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STRUCTURAL RELIABILITY THEORY
PAPER NO. 167

Submitted to ICOSSAR '97

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Dynamic Systems Driven By Non-Poissonian Impulses: Markov Vector Approach

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ABSTRACT: Dynamic systems under random trains of impulses driven by renewal point processes are studied. Then the system state variables no longer form a Markov vector as it is in the case of Poisson impulses. A general format is given for replacing an ordinary renewal process by an equivalent Poisson process at the expense of the introduction of auxiliary state variables. A technique is devised for truncating the hierarchy of stochastic equations governing the auxiliary state variables. For the generalized Erlang process, suitable for approximating a wide class of renewal processes, the technique is developed for modelling it, via suitable choice of parameters, with the help of a Poisson driven process. The theory is illustrated for a Duffing oscillator under the impulses driven by the generalized Erlang process of the order $k = 2$ approximating an original renewal process with a lognormally distributed interarrival times. The moment equations for the augmented Poisson driven system are derived and closed by an ordinary cumulant neglect closure at the order $N = 4$. The obtained moments are compared with those obtained by Monte Carlo simulations for both the original process with lognormally distributed interarrival times and for the approximating generalized Erlang process of the order $k = 2$.

Keywords: Stochastic Dynamics, Renewal Processes, Markov Vector Approach, Moment Technique.

1 INTRODUCTION

In some problems of engineering the discontinous part of the excitation may be considered as a train of impulses arriving at random instants of time. The dynamic response of non-linear systems with polynomial non-linearities subjected to a compound Poisson process have previously been considered by the authors in combination with ordinary cumulant neglect closure scheme, [1], or with a modified cumulant neglect closure scheme taking the discrete probability into account that no impulses have yet arrived, [2].

In ref. [3] the moment equation technique was extended to a class of Erlang renewal process driven systems. Then the interarrival time, I, is Gamma distributed $I \sim G(k-1, \nu)$, where k is an integer. The renewal driven system was reduced to a Poisson process driven system at the expense of the introduction of $k-1$ auxiliary state variables. The augmented state vector consisting of the structural state variables and the auxiliary state variables then form a Markov vector. In [4] the probability density functions of the response of such a system is obtained by path integration.

The main drawback of the method described in [3] is that only a rather limited number of interarrival time distributions may be modelled, because only two free parameters are available, of which k moreover is integer valued. In the present paper a generalization of the Erlang process is considered where the k subintervals of the interarrival time between impulses are assumed to be exponentially distributed with different parameters. Then a much larger class of interarrival time distributions can be modelled, simply because k free parameters are available. A general format for reducing a regular counting process to a compound Poisson process is derived, and it is shown how the generalized Erlang process fits into this format. The auxiliary state variables introduced in the format turn out to be a linear transformation of those applied in [3], when the indicated generalized Erlang process reduces to an ordinary one. The applicability of the described technique is illustrated by an example problem, where an original renewal process with lognormally distributed interarrival times is modelled by a generalized Erlang process of the order $k = 2$.

2 STATEMENT OF THE PROBLEM FOR ARBITRARY RENEWAL DRIVEN PROCESSES

Consider a general multi-degree-of-freedom nonlinear dynamical system under a random train of impulses driven by a renewal point process. The structural state vector, $\mathbf{Z}_1(t)$, consisting of the generalized displacements and velocities, is governed by the set of equations of motion

$$\begin{aligned} &\frac{d}{dt}\mathbf{Z}_1(t) = \mathbf{c}_1\left(\mathbf{Z}_1(t), t\right) \\ &+\mathbf{d}_1(\mathbf{Z}_1(t), t)\sum_{i=1}^{R(t)} P_{i,R}\delta\left(t - t_{i,R}\right), \quad t > 0, \\ &\mathbf{Z}_1(0) = \mathbf{z}_{1,0}, \end{aligned} \tag{1}$$

where $\mathbf{c}_1\big(\mathbf{Z}_1(t), t\big)$ is the drift vector, $\mathbf{d}_1\big(\mathbf{Z}_1(t), t\big)$ is an analogue of the diffusion vector in white noise driven problems and $\mathbf{z}_{1,0}$ denotes the vector of initial values of the structural variables $\mathbf{Z}_1(t)$ at the initial time 0.

The occurrence times $t_{i,R}$ of Dirac delta impulses are distributed according to an ordinary renewal counting process $\{R(t), t \in [0,\infty[\}, \Pr\{R(0) = 0\} = 1$. The mark variables $P_{i,R}$ are assumed to be independent, identically distributed, random variables, independent of the occurrence times $t_{i,R}$ and having the distribution as a random variable P. It is obvious that since the renewal process is not a process with independent increments, the state vector $\mathbf{Z}_1(t)$ governed by the equation (1) is not a Markov process.

Consider an arbitrary regular renewal counting process $R(t)$. Assume that its increments can be expressed as

$$dR(t) = \rho(N(t))dN(t), \tag{2}$$

where $N(t)$ is a homogeneous Poisson counting process and $\rho(N(t))$ is a suitable zero-memory transformation to be determined later. The nth degree product density $f_n(t_1, \ldots, t_n)$ of the point process $R(t)$ is defined as, [5]

$$\begin{aligned} &f_n(t_1, \ldots, t_n)dt_1 \ldots dt_n) = E\big[dR(t_1)\cdots dR(t_n)\big] \\ &= E\Big[\rho(N(t_1))dN(t_1)\cdots \rho(N(t_n))dN(t_n)\Big]. \end{aligned} \tag{3}$$

It may be shown that by suitable splitting of expectations the expression (3) for any renewal process may be recast as

$$\begin{aligned} &f_n(t_1, \ldots, t_n)dt_1 \cdots dt_n = \\ &E\big[\rho(N(t_1))\big] \cdots E\big[\rho(N(t_n - t_{n-1}))\big]\nu^n dt_1 \cdots dt_n. \end{aligned} \tag{4}$$

Seeing that $\nu E\big[\rho(N(t))\big] = h_o(t)$, which is the ordinary renewal density, the known result for the nth degree product density of the ordinary renewal process is obtained

$$f_n(t_1, \ldots, t_n) = h_o(t_1)h_o(t_2 - t_1)\cdots h_o(t_n - t_{n-1}) \tag{5}$$

Let us introduce

$$Y_j(t) = \rho(N(t) + j - 1) \quad , \quad j = 1, 2, \ldots, k \tag{6}$$

as new auxiliary state variables. The stochastic differential equations governing the time evolution of these state variables can be written as

$$\begin{aligned} &dY_j(t) = \rho((N(t)) + j - 1 + dN(t)) \\ &-\rho\big(N(t) + j - 1\big) \\ &= \Big(\rho((N(t)) + j)) - \rho\big(N(t) + j - 1\big)\Big)dN(t) \quad \Rightarrow \end{aligned}$$

$$\left.\begin{aligned} dY_1(t) &= \Big(Y_2(t) - Y_1(t)\Big)dN(t) \\ dY_2(t) &= \Big(Y_3(t) - Y_2(t)\Big)dN(t) \\ &\vdots \\ dY_{k-1}(t) &= \Big(Y_k(t) - Y_{k-1}(t)\Big)dN(t) \end{aligned}\right\} \tag{7}$$

Validity of the statements given in the first part of eqs. (7) is proved by showing that it is valid for either of $dN(t) = 0$ or $dN(t) = 1$. The hierarchy of stochastic differential equations (7) cannot be closed unless $Y_k(t)$ can be expressed in terms of the previous auxiliary state variables. The following linear dependency is adopted, which can be shown to be valid in the case of an ordinary Erlang process

$$Y_k(t) = -\frac{1}{a_k}\Big(a_0 + a_1 Y_1(t) + \cdots + a_{k-1}Y_{k-1}(t)\Big), \tag{8}$$

where $a_j \in R$. Equations (7) then attain the form

$$\left.\begin{aligned} dY_1(t) &= \Big(Y_2(t) - Y_1(t)\Big)dN(t) \\ dY_2(t) &= \Big(Y_3(t) - Y_2(t)\Big)dN(t) \\ &\vdots \\ dY_{k-1}(t) &= -\Big(\tfrac{a_0}{a_k} + \tfrac{a_1}{a_k}Y_1(t) + \cdots + \tfrac{a_{k-2}}{a_k}Y_{k-2}(t) + \\ &\qquad \Big(1 + \tfrac{a_{k-1}}{a_k}\Big)Y_{k-1}(t)\Big)dN(t) \end{aligned}\right\} \tag{9}$$

Equation (8) implies that $\rho(N(t))$ must fulfill the difference equation

$$a_0 + \sum_{j=1}^{k} a_j \rho(N(t)+j-1) = 0. \tag{10}$$

The solution of (10) is

$$\rho(N(t)) = b_0 + \sum_{j=1}^{k-1} b_j \lambda_j^{N(t)} = \sum_{j=0}^{k-1} b_j \lambda_j^{N(t)}, \tag{11}$$

where $\lambda_0 = 1$ and

$$b_0 = -\frac{\frac{a_0}{a_k}}{\frac{a_1}{a_k} + \cdots + \frac{a_{k-1}}{a_k} + 1}. \tag{12}$$

$b_j \in C,\ j = 1, \ldots, k-1$ are arbitrary constants and $\lambda_j \in C,\ j = 1, \ldots, k-1$ denote the solutions of the characteristic equation

$$\lambda^{k-1} + \frac{a_{k-1}}{a_k}\lambda^{k-2} + \cdots + \frac{a_2}{a_k}\lambda + \frac{a_1}{a_k} = 0. \tag{13}$$

If $\mid \lambda_j \mid \neq 1$, the corresponding term in (11) either extincts or explodes. Hence if the point process is assumed to be homogeneous, it is necessary that the eigenvalues all have the magnitude equal to 1, so

$$\lambda_j = \exp\Big(i\gamma_j\Big), \quad \gamma_j \in R/\{0\}, \quad j = 1, 2, \ldots, k-1 \tag{14}$$

The product density becomes, cf. (3) and (11)

$$f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n = \sum_{j_1=0}^{k-1} \cdots \sum_{j_n=0}^{k-1} b_{j_1} \cdots b_{j_n} \times E\Big[\lambda_{j_1}^{N(t_1)} \cdots \lambda_{j_n}^{N(t_n)} dN(t_1) \cdots dN(t_n)\Big]. \tag{15}$$

The expectation entering (15) may be recast as

$$E\Big[\lambda_{j_1}^{N(t_1)} \lambda_{j_2}^{N(t_1)+N(t_1,t_2)} \cdots \times \lambda_{j_n}^{N(t_1)+N(t_1,t_2)+\cdots+N(t_{n-1},t_n)} dN(t_1) \cdots dN(t_n)\Big] = E\Big[\big(\lambda_{j_1}\lambda_{j_2} \cdots \lambda_{j_n}\big)^{N(t_1)} \big(\lambda_{j_2} \cdots \lambda_{j_n}\big)^{N(t_1,t_2)} \cdots \times \big(\lambda_{j_{n-1}}\lambda_{j_n}\big)^{N(t_{n-2},t_{n-1})} \lambda_{j_n}^{N(t_{n-1},t_n)} dN(t_1) \cdots dN(t_n)\Big] \tag{16}$$

where $N(t_r, t_{r+1}) = N(t_{r+1}) - N(t_r)$. Splitting of the expectation, with due account of the overlapping of $n-1$ intervals, may be performed as follows

$$E\Big[\big(\lambda_{j_{r+1}} \cdots \lambda_{j_n}\big)^{N(t_r,t_{r+1})} dN(t_r)\Big] = E\Big[\big(\lambda_{j_{r+1}} \cdots \lambda_{j_n}\big)^{N(t_r,t_r+dt_r)+N(t_r+dt_r,t_{r+1})} dN(t_r)\Big] = E\Big[\big(\lambda_{j_{r+1}} \cdots \lambda_{j_n}\big)^{N(t_r,t_r+dt_r)} dN(t_r)\Big] \times E\Big[\big(\lambda_{j_{r+1}} \cdots \lambda_{j_n}\big)^{N(t_r+dt_r,t_{r+1})}\Big] = \lambda_{j_{r+1}} \cdots \lambda_{j_n} \nu E\Big[\big(\lambda_{j_{r+1}} \cdots \lambda_{j_n}\big)^{N(t_r+dt_r,t_{r+1})}\Big] dt_r, \tag{17}$$

for $r = 1, \ldots, n-1$. Consequently the expression for the product density becomes

$$f_n(t_1, \ldots, t_n) = \nu^n \sum_{j_1=1}^{k} \cdots \sum_{j_n=1}^{k} b_{j_1} \cdots b_{j_n} \times \lambda_{j_2}\lambda_{j_3}^2 \cdots \lambda_{j_{n-1}}^{n-2} \lambda_{j_n}^{n-1} \exp\Big(\nu(t_n - t_{n-1})(\lambda_{j_n} - 1)\Big) \times \exp\Big(\nu(t_{n-1} - t_{n-2})(\lambda_{j_n}\lambda_{j_{n-1}} - 1)\Big) \cdots \times \exp\Big(\nu(t_2 - t_1)(\lambda_{j_2} \cdots \lambda_{j_n} - 1)\Big) \times \exp\Big(\nu t_1(\lambda_{j_1}\lambda_{j_2} \cdots \lambda_{j_n} - 1)\Big) \tag{18}$$

3 MODELLING TECHNIQUE FOR RENEWAL PROCESSES

Assume that the interarrival time I is formed as a sum of k independent random variables E_j, i.e.

$$I = \sum_{j=1}^{k} E_j, \tag{19}$$

where $E_j \sim E(\nu_j)$ is assumed to be an exponentially distributed random variable with parameter ν_j. If $k = 1$, then $I \sim E(\nu)$ and a homogeneous Poisson counting process is obtained. If $\nu_1 = \nu_2 = \cdots = \nu_k = \nu$ then $I \sim \mathrm{G}(k-1, \nu)$ and an Erlang renewal process is obtained. The transformation of such a process to an equivalent Poisson process at the expense of introduction of extra auxiliary state variables similar to Y_j have been considered in previous papers by the authors [3,4]. Allowing for different parameters of E_j a much larger class of interarrival time distributions can be modelled by suitable choice of the parameters ν_j. The Laplace transform of the p.d.f. $f_I(t)$ of the interarrival time I is

$$f_I(s) = \prod_{j=1}^{k} f_{I_j}(s) = \prod_{j=1}^{k} \frac{\nu_j}{s + \nu_j}. \tag{20}$$

The renewal density then becomes , [5]

$$
\begin{aligned}
h_o(t) &= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{f_I(s)}{1-f_I(s)} e^{st} ds \\
&= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\prod_{j=1}^{k}\nu_j}{\prod_{j=1}^{k}(s+\nu_j)-\prod_{j=1}^{k}\nu_j} e^{st} ds \\
&= \prod_{j=1}^{k}\nu_j \cdot \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{e^{st}}{\prod_{j=0}^{k-1}(s-s_j)} ds \\
&= \prod_{j=1}^{k}\nu_j \cdot \sum_{l=0}^{k-1} \frac{e^{s_l t}}{\prod_{\substack{j=0\\ j\neq l}}^{k-1}(s_l-s_j)}, \qquad (21)
\end{aligned}
$$

where s_j, $j = 0,\dots,k-1$ are the roots of the denominator polynomial

$$
\prod_{j=1}^{k}(s+\nu_j) - \prod_{j=1}^{k}\nu_j = 0 \qquad (22)
$$

and β is chosen arbitrarily, so $\beta > \mathrm{Re}(s_j)$, $j = 0,\dots,k-1$. Notice that $s_0 = 0$ is always a root of (22). The last integral of (21) can be evaluated by the method of residues. In the same way the p.d.f of the interarrival time of impulses can be obtained via inverse Laplace transformation of (20)

$$
\begin{aligned}
f_I(t) &= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\prod_{j=1}^{k}\nu_j}{\prod_{j=1}^{k}(s+\nu_j)} e^{st} ds \\
&= \prod_{j=1}^{k}\nu_j \cdot \sum_{l=1}^{k} \frac{e^{-\nu_l t}}{\prod_{\substack{j=1\\ j\neq l}}^{k}(\nu_j-\nu_l)}. \qquad (23)
\end{aligned}
$$

The parameters $\nu_1,\dots,\nu_k$ should be chosen, so (23) fits a given target distribution, which will be illustrated in the example problem. Next, the conditions are investigated under which the present renewal process can be represented by a compound Poisson driven process as specified by eqs. (9). The first step is to fit the product densities of 1st degree following from (18)

$$
f_1(t) = \nu \sum_{l=0}^{k-1} b_l e^{\nu t(\lambda_l - 1)}. \qquad (24)
$$

According to (5) eqs. (21) and (24) should be equal, which provides the following solution for b_l and λ_l, $l = 0,\dots,k-1$

$$
b_l = \frac{1}{\nu}\frac{\prod_{j=1}^{k}\nu_j}{\prod_{\substack{j=0\\ j\neq l}}^{k-1}(s_l-s_j)} = \frac{1}{\nu}\frac{1}{\sum_{j=1}^{k}\frac{1}{s_l+\nu_j}}, \qquad (25)
$$

$$
\lambda_l = 1 + \frac{s_l}{\nu}. \qquad (26)
$$

In the last statement of (25) the following result has been utilized, cf. (22)

$$
\begin{aligned}
\prod_{\substack{j=0\\ j\neq l}}^{k-1}(s_l-s_j) &= \lim_{s\to s_l}\frac{\prod_{j=1}^{k}(s+\nu_j)-\prod_{j=1}^{k}\nu_j}{s-s_l} \\
&= \lim_{s\to s_l}\frac{d}{ds}\prod_{j=1}^{k}(s+\nu_j) = \sum_{j=1}^{k}\frac{\prod_{m=1}^{k}\nu_m}{s_l+\nu_j}.
\end{aligned} \qquad (27)
$$

Setting $N(0) = 0$ the initial values of (9) follow from (6), (11), (25) and (26) as

$$
Y_l(0) = \sum_{j=0}^{k-1} b_j\lambda_j^{l-1} = \frac{1}{\nu}\sum_{j=0}^{k-1}\frac{\left(1+\frac{s_j}{\nu}\right)^{l-1}}{\sum_{m=1}^{k}\frac{1}{s_j+\nu_m}}, \qquad (28)
$$

$l = 1,2,\dots,k-1$.

With known λ_l the final step is to determine the coefficients $\frac{a_0}{a_k}, \frac{a_1}{a_k}, \dots, \frac{a_{k-1}}{a_k}$ entering the differential equations (9). The latter of these fractions form the invariants of (13), which may be obtained from the expansion

$$
\prod_{j=1}^{k-1}(\lambda-\lambda_j) = \lambda^{k-1} + \frac{a_{k-1}}{a_k}\lambda^{k-2} + \dots + \frac{a_1}{a_k}, \qquad (29)
$$

where λ_j are given by (26). This requires the solution of (22). However there is no need to solve (22) for getting the coefficients $\frac{a_j}{a_k}$, $j = 1,\dots,k-1$. Actually, upon insertion of (26) in the left hand side of (29) one has from (22)

$$
\begin{aligned}
\prod_{j=1}^{k-1}\left(\lambda - 1 - \frac{s_j}{\nu}\right) &= \frac{1}{\nu^k(\lambda-1)}\prod_{j=0}^{k-1}\left(\nu(\lambda-1)-s_j\right) \\
&= \frac{1}{\nu^k(\lambda-1)}\left(\prod_{j=1}^{k}\left(\nu(\lambda-1)+\nu_j\right) - \prod_{j=1}^{k}\nu_j\right).
\end{aligned} \qquad (30)
$$

If the right hand side of (30) is expanded and compared with the right hand side of (29), a direct solution for $\frac{a_j}{a_k}$, $j = 1, \ldots, k-1$ is obtained. $\frac{a_0}{a_k}$ can finally be calculated from (12), (25).

The state vector augmented by the auxiliary variables is governed by the stochastic differential equations

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t), t\big)dt + \mathbf{d}\big(\mathbf{Z}(t), t, P(t)\big)dN(t) \quad (31)$$

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{Z}_1(t) \\ \mathbf{Z}_2(t) \end{bmatrix}, \quad \mathbf{c}\big(\mathbf{Z}(t), t\big) = \begin{bmatrix} \mathbf{c}_1\big(\mathbf{Z}_1(t), t\big) \\ \mathbf{0} \end{bmatrix},$$
$$\mathbf{d}\big(\mathbf{Z}(t), t, P(t)\big) = \begin{bmatrix} \mathbf{d}_1\big(\mathbf{Z}_1(t), t\big) Y_1(t) P(t) \\ \mathbf{d}_2\big(\mathbf{Z}_2(t), t\big) \end{bmatrix}. \quad (32)$$

$$\mathbf{Z}_2(t) = \begin{bmatrix} Y_1(t) \\ \vdots \\ Y_{k-2}(t) \\ Y_{k-1}(t) \end{bmatrix}.$$
$$\mathbf{d}_2\big(\mathbf{Z}_2(t)\big) =$$
$$\begin{bmatrix} Y_2(t) - Y_1(t) \\ \vdots \\ Y_{k-1}(t) - Y_{k-2}(t) \\ -\left(\frac{a_0}{a_k} + \frac{a_1}{a_k} Y_1(t) + \cdots + \frac{a_{k-2}}{a_k} Y_{k-2}(t) + \left(1 + \frac{a_{k-1}}{a_k}\right) Y_{k-1}(t)\right) \end{bmatrix} \quad (33)$$

$P(t)$ assumes the values $P(t_i) = P_i$ at the times t_i of the Poisson events and P_i are mutually independent and identically distributed as P. The equations for the mean values and joint centralized moments of 2nd, 3rd and 4th order are written as, [3]

$$\dot{\mu}_i(t) = E\Big[c_i\big(\mathbf{Z}(t), t\big)\Big] + \nu E\Big[d_i\big(\mathbf{Z}(t), t, P\big)\Big] \quad (34)$$

$$\dot{\kappa}_{ij}(t) =$$
$$2\Big\{E\Big[Z_i^0(t)\Big(c_j^0\big(\mathbf{Z}(t), t\big) + \nu d_j^0\big(\mathbf{Z}(t), t, P\big)\Big)\Big]\Big\}_s$$
$$+\nu E\Big[d_i\big(\mathbf{Z}(t), t, P\big) d_j\big(\mathbf{Z}(t), t, P\big)\Big] \quad (35)$$

$$\dot{\kappa}_{ijk}(t) =$$
$$3\Big\{E\Big[Z_i^0(t) Z_j^0(t)\Big(c_k^0\big(\mathbf{Z}(t), t\big) + \nu d_k^0\big(\mathbf{Z}(t), t, P\big)\Big)\Big]\Big\}_s$$
$$+\nu \cdot 3\Big\{E\Big[Z_i^0(t) d_j\big(\mathbf{Z}(t), t, P\big) d_k\big(\mathbf{Z}(t), t, P\big)\Big]\Big\}_s$$
$$+\nu E\Big[d_i\big(\mathbf{Z}(t), t, P\big) d_j\big(\mathbf{Z}(t), t, P\big) d_k\big(\mathbf{Z}(t), t, P\big)\Big] \quad (36)$$

$$\dot{\kappa}_{ijkl}(t) = 4\Big\{E\Big[Z_i^0(t) Z_j^0(t) Z_k^0(t)\Big(c_l^0\big(\mathbf{Z}(t), t\big) +$$
$$\nu d_l^0\big(\mathbf{Z}(t), t, P\big)\Big)\Big]\Big\}_s +$$
$$\nu \cdot 6\Big\{E\Big[Z_i^0(t) Z_j^0(t) d_k\big(\mathbf{Z}(t), t, P\big) d_l\big(\mathbf{Z}(t), t, P\big)\Big]\Big\}_s$$
$$+\nu \cdot 4\Big\{E\Big[Z_i^0(t) d_j\big(\mathbf{Z}(t), t, P\big) d_k\big(\mathbf{Z}(t), t, P\big)$$
$$\times d_l\big(\mathbf{Z}(t), t, P\big)\Big]\Big\}_s +$$
$$\nu E\Big[d_i\big(\mathbf{Z}(t), t, P\big) d_j\big(\mathbf{Z}(t), t, P\big) d_k\big(\mathbf{Z}(t), t, P\big)$$
$$\times d_l\big(\mathbf{Z}(t), t, P\big)\Big], \quad (37)$$

where $\{\cdot\}_s$ denotes the Stratonovich permutation symbol, [6], $Z_i^0(t) = Z_i(t) - \mu_i(t)$, $c_i^0\big(\mathbf{Z}(t), t, P\big) = c_i\big(\mathbf{Z}(t), t, P\big) - E\big[c_i\big(\mathbf{Z}(t), t, P\big)\big]$, $d_i^0\big(\mathbf{Z}(t), t, P\big) = d_i\big(\mathbf{Z}(t), t, P\big) - E\big[d_i\big(\mathbf{Z}(t), t, P\big)\big]$ are the centralized state vector components, drift vector components and diffusion vector components and $\kappa_{i_1 \cdots i_n}(t)$ denotes the nth order zero time-lag joint centralized moment.

If the drift vector $\mathbf{c}_1\big(\mathbf{Z}_1(t), t\big)$ and diffusion vector $\mathbf{d}_1\big(\mathbf{Z}_1(t), t\big)$ are polynomial nonlinear functions of the structural state vector $\mathbf{Z}_1(t)$, joint central moments of higher order than the provided moment equations (34-37) appear at the right hand side of these equations. Then a cumulant neglect closure at the order $N = 4$ will be used. In case of dense pulse arrivals an ordinary cumulant neglect closure scheme may be applied, whereas in case of sparse pulse arrivals a modified scheme may improve the stability and accuracy during the transient initial phase, [2].

4 EXAMPLE PROBLEM

A Duffing oscillator subjected to a compound renewal process is considered. Then

$$\mathbf{Z}_1(t) = \begin{bmatrix} X(t) \\ \dot{X}(t) \end{bmatrix},$$
$$\mathbf{c}_1\big(\mathbf{Z}_1(t)\big) = \begin{bmatrix} \dot{X}(t) \\ -2\zeta\omega_0 \dot{X}(t) - \omega_0^2\big(X(t) + \varepsilon X^3(t)\big) \end{bmatrix},$$
$$\mathbf{d}_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (38)$$

where $X(t)$ and $\dot{X}(t)$ denote the displacement and velocity response of the oscillator, ω_0 and ζ are the circular eigenfrequency and damping ratio of the corresponding linear oscillator, and ε is the nonlinearity parameter. The actual renewal process will be replaced by a generalized Erlang process of the order $k = 2$. Then the p.d.f., mean value μ_I and

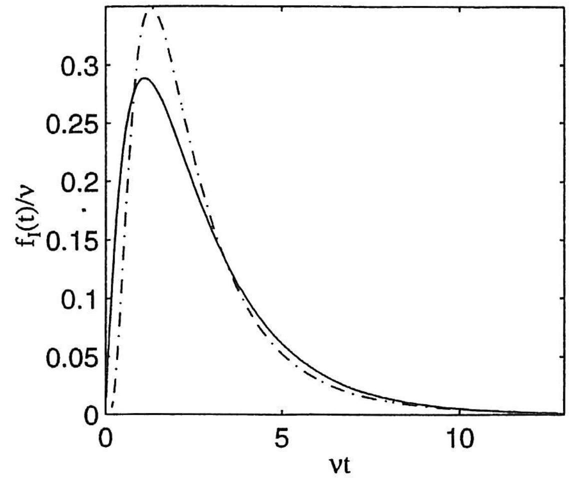

Fig. 1: P.d.f. of interarrival time of renewal processes with $\nu\mu_I = \frac{4}{3}$, $V_I = \sqrt{\frac{5}{8}}$.

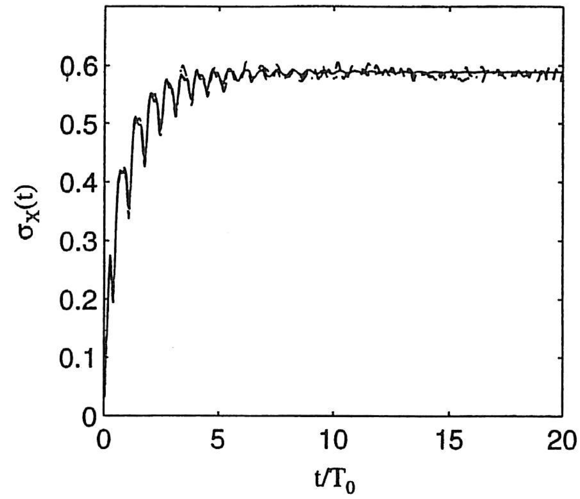

Fig. 2: Mean value $\mu_X(t)$ of displacement.

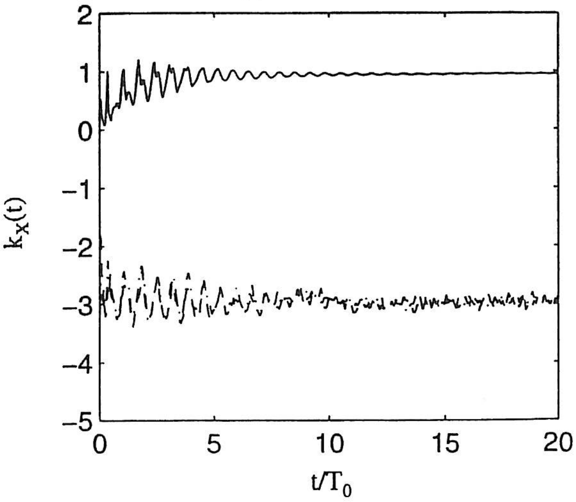

Fig. 3: Standard deviation $\sigma_X(t)$ of displacement.

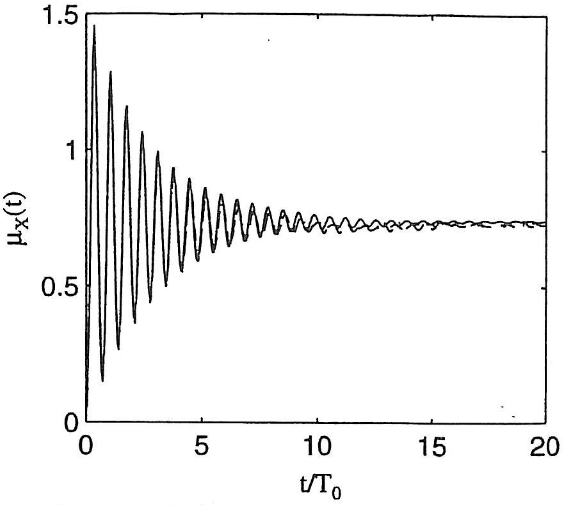

Fig. 4: Skewness $S_X(t)$ of displacement.

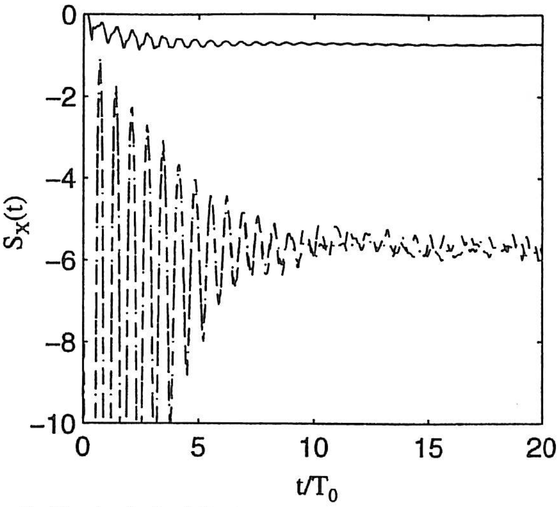

Fig. 5: Kurtosis $k_X(t)$ of displacement.

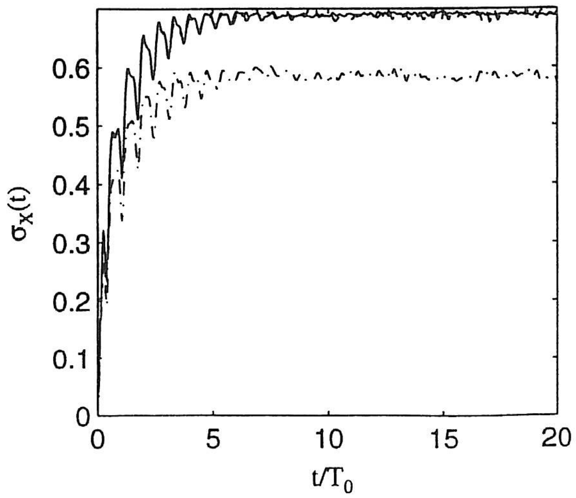

Fig. 6: Standard deviation $\sigma_X(t)$ of displacement, for various equivalent excitations.

variational coefficient V_I of I become, cf. (23)

$$f_I(t) = \frac{\nu_1\nu_2}{\nu_2-\nu_1}\Big(\mathrm{e}^{-\nu_1 t} - \mathrm{e}^{-\nu_2 t}\Big), \qquad (39)$$

$$\mu_I = E[I] = \frac{\nu_1+\nu_2}{\nu_1\nu_2},\ V_I = \frac{\sigma_I}{\mu_I} = \frac{\sqrt{\nu_1^2+\nu_2^2}}{\nu_1+\nu_2}. \qquad (40)$$

Assume that the actual renewal process has lognormally distributed interarrival times, so $\nu I \sim LN(\mu,\sigma^2)$, where μ and σ are non-dimensional parameters. The mean value and variational coefficient of I then become

$$\mu_I = \frac{1}{\nu}\exp\left(\mu+\frac{1}{2}\sigma^2\right),\ V_I = \sqrt{\exp\left(\sigma^2\right)-1}. \qquad (41)$$

With μ_I and V_I of the actual distribution given, ν_1 and ν_2 can be determined from (40) and (41). Assume $\nu\mu_I = \frac{4}{3}$, $V_I = \sqrt{\frac{5}{8}} \Rightarrow \nu_1 = 0.5\nu$, $\nu_2 = 1.5\nu$ and $\mu = \ln\sqrt{\frac{512}{117}}$, $\sigma = \sqrt{\ln\frac{13}{8}}$. In Fig. 1 the actual lognormal p.d.f. for I is denoted by -.-.-.-, and the approximating generalized Erlang p.d.f. resulting from this calibration procedure – by —. The data used in the example are $\omega_0 = 1s^{-1}$, $\zeta = 0.05$, $\varepsilon = 0.5$. The renewal density of the generalized Erlang process, with $k=2$ is obtained as

$$h_o(t) = \frac{\nu_1\nu_2}{\nu_1+\nu_2}\Big(1-\exp\big(-(\nu_1+\nu_2)t\big)\Big), \qquad (42)$$

hence the asymptotic mean arrival rate of impulses is $\frac{\nu_1\nu_2}{\nu_1+\nu_2}$. This quantity is chosen as $10\omega_0$, corresponding to an average number 20π of impulses of the generalized Erlang process per linear eigenperiod T_0, which means a very dense impulse train. The strengths of the impulses are assumed to be Rayleigh distributed, with parameter σ_P chosen in such a way that the original lognormal distributed and generalized Erlang renewal impulse processes have both the same mean square excitation level as the Poisson impulse process with the same mean arrival rate. Hence it is required that the response variance of the corresponding linear oscillator subjected to a Poisson impulse process with the mean arrival rate $\frac{\nu_1\nu_2}{\nu_1+\nu_2}$ should have unit value, i.e. (cf. [1,2]) $\frac{\nu_1\nu_2}{\nu_1+\nu_2}\frac{E[P^2]}{4\zeta\omega_0^3} = \frac{\nu_1\nu_2}{\nu_1+\nu_2}\frac{\sigma_P^2}{2\zeta\omega_0^3} = 1 \Rightarrow \sigma_P = 0.1$. In the analytical technique the parameters b_l and λ_l of the Poisson driven process are assumed in such a way that first degree product density $f_1(t)$ as given by (24) equals $h_o(t)$ given by (42). This allows the evaluation of the coeffficients $\frac{a_0}{a_2}$ and $\frac{a_1}{a_2}$ of the stochastic equation for the auxiliary state variable. Next the mean arrival rate ν of the underlying Poisson process is assumed as $\nu = (\nu_1+\nu_2)/2 = \frac{80}{3}\omega_0$. Then it appears that the second and higher degrees product densities as given by (18) split to the product form (5). Hence, the Poisson driven process becomes a renewal process with a renewal density

$$h_o(t) = \frac{\nu_1\nu_2}{2\nu}\Big(1-\exp(-2\nu t)\Big), \qquad (43)$$

hence the obtained process is a kind of an Erlang impulse process with strengths reduced by $\frac{\nu_1\nu_2}{\nu^2}$. An Erlang impulse process with the same mean square excitation level should have the impulses strengths evaluated from the condition $\frac{\nu}{2}\frac{\sigma_P^2}{2\zeta\omega_0^3} = 1 \Rightarrow \sigma_P = \sqrt{3}/20$. However since the strengths of the obtained process are reduced with respect to Erlang process, the assumed value of σ_P must be multiplied by $\frac{\nu^2}{\nu_1\nu_2}$ which yields $\sigma_P = \sqrt{3}/15 = 0.1154667$.

Since here the centralized drift vector components are cubic and the diffusion vector components are linear forms in the state variables, 5th and 6th order moments appear in the derived equations (34-37) for joint central moments of order up to fourth. Because of the high mean arrival rate $\nu = \frac{80}{3}\omega_0$ of impulses, these moments have been evaluated by means of an ordinary cumulant neglect closure scheme, [1].

In order to evaluate the level of approximation introduced both at the replacement of the actual renewal process with a generalized Erlang process and at the cumulant neglect closure procedure, Monte Carlo simulations have been performed, both for lognormally and Erlang distributed interarrival times, (cf [7]). The simulated results are obtained based on averaging over 500000 independent response curves, each obtained by numerical integration of the governing equations of motion (31), (32), (33), (38).

The results for the mean value $\mu_X(t) = \mu_1(t)$, the standard deviation $\sigma_X(t) = \sqrt{\kappa_{11}(t)}$, the skewness $S_X(t) = \frac{\kappa_{111}(t)}{\sigma_X^3(t)}$ and the kurtosis $k_X(t) = \frac{\kappa_{1111}(t)}{\sigma_X^4(t)} - 3$ are shown in Figs. 2-5 as a function of the non-dimensional time $\frac{t}{T_0}$. In the Figures -.-.-.- and - - - - denote Monte Carlo simulation results for lognormally distributed interarrival times, for the generalized Erlang process of order $k=2$, respectively, and — represents the analytical results for the Poisson driven process obtained for a generalized Erlang process. As seen the agreement between the Monte Carlo simulation results obtained for both renewal processes is very good. Although only the mean value and variance of these distributions are alike, identical response moments within acceptable accuracy are obtained up to and including order 4. Obviously even better results can be obtained if a generalized Erlang process of higher order than $k = 2$ is applied and suitably calibrated. The agreement between the analytical and the Monte Carlo results is very good in the case of the mean values and standard deviations, but it is less satisfactory in the case of skewness and kurtosis coefficients (higher order moments), which can be attributed to the closure of the hierarchy of moment equations at the order $N = 4$. More accurate analytical solutions for these quantities require closure at higher order.

In Fig. 6 is shown the displacement standard deviation obtained from the Monte Carlo simulation in cases where the system is subjected to a non-zero mean Gaussian white noise excitation $F(t) = \frac{\nu_1\nu_1}{\nu_1+\nu_2}E[P] + \sqrt{\frac{\nu_1\nu_1}{\nu_1+\nu_2}E[P^2]}W(t)$ (unbroken line —), to a compound Poisson process with the mean arrival rate $\frac{\nu_1\nu_1}{\nu_1+\nu_2}$ (- - - -) and to a renewal process with the lognormally distributed interarrival times (-.-.-.-.). A zero-mean, unit intensity Gaussian white noise process $W(t)$, was generated by the method of Penzien, [8]. In all cases 500000 sample curves are used. Because of the high mean arrival rate of impulses, the non-zero mean white noise process and the compound Poisson process produce, as expected, almost identical results. In contrast, the standard deviation for the renewal process is significantly different. In combination to the results obtained in Figs. 2-5 it is then concluded that the distribution of the interarrival time affects the response moments significantly, but primarily through its mean value and the variance, whereas higher order moments of the interarrival time distribution seem to have less influence.

5. CONCLUDING REMARKS

The moment equation technique has been devised for non-linear dynamic systems subjected to random trains of impulses driven by an ordinary renewal point process. The idea is to replace the actual renewal process with an approximating generalized Erlang process, and to convert a system into a Poisson driven system at the expense of introduction of auxiliary state variables. Thus the hierarchy of stochastic equations governing the introduced auxiliary variables and the equations for response moments are obtained.

As an example, a Duffing oscillator subjected to a renewal impulse process with log-normally distributed interarrival times is considered. The actual renewal process has been approximated by generalized Erlang process of order $k = 2$. Comparison with Monte Carlo simulation results shows that the analytical technique of equations for moments of the obtained Poisson driven system provides accurate first and second order moments. Better results could be expected if an approximation of order $k > 2$ was used.

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