Aalborg Universitet

Perturbation of near threshold eigenvalues: Crossover from exponential to non-exponential decay laws

Dinu, Victor; Jensen, Arne; Nenciu, Gheorghe

Publication date:
2010

Document Version
Early version, also known as pre-print

Link to publication from Aalborg University

Citation for published version (APA):
Dinu, V., Jensen, A., & Nenciu, G. (2010). *Perturbation of near threshold eigenvalues: Crossover from exponential to non-exponential decay laws*. Department of Mathematical Sciences, Aalborg University. Research Report Series No. R-2010-04

AALBORG UNIVERSITY

Perturbation of near threshold eigenvalues: Crossover from exponential to non-exponential decay laws

by

Victor Dinu, Arne Jensen and Gheorghe Nenciu

R-2010-04 June 2010

Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Perturbation of Near Threshold Eigenvalues: Crossover from Exponential to Non-Exponential Decay Laws

Victor Dinu
CAQP, Faculty of Physics, University of Bucharest
P.O. Box MG 11, RO-077125 Bucharest, Romania
Arne Jensen
Department of Mathematical Sciences, Aalborg University
Fr. Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark
Gheorghe Nenciu
Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, RO-014700 Bucharest, Romania
and
Department of Mathematical Sciences, Aalborg University
Fr. Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark

Abstract

For a two-channel model of the form

$$H_\varepsilon = \begin{bmatrix} H_{\mathrm{op}} & 0 \\ 0 & E_0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & W_{12} \\ W_{21} & 0 \end{bmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{H}_{\mathrm{op}} \oplus \mathbf{C},$$

appearing in the study of Feshbach resonances, we continue the rigorous study, begun in our paper [J. Math. Phys. **50** (2009), 013516], of the decay laws for resonances produced by perturbation of unstable bound states close to a threshold. The operator H_{op} is assumed to have the properties of a Schrödinger operator in odd dimensions, with a threshold at zero. We consider for ε small the survival probability $|\langle \Psi_0, e^{-itH_\varepsilon}\Psi_0\rangle|^2$, where Ψ_0 is the eigenfunction corresponding to E_0 for $\varepsilon = 0$. For E_0 in a small neighborhood of the origin *independent of* ε, the survival probability amplitude is expressed in terms of some special functions related to the error function, up to error terms vanishing as $\varepsilon \to 0$. This allows for a detailed study of the crossover from exponential to non-exponential decay laws, and then to the bound state regime, as the position of the resonance is tuned across the threshold.

1 Introduction

The problem of the decay laws for resonances produced by perturbation of unstable bound states has a long and distinguished history in quantum mechanics. There is an extensive body of literature about decay laws for resonances in general, both at the level of theoretical physics (see e.g. [4, 10, 11, 22, 27, 28, 29] and references therein), and at the level of rigorous mathematical physics (see e.g. [3, 5, 6, 7, 9, 12, 16, 17, 18, 19, 24, 31, 32] and references therein). It started with the computation by Dirac of the decay rate in second order time-dependent perturbation theory, leading to the well known exponential decay law, $e^{-\Gamma t}$. Here Γ is given by the famous "Fermi Golden Rule" (FGR), $\Gamma \sim |\langle \Psi_0, \varepsilon W \Psi_{\mathrm{cont},E_0}\rangle|^2$, where Ψ_0, E_0 are the unperturbed bound state eigenfunction and energy, respectively, and Ψ_{cont,E_0} is the continuum "eigenfunction" degenerate in energy with the bound state. The FGR formula met with a fabulous success, and as a consequence, the common wisdom is that the decay law for the resonances produced by perturbation of non-degenerate bound states is exponential, at least in the leading non-trivial order in the perturbation strength (for degenerate bound states Rabi type exponentially decaying oscillations can appear).

However, it has been known for a long time, at least for semi-bounded Hamiltonians, that the decay law cannot be purely exponential; there must be deviations at least at short and long times. This implies that, in more precise terms, the question is whether the decay law is exponential up to errors vanishing as the perturbation strength tends to zero. So at the rigorous level the crucial problem is the estimation of the errors. This proved to be a hard problem, and only during the past decades consistent rigorous results have been obtained. The generic result is that (see [3, 5, 12, 16, 24] and references therein) the decay law is indeed (quasi)exponential, i.e. exponential up to error terms vanishing in the limit $\varepsilon \to 0$, as long as the resolvent of the unperturbed Hamiltonian is sufficiently smooth, when projected onto the subspace orthogonal to the eigenvalue under consideration. For most cases of physical interest this turns out to be the case, as long as the unperturbed eigenvalue lies in the continuum, far away from the energetic thresholds, and this explain the tremendous success of the FGR formula.

The problem with the exponential decay law appears for bound states situated near a threshold, since in this case the projected resolvent might not be smooth, or may even blow up, when there is a zero resonance[1] at the threshold, see e.g. [14, 15, 16] and references therein. As it has been pointed out in [2], at threshold the FGR formula does not apply. Moreover, the fact that the non-smoothness of the resolvent opens the possibility of a non-exponential decay at all times has been mentioned at the heuristic level [20, 22] (although this possibility for the non-degenerate case has been sometimes denied [25]).

Let us mention that the question of the decay law for near threshold bound states is more than an academic one. While having the bound state in the very neighborhood of a threshold is a non-generic situation, recent advances in experimental technique have made it possible to realize this case for the so-called Feshbach resonances, where (with the aid of a magnetic field) it is possible to tune the energy of the bound state (and then the resonance position) throughout a neighborhood of the threshold energy.

The decay law for the case, when the resonance position is close to the threshold, has been considered at the rigorous level in [16, 17, 18, 19]. More precisely, in [16, 17, 18] the threshold bound states were considered, but under the condition that the shift in the energy due to perturbation (see [16, (3.1)]) is sufficiently large, such that the resonance position is at a distance of order ε from the threshold. In this case it turns out that the decay law is still exponential, but the FGR has to be modified. It is interesting to note that since in this case the ε dependence of the decay rate is a fractional power, the modified FGR cannot be obtained by naive perturbation theory.

The other case, when the resonance position is very close to the threshold (in a neighborhood of the threshold, shrinking as $\varepsilon \to 0$), has been considered in [8] for a two channel model Hamiltonian with the structure used in Feshbach resonance theory [21, 30]. The main result is the proof at the rigorous level that for some energy ranges the decay law is definitely non-exponential. More precisely, we proved that the survival probability, up to some error terms vanishing in the limit $\varepsilon \to 0$, can be written as an explicit integral, which has been analyzed numerically. The numerical study revealed that a remarkable variety, depending upon the values of the parameters involved, of different decay laws appear: Close to an exponential one, definitely non-exponential, or bound state like.

The present paper is a continuation of [8]. The setting is the same, but we add two important things. First, using an appropriate ansatz, close in the spirit to the well known Lorentzian (Breit-Wigner) approximation for per-

[1]To clarify the terminology, $H = -\Delta + V$ on $L^2(\mathbf{R}^d)$, $d = 1, 2, 3$, with $V(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2-\delta}|)$ as $|\mathbf{x}| \to \infty$, is said to have a zero resonance, if $H\Phi = 0$ has a solution, which is not in $L^2(\mathbf{R}^d)$, but in a slightly larger space, see e.g. [14, 15]. A zero resonance is also called a half-bound state.

turbed eigenvalues far from the threshold, but with a functional form taking into account the threshold behavior of the resolvent near threshold, we are able to cover a small ε-*independent* neighborhood of the threshold, improving at the same time the error term. Secondly, we express the approximated survival probability amplitude in terms of some special functions, related to the error function, replacing the exponential function in the decay law. As a result, we are able to obtain a rigorous and detailed description of the crossover of the decay law, as the resonance position is tuned through the threshold from positive to negative energies via tuning of E_0: Exponential decay with the usual FGR decay rate, to exponential decay with the modified FGR decay rate, then to non-exponential decay, and finally to bound state behaviour.

The contents of the paper is as follows. In Section 2 we recall from [8] the model Hamiltonian and its properties. Section 3 contains the guiding heuristics discussion, and a detailed description of the results. Section 4 contains the proofs. In the Appendix we discuss the properties of the functions appearing in the expression for the approximate survival probability amplitude, and their relations with error function and related special functions.

2 Notation and Assumptions

The setting is the same as in [8]. We repeat it below for the reader's convenience. We develop the theory in a somewhat abstract setting, which is applicable to two channel Schrödinger operators in odd dimensions, as they appear for example in the theory of Feshbach resonances (see e.g. [21, 30], and references therein).

Consider

$$H = \begin{bmatrix} H_{\mathrm{op}} & 0 \\ 0 & H_{\mathrm{cl}} \end{bmatrix} \quad \text{on } \mathcal{H} = \mathcal{H}_{\mathrm{op}} \oplus \mathcal{H}_{\mathrm{cl}}.$$

In concrete cases $\mathcal{H}_{\mathrm{op}} = L^2(\mathbf{R}^3)$ (or $L^2(\mathbf{R}_+)$ in the spherically symmetric case), and $H_{\mathrm{op}} = -\Delta + V_{\mathrm{op}}$ with $\lim_{|\mathbf{x}|\to\infty} V_{\mathrm{op}}(\mathbf{x}) = 0$. H_{op} describes the "open" channel. As for the "closed" channel, one starts again with a Schrödinger operator, but with $\lim_{|\mathbf{x}|\to\infty} V_{\mathrm{cl}}(\mathbf{x}) = V_{\mathrm{cl},\infty} > 0$. One assumes that H_{cl} has bound states below $V_{\mathrm{cl},\infty}$, which may be embedded in the continuum spectrum of H_{op}. Only these bound states are relevant for the problem at hand. Thus one can retain only one isolated eigenvalue (or a group of almost degenerate eigenvalues isolated from the rest of the spectrum); the inclusion of the rest of the spectrum of H_{cl} merely "renormalizes" the values of some coefficients, without changing the qualitative picture. In this paper we shall consider only non-degenerate eigenvalues, i.e. we shall take $H_{\mathrm{cl}} = E_0$

in $\mathcal{H}_{\rm cl} = \mathbf{C}$, such that

$$H = \begin{bmatrix} H_{\rm op} & 0 \\ 0 & E_0 \end{bmatrix}, \tag{2.1}$$

on

$$\mathcal{H} = \mathcal{H}_{\rm op} \oplus \mathbf{C} = \left\{ \Psi = \begin{bmatrix} \psi \\ \beta \end{bmatrix} \,\Big|\, \psi \in \mathcal{H}_{\rm op},\, \beta \in \mathbf{C} \right\}.$$

Apart from the spectrum of $H_{\rm op}$, H has a bound state

$$\Psi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{such that} \quad H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = E_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.2}$$

The problem is to study the fate of E_0, when an interchannel perturbation

$$\varepsilon W = \varepsilon \begin{bmatrix} 0 & W_{12} \\ W_{21} & 0 \end{bmatrix} \tag{2.3}$$

is added to H, i.e. the total Hamiltonian is

$$H_\varepsilon = H + \varepsilon W. \tag{2.4}$$

Throughout the paper we assume, without loss of generality, that $\varepsilon > 0$. For simplicity, we assume that W is a bounded self-adjoint operator on $\mathcal{H}$.

As already said in the Introduction, the quantity to be studied is the so-called survival probability amplitude

$$A_\varepsilon(t) = \langle \Psi_0, e^{-itH_\varepsilon} \Psi_0 \rangle. \tag{2.5}$$

As in [16, 8] we shall use the stationary approach to write down a workable formula for $A_\varepsilon(t)$. For this purpose we use the Stone formula to express the compressed evolution in terms of the compressed resolvent, and then we use the Schur-Livsic-Feshbach-Grushin (SLFG) partition formula to express the compressed resolvent as an inverse (for details, further references, and historical remarks about the SLFG formula, we send the reader to [16]). More precisely, by using the Stone formula and the SLFG formula, one arrives at the following basic formula for $A_\varepsilon(t)$, which often appears in the physics literature and is a particular case of the general formula in [16].

$$A_\varepsilon(t) = \lim_{\eta \searrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-itx} \,\mathrm{Im}\, F(x + i\eta, \varepsilon)^{-1} dx \tag{2.6}$$

with

$$F(z, \varepsilon) = E_0 - z - \varepsilon^2 g(z), \tag{2.7}$$

where

$$g(z) = \langle \Psi_0, WQ^*(H_{\text{op}} - z)^{-1} QW\Psi_0 \rangle, \tag{2.8}$$

and Q is the orthogonal projection on $\mathcal{H}_{\text{op}}$, considered as a map from $\mathcal{H}$ to $\mathcal{H}_{\text{op}}$.

Since we are interested in the form of $A_\varepsilon(t)$, when E_0 is near a threshold of H_{op}, we shall assume that 0 is a threshold of H_{op}, and that E_0 is close to zero.

The following assumption is imposed in the sequel and will not be repeated. Condition (iii) is imposed to exclude the trivial case.

Assumption 2.1. (i) *There exists $a > 0$, such that $(-a, 0) \subset \rho(H_{\text{op}})$ (the resolvent set) and $[0, a] \subset \sigma_{\text{ess}}(H_{\text{op}})$.*

(ii) $|E_0| \leq \frac{1}{2}$.

(iii) *We have $QW\Psi_0 \neq 0$.*

From Assumption 2.1 and (2.8) we get the following result.

Proposition 2.2. (i) *$g(z)$ is analytic in $\mathbf{C} \setminus \{(-\infty, -a] \cup [0, \infty)\}$.*

(ii) $g(\overline{z}) = \overline{g(z)}$.

(iii) *$g(z)$ is strictly increasing on $(-a, 0)$.*

(iv) $\operatorname{Im} g(z) > 0$ *for* $\operatorname{Im} z > 0$.

The aim of this paper is to consider at the rigorous mathematical physics level the problem of the decay law, for the case that E_0 is tuned past the threshold. For that purpose we need assumptions about the behavior of the function $g(z)$ in the neighborhood of the origin. In stating this assumption we use the notation from [8, 16] to facilitate reference to those papers.

Assumption 2.3. *For $\operatorname{Re} \kappa \geq 0$ and $z \in \mathbf{C} \setminus [0, \infty)$ we let*

$$\kappa = -i\sqrt{z}, \quad z = -\kappa^2. \tag{2.9}$$

Let for $a > 0$

$$D_a = \{z \in \mathbf{C} \setminus [0, \infty) \,|\, |z| < a\}. \tag{2.10}$$

Then for $z \in D_a$

$$g(z) = \sum_{j=-1}^{4} \kappa^j g_j + \kappa^5 r(\kappa), \tag{2.11}$$

$$\frac{d}{dz}g(z) = -\frac{1}{2\kappa}\sum_{j=-1}^{4} j\kappa^{j-1}g_j + \kappa^3 s(\kappa), \tag{2.12}$$

$$\sup_{z\in D_a}\{|r(\kappa)|, |s(\kappa)|\} < \infty. \tag{2.13}$$

Furthermore, we assume that $\lim_{\operatorname{Im} z\searrow 0}(g(z) - g_{-1}\kappa^{-1})$ *exists and is continuous on* $(-a, a)$.

As already explained, Assumption 2.3 includes the case, when $H_{\rm op} = -\Delta + V_{\rm op}$ in odd dimensions. The expansions for the resolvent of $-\Delta + V_{\rm op}$ leading to (2.11) are provided in [14, 13, 26, 15, 16, 17]. Taking into account that (at least formally)

$$\frac{d}{dz}g(z) = \langle \Psi_0, WQ^*(H_{\rm op} - z)^{-2}QW\Psi_0\rangle,$$

the result (2.12) can be derived in the same manner. More precisely, it can be shown that the expansion (2.11) is differentiable, see [14, 26, 33]. Examples of expansions with the corresponding explicit expressions for coefficients g_j are given in the Appendix to [16], with references to the literature.

Since the form of the decay law depends strongly upon the behaviour of $g(z)$ near 0, we divide the considerations into three cases.

(i) The *singular* case, in which $g_{-1} \neq 0$. In the Schrödinger case this corresponds to the situation, when $H_{\rm op}$ has a zero resonance at the threshold (see e.g. [14, 16]). Let us recall that the free particle in one dimension belongs to this class. From Proposition 2.2(iv) follows that

$$g_{-1} > 0. \tag{2.14}$$

(ii) The *regular* case, in which $g_{-1} = 0$ and $g_1 \neq 0$. We note that $g_{-1} = 0$ is the generic case for Schrödinger operators in one and three dimensions. Again from Proposition 2.2(iv) one has

$$g_1 < 0. \tag{2.15}$$

Let us remark that the behavior $\operatorname{Im} g(x+i0) \sim x^{1/2}$ as $x \to 0$ is nothing but the famous Wigner threshold law [30, 23].

(iii) The *smooth* case, in which $g_{-1} = g_1 = 0$. This case occurs for free Schrödinger operators in odd dimensions larger that three, and in the spherical symmetric case for partial waves $\ell \geq 1$, see [16, 17]. Notice that in this case $\frac{d}{dz}g(z)$ is uniformly bounded in D_a.

Let for $x \in (-a, a)$, $x \neq 0$,

$$\lim_{\eta \searrow 0} F(x + i\eta, \varepsilon) = F(x + i0, \varepsilon) = R(x, E_0, \varepsilon) + iI(x, E_0, \varepsilon). \tag{2.16}$$

Due to (2.7) and Assumption 2.3, $R(x, E_0, \varepsilon)$ is continuous and strictly decreasing, for sufficiently small ε. Thus for ε and E_0 small enough, the equation $R(x, E_0, \varepsilon) = 0$ has a unique solution $x_0(E_0, \varepsilon)$ on $(-a, a)$:

$$R(x_0(E_0, \varepsilon), E_0, \varepsilon) = 0. \tag{2.17}$$

To simplify the notation we omit the dependence of $R(x, E_0, \varepsilon)$, $I(x, E_0, \varepsilon)$, and $x_0(E_0, \varepsilon)$ on E_0 and ε. Throughout the paper H_{op} and W are kept fixed, while E_0 and ε are parameters; ε is positive and small, and E_0 is tuned past the threshold, i.e. takes values in a neighborhood of the origin.

A finite number of constants will appear; they are strictly positive, finite and independent of the parameters ε and E_0. We introduce the following notation:

Notation.

(i) $A \lesssim B$ means that there exists a constant c such that $A \leq cB$. An analogous definition holds for $A \gtrsim B$.

(ii) $A \simeq B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

(iii) $A \cong B$ means that A and B are equal to leading order in a parameter, e.g. $A = B + \delta(\varepsilon)$ with $\lim_{\varepsilon \searrow 0} \delta(\varepsilon) = 0$.

3 Heuristics and the results

We first give the heuristics, and then we state our results.

3.1 Heuristics

For E_0 outside a small (possibly ε-dependent) neighborhood of the origin, the situation is well understood, both at the heuristic level, and at the rigorous level. Indeed, for negative E_0, using the analytic perturbation theory, one can show that

$$|A_\varepsilon(t) - e^{-itE_\varepsilon}| \lesssim \varepsilon^2, \tag{3.1}$$

where E_ε is the perturbed eigenvalue, which coincides with E_0 in the limit $\varepsilon \to 0$. As a consequence, the survival probability remains close to one uniformly in time.

On heuristic grounds, if E_0 is positive, i.e. embedded in the essential spectrum of $H_{\rm op}$, Ψ_0 turns into a metastable decaying state. The main problem is to compute the "decay law", i.e. $|A_\varepsilon(t)|^2$, up to error terms vanishing in the limit $\varepsilon \to 0$. For eigenvalues embedded in the continuum spectrum the heuristics for the exponential decay law[2] $|A_\varepsilon(t)|^2 \cong e^{-2\Gamma(\varepsilon)t}$ runs as follows.

Suppose $F(z,\varepsilon)$ is sufficiently smooth, as z approaches the real line from above, $F(x+i0,\varepsilon)$, for x in a neighborhood of E_0. Let $F(x+i0,\varepsilon) = R(x)+iI(x)$. Then the equation $R(x)=0$ has a solution x_0 nearby E_0. The idea is that the main contribution to the integral in (2.6) comes from the neighborhood of x_0, and in this neighborhood

$$F(x+i0,\varepsilon) \cong x_0 - x + iI(x_0), \tag{3.2}$$

and then

$$\operatorname{Im} F(x,\varepsilon)^{-1} \cong \frac{-I(x_0)}{(x-x_0)^2 + I(x_0)^2}, \tag{3.3}$$

i.e. it has a Lorentzian peak shape leading to

$$|A_\varepsilon(t)|^2 \cong e^{-2|I(x_0)|t}. \tag{3.4}$$

For mathematical substantiation of this heuristics in the case, where either $E_0 > 0$ (embedded eigenvalues) or $E_0 = 0$ (threshold eigenvalues) but the perturbation "pushes" the eigenvalue sufficiently "far" into the continuum spectrum such that $x_0 \gtrsim \varepsilon$, we send the reader to [16] and references therein. In the cases where the resolvent has an analytic continuation through the positive semi-axis, $z_0 \cong x_0 + iI(x_0)$ is nothing but the position of the "resonance pole" and x_0 and $-I(x_0)$ are called resonance *position* and *width*, respectively. In this setting the exponential decay law comes from the resonance pole contribution, while the error term comes from the contribution of the "background" integral, see [12]. It is well known that irrespective of the approach the *main technical difficulty is to estimate the error term.*

The problem with the energies near the threshold is that $F(x+i0,\varepsilon)$ might not be smooth and can even blow up (see Assumption 2.3), if the open channel has a zero resonance at the threshold. Then a Lorentzian approximation might break down. For the case at hand, elaborating on a heuristic argument in [20], one can quantify at the heuristic level how far from the origin $x_0 > 0$ must be in order to have a chance for an exponential

[2] A better name is probably quasi-exponential decay law, in order to emphasize the fact that the equality is up to errors vanishing as $\varepsilon \to 0$.

decay law: The contribution of the tail at negative x of the Lorentzian must be negligible. Since

$$\int_{-\infty}^{0} \frac{|I(x_0)|}{(x-x_0)^2 + I(x_0)^2} dx \simeq \frac{|I(x_0)|}{x_0}, \tag{3.5}$$

one gets the condition

$$|I(x_0)| \ll x_0. \tag{3.6}$$

Consider first the condition (3.6) in the singular case. For $x > 0$ small enough

$$I(x) \cong -g_{-1}\varepsilon^2 x^{-1/2},$$

and the condition (3.6) gives $g_{-1}\varepsilon^2 x_0^{-1/2} \ll x_0$, i.e.

$$x_0 \gg \varepsilon^{4/3}. \tag{3.7}$$

If we take (by adjusting E_0!)

$$x_0 = b\varepsilon^p, \tag{3.8}$$

then one obtains, for $0 \leq p < 4/3$, the exponential decay law (see (3.4))

$$|A_\varepsilon(t)|^2 \cong e^{-2g_{-1}b^{-1/2}\varepsilon^{2-p/2}t}. \tag{3.9}$$

Notice that for $p = 0$ (i.e. the resonance stays away from the threshold as $\varepsilon \to 0$), (3.9) is nothing but the usual Fermi Golden Rule (FGR) formula. However, for $p > 0$ but not very large (i.e. the resonance position approaches zero as $\varepsilon \to 0$, but not too fast) one gets a "modified FGR formula " for which the ε-dependence of the resonance width is $\varepsilon^{2-p/2}$ instead of the usual ε^2-dependence.

For the regular case, a similar argument leads to the condition

$$x_0 \gg \varepsilon^4, \tag{3.10}$$

and a decay law

$$|A_\varepsilon(t)|^2 \cong e^{-2|g_1|b^{1/2}\varepsilon^{2+p/2}t}. \tag{3.11}$$

Finally, in the smooth case the condition (3.6) reads

$$\varepsilon^2 x_0^{1/2} \ll 1, \tag{3.12}$$

which holds true irrespective of how close to zero x_0 is. In other words, in the smooth case one observes an exponential decay law (with a resonance width vanishing as $x_0 \to 0$), as the resonance position is tuned past the threshold, via the tuning of the eigenvalue E_0.

3.2 Reduction to the case $g_2 = 0$

We argue that it is sufficient to consider the case, when $g_2 = 0$, i.e. (2.11) is replaced with

$$g(z) = \sum_{\substack{j\geq -1\\ j\neq 2}}^{N} \kappa^j g_j + \kappa^{N+1} r(\kappa), \tag{3.13}$$

which leads to a significant simplification of the proofs. Indeed, let

$$\tilde{F}(z,\varepsilon) = \frac{F(z,\varepsilon)}{1-\varepsilon^2 g_2}. \tag{3.14}$$

On the one hand, notice that if

$$\tilde{A}_\varepsilon(t) = \lim_{\eta\searrow 0} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx}\,\mathrm{Im}\,\tilde{F}(x+i\eta,\varepsilon)^{-1}dx, \tag{3.15}$$

then for sufficiently small ε (e.g. $|\varepsilon^2 g_2| \leq \frac{1}{2}$) we have

$$|\tilde{A}_\varepsilon(t) - A_\varepsilon(t)| \lesssim \varepsilon^2. \tag{3.16}$$

On the other hand,

$$\tilde{F}(z,\varepsilon) = \tilde{E}_1 - z - \varepsilon^2(\tilde{g}_{-1}\kappa^{-1} + \tilde{g}_1\kappa + \tilde{g}_3\kappa^{3/2} + \cdots), \tag{3.17}$$

with $\tilde{E}_1 = \dfrac{E_0 - \varepsilon^2 g_0}{1-\varepsilon^2 g_2}$, and $\tilde{g}_j = \dfrac{g_j}{1-\varepsilon^2 g_2}$, which is exactly of the same form as $F(z,\varepsilon)$, but without the linear term in the expansion of $g(z)$, and the other coefficients slightly "renormalized". In the sequel we consider only $F(z,\varepsilon)$ with $g(z)$ satisfying (3.13).

3.3 Resonance and bound state positions

Summing up, the heuristics predicts that if the resonance position is *outside* an energy window of size $\varepsilon^{4/3}$ and ε^4 in the singular and regular case, respectively, then the decay law is exponential with a decay rate depending on, how rapidly the resonance position approaches zero as $\varepsilon \to 0$. Moreover, it suggests that for the resonance position *inside* the above energy windows, the decay law is not exponential, but gives no hint about its actual form.

We proceed to the rigorous substantiation of the above heuristics, and to the derivation of decay laws. As the heuristics suggests, the zeroes of $R(x)$ (which for $x < 0$ coincide with those of $F(x+i0,\varepsilon)$) play a central rôle. The zero on the positive semi-axis, x_0, gives the resonance position, while the zero on negative semi-axis, x_b, gives the position of the bound state. The propositions below give estimates on x_0 and x_b in terms of the parameters appearing in $F(x+i0,\varepsilon)$, as given by (2.7) and (2.8).

In the remainder of this paper we shall take $a>0$ small enough, such that Assumption 2.1 holds true, and in addition the terms in $g(z)$ with $j \geq 3$ can be treated as perturbations. Let

$$E_1 = E_0 - \varepsilon^2 g_0 \quad \text{with} \quad |E_1| \leq \frac{a}{2}. \tag{3.18}$$

Proposition 3.1. *For $E_1 > 0$ and $\varepsilon > 0$ sufficiently small, the equation $R(x) = 0$ on $(0,a)$ has a unique solution x_0, and*

$$x_0 = E_1 + \mathcal{O}(\varepsilon^2 x_0^2). \tag{3.19}$$

In particular,

$$\lim_{E_1 \searrow 0} x_0 = 0. \tag{3.20}$$

Proposition 3.2. (i) *Assume $g_{-1} \neq 0$. Then for $\varepsilon > 0$ sufficiently small (and irrespective of the value of E_1) the equation $F(x,\varepsilon) = 0$ has a unique solution on $(-a,0)$ and*

$$|x_b| \simeq \begin{cases} \varepsilon^{4/3}, & \text{if } 0 \lesssim E_1 \lesssim \varepsilon^{4/3}, \\ \dfrac{\varepsilon^4}{E_1^2}, & \text{if } E_1 \gtrsim \varepsilon^{4/3}, \end{cases} \tag{3.21}$$

$$|x_b| \lesssim \varepsilon^{4/3} + |E_1|, \text{for } E_1 \leq 0. \tag{3.22}$$

(ii) *Assume $g_{-1} = 0$, $g_1 \neq 0$. Then for $E_1 \geq 0$, the equation $F(x,\varepsilon) = 0$ has no solutions on $(-\infty, 0)$. For $-a/2 \leq E_1 < 0$ and ε sufficiently small the equation $F(x,\varepsilon) = 0$ has a unique solution x_b on $(-a,0)$ and*

$$|x_b| \leq |E_1|. \tag{3.23}$$

3.4 Some previous results

In the case, where $F(x+i0,\varepsilon)$ is sufficiently smooth in a neighborhood of x_0, the mathematical substantiation of the quasi-exponential decay law (i.e. exponential decay up to errors vanishing as $\varepsilon \to 0$) follows from the results in [16] (see further references in this paper). In particular, for the smooth case, as well as for $x_0 \simeq \varepsilon$ (i.e. $p = 1$ in (3.8)), in the singular and regular case, one still has (quasi)-exponential decay. Let us stress here that in these cases the ansatz (3.2) is nothing but the approximation of $F(z,\varepsilon)$ with a linear function $L(z) = \alpha + i\beta - z$, where the constants α and β are fixed by the condition that F and L coincide at $x_0(\varepsilon)$:

$$F(x_0 + i0, \varepsilon) = L(x_0 + i0). \tag{3.24}$$

In the (non-smooth) threshold case it has been proved in [8] that indeed in some energy windows, which depend on the spectral properties of the unperturbed Hamiltonian at the threshold, the decay law is definitely non-exponential for all times. The main idea in [8] is that in the neighborhood of $z = 0$ one can replace $F(z, \varepsilon)$ by the following model function

$$F(z, \varepsilon) \cong E_0 - z - \varepsilon^2 \sum_{j=-1}^{N} \kappa^j g_j \equiv H(z, \varepsilon), \tag{3.25}$$

which leads to non-exponential decay laws. As an example we reproduce below the main result in [8] for the regular case. In this case the model function is (3.25) with $N = 2$ (and $g_{-1} = 0$),

$$H_r(z, \varepsilon) = E_0 - z - \varepsilon^2(g_0 - ig_1\sqrt{z} - g_2 z) = d(\tilde{E} - z + i\tilde{g}_1\sqrt{z}), \tag{3.26}$$

where

$$\tilde{E} = \frac{E_0 - \varepsilon^2 g_0}{1 - \varepsilon^2 g_2} \quad \text{and} \quad \tilde{g}_1 = \frac{g_1}{1 - \varepsilon^2 g_2}. \tag{3.27}$$

It is assumed that ε is sufficiently small, such that d is close to one.

Theorem 3.3 ([8, Theorem 2.8])**.** *Suppose $\tilde{E} \in [-a/2, (c/2)\varepsilon^{3/4}]$ for some $c > 0$. Then for all $t \geq 0$, and for sufficiently small ε, we have the following results.*
(i) *For $\tilde{E} \geq 0$ we have*

$$\left| A_\varepsilon(t) - \frac{1}{\pi} \int_0^\infty \frac{y^{1/2}}{(\tilde{f} - y)^2 + y} e^{-i\tilde{s}y} dy \right| \lesssim \varepsilon^{4/3}, \tag{3.28}$$

where

$$\tilde{s} = (\varepsilon^2 \tilde{g}_1)^2 t \quad \textit{and} \quad \tilde{f} = (\varepsilon^2 \tilde{g}_1)^{-2} \tilde{E}. \tag{3.29}$$

(ii) *For $\tilde{E} \leq 0$ we have*

$$\left| A_\varepsilon(t) - \frac{\sqrt{1 + 4|\tilde{f}|} - 1}{\sqrt{1 + 4|\tilde{f}|}} e^{-itx_b} - \frac{1}{\pi} \int_0^\infty \frac{y^{1/2}}{(\tilde{f} - y)^2 + y} e^{-i\tilde{s}y} dy \right| \lesssim \varepsilon^{4/3}. \tag{3.30}$$

Remark 3.4. In terms of $\tilde{f}$ the scaling in (3.8) can be written as

$$x_0 \cong \tilde{f}\varepsilon^{p-4}, \tag{3.31}$$

so according to the heuristics $\tilde{f}$ = const. is just the borderline between exponential and non-exponential decay laws. This is substantiated by the numerical computations presented in [8], as well as by the rigorous results in [16] for $p = 1$ i.e. $\tilde{f} \simeq \varepsilon^{-3}$.

Remark 3.5. Again in terms of the scaling (3.8), the interval $p \in (0, 3/4)$ is not covered by the results in [8]. One of the main goals here is to fill this gap, in order to have a complete picture of the crossover from exponential to non-exponential decay laws.

3.5 The model functions

We recall first that in the case of embedded eigenvalues (i.e. $p = 0$) the "model function" approximating $F(z, \varepsilon)$ is the linear approximation (3.2), determined by the condition (3.24). The ansatz we shall adopt in this paper for the "model function" approximating $F(z, \varepsilon)$ for all $p \in (0, \infty)$, see (3.8), is to replace $F(z, \varepsilon)$ by a function, $H(z)$, resembling the expansion of $F(z, \varepsilon)$ around the threshold, whose free parameters are fixed by a condition similar to (3.24).

More precisely, in the *singular case*, $g_{-1} \neq 0$,

$$H_s(z) = \alpha - z - \varepsilon^2\beta\kappa^{-1}, \tag{3.32}$$

and in the *regular case*, $g_{-1} = 0, g_1 \neq 0$,

$$H_r(z) = \alpha - z + \varepsilon^2\beta\kappa. \tag{3.33}$$

The signs in (3.32) and (3.33) are chosen to ensure that in all cases $\beta \geq 0$. The condition $\beta > 0$ should be compared with the Fermi Golden Rule condition, imposed in the case of an embedded eigenvalue ($p = 0$ case).

Thus in both cases there are two real parameters α and β, to be determined. In the case $E_1 \geq 0$ the condition used is $F(x_0, \varepsilon) = H_\iota(x_0)$, $\iota \in \{s, r\}$. Equality of the real and imaginary parts gives the two equations used to determine α and β. In the case $E_1 < 0$ the functions $F(x)$ and $H(x)$ are real-valued. The conditions used are $F(x_b, \varepsilon) = H_\iota(x_b)$, $\iota \in \{s, r\}$, together with

$$\frac{d}{dx}F(x_b, \varepsilon) = \frac{d}{dx}H_\iota(x_b), \quad \iota \in \{s, r\}. \tag{3.34}$$

Thus in the case $E_1 < 0$ our conditions determining α and β give as a result that the residues at the pole x_b of $\frac{1}{F(x,\varepsilon)}$ and $\frac{1}{H_\iota(x)}$, $\iota \in \{s, r\}$, are equal.

It is clear from this discussion that the parameters α and β take values depending on which case is being considered. These values will be given in connection with the proofs, since they are only needed there.

Remark 3.6. Let us note that the determination of α leads to

$$\alpha = \begin{cases} x_0, & \text{for } E_1 \geq 0, \\ x_b + \text{small term}, & \text{for } E_1 < 0. \end{cases} \tag{3.35}$$

The precise form of the small term for $E_1 < 0$ depends on the case being considered.

3.6 Main results; error analysis

We are now in a position to formulate the main technical results of this paper: For ε sufficiently small and E_0 in an ε-independent neighborhood of the threshold, the error in $A_\varepsilon(t)$ due to the replacement of $F(z,\varepsilon)$ with the model functions $H_\iota(z)$, $\iota \in \{s, r\}$ (as given by (3.32), (3.33), (3.35), and (3.34)) vanishes in the limit $\varepsilon \to 0$. In other words, we have to control $|A_\varepsilon(t) - \lim_{\eta\searrow 0} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx} \operatorname{Im} H_\iota(x+i\eta)^{-1}dx|$ as $\varepsilon \to 0$.

The contribution of the negative semi-axis in $\lim_{\eta\searrow 0} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx} \operatorname{Im} H_\iota(x+i\eta)^{-1}dx$ is just the residue at the zero, $\tilde{x}_b$, of $H_\iota(z)$ (when it exists) and equals $-\frac{1}{\frac{d}{dx}H_\iota(\tilde{x}_b)}e^{-it\tilde{x}_b}$. Accordingly

$$\lim_{\eta\searrow 0} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx} \operatorname{Im} H_\iota(x+i\eta)^{-1}dx = -\frac{1}{\frac{d}{dx}H_\iota(\tilde{x}_b)}e^{-it\tilde{x}_b} + \hat{A}_{\varepsilon,\iota}(t) \tag{3.36}$$

with

$$\hat{A}_{\varepsilon,\iota}(t) \equiv \frac{1}{\pi}\int_{0}^{\infty} e^{-itx} \operatorname{Im} H_\iota(x+i0)^{-1}dx, \quad \iota \in \{s, r\}. \tag{3.37}$$

Notice that by definition of $H_\iota(z)$, for $E_1 < 0$, we have $\tilde{x}_b = x_b$, and the contribution of the negative semi-axis in $\lim_{\eta\searrow 0} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx} \operatorname{Im} H_\iota(x+i\eta)^{-1}dx$ and in $\lim_{\eta\searrow 0} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx} \operatorname{Im} F(x+i\eta,\varepsilon)^{-1}dx$ are equal. For $\iota = s$, x_b exists also for $E_1 \geq 0$, and in this case $\tilde{x}_b \neq x_b$, but still as $\varepsilon \to 0$, we have $\frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)} \cong \frac{1}{\frac{d}{dx}F(x_b,\varepsilon)}$ (see the proof of Theorem 3.8).

Theorem 3.7 (Regular case)**.** *Let $g_{-1} = 0$, $g_1 \neq 0$. There exists c, $\frac{a}{2} \geq c > 0$, such that for sufficiently small ε, $|E_0| \leq c$, and all $t \geq 0$, we have*

$$\Big|A_\varepsilon(t) - \lim_{\eta\searrow 0} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx} \operatorname{Im} H_r(x+i\eta)^{-1}dx\Big| \lesssim \begin{cases} \varepsilon^2(1 + x_0^{1/2}|\ln\varepsilon|), & \text{for } E_1 > 0, \\ \varepsilon^2, & \text{for } E_1 \leq 0. \end{cases} \tag{3.38}$$

Theorem 3.8 (Singular case)**.** *Let $g_{-1} \neq 0$. There exists c, $\frac{a}{2} \geq c > 0$, such that for sufficiently small ε, $|E_0| \leq c$, and all $t \geq 0$, we have*

(i) *Let $E_1 \geq 0$, and let $\tilde{x}_b$ be the unique solution of $H_s(x) = 0$ on the negative semi-axis. Then*

$$\Big| A_\varepsilon(t) + \frac{1}{\frac{d}{dx} H_s(\tilde{x}_b)} e^{-itx_b} - \lim_{\eta \searrow 0} \frac{1}{\pi} \int_0^\infty e^{-itx} \operatorname{Im} H_s(x + i\eta)^{-1} dx \Big| \lesssim \begin{cases} \frac{\varepsilon^2}{x_0^{1/2}} |\ln \varepsilon|, & \text{for } x_0 \gtrsim \varepsilon^{4/3}, \\ \varepsilon^{4/3}, & \text{for } x_0 \lesssim \varepsilon^{4/3}. \end{cases} \tag{3.39}$$

(ii) *Let $E_1 < 0$. Then*

$$\Big| A_\varepsilon(t) + \frac{1}{\frac{d}{dx} H_s(x_b)} e^{-itx_b} - \lim_{\eta \searrow 0} \frac{1}{\pi} \int_0^\infty e^{-itx} \operatorname{Im} H_s(x + i0)^{-1} dx \Big| \lesssim \begin{cases} \frac{\varepsilon^2}{|E_1|^{1/2}}, & \text{for } |E_1| \gtrsim \varepsilon^{4/3}, \\ \varepsilon^{4/3}, & \text{for } |E_1| \lesssim \varepsilon^{4/3}. \end{cases} \tag{3.40}$$

Remark 3.9. For the smooth case ($g_{-1} = g_1 = 0$), see [8, Theorem 2.10]. This result gives an exponential decay law irrespective of the value of E_0.

Remark 3.10. We compare Theorem 3.3 with Theorem 3.7. The latter result is valid in an ε-independent neighborhood of zero, and has a better error estimate. This is due to the choice of α and β. The disadvantage is that these coefficients are not given in terms of expansion coefficients in $g(z)$, but are solutions to equations, which can be solved by perturbative methods.

3.7 Error function analysis: Crossover from exponential to non-exponential decay laws

As shown in the previous section, the bound state contribution has a simple form. Thus it remains to compute $\hat{A}_{\varepsilon,\iota}(t)$ as given by (3.37). Since H_ι, $\iota \in \{s, r\}$, have a simple functional form with only two free parameters, the integral in the r.h.s of (3.37) can be evaluated numerically or expressed in closed form in terms of some special functions. Some examples of numerical computations substantiating the heuristics presented in previous subsections have been presented in [8], but a detailed analytical study of the asymptotics was postponed. One of the main goals of this paper is to perform this analysis. The main point is that $A_\varepsilon(t)$, as well as its asymptotics, can be expressed in terms of some special functions

$$\mathcal{I}_1(z) = \frac{2z}{i\pi} \int_0^\infty \frac{e^{-ix^2}}{x^2 - z^2} dx, \tag{3.41}$$

$$\mathcal{I}_p(z) = \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \mathcal{I}_1(z), \quad p = 2, 3, \ldots, \tag{3.42}$$

closely related to the error function (see Appendix A for a detailed study of $\mathcal{I}_p(z)$).

3.7.1 The regular case

We begin with the (simpler) regular case. Here $H_r(z) = \alpha - z + \varepsilon^2\beta\kappa$, where α lies in a small neighborhood of the origin and $\beta > 0$. Let

$$\hat{A}_{\varepsilon,r}(t) \equiv \frac{1}{\pi} \int_0^\infty e^{-itx} \operatorname{Im} H_r(x + i0)^{-1} dx. \tag{3.43}$$

Passing to the variable $k = \sqrt{z} = i\kappa$ we get

$$\hat{A}_{\varepsilon,r}(t) = \frac{1}{i\pi} \int_{-\infty}^\infty \frac{e^{-itk^2}}{P_r(\kappa)} k dk, \tag{3.44}$$

where

$$P_r(\kappa) = \kappa^2 + \varepsilon^2\beta\kappa + \alpha. \tag{3.45}$$

The integral on the r.h.s. of (3.44) is to be understood as $\lim_{A\to\infty} \int_{-A}^{A} \frac{e^{-itk^2}}{P_r(\kappa)} kdk$. When the zeroes of P_r are distinct, a partial fraction decomposition yields the following result.

Proposition 3.11 (Regular case)**.**

$$\hat{A}_{\varepsilon,r}(t) = -\sum_{j=1}^{2} q_j \mathcal{I}_1(i\kappa_j\sqrt{t}), \tag{3.46}$$

where

$$\kappa_j = \tfrac{1}{2}\big(-\varepsilon^2\beta - (-1)^j\sqrt{\varepsilon^4\beta^2 - 4\alpha}\big), \quad j = 1, 2, \tag{3.47}$$

are the roots of $P_r(\kappa)$*, and*

$$q_j = \tfrac{1}{2}\Big(1 + (-1)^j \frac{\varepsilon^2\beta}{\sqrt{\varepsilon^4\beta^2 - 4\alpha}}\Big), \quad j = 1, 2, \tag{3.48}$$

are the corresponding residues of $\dfrac{\kappa}{P_r(\kappa)}$.

Remark 3.12. For $\alpha = \frac{\beta^2\varepsilon^4}{4}$, the κ_j, $j = 1, 2$, coincide and the coefficients q_j become infinite. One can show that as $\alpha \to \frac{\beta^2\varepsilon^4}{4}$. the formula (3.46) has a limit

$$\hat{A}_{\varepsilon,r}(t) = i\mathcal{I}_3(i\frac{\beta\varepsilon^2}{2}\sqrt{t}). \tag{3.49}$$

We shall not make use of (3.49) in what follows, since the case $\alpha = \frac{\beta^2\varepsilon^4}{4}$ belongs to the crossover regime (see below), when $\hat{A}_{\varepsilon,r}$ is given also by (3.67), which has been analyzed in [8].

As a consequence of Proposition 3.11 we can now discuss the various regimes.

The exponential regime

According to the heuristics, if we set

$$\alpha = b\varepsilon^p, \ \ b > 0, \tag{3.50}$$

then for

$$p \in [0, 4) \tag{3.51}$$

the decay law is still exponential. Fix $p \in [0, 4)$. Notice that as $\varepsilon \to 0$

$$\frac{\varepsilon^2}{\sqrt{\alpha}} \simeq \varepsilon^{2-\frac{p}{2}} \tag{3.52}$$

and then (see Remark 3.12) one can use Proposition 3.11.

In this case there is no bound state contribution. Using the properties of $\mathcal{I}_p(z)$ (see Appendix A), one obtains

Proposition 3.13 (Regular case)**.** (i) *Using* (3.47) *and* (3.48)*, we have*

$$\hat{A}_{\varepsilon,r}(t) = 2q_2 e^{i\kappa_2^2 t} - \frac{\beta}{2}\frac{\varepsilon^2}{\sqrt{\alpha}}\overline{\mathcal{I}_3(i\sqrt{\alpha t})} + \mathcal{O}(\varepsilon^{4-p}), \tag{3.53}$$

and up to error terms as in Theorem 3.7 *we have*

$$A_\varepsilon(t) = \hat{A}_{\varepsilon,r}(t). \tag{3.54}$$

(ii) *For* $p \in [0, 4)$ *we have*

$$|A_\varepsilon(t)|^2 = e^{-2\beta b^{1/2}\varepsilon^{2+\frac{p}{2}}t}, \tag{3.55}$$

and for $p \in (0, 4)$ *we have*

$$|A_\varepsilon(t)|^2 = e^{-2|g_1|b^{1/2}\varepsilon^{2+\frac{p}{2}}t}, \tag{3.56}$$

in both cases up to errors vanishing as $\varepsilon \to 0$.

The formula (3.56) agrees with the heuristic formula (3.11), as well as with the rigorous result in [16] for $p = 1$. Notice that, as expected, as p approaches 4 from below, the exponential decay law becomes less and less accurate, so one needs to compute corrections. Proposition 3.13 gives only the first order correction in $\varepsilon^2/\sqrt{|\alpha|}$, but the method of proof provides also the higher order corrections.

The bound state regime

If

$$\alpha = -b\varepsilon^p, \quad b > 0, \tag{3.57}$$

then for

$$p \in [0, 4) \tag{3.58}$$

one expects (see the heuristics) a bound state regime i.e. to leading order the contribution comes from the bound state. The result below provides the mathematical substantiation as well as the first order correction. Again the proof gives the means to compute higher order corrections.

As in the previous case, as $\varepsilon \to 0$, we have

$$\frac{\varepsilon^2}{\sqrt{|\alpha|}} \simeq \varepsilon^{2-\frac{p}{2}}. \tag{3.59}$$

Note that $\kappa_1 > 0$, and there is a contribution from the pole of $\dfrac{1}{F(z,\varepsilon)}$ at $x_b = -\kappa_1^2$.

The analogue of Proposition 3.13 reads

Proposition 3.14 (Regular case)**.**

$$\hat{A}_{\varepsilon,r}(t) = -i\frac{\beta}{2}\frac{\varepsilon^2}{\sqrt{|\alpha|}}\mathcal{I}_3(i\sqrt{|\alpha|t}) + \mathcal{O}(\varepsilon^{(4-p)}), \tag{3.60}$$

and up to error terms as in Theorem 3.7 *we have*

$$A_\varepsilon(t) = \hat{A}_{\varepsilon,r}(t) + \Big(1 - i\frac{\beta}{2}\frac{\varepsilon^2}{\sqrt{|\alpha|}}\Big)e^{i\kappa_1^2 t}. \tag{3.61}$$

To leading order one obtains the bound state behavior

$$|A_\varepsilon(t)|^2 = 1. \tag{3.62}$$

The non-exponential regime

We come now to the most interesting part of our analysis, when

$$|\alpha| = b\varepsilon^p, \quad b > 0, \quad \text{with} \quad p \geq 4. \tag{3.63}$$

According to the heuristics, for these values of p the decay law is neither (quasi)-exponential nor bound state like. We consider two cases separately.

Case 1: The threshold regime given by $p > 4$.

In this case the survival probability amplitude is given by

Proposition 3.15 (Regular case)**.** *Up to errors as in Theorem* 3.7 *we have*

$$A_\varepsilon(t) = \mathcal{I}_1(i\varepsilon^2\beta\sqrt{t}) + \mathcal{O}(\varepsilon^{p-4}). \tag{3.64}$$

The result (3.64) implies that the decay law is non-exponential for all $p > 4$.

Remark 3.16. The error term becomes more and more important, as p approaches the critical value $p = 4$. As before, higher order corrections to the leading term can be computed in terms of $\mathcal{I}_p$. The leading term is independent of α and equals the threshold case $x_0 = E_1 = 0$. Since $\mathcal{I}_1$ can be expressed (see Appendix A) in terms of the error function, one can rewrite (3.64) as follows:

$$A_\varepsilon(t) = \mathrm{w}(e^{i3\pi/4}\varepsilon^2\beta\sqrt{t}) + \mathcal{O}(\varepsilon^{p-4}) = e^{is}\big(1 - \mathrm{erf}(e^{i\pi/4}s^{1/2})\big) + \mathcal{O}(\varepsilon^{p-4}), \tag{3.65}$$

where

$$s = \beta^2\varepsilon^4 t. \tag{3.66}$$

In particular, (3.66) implies that the threshold decay time scale in the regular case is $t \sim \varepsilon^{-4}$.

Case 2: The "crossover regime", which is given by $p = 4$.

This case has been considered in [8]. Indeed, in this case in scaled variables $s = \varepsilon^4\beta^2 t$, $f = \dfrac{\alpha}{\varepsilon^4\beta^2}$ (note that for $p = 4$, f =const.) we have directly from (3.43) and (3.33):

$$\hat{A}_\varepsilon(t) = \frac{1}{\pi}\int_0^\infty \frac{y^{1/2}}{(f-y)^2 + y} e^{-isy} dy, \tag{3.67}$$

and the integral has been analyzed numerically in [8]. The decay law is non-exponential for finite f, while as $f \to \pm\infty$, one reaches the exponential and bound state behaviour, respectively.

3.7.2 The singular case

We now turn to the singular case, where the function approximating $F(z,\varepsilon)$ is given by $H_s(z) = \alpha - z - \varepsilon^2\beta/\kappa$, with α lying in a small neighborhood of the origin, and $\beta > 0$. The results are similar to those in the regular case, but a bit more complicated, due to the singular behavior of $H_s(z)$.

We recall (3.37) that

$$\hat{A}_{\varepsilon,s}(t) = \frac{1}{\pi}\int_0^\infty e^{-itx}\,\mathrm{Im}\,H_s(x+i0)^{-1}dx. \tag{3.68}$$

Passing to the variable $k = \sqrt{z} = i\kappa$, we can write it as

$$\hat{A}_{\varepsilon,s}(t) = \frac{-1}{\pi}\int_{-\infty}^\infty \frac{k^2 e^{-itk^2}}{P_s(\kappa)}dk, \tag{3.69}$$

where

$$P_s(\kappa) = \kappa^3 + \alpha\kappa - \varepsilon^2\beta, \tag{3.70}$$

and the integral on the r.h.s. of (3.69) is to be understood as the improper integral $\lim_{A\to\infty}\int_{-A}^{A}\frac{e^{-itk^2}}{P_s(\kappa)}k^2dk$.

If the zeroes of P_s are distinct, the partial fraction decomposition leads to the following result.

Proposition 3.17 (Singular case)**.**

$$\hat{A}_{\varepsilon,s}(t) = -\sum_{j=1}^{3} q_j\mathcal{I}_1(i\kappa_j t) \tag{3.71}$$

where κ_j, $j = 1,2,3$, are the roots of $P_s(\kappa)$ (as given by the Cardano formula, see (4.82) *below), and*

$$q_j = \frac{\kappa_j^2}{3\kappa_j^2+\alpha} \tag{3.72}$$

are the corresponding residues of $\frac{\kappa^2}{P_s(\kappa)}$ at $\kappa = \kappa_j$, $j = 1,2,3$.

Without using the explicit formulae for κ_j we can get the following properties of the roots of $P_s(\kappa)$:

i. $P_s(\kappa) = 0$ always has a positive solution, which we label κ_3. It corresponds to the bound state at $\tilde{x}_b = -\kappa_3^2$.

ii. For
$$\alpha \neq \alpha_0(\varepsilon) \equiv -3\left(\varepsilon^2\beta/2\right)^{2/3} \tag{3.73}$$
all κ_j are distinct. Note again that the case $\alpha = \alpha_0(\varepsilon)$ belongs to the crossover regime.

iii. For $\alpha > \alpha_0(\varepsilon)$ the other two solutions κ_1 and κ_2 are complex conjugates with real part equal to $-\frac{\kappa_3}{2}$. We label by κ_1 the one with positive imaginary part.

iv. For $\alpha < \alpha_0(\varepsilon)$, κ_1 and κ_2 are real, $\kappa_1, \kappa_2 < 0$, and $\kappa_1 + \kappa_2 = -\kappa_3$.

We recall that we take $|\alpha| = b\varepsilon^p$. For $p \neq \frac{4}{3}$ the somewhat complicated expressions for κ_j and q_j have simple expansions in the limit $\varepsilon \to 0$. Combining these expansions with the properties of $\mathcal{I}_p$, one arrives at results, which are very similar to those in the regular case. As before, we give only "first order" corrections, but higher order corrections can be computed. These expressions are much more complicated.

The exponential regime

If we have

$$\alpha = b\varepsilon^p,\ b > 0;\ p \in [0, 4/3), \tag{3.74}$$

the decay law is still exponential at small ε. Notice that as $\varepsilon \to 0$

$$\frac{\varepsilon^2}{\sqrt{\alpha^3}} \simeq \varepsilon^{2-\frac{3p}{2}}. \tag{3.75}$$

Using the properties of $\mathcal{I}_p(z)$ one gets the following result.

Proposition 3.18 (Singular case)**.** (i) *We have*

$$\hat{A}_\varepsilon(t) = 2q_2 e^{i\kappa_2^2 t} - i\frac{\beta\varepsilon^2}{2\alpha^{3/2}}\overline{\left(\mathcal{I}_1(i\sqrt{\alpha t}) - i\sqrt{\alpha t}\mathcal{I}_2(i\sqrt{\alpha t})\right)} + \mathcal{O}\left(\varepsilon^{2(2-3p/2)}\right) \tag{3.76}$$

(ii) *For $p \in [0, 4/3)$, we have*

$$|A_\varepsilon(t)|^2 = e^{-2\beta b^{1/2}\varepsilon^{2-\frac{p}{2}} t}, \tag{3.77}$$

and for $p \in (0, 4)$ we have

$$|A_\varepsilon(t)|^2 = e^{-2g_{-1}b^{1/2}\varepsilon^{2-\frac{p}{2}} t}, \tag{3.78}$$

in both cases up to errors vanishing as $\varepsilon \to 0$.

A remark is in order here. In spite of the fact that there is a bound state at $\tilde{x}_b = -\kappa_3^2$ for "large" positive α, its contribution is small and can be absorbed in the error term.

The bound state regime

In the case

$$\alpha = -b\varepsilon^p, \quad b > 0; \quad p \in [0, 4/3), \tag{3.79}$$

the heuristics predicts a bound state regime: The leading order contribution comes only from the bound state at $\tilde{x}_b = -\kappa_3^2$. The result below, similar to Proposition 3.14, provides the mathematical substantiation, as well as the first order correction.

Proposition 3.19 (Singular case)**.**

$$\hat{A}_\varepsilon(t) = \frac{\beta\varepsilon^2}{2\,|\alpha|^{3/2}}\big(\mathcal{I}_1(i\sqrt{|\alpha|t}) - i\sqrt{|\alpha|t}\mathcal{I}_2(i\sqrt{|\alpha|t})\big) + \mathcal{O}(\varepsilon^{2(2-\frac{p}{2})}), \tag{3.80}$$

and up to error terms as in Theorem 3.8

$$A_\varepsilon(t) = \hat{A}_{\varepsilon,s}(t) + 2q_3 e^{i\kappa_3^2 t}. \tag{3.81}$$

To leading order one obtains the bound state behavior

$$|A_\varepsilon(t)|^2 = 1. \tag{3.82}$$

The non-exponential regime

As in the regular case, according to the heuristics for $\alpha = b\varepsilon^p$, $b \in \mathbf{R}$, $p \geq 4/3$, the decay is neither exponential nor bound state like. As in the regular case, we shall distinguish between two cases.

Case 1: The threshold regime, which is $p > 4/3$. Up to errors as in Theorem 3.8 we have the result.

Proposition 3.20 (Singular case)**.** *Assume* $p > 4/3$. *Then we have*

$$A_\varepsilon(t) = \frac{2}{3}e^{i\kappa_3^2 t} - \frac{1}{3}\sum_{j=1}^{3}\mathcal{I}_1(i\rho_j\beta^{1/3}\varepsilon^{2/3}\sqrt{t}) + \mathcal{O}(\varepsilon^{p-\frac{4}{3}}). \tag{3.83}$$

Notice that in contrast to the regular case, there is a non-vanishing contribution from the bound state.

Case 2: The "crossover regime", which in the singular case takes place at the value $p = 4/3$. When $p = 4/3$, $f = \beta^{-2/3}b$, $b \in \mathbf{R}$ does not depend on ε, so there is no useful expansions for κ_j, q_j, $j = 1, 2, 3$. Accordingly (it has been done in [8] using the scaled variable $s = \varepsilon^{4/3}\beta^{2/3}t$) one writes (3.37) as

$$\hat{A}_\varepsilon(t) = \frac{1}{\pi}\int_0^\infty \frac{y^{1/2}}{y(f-y)^2 + 1}e^{-isy}dy,$$

and the integral can be analyzed numerically. In accordance with the above results, as $f \to \pm\infty$ one reaches the exponential and bound state behavior respectively; we refer to [8] for details.

4 The proofs

We now give the proofs of the results stated in the previous section.

4.1 Proof of Proposition 3.1

Assumption 2.3 and (2.7) (see also (3.13)) imply that for $0 < x < a$ we can write R in the form

$$R(x) = E_1 - x - \varepsilon^2 x^2 f_1(x) \quad \text{with} \quad \sup_{0<x<a} |f_1(x)| < \infty. \tag{4.1}$$

We have $R(0) = E_1$. For a sufficiently small ε we have $R(a) < 0$, and also that $R(x)$ is strictly decreasing. Thus it follows that the equation $R(x) = 0$ has a unique solution, $x_0 \in (0, a)$, which satisfies

$$x_0 = E_1 - \varepsilon^2 x_0^2 f_1(x_0), \tag{4.2}$$

which together with (4.1) finishes the proof.

4.2 Proof of Proposition 3.2

Part (i). One can obtain (3.21) and (3.22) by a rather tedious perturbation procedure for solving the equation for x_b; instead we shall give below a simple geometric argument. Consider first, for $m \geq 0$ and $n \simeq 1$, the (unique) positive solution $\tilde{y}$ of

$$f_{m,n}(y) \equiv m + y - \frac{n\varepsilon^2}{y^{1/2}} = 0. \tag{4.3}$$

Using Cardano's formulae one can see that

$$\tilde{y} \simeq \begin{cases} \varepsilon^{4/3}, & \text{for } 0 \leq m \leq \frac{n^{2/3}}{2}\varepsilon^{4/3}, \\ \dfrac{\varepsilon^4}{m^2}, & \text{for } m \gtrsim \varepsilon^{4/3}. \end{cases} \tag{4.4}$$

A simpler argument for (4.4) is to argue as follows. Let $\tilde{\tilde{y}} = (n\varepsilon^2)^{2/3}$, such that $f_{m,n}(\tilde{\tilde{y}}) = m$. Then

$$\frac{d}{dy} f_{m,n}(y) \geq 1 \tag{4.5}$$

implies that $\tilde{\tilde{y}} - m \leq \tilde{y} \leq \tilde{\tilde{y}} \simeq \varepsilon^{4/3}$, which gives the first part of (4.4).

Assume $m \gtrsim \varepsilon^{4/3}$. Then for $y \lesssim \varepsilon^{4/3}$ we have

$$m - \frac{n\varepsilon^2}{y^{1/2}} \leq m + y - \frac{n\varepsilon^2}{y^{1/2}} \leq \text{const.} m - \frac{n\varepsilon^2}{y^{1/2}},$$

which implies

$$(\frac{n\varepsilon^2}{\text{const}.m})^2 \le \tilde{y} \le (\frac{n\varepsilon^2}{m})^2,$$

and thus the second part of (4.4).

Consider now $F(x,\varepsilon)$ for $x<0$ and $|x|$ sufficiently small. Recall that for $x<0$ we have $\kappa = |x|^{1/2}$. Then

$$E_1 - x - \varepsilon^2 \frac{2g_{-1}}{|x|^{1/2}} \le F(x,\varepsilon) \le E_1 - x - \varepsilon^2 \frac{g_{-1}}{2|x|^{1/2}}, \tag{4.6}$$

and the estimates (4.4) lead to (3.21).

The argument for (3.22) is similar. Consider $f_{m,n}(y)$ for $m \le 0$. Notice that (4.5) still holds true and implies

$$\tilde{\tilde{y}} \le \tilde{y} \le |m| + \tilde{\tilde{y}}.$$

Use again (4.6).

Part (ii). Note that in this case $F(0,\varepsilon) = E_1$, and as always (see (2.7) and Proposition 2.2) for $x<0$ we have $\frac{d}{dx}F(x,\varepsilon) \le -1$. This implies the non-existence of bound states for $E_1 \ge 0$, the existence and uniqueness of the solution for $E_1 \le 0$ (recall that for ε sufficiently small and $-\frac{a}{2} \le E_1$ we have $F(-a,\varepsilon) > 0$), as well as (3.23).

4.3 Proof of Theorem 3.7

Consider first the case $E_1 \ge 0$, such that $x_0 \ge 0$ exists.

Lemma 4.1. *Assume $E_1 \ge 0$. For $c>0$ sufficiently small, and $0 \le x_0 \le \frac{c}{2}$ we have*

$$\int_0^c \Big| \frac{1}{F(x+i0,\varepsilon)} - \frac{1}{H_r(x+i0)} \Big| dx \lesssim \begin{cases} \varepsilon^2 \ln \frac{1}{\varepsilon}, & \text{for } 0 \le x_0 \le \frac{c}{2} \\ \varepsilon^2, & \text{for } 0 \le x_0 \lesssim \varepsilon^p, \text{ any } p>0. \end{cases} \tag{4.7}$$

Proof. We recall that we always have $c \le a/2$. Furthermore, recall also that $g_1 < 0$. Then Assumption 2.3 and (3.13) imply

$$R(x) \equiv E_1 - x - \varepsilon^2 x^2 f_1(x), \quad I(x) = \varepsilon^2 \sqrt{x}(g_1 + \mathcal{O}(x)) \equiv -\varepsilon^2 \sqrt{x} f_2(x), \tag{4.8}$$

with f_j uniformly Lipschitz on $[0,c]$, $j=1,2$. Taking c small enough, (4.8) implies

$$|R(x)| \ge \tfrac{1}{2}|x-x_0|, \quad |I(x)| \ge \tfrac{1}{2}\varepsilon^2 \sqrt{x}|g_1|. \tag{4.9}$$

By definition of the model function (see Section 3.5) we have

$$\alpha = E_1 - \varepsilon^2 x_0^2 f_1(x_0), \quad \beta = f_2(x_0). \tag{4.10}$$

Since both f_1 and f_2 are uniformly Lipschitz, we get

$$\begin{aligned} &|F(x+i0,\varepsilon) - H_r(x+i0)| \\ &\qquad \le \varepsilon^2\big(|x^2 f_1(x) - x_0^2 f_1(x_0)| + \sqrt{x}|f_2(x) - f_2(x_0)|\big) \\ &\qquad\qquad \lesssim \varepsilon^2 |x - x_0|(\hat{x} + \sqrt{x}), \end{aligned} \tag{4.11}$$

where $\hat{x}$ lies between x and x_0. Putting together (4.9) and (4.11) we get

$$\int_0^c \Big| \frac{1}{F(x+i0,\varepsilon)} - \frac{1}{H_r(x+i0)} \Big| dx \lesssim \varepsilon^2 \int_0^c \frac{|x-x_0|(\hat{x}+\sqrt{x})}{|x-x_0|^2 + \varepsilon^4 x} dx. \tag{4.12}$$

We estimate the integral on the right hand side in (4.12) on three subintervals. Consider first $\int_{2x_0}^c$. In this case $\hat{x} \le x \lesssim \sqrt{x}$, $|x - x_0| < x$, $|x - x_0| > x/2$, and then

$$\varepsilon^2 \int_{2x_0}^c \frac{|x-x_0|(\hat{x}+\sqrt{x})}{|x-x_0|^2 + \varepsilon^4 x} dx \lesssim \varepsilon^2 \int_{2x_0}^c \frac{\sqrt{x}}{x + \varepsilon^4} dx \lesssim \varepsilon^2. \tag{4.13}$$

Consider now $\int_0^{x_0}$. Here we have $\hat{x} < x_0$, such that

$$\begin{aligned} \int_0^{x_0} \frac{|x-x_0|(\hat{x}+\sqrt{x})}{|x-x_0|^2 + \varepsilon^4 x} dx &\lesssim \sqrt{x_0} \int_0^{x_0} \frac{x_0 - x}{|x-x_0|^2 + \varepsilon^4 x} dx \\ &= \sqrt{x_0} \int_0^1 \frac{y}{y^2 + \frac{\varepsilon^4}{x_0}(1-y)} dy. \end{aligned} \tag{4.14}$$

In estimating the last integral in (4.14) we use the notation $m = \frac{\varepsilon^4}{x_0} \gtrsim \varepsilon^4$. We get

$$\int_0^{1/2} \frac{y}{y^2 + m(1-y)} dy \lesssim \int_0^{1/2} \frac{y}{y^2 + m} dy = \frac{1}{2} \ln(1 + \frac{1}{4m}), \tag{4.15}$$

$$\int_{1/2}^1 \frac{y}{y^2 + m(1-y)} dy \lesssim \int_{1/2}^1 \frac{1}{1 + m(1+y)} dy \frac{1}{m} \ln(1 + \frac{m}{2}). \tag{4.16}$$

Since $\sup_{u>0} u \ln(1 + \frac{1}{2u}) < \infty$, the estimates (4.15) and (4.16) imply

$$\int_0^1 \frac{y}{y^2 + \frac{\varepsilon^4}{x_0}(1-y)} dy \lesssim \ln \frac{1}{\varepsilon}. \tag{4.17}$$

Finally consider

$$\int_{x_0}^{2x_0} \frac{|x-x_0|(\hat{x}+\sqrt{x})}{|x-x_0|^2+\varepsilon^4 x}dx \lesssim \sqrt{x_0}\int_{x_0}^{2x_0} \frac{x-x_0}{|x-x_0|^2+\varepsilon^4 x}dx$$
$$\leq \sqrt{x_0}\int_0^{x_0} \frac{u}{u^2+\varepsilon^4 u}du = \sqrt{x_0}\ln(1+\frac{x_0}{\varepsilon^4}) \lesssim \sqrt{x_0}\ln\frac{1}{\varepsilon} \quad (4.18)$$

Putting together (4.13), (4.14), (4.17), and (4.18), one obtains (4.7) and the proof of Lemma 4.1 is finished. □

Consider now $E_1 < 0$. In this case we claim the following result.

Lemma 4.2. *Assume $E_1 < 0$. For $c > 0$ sufficiently small, and $-\frac{c}{2} < E_1 < 0$ we have*

$$\int_0^c \Big|\frac{1}{F(x+i0,\varepsilon)} - \frac{1}{H_r(x+i0)}\Big|dx \lesssim \varepsilon^2. \quad (4.19)$$

Proof. In this case we write for $x < 0$

$$F(x,\varepsilon) = E_1 - x - g_1\varepsilon^2\sqrt{|x|} + \varepsilon^2|x|^{3/2}f(x). \quad (4.20)$$

From Assumption 2.3 (recall that for $F(x,\varepsilon)$ is analytic for $x < 0$) follows that $||x|^{-1/2}\frac{d}{dx}(|x|^{3/2}f(x))| \lesssim 1$. The equation for β (the equality of the derivatives of F and H_r at x_b) leads to

$$\beta = -g_1 + 2\sqrt{|x_b|}\frac{d}{dx}(|x|^{3/2}f(x))\Big|_{x_b},$$

which implies

$$|\beta + g_1| \lesssim |x_b|. \quad (4.21)$$

The equation for α is

$$\alpha - x_b + \varepsilon^2\sqrt{|x_b|}\beta = F(x_b,\varepsilon) = 0.$$

Using (4.21) we get

$$\alpha = x_b - \varepsilon^2\sqrt{|x_b|}\beta = x_b + \varepsilon^2\sqrt{|x_b|}g_1 + \varepsilon^2\mathcal{O}(|x_b|^{3/2}). \quad (4.22)$$

On the other hand, $F(x_b,\varepsilon) = 0$ and (4.20) give

$$E_1 = x_b + \varepsilon^2\sqrt{|x_b|}g_1 + \varepsilon^2\mathcal{O}(|x_b|^{3/2}), \quad (4.23)$$

which together with (4.21) yields

$$|\alpha - E_1| \lesssim \varepsilon^2|x_b|^{3/2}. \quad (4.24)$$

As in the proof of the previous lemma, we need estimates on $|F|$ and $|H_r|$. We claim that

$$\min\{|F(x+i0,\varepsilon)|, |H_r(x+i0)|\} \geq \frac{1}{2}((|E_1|+x)^2 + g_1^2\varepsilon^4 x)^{1/2}. \qquad (4.25)$$

Concerning $|F|$, the estimate for the imaginary part is the same as in the previous lemma, i.e. $|I(x)| \geq \frac{1}{2}|g_1|\varepsilon^2\sqrt{x}$. As for $|R(x)|$, (4.8) is written as $R(x) = E_1 - x(1+\varepsilon^2 x f_1(x))$, such that $R(x) \leq E_1 - \frac{x}{2}$, and then $|R(x)| \geq \frac{1}{2}(|E_1|+x)$. As for $|H_r|$, from its definition follows

$$|H_r(x+i0)|^2 = |\alpha - x|^2 + \varepsilon^4\beta^2 x. \qquad (4.26)$$

For a sufficiently small ε (and using $E_1 < 0$), the results (4.24) and (3.23) imply

$$\alpha \leq E_1 + |\alpha - E_1| = E_1 + \mathcal{O}(\varepsilon^2|E_1|^{3/2}) \leq \frac{E_1}{2}. \qquad (4.27)$$

On the other hand, for $|x_b|$ sufficiently small, we get from (4.21) the estimate

$$\beta = -g_1 + \beta + g_1 \geq |g_1| - \mathcal{O}(|x_b|) \geq \frac{|g_1|}{2}. \qquad (4.28)$$

Putting together (4.26), (4.27), and (4.28), one obtains (4.25). Furthermore, from (4.20), (3.33), (4.21), and (4.24), we get

$$\begin{aligned}|F(x+i0) - H_r(x+i0)| &= |E_1 - \alpha + i\varepsilon^2\sqrt{|x|}(g_1+\beta)| + \mathcal{O}(\varepsilon^2|x|^{3/2}) \\ &\lesssim \varepsilon^2(|x_b|^{3/2} + |x_b|\sqrt{|x|} + |x|^{3/2}), \quad (4.29)\end{aligned}$$

which together with (4.25) gives

$$\int_0^c \left|\frac{1}{F(x+i0,\varepsilon)} - \frac{1}{H_r(x+i0)}\right| dx \lesssim \varepsilon^2 \int_0^c \frac{|x_b|^{3/2} + |x_b|\sqrt{|x|} + |x|^{3/2}}{(|E_1|+x)^2 + g_1^2\varepsilon^4 x}. \quad (4.30)$$

Consider now various terms in (4.30). Due to (3.23) we have the following three estimates.

$$|x_b|^{3/2}\int_0^c \frac{1}{(|E_1|+x)^2 + g_1^2\varepsilon^4 x}dx \leq |x_b|^{3/2}\int_0^c \frac{1}{(|E_1|+x)^2}dx \lesssim \frac{|x_b|^{3/2}}{|E_1|} \lesssim 1,$$

$$|x_b|\int_0^c \frac{\sqrt{x}}{(|E_1|+x)^2 + g_1^2\varepsilon^4 x}dx \leq |x_b|\int_0^c \frac{\sqrt{x}}{(|E_1|+x)^2}dx \lesssim \frac{|x_b|}{|E_1|} \lesssim 1,$$

$$\int_0^c \frac{x^{3/2}}{(|E_1|+x)^2 + g_1^2\varepsilon^4 x}dx \leq \int_0^c \frac{1}{\sqrt{x}}dx \lesssim 1,$$

which together with (4.30) gives (4.19). □

Proof of Theorem 3.7. From this point onwards the proof of Theorem 3.7 follows closely the proof of Theorem 3.3 in [8]. For the convenience of the reader we outline it for the case $E_1 \geq 0$. We take E_1 sufficiently small (e.g. $E_1 \leq \frac{c}{4}$), such that $\alpha \leq \frac{c}{2}$. Then for $x \geq c$ we have $|H_r(x+i0)|^2 \gtrsim x^2 + \varepsilon^4 x$. Using (3.33) we get $|\operatorname{Im} H_r(x+i0)| \lesssim \varepsilon^2 \sqrt{x}$. Thus

$$\int_c^\infty \frac{|\operatorname{Im} H_r(x+i0)|}{|H_r(x+i0)|^2} dx \lesssim \varepsilon^2 \int_c^\infty \frac{\sqrt{x}}{x^2+\varepsilon^4 x} dx \simeq \varepsilon^2. \tag{4.31}$$

Let now

$$A_{c,\varepsilon}(t) = \lim_{\eta \searrow 0} \frac{1}{\pi} \int_0^c e^{-itx} \operatorname{Im} F(x+i\eta, \varepsilon)^{-1} dx.$$

Due to Assumption 2.3 the limit $\eta \searrow 0$ can be taken, such that

$$A_{c,\varepsilon}(t) = \frac{1}{\pi} \int_0^c e^{-itx} \operatorname{Im} F(x+i0, \varepsilon)^{-1} dx. \tag{4.32}$$

Lemma 4.1 and (4.31) imply that

$$|A_{c,\varepsilon}(t) - \frac{1}{\pi} \int_0^\infty e^{-itx} \operatorname{Im} H_r(x+i0)^{-1} dx| \lesssim \begin{cases} \varepsilon^2 \ln \frac{1}{\varepsilon}, & \text{for } 0 \leq x_0 \leq \frac{c}{2}, \\ \varepsilon^2, & \text{for } 0 \leq x_0 \lesssim \varepsilon^p, \text{ any } p > 0. \end{cases} \tag{4.33}$$

We finish the proof by using "Hunziker's trick", see[12]. More precisely, observe that

$$|A_{c,\varepsilon}(t) - A_\varepsilon(t)| \leq |A_{c,\varepsilon}(0) - A_\varepsilon(0)| = |A_{c,\varepsilon}(0) - 1|. \tag{4.34}$$

From [8, Lemma 3.4(i)] we get

$$\frac{1}{\pi} \int_0^\infty \operatorname{Im} H_r(x+i0)^{-1} dx = 1. \tag{4.35}$$

Putting together (4.33), (4.34), (4.35), and (4.33) for $t = 0$, finishes the proof of Theorem 3.7 for the case $E_1 \geq 0$.

The proof for the case is $E_1 < 0$ similar: use Lemma 4.2 and (see Lemma 3.4(ii) in [8])

$$-\frac{1}{\frac{d}{dx} H_r(x_b)} + \frac{1}{\pi} \int_0^\infty \operatorname{Im} H_r(x+i0)^{-1} dx = 1. \quad \square \tag{4.36}$$

4.4 Proof of Theorem 3.8

The proof consists of the same steps as in the proof of Theorem 3.7. In this case $I(x) = \varepsilon^2 x^{-1/2}(g_{-1} + \mathcal{O}(x)) \equiv \varepsilon^2 x^{-1/2} f_2(x)$ (not the same function that was also denoted by f)2 above). Consider first the case $E_1 \geq 0$, such that $x_0 \geq 0$ exists.

Lemma 4.3. *For $c > 0$ sufficiently small and $0 \leq x_0 \leq \frac{c}{2}$ we have*

$$\int_0^c \left| \frac{1}{F(x+i0,\varepsilon)} - \frac{1}{H_s(x+i0)} \right| dx \lesssim \begin{cases} \frac{\varepsilon^2}{x_0^{1/2}} \ln \frac{1}{\varepsilon}, & \text{for } \varepsilon^{4/3} \lesssim x_0 \leq \frac{c}{2}, \\ \varepsilon^{4/3}, & \text{for } 0 \leq x_0 \lesssim \varepsilon^{4/3}. \end{cases} \tag{4.37}$$

A computation similar to the one in the proof of Lemma 4.1 leads to

$$\int_0^c \left| \frac{1}{F(x+i0,\varepsilon)} - \frac{1}{H_s(x+i0)} \right| dx \lesssim \varepsilon^2 \int_0^c \frac{|x-x_0|(\hat{x} + x^{-1/2})}{|x-x_0|^2 + \varepsilon^4 x^{-1}} dx, \tag{4.38}$$

where $\hat{x}$ lies between x and x_0. The term containing $\hat{x}$ is the same as in the regular case, so we have to consider only $\varepsilon^2 \int_0^c \frac{|x-x_0|x^{-1/2}}{|x-x_0|^2+\varepsilon^4 x^{-1}} dx$.

Consider first the case $x_0 \gtrsim \varepsilon^{4/3}$. Observe that

$$\varepsilon^2 \frac{|x-x_0|x^{-1/2}}{|x-x_0|^2 + \varepsilon^4 x^{-1}} \leq \frac{1}{2}, \tag{4.39}$$

and then

$$\varepsilon^2 \int_0^{\varepsilon^2} \frac{|x-x_0|x^{-1/2}}{|x-x_0|^2 + \varepsilon^4 x^{-1}} dx \leq \varepsilon^2. \tag{4.40}$$

We use the following estimate

$$\begin{aligned} x^{-1/2} \leq |x^{-1/2} - x_0^{-1/2}| + x_0^{-1/2} &= \frac{1}{x^{-1/2} + x_0^{-1/2}} \frac{|x-x_0|}{x_0 x} + x_0^{-1/2} \\ &\leq \frac{|x-x_0|}{x x_0^{1/2}} + x_0^{-1/2}. \end{aligned}$$

We then get

$$
\begin{aligned}
\varepsilon^2 \int_{\varepsilon^2}^{c} \frac{|x-x_0|x^{-1/2}}{|x-x_0|^2+\varepsilon^4 x^{-1}}dx &\le \frac{\varepsilon^2}{x_0^{1/2}} \int_{\varepsilon^2}^{c} \frac{|x-x_0|^2}{x|x-x_0|^2+\varepsilon^4}dx \\
&\quad + \frac{\varepsilon^2}{x_0^{1/2}} \int_{\varepsilon^2}^{c} \frac{|x-x_0|}{|x-x_0|^2+\varepsilon^4 x^{-1}}dx \\
&\lesssim \frac{\varepsilon^2}{x_0^{1/2}} \int_{\varepsilon^2}^{c} x^{-1} + \frac{\varepsilon^2}{x_0^{1/2}} \int_{\varepsilon^2}^{c} \frac{|x-x_0|}{|x-x_0|^2+\varepsilon^4}dx \\
&\lesssim \frac{\varepsilon^2}{x_0^{1/2}} \ln\frac{1}{\varepsilon}.
\end{aligned}
$$

Consider now the case $x_0 \lesssim \varepsilon^{4/3}$. Due to (4.39) it remains to estimate $\int_{\text{const. } \varepsilon^{4/3}}^{c}$. Here one has $\frac{x}{2} \le x - x_0 \le x$ and then

$$
\begin{aligned}
\int_{\text{const. } \varepsilon^{4/3}}^{c} \frac{|x-x_0|x^{-1/2}}{|x-x_0|^2+\varepsilon^4 x^{-1}}dx &\le \varepsilon^2 \int_{\text{const. } \varepsilon^{4/3}}^{c} \frac{x^{1/2}}{x^2+\varepsilon^4 x^{-1}}dx \\
&\lesssim \int_{\text{const. } \varepsilon^{4/3}}^{c} x^{-3/2}dx \lesssim \varepsilon^{4/3}, \qquad (4.41)
\end{aligned}
$$

and the proof of Lemma 4.3 is finished.

The next step (still for the case $E_1 \ge 0$) is to control the error when F is replaced by H_s in the bound state contribution.

Lemma 4.4. *Assume $E_1 \ge 0$. Let $\tilde{x}_b$ be the unique solution on $(-\infty, 0)$ of the equation $H_s(x) = 0$. Then*

$$
\Big|\frac{1}{\frac{d}{dx}F(x_b)} - \frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)}\Big| \lesssim \begin{cases} \dfrac{\varepsilon^4}{x_0^2}, & \text{for } x_0 \gtrsim \varepsilon^{4/3} \\ \varepsilon^{4/3}, & \text{for } x_0 \lesssim \varepsilon^{4/3}. \end{cases} \qquad (4.42)
$$

Proof.

$$
\begin{aligned}
\Big|\frac{1}{\frac{d}{dx}F(x_b)} - \frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)}\Big| &\le \Big|\frac{1}{\frac{d}{dx}F(x_b)} - \frac{1}{\frac{d}{dx}H_s(x_b)}\Big| \\
&\quad + \Big|\frac{1}{\frac{d}{dx}H(x_b)} - \frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)}\Big|. \qquad (4.43)
\end{aligned}
$$

Consider the first term in (4.43). Assumption 2.3 and (3.32) follows that

$$
\Big|\frac{d}{dx}F(x_b) - \frac{d}{dx}H_s(x_b)\Big| \lesssim |\beta - g_{-1}|\frac{\varepsilon^2}{|x_b|^{3/2}} + \frac{\varepsilon^2}{|x_b|^{1/2}}, \qquad (4.44)
$$

$$\Big|\frac{d}{dx}F(x_b)\Big| \simeq |\frac{d}{dx}H_s(x_b)| \simeq 1 + \frac{\varepsilon^2}{|x_b|^{3/2}}. \tag{4.45}$$

Furthermore, the equation for β is

$$\frac{\beta\varepsilon^2}{|x_0|^{1/2}} = \frac{g_{-1}\varepsilon^2}{|x_0|^{1/2}} + \varepsilon^2\mathcal{O}(x_0^{1/2}),$$

which implies

$$\beta = g_{-1} + \mathcal{O}(x_0). \tag{4.46}$$

Using (4.44), (4.45), and (4.46), we get

$$\Big|\frac{1}{\frac{d}{dx}F(x_b)} - \frac{1}{\frac{d}{dx}H_s(x_b)}\Big| \lesssim \Big(\frac{x_0\varepsilon^2}{|x_b|^{3/2}} + \frac{\varepsilon^2}{|x_b|^{1/2}}\Big)\Big(1 + \frac{\varepsilon^2}{|x_b|^{3/2}}\Big)^{-2}. \tag{4.47}$$

From (3.21) and (2.17) one has

$$|x_b| \simeq \begin{cases} \varepsilon^{4/3}, & \text{if } 0 \le x_0 \lesssim \varepsilon^{4/3}, \\ \dfrac{\varepsilon^4}{E_1^2}, & \text{if } x_0 \gtrsim \varepsilon^{4/3}. \end{cases} \tag{4.48}$$

Combining (4.47) with (4.48) one obtains

$$\Big|\frac{1}{\frac{d}{dx}F(x_b)} - \frac{1}{\frac{d}{dx}H_s(x_b)}\Big| \lesssim \begin{cases} \varepsilon^{4/3}, & \text{for } 0 \le x_0 \lesssim \varepsilon^{4/3}, \\ \dfrac{\varepsilon^4}{x_0^2}, & \text{for } x_0 \gtrsim \varepsilon^{4/3}. \end{cases} \tag{4.49}$$

The last step in proving Lemma 4.4 is to estimate $\Big|\frac{1}{\frac{d}{dx}H(x_b)} - \frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)}\Big|$. Taylor's theorem implies

$$\Big|\frac{1}{\frac{d}{dx}H(x_b)} - \frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)}\Big| \lesssim \frac{|\frac{d^2}{dx^2}H_s(u)|}{|\frac{d}{dx}H_s(u)|^2}|x_b - \tilde{x}_b|, \tag{4.50}$$

where u lies between $\tilde{x}_b$ and x_b. Since $H_s(\tilde{x}_b) = 0$ and $|\frac{d}{dx}H(x)| \ge 1$, we get

$$|x_b - \tilde{x}_b| \le |H_s(x_b)|.$$

Since $F(x_b) = 0$, one has from (2.7), (3.32), (3.35), (3.19), and (4.46) that

$$\begin{aligned} |H_s(x_b)| &= |H_s(x_b) - F(x_b)| \\ &\lesssim |x_0 - E_1| + \varepsilon^2|g_{-1} - \beta||x_b|^{-1/2} + \varepsilon^2|x_b|^{1/2} \\ &\lesssim \varepsilon^2(x_0^2 + x_0|x_b|^{-1/2} + |x_b|^{1/2}). \end{aligned} \tag{4.51}$$

Again we have to consider two cases separately. First we consider the case $x_0 \lesssim \varepsilon^{4/3}$. The result (3.35) and the proof of Proposition 3.2, together with (4.48) leads to

$$|x_b| \simeq |\tilde{x}_b| \simeq \varepsilon^{4/3}.$$

Since u lies between x_b and $\tilde{x}_b$, we get $|\frac{d}{dx}H(u)| \simeq 1$ and $|\frac{d^2}{dx^2}H(u)| \simeq \varepsilon^{-4/3}$. Inserting these results into (4.51) one gets from (4.50) the result

$$\Big|\frac{1}{\frac{d}{dx}H(x_b)} - \frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)}\Big| \lesssim \varepsilon^{4/3}. \qquad (4.52)$$

Consider now the case $x_0 \gtrsim \varepsilon^{4/3}$. By the same argument as before

$$|x_b| \simeq |\tilde{x}_b| \simeq |u| \simeq \frac{\varepsilon^4}{x_0^2}.$$

Then $|\frac{d}{dx}H(u)| \simeq 1 + \frac{x_0^3}{\varepsilon^4}$ and $|\frac{d^2}{dx^2}H(u)| \simeq \frac{x_0^5}{\varepsilon^8}$, which together with (4.51) and (4.50) leads to

$$\Big|\frac{1}{\frac{d}{dx}H(x_b)} - \frac{1}{\frac{d}{dx}H_s(\tilde{x}_b)}\Big| \lesssim \frac{\varepsilon^4}{x_0^2}. \qquad (4.53)$$

Putting together (4.49), (4.52), and (4.53) finishes the proof of the Lemma 4.4. □

We are left with the estimate of the error for the case $E_1 < 0$. Since the pole positions and the residues for F and H_s coincide by the definition of H_s (see Section 3.5) the error comes only from the positive semi-axis integral.

Lemma 4.5. *For $c > 0$ sufficiently small and $-\frac{c}{2} < E_1 < 0$,*

$$\int_0^c \Big|\frac{1}{F(x+i0,\varepsilon)} - \frac{1}{H_s(x+i0)}\Big| dx \lesssim \begin{cases} \dfrac{\varepsilon^2}{|E_1|^{1/2}}, & \text{for } |E_1| \gtrsim \varepsilon^{4/3} \\ \varepsilon^{4/3}, & \text{for } |E_1| \lesssim \varepsilon^{4/3}. \end{cases} \qquad (4.54)$$

Proof. The arguments that lead to (4.21) (4.24) and (4.11), can be applied in this case and yield

$$|\beta - g_{-1}| \lesssim |x_b| \quad \text{and} \quad |\alpha - E_1| \lesssim \varepsilon^2 |x_b|^{1/2}, \qquad (4.55)$$

and furthermore

$$|F(x+i0) - H_s(x+i0)| = |E_1 - \alpha + i\varepsilon^2\sqrt{|x|}(g_{-1} - \beta)| + \varepsilon^2\mathcal{O}(|x|^{3/2}). \quad (4.56)$$

Using (4.55) in (4.56), one gets

$$|F(x+i0)-H_s(x)| \lesssim \varepsilon^2(|x_b|^{1/2}+|x_b||x|^{-1/2}+|x|^{1/2}). \tag{4.57}$$

Since $E_1<0$, one has $|R(x)|\geq(|E_1|+x)/2$. Furthermore,

$$|I(x)|=\varepsilon^2 x^{-1/2}(g_{-1}+\mathcal{O}(x)),$$

and then

$$|F(x+i0)| \gtrsim \big((|E_1|+x)^2+\frac{\varepsilon^4}{x}\big)^{1/2}. \tag{4.58}$$

The result (3.32) leads to

$$|H_s(x+i0)| \gtrsim \big((\alpha-x)^2+\frac{\varepsilon^4}{x}\big)^{1/2}. \tag{4.59}$$

The problem with $|H_s|$ is that $\alpha-x$ might vanish for some $x>0$. However, for $x\gtrsim\varepsilon^2$, we can use (4.55) to get

$$x-\alpha=\frac{x}{2}-E_1+\frac{x}{2}+E_1-\alpha\geq\frac{x}{2}-E_1\geq\frac{x+|E_1|}{2},$$

and then from (4.57), (4.58) and (4.59) one has

$$\int_{\text{const. }\varepsilon^2}^{c}\Big|\frac{1}{F(x+i0,\varepsilon)}-\frac{1}{H_s(x+i0)}\Big|dx \lesssim \varepsilon^2\int_{\text{const. }\varepsilon^2}^{c}\frac{|x_b|^{1/2}+|x_b|x^{-1/2}+x^{1/2}}{(|E_1|+x)^2+\frac{\varepsilon^4}{x}}dx. \tag{4.60}$$

We estimate (4.60) for $|E_1|\lesssim\varepsilon^{4/3}$. In this case we get Proposition 3.2(i) $|x_b|\lesssim\varepsilon^{4/3}$, and then

$$\varepsilon^2\int_{\text{const. }\varepsilon^2}^{c}\frac{|x_b|^{1/2}+|x_b|x^{-1/2}+x^{1/2}}{(|E_1|+x)^2+\frac{\varepsilon^4}{x}}dx \lesssim \varepsilon^2\int_0^c x\frac{\varepsilon^{2/3}+\varepsilon^{4/3}x^{-1/2}+x^{1/2}}{x^3+\varepsilon^4}dx \lesssim \varepsilon^{4/3}\int_0^\infty\frac{1+y^{1/2}+y^{-1/2}}{1+y^3}dy. \tag{4.61}$$

Using Proposition 3.2(i) once more, we get for $|E_1|\gtrsim\varepsilon^{4/3}$ that $|x_b|\lesssim|E_1|$, and then that

$$
\begin{aligned}
\varepsilon^2 \int_{\text{const. } \varepsilon^2}^{c} & \frac{|x_b|^{1/2} + |x_b| x^{-1/2} + x^{1/2}}{(|E_1| + x)^2 + \frac{\varepsilon^4}{x}} dx \\
& \lesssim \varepsilon^2 \int_0^c \frac{|E_1|^{1/2} + |E_1| x^{-1/2} + x^{1/2}}{(x + |E_1|)^2} dx \\
& \lesssim \frac{\varepsilon^2}{|E_1|^{1/2}} \int_0^\infty \frac{1 + y^{1/2} + y^{-1/2}}{(1+y)^2} dy \lesssim \varepsilon^{4/3}. \qquad (4.62)
\end{aligned}
$$

The last step is to estimate

$$
\varepsilon^2 \int_0^{\text{const. } \varepsilon^2} \frac{|x_b|^{1/2} + |x_b| x^{-1/2} + x^{1/2}}{(|E_1| + x)^2 + \frac{\varepsilon^4}{x}} dx.
$$

As before, Proposition 3.2(i) implies for $|E_1| \lesssim \varepsilon^{4/3}$ that $|x_b| \lesssim \varepsilon^{4/3}$, and then

$$
\begin{aligned}
\varepsilon^2 \int_0^{\text{const. } \varepsilon^2} & \frac{|x_b|^{1/2} + |x_b| x^{-1/2} + x^{1/2}}{(|E_1| + x)^2 + \frac{\varepsilon^4}{x}} dx \\
& \lesssim \varepsilon^2 \int_0^{\text{const. } \varepsilon^2} \frac{\varepsilon^{2/3} + \varepsilon^{4/3} x^{-1/2} + x^{1/2}{}^{1/2}}{x^3 + \varepsilon^4} x dx \\
& \lesssim \varepsilon^{4/3} \int_0^{\text{const. } \varepsilon^{2/3}} \frac{1 + y^{1/2} + y^{-1/2}}{(1+y^3)^{1/2}} dy. \qquad (4.63)
\end{aligned}
$$

For $|E_1| \gtrsim \varepsilon^{4/3}$, again from Proposition 3.2(i), $|x_b| \lesssim |E_1|$, and then

$$
\begin{aligned}
\varepsilon^2 \int_0^{\text{const. } \varepsilon^2} & \frac{|x_b|^{1/2} + |x_b| x^{-1/2} + x^{1/2}}{(|E_1| + x)^2 + \frac{\varepsilon^4}{x}} dx \\
& \lesssim \varepsilon^2 \int_0^{\text{const. } \varepsilon^2} \frac{|E_1|^{1/2} + |E_1| x^{-1/2} + x^{1/2}}{(x + |E_1|) \frac{\varepsilon^2}{x^{1/2}}} dx \\
& = \int_0^{\text{const. } \frac{\varepsilon^2}{|E_1|}} \frac{1 + y^{1/2} + y^{-1/2}}{1+y} |E_1| dy \simeq \varepsilon^2. \qquad (4.64)
\end{aligned}
$$

Now (4.54) follows from (4.61) and (4.64), and the proof of the lemma is finished. □

Proof of Theorem 3.8. From this point the proof is the same as the proof of Theorem 3.7: use Lemmas 4.4 -4.5 and (see the proof of Lemma 3.2 in [8])

$$
-\frac{1}{\frac{d}{dx} H_r(x_b)} e^{-itx_b} + \frac{1}{\pi} \int_0^\infty e^{-itx} \operatorname{Im} H_r(x + i0)^{-1} dx = 1. \quad \square \qquad (4.65)
$$

4.5 Proof of Proposition 3.11

Straightforward computation, which we omit.

4.6 Proof of Proposition 3.13

With the notation

$$r = \sqrt{\alpha - \frac{\varepsilon^4\beta^2}{4}}. \tag{4.66}$$

we can write (3.47) and (3.48) as follows.

$$\kappa_j = (-1)^{j+1} ir - \frac{\varepsilon^2\beta}{2}, \quad q_j = \frac{1}{2} + (-1)^{j+1} i\frac{\varepsilon^2\beta}{4r}, \quad j = 1, 2. \tag{4.67}$$

Note that $\frac{\varepsilon^2}{r} \simeq \varepsilon^{2-\frac{p}{2}}$. we have $\kappa_1 = \overline{\kappa}_2$. Thus $i\kappa_1$ and $i\kappa_2$ lie in the third and fourth quadrants respectively. Using (A.8) and (4.67), one obtains by direct computation

$$\begin{aligned}\hat{A}_{\varepsilon,r} = 2q_2 e^{it\kappa_2^2} - \frac{1}{2}\overline{(\mathcal{I}_1(\kappa_1\sqrt{t}) + \mathcal{I}_1(\overline{\kappa_1}\sqrt{t})} \\ + i\frac{\beta\varepsilon^2}{4r}\overline{(\mathcal{I}_1(\kappa_1\sqrt{t}) - \mathcal{I}_1(\overline{\kappa_1}\sqrt{t})}.\end{aligned} \tag{4.68}$$

Now using (4.67) and the Taylor expansion for $\mathcal{I}_1(\kappa_1\sqrt{t})$ around $ir\sqrt{t}$ one gets (with a slight abuse of notation)

$$\mathcal{I}_1(\kappa_1\sqrt{t}) = \mathcal{I}_1(ir\sqrt{t}) - \frac{\beta\varepsilon^2}{2}\sqrt{t}\frac{d}{dz}\mathcal{I}_1(ir\sqrt{t}) + \Big(\frac{\beta\varepsilon^2}{2}\sqrt{t}\Big)^2 \frac{d^2}{dz^2}\mathcal{I}_1(\tilde{z}),$$

where $\operatorname{Im}\tilde{z} = r\sqrt{t}$. This expansion (and the corresponding expansion for $\mathcal{I}_1(\overline{\kappa_1}\sqrt{t})$) gives, together with (A.5), (A.6), (A.7), and Lemma A.4 the result

$$\hat{A}_{\varepsilon,r} = 2q_2 e^{it\kappa_2^2} - \frac{\beta\varepsilon^2}{2r}\overline{\mathcal{I}_3(ir\sqrt{t})} + \mathcal{O}\Big(\frac{\varepsilon^4}{r^2}\Big). \tag{4.69}$$

Expand once again $\mathcal{I}_3(ir\sqrt{t})$, this time around $i\sqrt{\alpha t}$, see (4.66), and the proof of the first part of the proposition is concluded.

For the second part of the proposition, we first note that $\mathcal{I}_p$ are uniformly bounded on the imaginary axis (see Lemma A.5). In order to compute $|A_\varepsilon(t)|^2$ up to errors vanishing as $\varepsilon \to 0$, one need only to consider the first term on the r.h.s. of (3.53). By (3.47) we have $\operatorname{Im}\kappa_2^2 = \beta\varepsilon^2 r$, such that neglecting the terms vanishing as $\varepsilon \to 0$ in (3.48) we get

$$|A_\varepsilon(t)|^2 = e^{-2\beta r\varepsilon^2 t}. \tag{4.70}$$

To further simplify (4.70), we employ the following elementary inequality. For $a > 0$, $|b| \le \frac{a}{2}$ we have

$$\sup_{t\ge 0} |e^{-(a+b)t} - e^{-at}| \ge \frac{2|b|}{a} \sup_{y\ge 0} ye^{y}. \tag{4.71}$$

Now (3.55) follows from (4.70), (4.71), and the fact that

$$r = \sqrt{\alpha}\big(1 + \mathcal{O}\big(\varepsilon^{4-p}\big)\big). \tag{4.72}$$

Finally, the definition of β implies that

$$\beta = -g_1 + \mathcal{O}(\varepsilon^p), \tag{4.73}$$

which together with (3.55) and (4.71) gives (3.56), and the proof is finished.

4.7 Proof of Proposition 3.14

For the computation of $\hat{A}_{\varepsilon,r}(t)$ we proceed exactly as in the proof of Proposition 3.13, with the difference that here there is no need to use Lemma A.3. In this case κ_1 and κ_2 are real,

$$\kappa_j = \pm r' - \frac{\beta\varepsilon^2}{2}, \quad r' = \sqrt{|\alpha| + \frac{\beta^2\varepsilon^4}{4}}, \quad j = 1, 2,$$

and a computation similar to the one leading to (4.69) gives

$$\hat{A}_{\varepsilon,r} = -i\frac{\beta\varepsilon^2}{2r'}\mathcal{I}_3(ir'\sqrt{t}) + \mathcal{O}\Big(\frac{\varepsilon^4}{r'^2}\Big). \tag{4.74}$$

The contribution of the bound state at $x_b = -\kappa_1^2$ (see (3.36)) reads after a straightforward computation

$$-\frac{1}{\frac{d}{dx}H_r(\tilde{x}_b)}e^{-it\tilde{x}_b} = \Big(1 - \frac{\beta\varepsilon^2}{2r'}\Big)e^{it\kappa_1^2}. \tag{4.75}$$

Recall that $|\alpha| = \varepsilon^p$. Thus

$$r' = \sqrt{|\alpha|}\Big(1 + \mathcal{O}(\frac{\varepsilon^4}{|\alpha|})\Big) = \sqrt{|\alpha|}\big(1 + \mathcal{O}(\varepsilon^{4-p})\big).$$

Expanding in (4.74) $\mathcal{I}_3(ir'\sqrt{t})$ around $i\sqrt{|\alpha|t}$, and using an argument similar to the one leading to (4.69), one obtains (3.61). Finally (3.62) is a direct consequence of (3.61).

4.8 Proof of Proposition 3.15

Recall that $|\alpha| = b\varepsilon^p$, $p > 4$. As $\varepsilon \to 0$, and $|\alpha| << \frac{\varepsilon^4\beta^2}{2}$, we get $\kappa_j = \overline{\kappa}_j$. Expanding in (3.47) and (3.48), one gets

$$\kappa_1 = -\frac{\alpha}{\varepsilon^2\beta}\left(1 + \mathcal{O}(\varepsilon^{p-4})\right), \quad \kappa_2 = -\varepsilon^2\beta\left(1 + \mathcal{O}(\varepsilon^{p-4})\right), \tag{4.76}$$

$$q_1 = \mathcal{O}(\varepsilon^{p-4}), \quad q_2 = 1 + \mathcal{O}(\varepsilon^{p-4}). \tag{4.77}$$

An argument similar to the one leading to (4.69) implies that (see (3.46) and (A.7))

$$\hat{A}_{\varepsilon,r} = \mathcal{I}_1(i\varepsilon^2\beta\sqrt{t}) + \mathcal{O}(\varepsilon^{p-4}). \tag{4.78}$$

Thus (4.77) implies that the contribution (when it exists) of the bound state is also of the order $\mathcal{O}(\varepsilon^{p-4})$, which together with (4.78) finishes the proof of (3.64).

4.9 Proof of Proposition 3.17

As the proof of Proposition 3.11 this is a straightforward computation.

4.10 Proof of Proposition 3.18

In what follows we need the formulae for κ_j. Using the scaled quantities

$$f = (\varepsilon^2\beta)^{-2/3}\alpha, \quad y = (\varepsilon^2\beta)^{-1/3}\kappa, \tag{4.79}$$

one has

$$\kappa_j = (\varepsilon^2\beta)^{1/3}y_j \tag{4.80}$$

where y_j are the solutions of

$$y^3 + fy - 1. \tag{4.81}$$

The Cardano formula gives (for the labelling of κ_j see the discussion following Proposition 3.17).

$$y_j = \rho_j\big(r - \frac{f}{3r}\big) \tag{4.82}$$

where

$$\rho_j = e^{\frac{2\pi i}{3}(j-3)}, \quad r = \Big(\frac{1 + (1 + \frac{4f^3}{27})^{1/2}}{2}\Big)^{1/3}. \tag{4.83}$$

In (4.83) the principal determination of fractional powers is taken.

We recall that we are assuming $\alpha = b\varepsilon^p$ and $0 < p < 4/3$. Thus $f = b\beta^{-2/3}\varepsilon^{-(\frac{4}{3}-p)} \to \infty$ as $\varepsilon \to \infty$. Expanding in (4.82) one obtains

$$\begin{aligned}
\kappa_1 &= i\sqrt{\alpha}\Big(1 + i\frac{\beta\varepsilon^2}{2\alpha^{3/2}} + \mathcal{O}(f^{-3})\Big) \\
\kappa_2 &= \overline{\kappa_1} \\
\kappa_3 &= \frac{\beta\varepsilon^2}{\alpha}\left(1 + \mathcal{O}(f^{-3/2})\right)
\end{aligned} \tag{4.84}$$

and

$$\begin{aligned}
q_1 &= \frac{1}{2} - i\frac{\beta\varepsilon^2}{4\alpha^{3/2}} + \mathcal{O}(f^{-3}) \\
q_2 &= \overline{q_1} \\
q_3 &= \mathcal{O}(f^{-3}).
\end{aligned} \tag{4.85}$$

Now (4.85) and Lemma A.5 imply that

$$q_3\mathcal{I}_1(i\kappa_3\sqrt{t}) = \mathcal{O}(f^{-3})$$

uniformly in $t > 0$, such that up to errors of order ε^{4-3p}, we have

$$\hat{A}_{\varepsilon,s}(t) = -\sum_{j=1}^{2} q_j\mathcal{I}_1(i\kappa_j\sqrt{t}).$$

From this point the proof of Proposition 3.18 is a repetition of the proof of Proposition 3.13.

4.11 Proof of Proposition 3.19

In this case all κ_j, q_j are real, and as $\varepsilon \to 0$ we have

$$\begin{aligned}
\kappa_2 &= \frac{\beta\varepsilon^2}{|\alpha|}\Big(1 + \mathcal{O}(|f|^{-3/2})\Big) \\
\kappa_1 &= |\alpha|^{1/2}\Big(-1 + \frac{\beta\varepsilon^2}{2|\alpha|^{3/2}} + \mathcal{O}(|f|^{-3})\Big) \\
\kappa_3 &= |\alpha|^{1/2}\left(1 + \frac{\beta\varepsilon^2}{2|\alpha|^{3/2}} + \mathcal{O}(|f|^{-3})\right)
\end{aligned} \tag{4.86}$$

and

$$q_2 = \mathcal{O}(|f|^{-3})$$

$$q_3 = \frac{1}{2} - \frac{\beta\varepsilon^2}{4|\alpha|^{3/2}} + \mathcal{O}(|f|^{-3}) \tag{4.87}$$

$$q_1 = \frac{1}{2} + \frac{\beta\varepsilon^2}{4|\alpha|^{3/2}} + \mathcal{O}(|f|^{-3}).$$

Thus we have

$$q_2\mathcal{I}_1(i\kappa_2\sqrt{t}) = \mathcal{O}(|f|^{-3}),$$

uniformly in $t > 0$, such that up to errors of order ε^{4-3p},

$$\hat{A}_{\varepsilon,s}(t) = -\sum_{j=1,3} q_j\mathcal{I}_1(i\kappa_j\sqrt{t}).$$

Repetition of the proof of (3.60) leads to (3.80). Adding the contribution of the bound state one obtains (3.81). Finally, taking into account (4.87) one obtains (3.82) and the proof of Proposition 3.19 is finished.

4.12 Proof of Proposition 3.20

In this case we have $f \to 0$ as $\varepsilon \to 0$. Expanding in (4.82) one obtains

$$\kappa_j = \rho_j\beta^{1/3}\varepsilon^{2/3}(1 + \mathcal{O}(\varepsilon^{p-\frac{4}{3}}), \;\; q_j = \frac{1}{3} + \mathcal{O}(\varepsilon^{p-\frac{4}{3}}). \tag{4.88}$$

Using the expansion of $\mathcal{I}_1(i\kappa_j\sqrt{t})$ around $i\rho_j\beta^{1/3}\varepsilon^{2/3}\sqrt{t}$, one obtains from (3.71) (by an argument similar to the one leading to (4.69))

$$\hat{A}_{\varepsilon,s}(t) = -\frac{1}{3}\sum_{j=1}^{3}\mathcal{I}_1(i\rho_j\beta^{1/3}\varepsilon^{2/3}\sqrt{t}) + \mathcal{O}(\varepsilon^{p-\frac{4}{3}}).$$

Adding the bound state contribution and using (4.88) in the form

$$2q_3e^{i\kappa_3 t} = \frac{2}{3}e^{i\kappa_3 t} + \mathcal{O}(\varepsilon^{p-\frac{4}{3}}),$$

one obtains (3.83).

A The Function $\mathcal{I}_p$

Define for integers $p \geq 1$, and for complex z with $\operatorname{Im} z \neq 0$, the function

$$\mathcal{I}_p(z) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{e^{-ix^2}}{(x-z)^p}dx. \tag{A.1}$$

The integral in (A.1) is absolutely convergent, if $p \geq 2$. For $p = 1$ one can define it by

$$\mathcal{I}_1(z) = \lim_{A\to\infty} \frac{1}{i\pi} \int_{-A}^{A} \frac{e^{-ix^2}}{x-z} dx. \tag{A.2}$$

One can also define

$$\mathcal{I}_0(z) = \lim_{A\to\infty} \frac{1}{i\pi} \int_{-A}^{A} e^{-ix^2} dx. \tag{A.3}$$

As an alternative to (A.2) one can use the formula

$$\mathcal{I}_1(z) = \frac{2z}{i\pi} \int_0^{\infty} \frac{e^{-ix^2}}{x^2 - z^2} dx. \tag{A.4}$$

The functions $\mathcal{I}_p$ for $p \geq 2$ are up to a multiplicative constant the derivatives of $\mathcal{I}_1$, since we have

$$\mathcal{I}_p(z) = \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \mathcal{I}_1(z) \tag{A.5}$$

Lemma A.1. *For $p \geq 2$ we have*

$$|\mathcal{I}_p(z)| \leq \frac{1}{\pi |\mathrm{Im}\, z|^{p-1}} \int_{-\infty}^{\infty} (1+x^2)^{-p/2} dx. \tag{A.6}$$

Proof. The result follows from a simple computation, which we omit. □

Lemma A.2. *For $p \geq 1$ and $b \in \mathbf{R} \setminus \{0\}$ we have*

$$\mathcal{I}_p(ib) = (-1)^p \mathcal{I}_p(-ib). \tag{A.7}$$

Proof. Using (A.4) we get

$$\mathcal{I}_1(ib) = \frac{2b}{\pi} \int_0^{\infty} \frac{e^{-ix^2}}{x^2 + b^2} dx = (-1)\mathcal{I}_1(-ib).$$

This result, together with (A.5), implies (A.7). □

Lemma A.3. *Assume that* $\mathrm{Re}\, z \neq 0$ *and* $\mathrm{Im}\, z \neq 0$*.Then we have the result*

$$\mathcal{I}_1(z) = \overline{\mathcal{I}_1(-\overline{iz})} - \big(\mathrm{sign}(\mathrm{Re}\, z) - \mathrm{sign}(\mathrm{Im}\, z)\big) e^{-iz^2}. \tag{A.8}$$

Proof. This result follows from the calculus of residues. We sketch the details. Consider the contour Γ_A in the complex plane, as shown in Figure 1. We

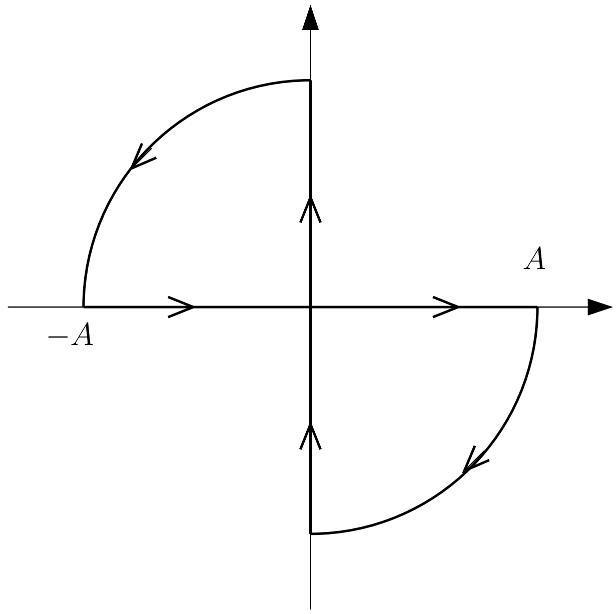

Figure 1: The integration path Γ_A

assume $A > |z|$. The residue theorem implies that we have

$$\int_{\Gamma_A} \frac{e^{-i\zeta^2}}{\zeta - z} d\zeta = -\big(\operatorname{sign}(\operatorname{Re} z) - \operatorname{sign}(\operatorname{Im} z)\big) i\pi e^{-iz^2}. \tag{A.9}$$

The contributions from the two circular arcs vanish as $A \to \infty$, since $\sin(2t)$ is negative for t satisfying $\pi/2 < t < \pi$ and $3\pi/2 < t < 2\pi$. Thus we have

$$\int_{-A}^{A} \frac{e^{-ix^2}}{x - z} dx + \int_{-A}^{A} \frac{e^{iy^2}}{y + iz} dy = -\big(\operatorname{sign}(\operatorname{Re} z) - \operatorname{sign}(\operatorname{Im} z)\big) i\pi e^{-iz^2} + o(1) \tag{A.10}$$

The result now follows by rewriting the second term on the left using complex conjugation, and by taking the limit $A \to \infty$.

□

Lemma A.4. *For* $\operatorname{Im} z \neq 0$ *and* $p \geq 2$ *we have*

$$\mathcal{I}_{p-1}(z) + z\mathcal{I}_p(z) = \frac{ip}{2}\mathcal{I}_{p+1}(z). \tag{A.11}$$

Proof. Follows from an integration by parts. The details are omitted. □

Lemma A.5. *We have for all $p \geq 1$ that*

$$\sup_{b>0} |\mathcal{I}_p(ib)| < \infty. \tag{A.12}$$

Proof. For $p = 1$ we use (A.4) to get

$$|\mathcal{I}_1(ib)| = \Big|\frac{2b}{\pi}\int_0^\infty \frac{e^{-it^2}}{t^2+b^2}dt\Big| \leq \frac{2}{\pi}\int_0^\infty \frac{x}{t^2+x^2}dt = 1.$$

For $p \geq 2$ we get a uniform estimate for $b \geq 1$ from (A.6). For $0 < b < 1$ we can deform the integration contour in $\mathcal{I}_p(ib)$ into the lower half plane, e.g. follow the real axis from $-\infty$ to -1, the unit circle in the lower half plane, and then the real axis from 1 to ∞. Then we get a uniform estimate also in this case. □

We now establish the connections with the error function. To this end we recall some definitions, see [1, page 297].

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt, \tag{A.13}$$

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z), \tag{A.14}$$

The integral in (A.13) can be taken along any path in the complex plane connecting 0 and z. We also need

$$\mathrm{w}(z) = e^{-z^2}\operatorname{erfc}(-iz) \quad \text{for } \operatorname{Im} z > 0, \tag{A.15}$$

see [1, 7.1.3]. Note that all these functions are entire functions.

We define the function

$$\hat{\mathrm{w}}(z) = \frac{i}{\pi}\int_{-\infty}^\infty \frac{e^{-t^2}}{z-t}dt = \frac{2iz}{\pi}\int_0^\infty \frac{e^{-t^2}}{z^2-t^2}dt \quad \text{for } \operatorname{Im} z \neq 0. \tag{A.16}$$

We have the result [1, (7.1.3)]

$$\mathrm{w}(z) = \hat{\mathrm{w}}(z) \quad \text{for } \operatorname{Im} z > 0. \tag{A.17}$$

In the lemma below, which gives the relation between $\mathcal{I}_1(z)$ and $\hat{\mathrm{w}}(z)$, we use $\operatorname{Arg} z$ to denote the determination of the argument taking values in $(0, 2\pi)$.

Lemma A.6. *Assume that* $\operatorname{Im} z \neq 0$ *and* $\operatorname{Im}(e^{i\pi/4}z) \neq 0$. *Then we have*

$$\mathcal{I}_1(z) = \sigma(z)e^{-iz^2} + \hat{\mathrm{w}}(e^{i\pi/4}z), \tag{A.18}$$

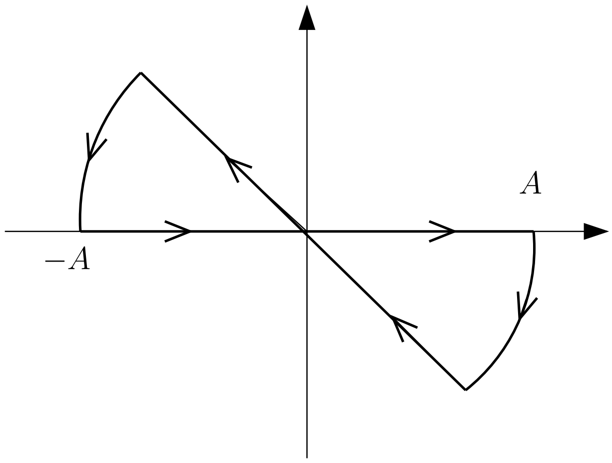

Figure 2: The integration path γ_A

where

$$\sigma(z) = \begin{cases} 2, & 3\pi/4 < \operatorname{Arg} z < \pi, \\ -2, & 7\pi/4 < \operatorname{Arg} z < 2\pi, \\ 0, & \textit{otherwise.} \end{cases} \tag{A.19}$$

Proof. Let z be fixed and satisfying the assumption in the Lemma. We denote by γ_A the path shown in Figure 2, where $A > |z|$.

The calculus of residues yields that

$$\int_{\gamma_A} \frac{e^{-i\zeta^2}}{\zeta - z} d\zeta = \sigma(z)\pi i e^{-iz^2},$$

where $\sigma(z)$ is defined in the Lemma.

On the other hand, using the explicit parameterization and the fact that the contributions from the two circular arcs tend to zero as $A \to \infty$, cf. the proof of Lemma A.3, we also have

$$\int_{\gamma_A} \frac{e^{-i\zeta^2}}{\zeta - z} d\zeta = \int_{-\infty}^{\infty} \frac{e^{-ix^2}}{x - z} dx - \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - e^{i\pi/4} z} dt + o(1).$$

Thus the result follows by taking the limit $A \to \infty$ and using the definitions of $\mathcal{I}_1(z)$ and $\hat{\mathrm{w}}(z)$, noting the choice of signs in these definitions. ☐

We finish by giving also the relation between $\mathcal{I}_1(z)$ and $\mathrm{w}(z)$. From (A.17) and Lemma A.3 one has the result

Lemma A.7. *Let for* $\operatorname{Im} z \neq 0$*,* $\Sigma(z) = \operatorname{sign}(\operatorname{Im} z)$*. Then*

$$\mathcal{I}_1(z) = \Sigma(z)\,\mathrm{w}(\Sigma(z)e^{i\pi/4}z). \tag{A.20}$$

Proof. For $0 < \operatorname{Arg} z < 3\pi/4$ (A.20) follows from (A.17) and Lemma A.3. By analytic continuation one obtains (A.20) for $\Sigma(z) > 0$. Similarly, for $7\pi/4 < \operatorname{Arg} z < 2\pi$, (A.20) follows from (A.17), Lemma A.3, and $\mathrm{w}(-z) = 2e^{-z^2} - \mathrm{w}(z)$, see [1, 7.1.11]. □

Acknowledgement. Arne Jensen was partially supported by the grant "Mathematical Physics" from the Danish Natural Science Research Council. Most of the work was carried out, while Gheorghe Nenciu was visiting professor at Department of Mathematical Sciences, Aalborg University. The support of the Department is gratefully acknowledged.

References

[1] M. Abramowitz and I. A. Stegun: *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] B. Baumgartner: Interchannel resonances at a threshold, J. Math. Phys. **37** (1996), 5928–5938.

[3] L. Cattaneo, G.M. Graf, and W. Hunziker: A general resonance theory based on Mourre's inequality, Ann. H. Poincaré **7** (2006), 583–614.

[4] C.B. Chiu, E.C.G. Sudarshan, and B. Misra: The evolution of unstable quantum states and a resolution of Zeno's paradox. Phys. Rev. D **16**, 520-529 (1976).

[5] O. Costin and A. Soffer: Resonance theory for Schrödinger operators, Comm. Math. Phys. **224** , 133–152 (2001).

[6] E.B. Davies: Resonances, spectral concentration and exponential decay. Lett. Math. Phys. **1**, 31-35 (1975).

[7] M. Demuth: Pole approximation and spectral concentration, Math. Nachr. **73** (1976), 65–72.

[8] V. Dinu, A. Jensen, and G. Nenciu: Non-exponential decay laws in perturbation theory of near threshold eigenvalues, J. Math. Phys. **50** (2009), 013516.

[9] P. Exner: *Open Quantum Systems and Feynman integrals*, Reidel, Dordrecht, 2002.

[10] L. Fonda, G.C. Ghirardi, and A. Rimini: Decay theory of unstable quantum systems. Rep. Prog. Phys. **41**, 587-631 (1978).

[11] P.T. Greenland: Seeking the non-exponential decay. Nature **335**, 298-299 (1988).

[12] W. Hunziker: Resonances, metastable states and exponential decay laws in perturbation theory, Comm. Math. Phys. **132** (1990), 177–188.

[13] A. Jensen: Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbf{R}^m)$, $m \geq 5$, Duke Math. J. **47** (1980), 57–80.

[14] A. Jensen and T. Kato: Spectral properties of Schrödinger operators and time-decay of the wave functions, Duke Math. J. **46** (1979), 583–611.

[15] A. Jensen and G. Nenciu: A unified approach to resolvent expansions at thresholds, Rev. Math. Phys. **13** (2001), no. 6, 717–754.

[16] A. Jensen and G. Nenciu: The Fermi golden rule and its form at thresholds in odd dimensions. Comm. Math. Phys. **261** (2006), 693–727.

[17] A. Jensen and G. Nenciu: Schrödinger operators on the half line: Resolvent expansions and the Fermi golden rule at thresholds, Proc. Indian Acad. Sci. (Math. Sci.) **116** (2006), 375–392.

[18] A. Jensen and G. Nenciu: On the Fermi golden rule: Degenerate eigenvalues, In: *Perspectives in Operator Algebras and Mathematical Physics*, pp 91-103. Theta, Bucharest 2008 (Proc. Conf. Operator Theory and Mathematical Physics, Bucharest, August 2005).

[19] A. Jensen and G. Nenciu: Exponential decay laws in perturbation theory of threshold and embedded eigenvalues. In: *New Trends in Mathematical Physics*, pp 525-539. Springer, 2009 (Proceedings: 15th International Congress on Mathematical Physics, August 5i–11, 2006 Rio de Janeiro.

[20] T. Jittoh, S. Masumoto, J. Sato, Y. Sato, and K. Takeda: Non-exponential decay of an unstable quantum system: Small Q-value s-wave decay. Phys. Rev. A **71**, 012109 (2005).

[21] T. Köhler, K. Gòral, and P. Julienne: Production of cold molecules via magnetically tunable Feshbach resonances, Rev. Mod. Phys. **78**, 1311-1361 (2006).

[22] M. Lewenstein, J. Zakrewski, T. Mossberg, and J. Mostowski: Non-exponential spontaneous decay in cavities and waveguides , J. Phys. B: At. Mol. Opt. Phys. **21**, L9-L14 (1988).

[23] B. Marcelis, E. van Kempen, B. Verhaar, and J. Kokkelmans: Feshbach resonances with large background scattering length: interplay with open-channel resonances. Phys. Rev. A **70**, 012701 (2004).

[24] M. Merkli and I.M. Sigal: A time-dependent theory of quantum resonances, Comm. Math. Phys. **201** (1999), 549–576.

[25] M. Miyamoto: Bound-state eigenenergy outside and inside the continuum for unstable multilevel systems. Phys. Rev. A **72**, 063405 (2005)

[26] M. Murata: Asymptotic expansions in time for solutions of Schrödinger-type equations, J. Funct. Anal. **49** (1982), no. 1, 10–56.

[27] H. Nakazato, M. Namiki, and S. Pascazio: Temporal behavior of quantum mechanical systems. International J. Modern Physics B **3**, 247-295 (1996).

[28] R. G. Newton: *Quantum Physics*, Springer-Verlag 2002.

[29] C. Nicolaides: Physical constraints on nonstationary states and nonexponential decay. Phys. Rev. A **66**, 022118 (2002)

[30] N. Nygaard, B. Schneider, and P. Julienne: Two-channel R-matrix analysis of magnetic-field-induced Feshbach resonances, Phys. Rev. A **73**, 042705 (2006)

[31] A. Orth: Quantum mechanical resonance and limiting absorption: the many body problem, Comm. Math. Phys. **126** (1990), 559–573.

[32] X. P. Wang: Embedded eigenvalues and resonances of Schrödinger operators with two channels, Ann. Fac. Sci. Toulouse Math. (6) **16** (2007), no. 1, 179–214.

[33] K. Yajima: The L^p boundedness of wave operators for Schrödinger operators with threshold singularities I, Odd dimensional case, J. Math. Sci. Univ. Tokyo **13** (2006), 43–93.