System Dynamics and Modified Cumulant Neglect Closure Schemes

Köylüoglu, H. Ugur; Nielsen, Søren R.K.

Publication date:
1996

Document Version
Publisher's PDF, also known as Version of record

Link to publication from Aalborg University

Citation for published version (APA):
Köylüoglu, H. U., & Nielsen, S. R. K. (1996). *System Dynamics and Modified Cumulant Neglect Closure Schemes*. Dept. of Building Technology and Structural Engineering. Structural Reliability Theory Vol. R9603 No. 151

INSTITUTTET FOR BYGNINGSTEKNIK
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING
AALBORG UNIVERSITET • AUC • AALBORG • DANMARK

STRUCTURAL RELIABILITY THEORY
PAPER NO. 151

Submitted to ASCE Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability, Worcester, USA, August 1996

H. U. KÖYLÜOĞLU & S. R. K. NIELSEN

SYSTEM DYNAMICS AND MODIFIED CUMULANT NEGLECT CLOSURE SCHEMES

JANUARY 1996 ISSN 1395-7953 R9603

INSTITUTTET FOR BYGNINGSTEKNIK
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING
AALBORG UNIVERSITET • AUC • AALBORG • DANMARK

STRUCTURAL RELIABILITY THEORY
PAPER NO. 151

Submitted to ASCE Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability, Worcester, USA, August 1996

H. U. KÖYLÜOĞLU & S. R. K. NIELSEN

SYSTEM DYNAMICS AND MODIFIED CUMULANT NEGLECT CLOSURE SCHEMES

JANUARY 1996 ISSN 1395-7953 R9603

System Dynamics and Modified Cumulant Neglect Closure Schemes

H. Uğur Köylüoğlu[1] and Søren R. K. Nielsen[2]

ABSTRACT

Dealing with multipeaked problems, the goal of the paper is to improve the quality of the approximations for the expectations appearing in the differential equations written for the statistical moments of the state vector, guided by insight in the system dynamics. For systems with polynomial non-linearities, modifications in the cumulant neglect closure scheme are suggested. The methodology is illustrated using the two wells oscillator. An error analysis is performed to compare the modified and ordinary cumulant neglect closure schemes applied at the second and fourth order levels with the exact results available.

1. INTRODUCTION

The equation of motion of non-linear dynamic systems driven by filtered white noise processes are basically non-linear stochastic differential equations. Because of the non-linearities, non-provided expectations appear in the differential equations written for the statistical moments of the state vector. These can be evaluated approximately by means of socalled closure schemes. A cumulant neglect closure scheme is such a closure scheme which can be applied with high efficiency in case of polynomial non-linearities generating single well potentials such as the Duffing oscillator with hardening spring stiffness. In these cases, the found joint probability density function (jpdf) of the state vector appears as monomodal and almost Gaussian. However, in some cases the jpdf displays multipeaks, generated by the system dynamics. In these problems, the cumulant neglect closure may give highly erroneous results and even become numerically unstable, Bergman et al. (1995). It is the idea of the paper to transform any state variable, which possesses multipeak behaviour, into an auxiliary variable that would behave monomodally, and then perform the cumulant neglect closure on the jpdf of these variables. The methodology is illustrated using a two wells oscillator, for which exact stationary jpdf is known. In what follows, closed form results for the variance of the displacement of the two wells oscillator are obtained using the ordinary and modified cumulant neglect closure schemes. Finally, the variance of the displacement obtained by the ordinary and modified cumulant neglect closure schemes applied at the 2nd and 4th order level are compared with the exact results.

[1]Koç University, TR-80860 İstinye, İstanbul, Turkey.
[2]Aalborg University, DK-9000, Aalborg, Denmark.

2. TWO WELLS OSCILLATOR

The equation of motion of a two wells oscillator driven by white noise can be written in state vector form as

$$d\mathbf{Z}(t) = \mathbf{c}(\mathbf{Z}(t))dt + \mathbf{d}dW(t) \quad , \quad \mathbf{Z}(0) = \mathbf{0} \tag{1}$$

$$\mathbf{Z}(t) = \begin{bmatrix} X(t) \\ \dot{X}(t) \end{bmatrix} \quad , \quad \mathbf{c}(\mathbf{Z}(t)) = \begin{bmatrix} \dot{X}(t) \\ -2\zeta\omega_0\dot{X}(t) - \frac{\partial V(X)}{\partial X} \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 0 \\ \sqrt{2\pi S_0} \end{bmatrix} \tag{2}$$

$$V(X) = -\omega_0^2 \frac{X^2}{2}\left(1 - \frac{1}{2}\left(\frac{X}{x_0}\right)^2\right) \tag{3}$$

where $\mathbf{c}(\mathbf{Z}(t))$ and $\mathbf{d}$ are the drift and diffusion vectors. ζ and ω_0 are parameters to define the system damping and stiffness. x_0 is the non-linear parameter denoting the locations of the peaks in the potential function and the pdf of the displacement $X(t)$ as shown in Figure 1. $W(t)$ is the unit Wiener process. The exact stationary jpdf $f_{X\dot{X}}(x, \dot{x})$ of the state vector $\mathbf{Z}(t)$ is known.

$$f_{X\dot{X}}(x, \dot{x}) = C \exp\left(- \frac{2\zeta\omega_0}{\pi S_0}(\frac{1}{2}\dot{x}^2 + V(x))\right) \tag{4}$$

where C is a constant determined from the normalization of the jpdf. All joint statistical moments of odd order are zero due to the antisymmetry of the drift vector. Using Itô's lemma, the statististical moment equations can be derived for the 2nd μ_{ij} and 4th μ_{ijkl} order statistical moments.

$$\dot{\mu}_{ij} = 2\{B_{im}\mu_{mj} + D_{imnp}\mu_{mnpj}\}_s + d_i d_j \quad , \quad \mu_{ij}(0) = 0 \tag{5}$$

$$\dot{\mu}_{ijkl} = 4\{B_{im}\mu_{mjkl} + D_{imnp}\mu_{mnpjkl}\}_s + 6\{d_i d_j \mu_{kl}\}_s \quad , \quad \mu_{ijkl}(0) = 0 \tag{6}$$

$\{\cdot\}_s$ is the symmetry operator, and the tensorial notation $c_i(\mathbf{Z}) = B_{im}Z_m + D_{imnp}Z_m Z_n Z_p$ of the cubic non-linear drift vector is employed. Closed form results for σ_X^2 are obtained from the quadratic and cubic equations given in (7) if the cumulant neglect closure is applied at the 2nd (this is tantamount to Gaussian closure) and 4th order level. Unprovided expectations μ_{ijkl} and μ_{ijklmn} appearing in (5) and (6) for the respective order of closure are approximated as given in (8), Stratonovich (1963).

$$\left.\begin{aligned} &\frac{\pi S_0}{2\zeta\omega_0^3} + \sigma_X^2 - \frac{3}{x_0^2}\sigma_X^4 = 0 \\ &\frac{\pi S_0}{2\zeta\omega_0^3} + \sigma_X^2 - \frac{1}{x_0^2}\left(\frac{30\sigma_X^6 + 3\frac{\pi S_0}{2\zeta\omega_0^3}x_0^2\sigma_X^2}{15\sigma_X^2 - x_0^2}\right) = 0 \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} \mu_{ijkl} &= 3\{\mu_{ij}\mu_{kl}\}_s \\ \mu_{ijklmn} &= 15\{\mu_{ij}\mu_{klmn}\}_s + 10\{\mu_{ijk}\mu_{lmn}\}_s - 30\{\mu_{ij}\mu_{kl}\mu_{mn}\}_s \end{aligned}\right\} \tag{8}$$

3. MODIFIED CUMULANT NEGLECT CLOSURE

Considering the multipeaked behaviour of the jpdf, a closure scheme using a multipeaked jpdf for $f_{X\dot{X}}(x, \dot{x})$ is proposed.

$$f_{X\dot{X}}(x,\dot{x}) = \frac{1}{2}\Big(f_{V\dot{X}}(x + x_0, \dot{x}) + f_{V\dot{X}}(-x + x_0, -\dot{x})\Big) \tag{9}$$

where the jpdf $f_{V\dot{X}}$ of the auxillary variable V and $\dot{X}$ is assumed to be monomodal. As seen, the setting (9) ensures that $f_{X\dot{X}}(x,\dot{x}) = f_{X\dot{X}}(-x, -\dot{x})$, which is caused by the anti-symmetry of the drift vector, $\mathbf{c}(\mathbf{Z}(t)) = -\mathbf{c}(-\mathbf{Z}(t))$. An ordinary cumulant neglect closure is now suitable with respect to this distribution. From (9), it follows that

$$\begin{aligned} &E[X^m\dot{X}^n] = \frac{1}{2}(1 + (-1)^{m+n})E[(V - x_0)^m\dot{X}^n] = \\ &\frac{1}{2}(1 + (-1)^{m+n})\sum_{l=0}^{m}\binom{m}{l}E[V^l\dot{X}^n](-x_0)^{m-l} \end{aligned} \tag{10}$$

For the application of (10) for closure at the 4th order level, first the expectations $E[V^l\dot{X}^n]$, $l + n = 2, 4$ appearing on the right hand side are expressed by the provided moments $E[X^m\dot{X}^n]$, $m+n = 2, 4$ on the left hand side. Next, moments $E[V^l\dot{X}^n]$, $l+n = 6$ are expressed by now known moments $E[V^l\dot{X}^n]$, $l+n = 2, 4$ using the cumulant neglect closure. Finally, $E[X^m\dot{X}^n]$, $m + n = 6$ are evaluated from (10). A similar procedure is applied for closure at the 2nd order. With these modifications in the moment equations, the following polynomial equations similar to (7) are obtained to solve for σ_X^2.

$$\left.\begin{aligned} &2x_0^2 + \frac{\pi S_0}{2\zeta\omega_0^3} + \sigma_X^2 - \frac{3}{x_0^2}\sigma_X^4 = 0 \\ &\frac{\pi S_0}{2\zeta\omega_0^3} + \sigma_X^2 - \frac{1}{x_0^2}\left(\frac{-16x_0^6 + 30\sigma_X^6 + 3\frac{\pi S_0}{2\zeta\omega_0^3}x_0^2\sigma_X^2}{15\sigma_X^2 - x_0^2}\right) = 0 \end{aligned}\right\} \tag{11}$$

As an effect of the proposed modifications of the cumulant neglect closure scheme, an additional $2x_0^2$ is observed in the first equation of (11) compared to (7), whereas $-16x_0^6$ is added to the numerator of the last term in the second equation. The improvements in terms of the pdf and σ_X^2 are illustrated for a wide range of non-linearities below.

4. NUMERICAL EXAMPLE

In what follows, $\frac{\pi S_0}{2\zeta\omega_0} = 1$ and $\omega_0 = 1$. Special attention is drawn to the value $x_0 = \sqrt{10}$ to compare the modified closure scheme results with the ones available in the literature, Bergman et al. (1995). First, the modified analysis capture the double peaked behaviour as shown in Figure 1. The corresponding stationary σ_X^2 calculated using the ordinary and modified closure methods is listed in Table 1 with the associated errors. Non-stationary $\sigma_X^2(t)$ is plotted in Figure 2. Modified Gaussian closure results of Figure 2b are better than the ordinary closure scheme results plotted in Figure 2a, Bergman et al. (1995). Finally, x_0 is varied in the domain of [$0.1 - 5.0$] which covers a wide range of non-linearities, and stationary σ_X^2 obtained using modified and ordinary cumulant neglect closure schemes are compared and it is found that modifications not only capture the peaked behaviour but also perform well for the 2nd moment analysis for a wide range of non-linearities.

5. CONCLUSION

Based on system dynamics, especially in multipeaked problems, modifications in the cumulant neglect closure schemes would provide better results in terms of statistical moments and the jpdf of the response.

6. REFERENCES

[1] Bergman, L.A., Wojtkiewicz, S.F., Johnson, E.A. and Spencer, Jr, B.F. (1995): *Some reflections on the efficiency of moment closure methods*, Computational Stochastic Mechanics '94, Ed: P.D. Spanos, pp. 87-95, A.A. Balkema, Rotterdam.

[2] Stratonovich, R.L. (1963): *Topics in the Theory of Random Noise*, Vol. I, Gordon and Beach Science Publishers, New York.

Table 1. Exact σ_X^2 and associated errors for $x_0^2 = 10$

Method	Stationary σ_X^2	Error
Exact	0.7136	-
Ordinary 2nd order	4.1387	52.5
Ordinary 4th order	5.0000	42.6
Modified 2nd order	10.1977	17.0
Modified 4th order	10.1799	16.8

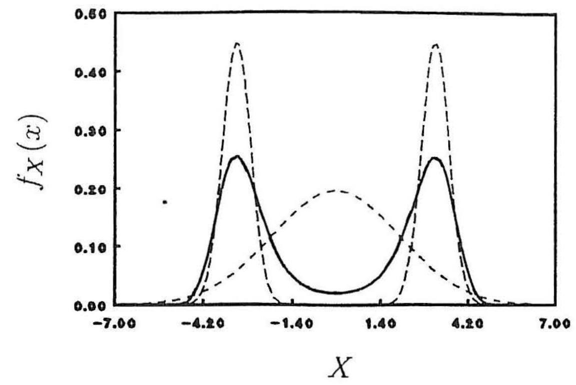

Figure 1. Exact (solid line) stationary pdf of X compared to ordinary (thin dashed line) and modified (thick dashed line) cumulant neglect closure of the 2nd order.

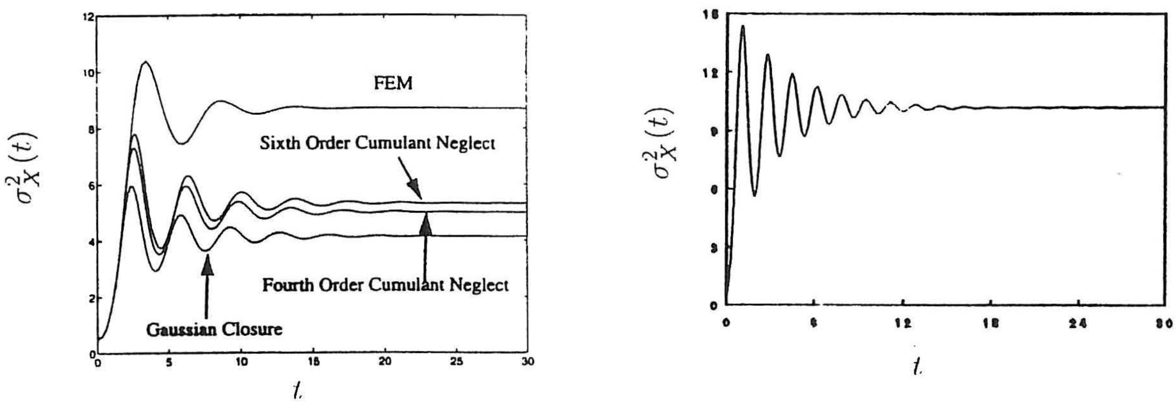

Figure 2. Non-stationary results. a) $\sigma_X^2(t)$ versus time for 2nd, 4th, 6th order closure, Bergman et al. (1995). b) σ_X^2 versus time for modified 2nd order closure.

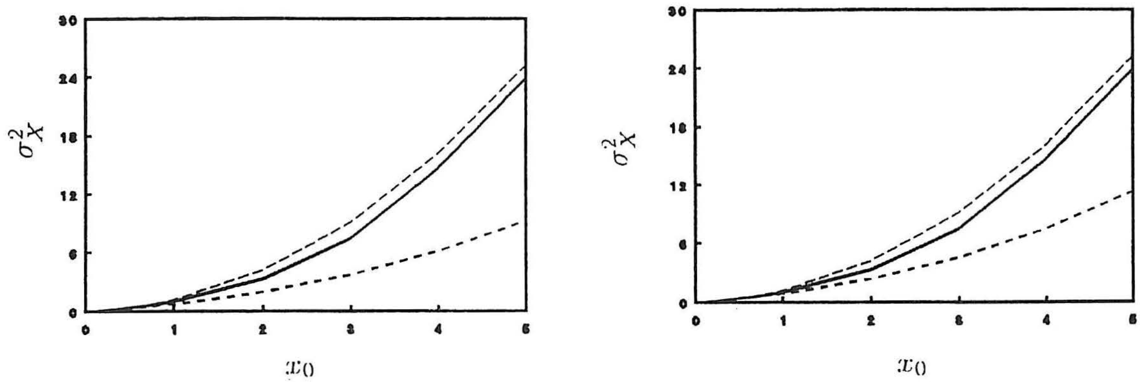

Figure 3. Stationary σ_X^2 versus x_0 a) Exact (solid line), ordinary (thick dashed line) and modified (thin dashed line) Gaussian closure results. b) Exact (solid line), ordinary (thick dashed line) and modified (thin dashed line) cumulant neglect closure of the 4th order.

STRUCTURAL RELIABILITY THEORY SERIES

PAPER NO. 121: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Applications of Interval Mapping for Structural Uncertainties and Pattern Loadings.* ISSN 0902-7513 R9411.

PAPER NO. 122: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Fast Cell-to-Cell Mapping (Path Integration) with Probability Tails for the Random Vibration of Nonlinear and Hysteretic Systems.* ISSN 0902-7513 R9410.

PAPER NO. 123: A. Aşkar, H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Faster Simulation Methods for the Nonstationary Random Vibrations of Nonlinear MDOF Systems.* ISSN 0902-7513 R9405.

PAPER NO. 125: H. I. Hansen, P. H. Kirkegaard & S. R. K. Nielsen: *Modelling of Deteriorating RC-Structures under Stochastic Dynamic Loading by Neural Networks.* ISSN 0902-7513 R9409.

PAPER NO. 126: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Reliability Approximations for MDOF Structures with Random Properties subject to Random Dynamic Excitation in Modal Subspaces.* ISSN 0902-7513 R9440.

PAPER NO. 127: H. U. Köylüoğlu, S. R. K. Nielsen and A. Ş. Çakmak: *A Faster Simulation Method for the Stochastic Response of Hysteretic Structures subject to Earthquakes.* ISSN 0902-7513 R9523.

PAPER NO. 128: H. U. Köylüoğlu, S. R. K. Nielsen, A. Ş. Çakmak & P. H. Kirkegaard: *Prediction of Global and Localized Damage and Future Reliability for RC Structures subject to Earthquakes.* ISSN 0901-7513 R9426.

PAPER NO. 129: C. Pedersen & P. Thoft-Christensen: *Interactive Structural Optimization with Quasi-Newton Algorithms.* ISSN 0902-7513 R9436.

PAPER NO. 130: I. Enevoldsen & J. D. Sørensen: *Decomposition Techniques and Effective Algorithms in Reliability-Based Optimization.* ISSN 0902-7513 R9412.

PAPER NO. 131: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Approximate Forward Difference Equations for the Lower Order Non-Stationary Statistics of Geometrically Non-Linear Systems subject to Random Excitation.* ISSN 0902-7513 R9422.

PAPER NO. 132: I. B. Kroon: *Decision Theory applied to Structural Engineering Problems.* Ph.D.-Thesis. ISSN 0902-7513 R9421.

PAPER 133: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Stochastic Dynamics of Nonlinear Structures with Random Properties subject to Random Stationary Excitation.* ISSN 0902-7513 R9520.

PAPER NO. 134: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Solution of Random Structural System subject to Non-Stationary Excitation: Transforming the Equation with Random Coefficients to One with Deterministic Coefficients and Random Initial Conditions.* ISSN 0902-7513 R9429.

PAPER NO. 135: S. Engelund, J. D. Sørensen & S. Krenk: *Estimation of the Time to Initiation of Corrosion in Existing Uncracked Concrete Structures.* ISSN 0902-7513 R9438.

STRUCTURAL RELIABILITY THEORY SERIES

PAPER NO. 136: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Solution Methods for Structures with Random Properties subject to Random Excitation.* ISSN 0902-7513 R9444.

PAPER NO. 137: J. D. Sørensen, M. H. Faber & I. B. Kroon: *Optimal Reliability-Based Planning of Experiments for POD Curves.* ISSN 0902-7513 R9455.

PAPER NO. 138: S.R.K. Nielsen & P.S. Skjærbæk, H.U. Köylüoğlu & A.Ş. Çakmak: *Prediction of Global Damage and Reliability based upon Sequential Identification and Updating of RC Structures subject to Earthquakes.* ISSN 0902-7513 R9505.

PAPER NO. 139: R. Iwankiewicz, S. R. K. Nielsen & P. S. Skjærbæk: *Sensitivity of Reliability Estimates in Partially Damaged RC Structures subject to Earthquakes, using Reduced Hysteretic Models.* ISSN 0902-7513 R9507.

PAPER NO 141: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Uncertain Buckling Load and Reliability of Columns with Uncertain Properties.* ISSN 0902-7513 R9524.

PAPER NO. 142: S. R. K. Nielsen & R. Iwankiewicz: *Response of Non-Linear Systems to Renewal Impulses by Path Integration.* ISSN 0902-7513 R9512.

PAPER NO. 145: H. U. Köylüoğlu, S. R. K. Nielsen, Jamison Abbott and A. Ş. Çakmak: *Local and Modal Damage Indicators for Reinforced Concrete Shear Frames subject to Earthquakes.* ISSN 0902-7513 R9521

PAPER NO. 146: P. H. Kirkegaard, S. R. K. Nielsen, R. C. Micaletti and A. Ş. Çakmak: *Identification of a Maximum Softening Damage Indicator of RC-Structures using Time-Frequency Techniques.* ISSN 0902-7513 R9522.

PAPER NO. 147: R. C. Micaletti, A. Ş. Çakmak, S. R. K. Nielsen & P. H. Kirkegaard: *Construction of Time-Dependent Spectra using Wavelet Analysis for Determination of Global Damage.* ISSN 0902-7513 R9517.

PAPER NO. 148: H. U. Köylüoğlu, S. R. K. Nielsen & A. Ş. Çakmak: *Hysteretic MDOF Model to Quantify Damage for TC Shear Frames subject to Earthquakes.* ISSN 1395-7953 R9601.

PAPER NO. 149: P. S. Skjærbæk, S. R. K. Nielsen & A. Ş. Çakmak: *Damage Location of Severely Damaged RC-Structures based on Measured Eigenperiods from a Single Response.* ISSN 0902-7513 R9518.

PAPER 151: H. U. Köylüoğlu & S. R. K. Nielsen: *System Dynamics and Modified Cumulant Neglect Closure Schemes.* ISSN 1395-7953 R9603.

Department of Building Technology and Structural Engineering
Aalborg University, Sohngaardsholmsvej 57, DK 9000 Aalborg
Telephone: +45 98 15 85 22 Telefax: +45 98 14 82 43