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STRUCTURAL RELIABILITY THEORY
PAPER NO. 175

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Stochastic Properties of Plasticity Based Constitutive Law for Concrete

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Abstract

The purpose of this paper is to obtain a stochastic model for the parameters in a constitutive model for concrete based on associated plasticity theory and with emphasis placed on the pre-failure range. The constitutive model is based on a Drucker Prager yield surface augmented by a Rankine cut-off criterion. The statistics of the material parameters are obtained by applying biaxial test results for plane concrete slabs to the constitutive model using the maximum likelihood method. Response surfaces are used to obtain the statistics for a parameter depending on other previously obtained parameters. Finally an illustrative example of a reliability analysis of a plane concrete panel is given.

1 Introduction

Over the last decade probabilistic numerical tools such as the Stochastic Finite Element Method have been made accessible, making the reliability of structural systems predictable. Among the early contributions mention can be made of (Kiureghian and Ke 1988; Deodatis 1991) all considering linear elastic problems. Material non-linearities have only in the last few years been considered extensively by the research community, e.g. by (Frangopol et al. 1996). Almost no effort has been put into actually obtaining the statistics of the material parameters.

It is the scope of this paper to obtain the stochastic model for the parameters in a relatively simple constitutive model for concrete based on associated plasticity theory. The parameter estimation procedure is based on the maximum likelihood method (Bard 1974), and response surface techniques (Myers 1971) are adopted to estimate the statistics of the material parameters, especially those which depend on other parameters.

2 Concrete constitutive model

In this paper an associated Drucker Prager yield function f augmented by a Rankine tension cut-off criterion

$$f = \begin{cases} \alpha\frac{I_1}{\tau} + \beta\frac{\sqrt{J_2}}{\tau} - 1 & \texttt{for} \quad \sigma_1 \leq \gamma f_c \\ \frac{\sigma_1}{f_c} - \gamma & \texttt{else} \end{cases} \tag{1}$$

where τ is an isotropic hardening parameter, σ_1 is the first principal stress, α, β and γ are material parameters, I_1 is the first stress invariant and J_2 is the second deviatoric stress invariant. The yield function is also used as a plastic potential function.

The hardening parameter τ is given by an isotropic strain hardening rule as proposed by (Labbane et al. 1993)

$$\tau = f_c[1.0 - 0.75\exp(-B\varepsilon_p)] \tag{2}$$

where f_c is the uniaxial compressive strength of the concrete material, B is a material parameter and ε_p is the effective plastic strain, which can be obtained from the incremental plastic strain, $d\varepsilon_{ij}^p$ by

$$d\varepsilon_p = \sqrt{d\varepsilon_{ij}^p d\varepsilon_{ij}^p} \tag{3}$$

3 Parameter estimation procedure

The constitutive model is fitted to the biaxial test results from (Kupfer and Hilsdorf 1969). They loaded a number of quadratic plane concrete specimens using proportional loading control and monitored the stress-strain relations for a concrete material with a mean value of the uniaxial compressive strength, $f_c = 32.06$ MPa. The test results also consist of a number of recorded biaxial failure stress conditions with the uniaxial compressive strength varying from 18.6 MPa to 57.6 MPa.

The statistics of the material parameters, mean values and covariance functions, are obtained using the maximum likelihood method (Bard 1974). With $\boldsymbol{\theta} = [\theta_1 \ldots \theta_k]^T$ defined as a vector of k material parameters to be estimated, the residual vector $\boldsymbol{e}_\mu(\boldsymbol{\theta})=[e_{\mu,1}(\boldsymbol{\theta}) \ldots e_{\mu,m}(\boldsymbol{\theta})]^T$, is introduced as the difference between a measured response and the response obtained by a constitutive model, consisting of m equations, for a the set of test data no. μ. Assuming a Gaussian distribution and a general, unknown covariance of $\boldsymbol{e}_\mu(\boldsymbol{\theta})$, the maximum likelihood estimate $\boldsymbol{\theta}^*$ of the parameters $\boldsymbol{\theta}$ is found by minimizing

$$\Phi(\boldsymbol{\theta}) = \frac{n}{2}\log[\det \boldsymbol{M}(\boldsymbol{\theta})] \tag{4}$$

where n is the number of available sets of test data and $\boldsymbol{M}(\boldsymbol{\theta})$ is the moment matrix of the residuals given as

$$\boldsymbol{M}(\boldsymbol{\theta}) = \sum_{\mu=1}^{n} \boldsymbol{e}_\mu(\boldsymbol{\theta})\boldsymbol{e}_\mu^T(\boldsymbol{\theta}) \tag{5}$$

The model uncertainty expressed as the covariance of the parameters is then obtained by the inverse of the Hessian matrix of (4),

$$\boldsymbol{V}(\boldsymbol{\theta}) = \boldsymbol{H}^{-1} \tag{6}$$

with the elements given as $H_{ij}=\partial^2\Phi(\boldsymbol{\theta})/\partial\theta_i\partial\theta_j|_{\boldsymbol{\theta}^*}$.

When several stress-strain relations are considered, each will be considered as one realization in the fitting procedure to avoid the statistics being too dependent on the number of digitized data. Based on the results from all stress-strain relations estimates of the mean values, $\boldsymbol{\theta}^*$ and covariances, $\boldsymbol{V}(\boldsymbol{\theta})$ of the parameters $\boldsymbol{\theta}$ are then obtained.

4 Linear elastic parameters

The linear elastic parameters E and ν are obtained for a pure linear elastic behaviour corresponding to $f < 0$. The error residual $e_{\mu}^{el}(E, \nu)$, for the set of test data no. μ, can then be obtained using

$$e_{\mu}^{el}(E, \nu) = \boldsymbol{\sigma}_{ij,\mu} - D_{ijkl}(E, \nu)\epsilon_{kl,\mu} \tag{7}$$

A stress cut-off level S is introduced below which the linear elastic parameters are fitted to the stress-strain relations. The cut-off level is illustrated for a set of test results where $\sigma_{11}/\sigma_{22}= -1/-0.52$ in figure 1.

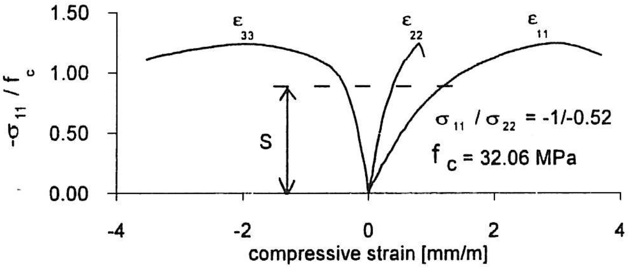

Figure 1: *Stress-strain relation from Kupfer tests, shown together with the threshold level* S.

In figure 2 $\sigma_E=\sqrt{V_{E,E}}$ and $\sigma_{\nu}=\sqrt{V_{\nu,\nu}}$ are shown as functions of the cut-off level S, for the test results shown in figure 1. For a low value of S, the number of digitized data points is insufficient to obtain an estimate of the statistics, and for a comparatively high value of S, the non-linear part of the stress-strain relation is falsely included in the fitting procedure. However, for a value of S of about 0.5 MPa the value of the model uncertainty calculated by (6) seems reasonably constant and corresponding to this, an estimate of the parameters is obtained. A similar trend exists for the remaining 8 stress-strain relation where the same procedure is carried out to obtain E_i^* and ν_i^* for each test. When obtaining the statistics among these results one obtains $[E^*, \nu^*]^T=[30.6{\cdot}10^9, 0.17]^T$, $[\sigma_E, \sigma_{\nu}]^T=[1.98 \cdot 10^9, 0.0044]^T$ and $\rho_{E,\nu} = 0.16$.

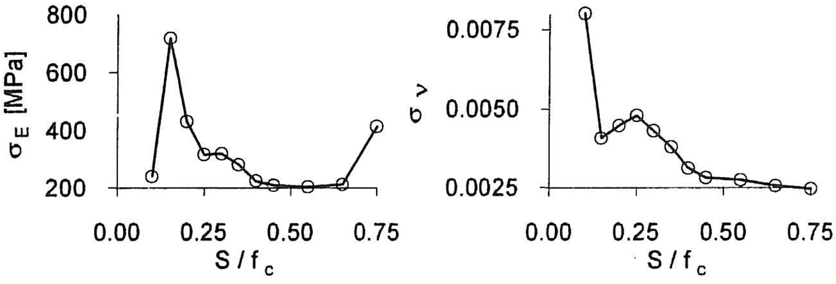

Figure 2: *Statistics* σ_E *and* σ_{ν} *plotted against cut-off level* S.

5 Estimation of the yield surface

The test data describing the stress state at failure are used to fit the yield surface parameters α, β and γ. By plotting the normalized stresses σ_{22}/f_c against σ_{11}/f_c, it can be seen from figure 3 that the parameters of the failure surface are almost independent of f_c. It is then reasonable to treat the uncertainty when fitting the failure surface as model uncertainty and to use (6) directly to estimate $V(\alpha, \beta, \gamma)$. f_c is considered to be deterministic when using these test results.

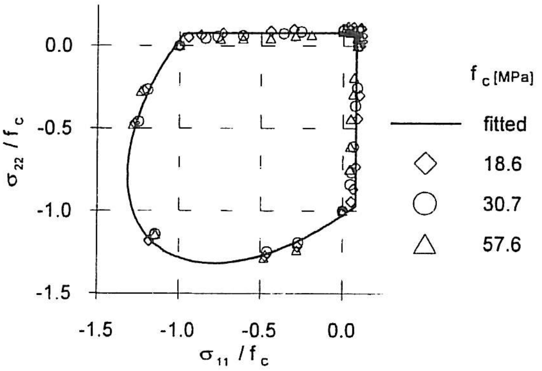

Figure 3: *Normalized failure data, Kupfer data and fitted results.*

The compressive part of the yield surface with parameters α and β is estimated, using the Drucker-Prager yield surface. In the tensile part γ is estimated using the Rankine criterion. The two parts are therefore assumed to be independent of each other, i.e. $\rho_{\alpha,\gamma} = \rho_{\beta,\gamma} = 0$ and the following error residuals are used

$$\begin{aligned} e^{yi}_{\mu,c} &= 1 - \alpha \tfrac{I_{1,\mu}}{f_{c,\mu}} - \beta \tfrac{\sqrt{J_{2,\mu}}}{f_{c,\mu}} \quad for \quad \sigma_{1,\mu} < 0 && (8) \\ e^{yi}_{\mu,t} &= \gamma - \tfrac{\sigma_{1,\mu}}{f_{c,\mu}} \quad for \quad \sigma_{1,\mu} > 0 && (9) \end{aligned}$$

Carrying out the analysis, the following results are obtained $[\alpha^*, \beta^*, \gamma^*]^T$= $[0.123, 1.902, 0.077]^T$, $[\sigma_\alpha, \sigma_\beta, \sigma_\gamma]^T$=$[0.017, 0.046, 0.0044]^T$ and $\rho_{\alpha,\beta}$=0.97. The failure surface corresponding to the mean values of the fitted parameters is shown as a solid line in figure 3.

6 Hardening parameter

The remaining parameter B in (2) can be obtained by formulating the following error residual

$$e^{ha}_{\mu}(B) = \tau_\mu - f_{c,\mu}[1.0 - 0.75 \exp(-B\varepsilon_{p,\mu})] \tag{10}$$

ε_p is obtained from (3) and the available test data using the plastic strain increment $d\varepsilon^p_{ij} = d\varepsilon_{ij} - D^{-1}_{ijkl} d\sigma_{ij}$. By doing this the variables E and ν, which are uncertain, have to be known. τ can be obtained by demanding that all data points in the elasto-plastic range, $\tau > 0.25 \cdot f_c$ are on the yield function, i.e. $f = 0$ using (1). This requires that α and β are known. In figure 4 τ is plotted as a function of ε_p. All curves are obtained using the mean values of E, ν, α and β. The dashed curves present the hardening behaviour for each of the stress-strain relations considered, whereas the solid curve represents the result of the fitting procedure using the mean value of the hardening parameter B.

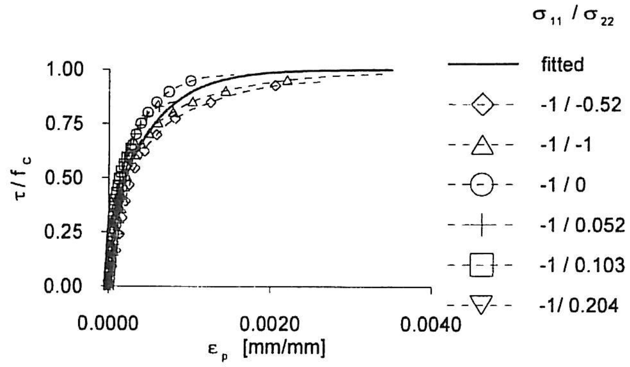

Figure 4: *Hardening behaviour, fitted and extracted from Kupfer tests.*

As the statistics of B are found conditional on the other parameters, the correlations $\rho_{E,B}$, $\rho_{\nu,B}$, $\rho_{\alpha,B}$ and $\rho_{\beta,B}$ are indirectly included, as B^* and σ_B are calculated as functions of E, ν, α and β.

The statistics of B are determined by response surfaces as functions of E, ν, α and β using 2. order polynomial approximations. The design points are chosen according to a CCD design space, a first order factorial design augmented by additional points to allow estimation of the coefficients of a second order surface (Myers 1971). An illustration of the results of this analysis is shown in figure 5, where B^* is plotted as a function of each of the four parameters with the remaining parameters kept at their mean values.

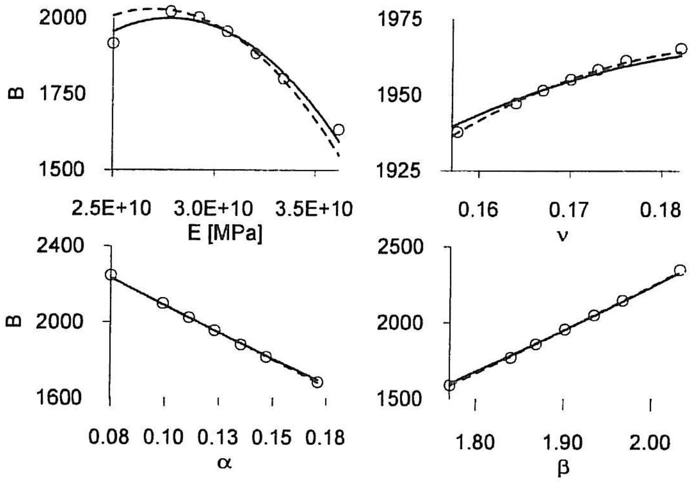

Figure 5: *CCD app. and calculated values of* B^*. *o:calculated; — : CCD with corner* 1σ *from mean; - - : CCD with corner* 2σ *from mean.*

It is observed from the figure that a good fit between the CCD approximation of B^* and the directly calculated value is observed in the intervals considered in the figure. A similar result is obtained, when a response surface approximation for σ_B is determined. B is the parameter in the constitutive model with the comparatively largest uncertainty, namely with a coefficient of variation of approximately 0.3. This can be attributed to the fact that the assumptions concerning associated flow, isotropic hardening and choice of hardening function τ are included in this parameter.

7 Stress-strain relations

To give an idea of the fit of the constitutive model when predicting stress-strain behaviour 2 different proportional loading conditions are simulated corresponding to the test results, $\sigma_{11}/\sigma_{22}=-1/-0.52$, compressive-compressive, $\sigma_{11}/\sigma_{22}=-1/0.052$, compressive-tensile, shown in the following figures, with mean values of all material parameters and $f_c = 32.06$ MPa.

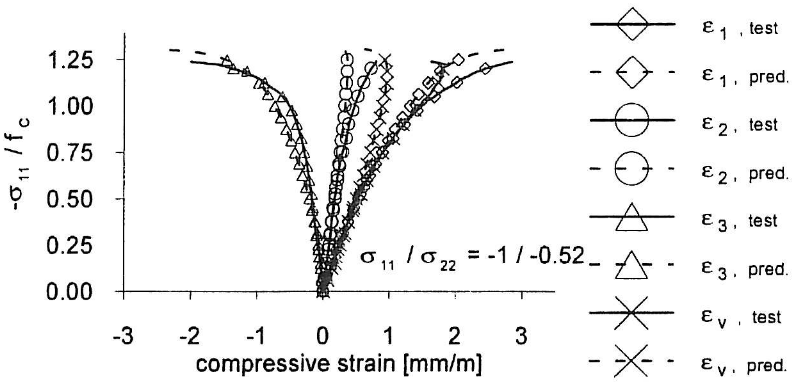

Figure 6: *Comparison of Kupfer test result with predicted stress-strain behaviour.*

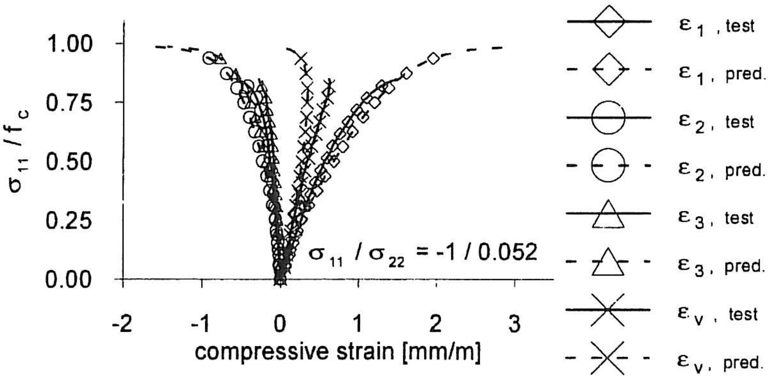

Figure 7: *Comparison of Kupfer test result with predicted stress-strain behaviour.*

A comparatively good prediction of the stress-strain relations is obtained, when the 3 strains, ε_{11}, ε_{22} and ε_{33} are considered. However, when the volumetric strain, $\varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ is calculated from both the tests and the predictions, a more crucial deviation exists. This can be contributed to the fact that an associated flow rule has been adapted, not accounting for volumetric dilatation.

8 Reliability analysis of concrete

In this section an example of a reliability analysis of a concrete structure is presented. The FORM reliability index β is obtained iteratively on the basis of a limit state function, which can be written g($\boldsymbol{x}$)=0, where $\boldsymbol{x} = (x_1, \ldots, x_n)$ are realizations of stochastic variables $\boldsymbol{X} = (X_1, \ldots, X_n)$ (Madsen et al. 1986). A 0.1 m thick quadratic concrete panel with side length 1.0 m is investigated, see figure 8. In this study, the limit state function is chosen as $g = 2 \cdot 10^3 - |U_{25}|$ where U_{25} is the vertical displacement at node 25.

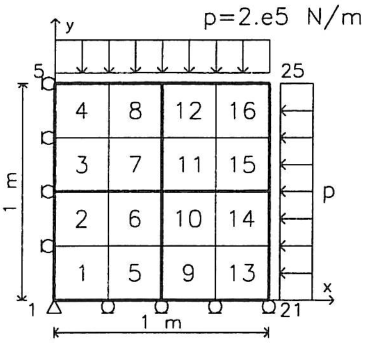

Figure 8: *Concrete Panel, deterministic and stochastic finite element mesh.*

The finite element mesh consists of 16 deterministic and 4 stochastic elements. The stochastic fields are modelled as Gaussian. It is assumed that the coefficient of variation of f_c is 0.1 with $\mu_{f_c} = 32.06$ MPa. All correlation coefficients which have not previously been mentioned are assumed to be 0. The model consists of 7 stochastic variables per element, E,ν,α,β,γ,B and f_c with a total of 28 stochastic variables.

The following cross correlation function is used (Frangopol et al. 1996)

$$\rho_{x_{i,b},x_{j,c}} = \exp[-(\Delta x/a)^2 - (\Delta y/a)^2] \tag{11}$$

where $x_{i,b}$ is the variable i in element b, Δx and Δy are the horizontal and vertical distance between elements b and c, and a is a correlation length parameter, which is assumed equal for all stochastic fields considered. The result of a reliability analysis performed with a varying value of a is shown on figure 9. As expected, it is seen, that the reliability index decreases with increasing values of a.

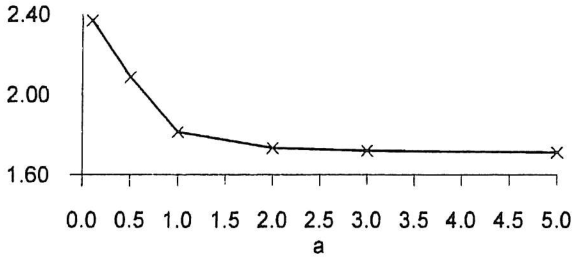

Figure 9: *Result of reliability analysis for concrete panel with varying correlation length parameter a.*

9 Conclusion

The statistics of the material parameters in a nonlinear plasticity based constitutive model for concrete have been obtained using the maximum likelihood method. The constitutive equations are fitted to available test data. First the statistics involving the linear elastic and the failure part of the constitutive relation have been obtained. The hardening behaviour has then been determined by response surface approximations.

Finally, an example of a reliability analysis of a concrete structure is given. Future developments could be to implement a fracture based softening model to cope with post peak behaviour. Further, Bayesian statistics can be used in order to include prior knowledge on the material parameters.

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