol{M}_u\boldsymbol{u}_k + h\varphi_k\right),\end{aligned} \tag{14}$$

where

$$\begin{aligned}\boldsymbol{M}_z &= 2\boldsymbol{q}(t)^T e^{\tilde{\boldsymbol{A}}h}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} e^{\hat{\boldsymbol{A}}h}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}\\ \boldsymbol{M}_u &= 2\boldsymbol{q}(t)^T e^{\tilde{\boldsymbol{A}}h}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} e^{\hat{\boldsymbol{A}}h}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}\end{aligned}$$

with

$$\hat{\boldsymbol{A}} = \begin{bmatrix}-\tilde{\boldsymbol{A}} & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}, \quad \tilde{\boldsymbol{A}} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix},$$

and the matrices $\boldsymbol{I}$ and $\boldsymbol{0}$ of appropriate dimensions.

Now, the quadratic term is discretized by using (7)

$$\begin{aligned}P_2(T) &= -\boldsymbol{z}(t)^T\boldsymbol{Q}\boldsymbol{z}(t)\\ &= -\sum_{k=0}^{N-1}\int_0^h \left(\boldsymbol{z}_k^T e^{\boldsymbol{A}^Tt} + \boldsymbol{u}_k^T\int_0^t \boldsymbol{B}^T e^{\boldsymbol{A}^T(t-s)}ds\right)\boldsymbol{Q}\\ &\qquad\left(e^{\boldsymbol{A}t}\boldsymbol{z}_k + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{B}ds\boldsymbol{u}_k\right)dt\\ &= -\sum_{k=0}^{N-1}\int_0^h \begin{bmatrix}\boldsymbol{z}_k^T & \boldsymbol{u}_k^T\end{bmatrix} e^{\tilde{\boldsymbol{A}}^Tt}\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{Q}\\ &\qquad\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} e^{\tilde{\boldsymbol{A}}t}\begin{bmatrix}\boldsymbol{z}_k\\ \boldsymbol{u}_k\end{bmatrix}dt\\ &= -\sum_{k=0}^{N-1}\begin{bmatrix}\boldsymbol{z}_k^T & \boldsymbol{u}_k^T\end{bmatrix} e^{\tilde{\boldsymbol{A}}^Th}\boldsymbol{Y}(h)e^{\tilde{\boldsymbol{A}}h}\begin{bmatrix}\boldsymbol{z}_k\\ \boldsymbol{u}_k\end{bmatrix}\end{aligned} \tag{15}$$

where $\tilde{\boldsymbol{A}}$ is as above and

$$\boldsymbol{Y}(h) = \int_0^h e^{-\tilde{\boldsymbol{A}}^T(h-t)}\bar{\boldsymbol{Q}}e^{-\tilde{\boldsymbol{A}}(h-t)}dt \tag{16}$$
$$\bar{\boldsymbol{Q}} = \begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{Q}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix}$$

The integral in (16) is on the form of the solution to a matrix differential equations which can be formulated as

$$\boldsymbol{Y}(h) = \int_0^h e^{-\tilde{\boldsymbol{A}}^T(h-t)}\bar{\boldsymbol{Q}}e^{-\tilde{\boldsymbol{A}}(h-t)}dt \Rightarrow$$
$$-\frac{d}{dh}\boldsymbol{Y}(h) = \tilde{\boldsymbol{A}}^T\boldsymbol{Y}(h) + \boldsymbol{Y}(h)\tilde{\boldsymbol{A}} - \bar{\boldsymbol{Q}}, \quad \boldsymbol{Y}(0) = \boldsymbol{0}. \tag{17}$$

Using the $\mathrm{Vec}(\cdot)$ notation which is defined as

$$\mathrm{Vec}(\boldsymbol{P}) = \begin{bmatrix}p_1\\ \vdots\\ p_n\end{bmatrix}, \tag{18}$$

where p_i is the columns of $\boldsymbol{P}$, it is possible to formulated (17) as

$$-\frac{d\mathrm{Vec}(\boldsymbol{Y}(h))}{dh} = \boldsymbol{F}\mathrm{Vec}(\boldsymbol{Y}(t)) - \mathrm{Vec}(\bar{\boldsymbol{Q}}) \tag{19}$$

where

$$\boldsymbol{F} = \left(\boldsymbol{I}\otimes\tilde{\boldsymbol{A}}^T + \tilde{\boldsymbol{A}}^T\otimes\boldsymbol{I}\right)$$

and $\otimes$ denotes the Kronecker product. By using the solution to standard vector differential equation and (8), the solution to (19) is given by

$$\begin{aligned}\mathrm{Vec}(\boldsymbol{Y}(h)) &= \int_0^h e^{\boldsymbol{F}(h-\tau)}d\tau\mathrm{Vec}(\bar{\boldsymbol{Q}})\\ &= e^{\boldsymbol{F}h}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} e^{\hat{\boldsymbol{F}}h}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}\mathrm{Vec}(\bar{\boldsymbol{Q}})\\ &= e^{\boldsymbol{F}h}\tilde{\boldsymbol{F}}\mathrm{Vec}(\bar{\boldsymbol{Q}}),\end{aligned}$$

where

$$\tilde{\boldsymbol{F}} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} e^{\hat{\boldsymbol{F}}h}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}, \quad \hat{\boldsymbol{F}} = \begin{bmatrix}-\boldsymbol{F} & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}.$$

That is (15) can be expressed as

$$P_2(T) = -\sum_{k=0}^{N-1}\begin{bmatrix}\boldsymbol{z}_k^T & \boldsymbol{u}_k^T\end{bmatrix}\begin{bmatrix}\boldsymbol{N}_{zz} & \boldsymbol{N}_{zu}\\ \boldsymbol{N}_{uz} & \boldsymbol{N}_{uu}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}_k\\ \boldsymbol{u}_k\end{bmatrix} \tag{20}$$

where

$$\boldsymbol{N}_{zz} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\tilde{\boldsymbol{A}}^T h} \mathrm{Vec}^{-1}\left(e^{\boldsymbol{F}h} \tilde{\boldsymbol{F}} \mathrm{Vec}(\bar{\boldsymbol{Q}}) \right) e^{\tilde{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{N}_{zu} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} e^{\tilde{\boldsymbol{A}}^T h} \mathrm{Vec}^{-1}\left(e^{\boldsymbol{F}h} \tilde{\boldsymbol{F}} \mathrm{Vec}(\bar{\boldsymbol{Q}}) \right) e^{\tilde{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}$$

$$\boldsymbol{N}_{uz} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} e^{\tilde{\boldsymbol{A}}^T h} \mathrm{Vec}^{-1}\left(e^{\boldsymbol{F}h} \tilde{\boldsymbol{F}} \mathrm{Vec}(\bar{\boldsymbol{Q}}) \right) e^{\tilde{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{N}_{uu} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} e^{\tilde{\boldsymbol{A}}^T h} \mathrm{Vec}^{-1}\left(e^{\boldsymbol{F}h} \tilde{\boldsymbol{F}} \mathrm{Vec}(\bar{\boldsymbol{Q}}) \right) e^{\tilde{\boldsymbol{A}}h} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}$$

with matrices $\boldsymbol{I}$ and $\boldsymbol{0}$ of appropriate dimensions, and $\mathrm{Vec}^{-1}\left(e^{\boldsymbol{F}h} \tilde{\boldsymbol{F}} \mathrm{Vec}(\bar{\boldsymbol{Q}}) \right)$, an $n \times n$ matrix, denoting the "inverse" of the Vec-operator in (18), i.e., reshaping the vector into a matrix.

Hence the optimization problem together with quadratic tracking error can be formulated as

$$\max_{\substack{u \in U \\ \alpha \geq 0}} \sum_{k=0}^{N-1} C_k$$
$$\text{subject to} \quad \boldsymbol{z}_{k+1} = \boldsymbol{\Phi} \boldsymbol{z}_k + \boldsymbol{\Gamma} \boldsymbol{u}_k,$$

where

$$C_k = \begin{bmatrix} \boldsymbol{z}_k^T & \boldsymbol{u}_k^T \end{bmatrix} \boldsymbol{N} \begin{bmatrix} \boldsymbol{z}_k \\ \boldsymbol{u}_k \end{bmatrix} + \boldsymbol{M}_z \boldsymbol{z}_k + \boldsymbol{M}_u \boldsymbol{u}_k + h\varphi_k,$$

with

$$\boldsymbol{N} = -\begin{bmatrix} \boldsymbol{N}_{zz} & \boldsymbol{N}_{zu} \\ \boldsymbol{N}_{uz} & \boldsymbol{N}_{uu} \end{bmatrix},$$

and the matrices $\boldsymbol{N}_{zz}$, $\boldsymbol{N}_{zu}$, $\boldsymbol{N}_{uz}$, $\boldsymbol{N}_{uu}$, $\boldsymbol{M}_z$, and $\boldsymbol{M}_u$ as given above.

The optimization problem above has been solved and the results are depicted in Figure 3. As seen in the figure

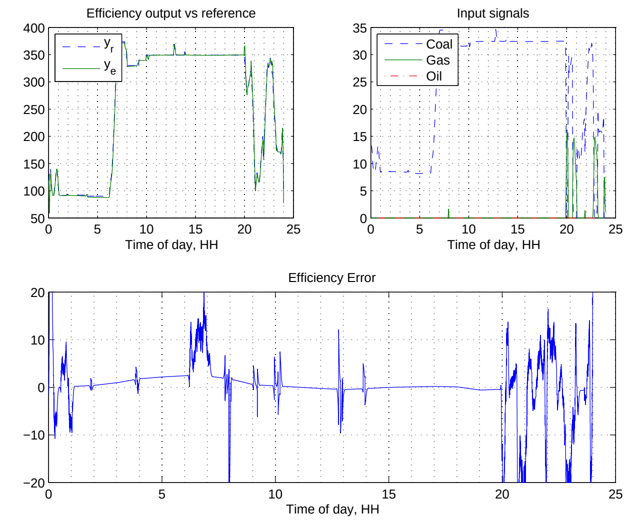

Fig. 3. Graphs of the efficiency output, input usage, and tracking error for the optimization problem with quadratic tracking error.

the reference is tracked with a small accumulated error caused by the quadratic error term in (12).

VI. Comparison of Optimization Methods

In this section we compare the two difference methods for solving the problem (9) with reference tracking.

Comparing the efficiency error of the three different methods it is noted that the mean of the error in the case of side constraints is less than the quadratic method. However, the fluctuations of the efficiency error when using side constraints are more frequent than the quadratic case.

The profit of the three different methods are almost identical and are therefore not included in this analysis. We note that this is also supported by the fact that the use of fuels in the three approaches are similar and by the fact that the efficiency error is small hence producing similar profits.

In table I the times for running the optimization are presented for the three solution strategies. Hence the

Method	Optimization Time	Solver
Side Constraint ($L = 5$)	896 s	SeDuMi
Side Constraint ($L = 1$)	168 s	SeDuMi
Quadratic	157 s	BPMPD

TABLE I

Comparison of optimization times between the three solution strategies.

quadratic method or side constraint method with $L = 1$ should be applied if only the optimization time is considered.

As the profits of each of the methods are the same the choice of methods should be based on the need for computation time and requirements on tracking performance, which depend entire on the specific control problem.

To this end we remark that if continuous time is considered, the quadratic method has the advantage of only having the dynamical system as side constraint, which eases application of the Pontryagin maximum principle.

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