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Simplicial models for trace spaces

by

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SIMPLICIAL MODELS FOR TRACE SPACES

MARTIN RAUSSEN

ABSTRACT. Directed Algebraic Topology studies topological spaces in which certain directed paths (d-paths) – in general irreversible – are singled out. The main interest concerns the spaces of directed paths between given end points – and how those vary under variation of the end points. The original motivation stems from certain models for concurrent computation. So far, spaces of d-paths and their topological invariants have only been determined in cases that were elementary to overlook.
In this paper, we develop a systematic approach describing spaces of directed paths – up to homotopy equivalence – as prodsimplicial complexes (with products of simplices as building blocks). This method makes use of certain poset categories of binary matrices and applies to a class of directed spaces that arise from a class of models of computation; still restricted but with a fair amount of generality. In the final section, we outline a generalization to model spaces known as Higher Dimensional Automata.
In particular, we describe algorithms that allow to determine not only the fundamental category of such a model space, but all homological invariants of spaces of directed paths within it. The prodsimplical complexes and their associated chain complexes are finite, but they will, in general, have a huge number of generators.

1. INTRODUCTION

1.1. **Background.** With motivations arising originally from concurrency theory within Computer Science, a new field of research, directed algebraic topology, has emerged; for a comprehensive overview, we refer to the recent book by M. Grandis [12]. Its main characteristic is, that it involves spaces of "directed paths" (or timed paths, executions): these directed paths can be concatenated, but in general *not* reversed; time is not reversable.

A particular model for concurrent computation, called Higher Dimensional Automata (HDA) was introduced by V. Pratt [21] back in 1991. Mathematically, HDAs can be described as (labelled) pre-cubical sets [2, 1] with a preferred set of directed paths (respecting the natural partial orders) in any of the cubes of the model; (di-)homotopies of such directed paths have to respect the order along a deformation [7].

Compared to other well-studied concurrency models like labelled transition systems, event sturctures, Petri nets etc. (for a survey on those cf. [28]), it has been shown by R.J. van Glabbeek [27] that Higher Dimensional Automata have the highest expressivity; on the other hand, they are certainly less studied and less often applied so far.

All concurrency models deal with sets of states and with sets of execution paths (with some further structure). The interest is mainly in the latter; but typically, it is difficult to get an overview and to infer valuable information about the execution space from the state space model.

A general framework for topological spaces with directed paths was defined and investigated as the category of d-spaces. The objects are topological spaces with a preferred set of d-paths; the morphisms are the continuous maps preserving d-paths; cf. in particular [11, 10, 12]. Grandis investigates in particular the fundamental category (generalising the fundamental group) of d-spaces. This made it interesting to investigate how spaces of d-paths (with given fixed end points) vary under variation of these end points [24] and how this gives rise to suitable decomposition of the state space into "components"[5, 9].

General topological properties of spaces of d-paths and of *traces* (=d-paths up to monotone reparametrizations [4, 25] in semi-cubical complexes were investigated in [26]. But so far, apart from low-dimensional examples with convincing drawings, there have been very few explicit examples of actual computations of spaces of such traces (for an attempt in dimension two, cf. [22]); let alone a general method to perform such computations.

It is the aim of this article to make the homotopy types of trace spaces computable for a restricted class of Higher Dimensional Automata – those arising from the semaphore or PV-models introduced by E.W. Dijkstra [3] back in 1968. The state spaces for such models are complements of a number of hyperrectangular "holes" in a partially ordered hypercube $\vec{I}^n$. We describe trace spaces for these models explicitly as finite-dimensional prodsimplicial complexes [18] (with products of simplices as their building blocks) with the nerve of a particular category as barycentric subdivision. This makes it – at least in principle, the complexes may have very many cells – possible to calculate algebraic topological invariants of such trace spaces. For applications in concurrency, it is already very important to know the Betti number β_0 and to get hold on the connected components of a trace space; traces in each component will always lead to the same result in a concurrent computation. We will finally hint on how to extend our results to general HDA.

The overall philosophy reminds a bit of the analysis of the topology of path spaces in CW-complexes in Milnor's article [20]: Also the spaces of d-paths in a pre-cubical complex with given end points are equi locally convex (ELCX) [26] and thus locally contractible; for the PV-models analysed here, the contractible subsets can be described explicitly (by a blend of order and combinatorics).

1.2. **Structure and overview of results.** PV-models [3] are a particular class of models for linear concurrent computations with *semaphores*, a particularly simple, but instructive class of Higher Dimensional Automata. These models are introduced in Section 2; the state space for such a model is embeddeded (including the partial order) in a hypercube $\vec{I}^n$. To get going, we define certain subspaces of the model spaces and show that the space of d-paths within each such subspace (for simplicity, from the bottom $\mathbf{0}$ to the top $\mathbf{1}$) is *empty or contractible* by a specific contraction making use of the partial order. Moreover we show, that every such d-path is contained in at least one of these subspaces.

This allows us in Section 3 to define a poset category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ indexing the spaces of d-paths in X (for simplicity, with paths starting at $\mathbf{0}$ and ending at $\mathbf{1}$ described above and their intersections. That category is naturally isomorphic to a subcategory of a *product* of a number of poset categories of non-empty subsets of the positive integers $[1:n]$ less than or equal to n. A topological realization of this subcategory can thus be modelled on products of simplices and gives rise to a *prodsimplicial complex* [18]. Using standard methods (nerve lemma, projection lemma etc., cf. [18]), we show that the space of d-paths (or rather traces, i.e, d-paths modulo monotone reparametrizations [4]) $\vec{T}(X)(\mathbf{0},\mathbf{1})$ in such a model space is in fact homotopy equivalent to an explicit prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ that arises as geometric realization of the poset category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ – with the nerve $\Delta(\mathcal{C}(X)(\mathbf{0},\mathbf{1}))$ as barycentric subdivision.

It is the aim of Section 4 to achieve an explicit description of the index category. To this end, it is necessary to decide, for every of the subspaces mentioned above, whether it is empty or not, i.e., whether there exists a d-path within it from bottom to top. Each such subspace can be described as the complement of a number of homothetic hyperrectangles (with faces parallel to the coordinate planes) extending the original holes; it turns out that it is enough to find out whether there exist deadlock points (the only d-path with a deadlock as source is trivial) in these extended models; a combinatorial search algorithm for deadlocks was described in [6]. The outcome of a systematic search for deadlocks (in all extended models) is a set $D(X)(\mathbf{0},\mathbf{1})$ of *minimal non-faces* – all of dimension $n-1$) – of the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ within the complex $(\Delta^{n-1})^l$. The *maximal faces* of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ can now be determined via minimal transversals in the associated hypergraph.

The explicit determination of the complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ thus achieved makes the calculation of algebraic topological invariants of the trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ possible. Even if, for complicated model spaces, the "curse of dimensionality" might prohibit explicit calculations, it will still be interesting and possible to study the change of invariants under change of end points (in rounds of computation; compare [14] and other sources in distributed computing for this point of view).

In Section 5, we describe for this purpose the changes that become necessary when one investigates spaces of traces with source and target different from top and bottom (either points or also subsets of sources and targets). This is essential for the study of topology change along a state space model and also for inductive calculations. A particularly important case, semaphores with arity one (the semaphore obstructions prohibit all but one process to proceed) are finally dealt with. These models come equipped with *discrete* trace spaces that can be described by sets of (order) *compatible* permutations within $(\Sigma_n)^k$ – for k such semaphores.

The final Section 6 takes first steps in generalizing the methods described so far. Dijkstra's PV-models can easily be generalized to a state space that is a product of digraphs with "hyperrectangular holes" modelling processes that may branch, merge and loop.

For these, the topology of the trace space can be determined in two steps: First determine (the components of) the traces in the product of digraphs (discrete, a product of trace spaces of the 1-dimensional digraphs) – without holes; then, for each of these components, one can pull back (or "unloop") to a state space, including holes, of the type previously investigated. It will still have to be investigated how to unloop in a coherent manner in order to reuse calculations (of deadlocks etc.) performed at earlier steps. For general HDA (modelled on pre-cubical sets), it is no longer possible to use the explicit contraction method for specific subspaces yielding local contractability used in this article. Instead, it is probably necessary to use the method described in [26] with a higher combinatorial complexity still to be sorted out.

2. MODELS OF COMPUTATION AND SUBSPACES

2.1. **A simple higher dimensional automaton.** To start with, we analyse trace spaces in the following simple situation: A (linear) schedule for each of a number of n individual processors $P_j, 1 \leq j \leq n$, is modelled on the directed interval $\vec{I}_j = [0,1]$. On subintervals $I_j^i \subseteq I_j, 1 \leq i \leq l$, there is potential conflict with the schedules of the othere processors. These subintervals are supposed to be open in the subspace topology; in particular, closed and half-closed intervals occur only with 0, resp. 1 as boundaries. We use the notation $a_j^i = \inf I_j^i, b_j^i = \sup I_j^i$.

The state space for concurrent executions of n linear processes is the space $X = \vec{I}^n \setminus F \subset \vec{I}^n$ with the *forbidden region* $F = \bigcup_{i=1}^l R^i$; each R^i is the "homothetic" open hyperrectangle $R^i = \prod_{j=1}^n I_j^i$ (with faces parallel to the coordinate hyperplanes). The forbidden region F models conflicts and may not be entered. The space X inherits a partial order $\leq$ from the componentwise partial order $\leq$ on $\vec{I}^n$.

We study compound schedules (execution paths) in such a state space X: A *d-path* in X is a continuous path $p : \vec{I} \to X$ that is continuous and order-preserving, i.e., each coordinate $\pi_j \circ p : \vec{I} \to X \subset \vec{I}^n \to \vec{I}, 1 \leq j \leq n$, is *weakly increasing*. The space $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ consists of all d-paths in X starting at $\mathbf{c} \in X$ and ending at $\mathbf{d} \in X$; in particular, these d-paths avoid the "forbidden region" $F \subset \vec{I}^n$. Consult e.g. [13, 6] for detailed descriptions.

As a topological space, $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ is given the subspace topology inherited from the space $P(X)(\mathbf{c}, \mathbf{d}) = [(I, 0, 1); (X, \mathbf{c}, \mathbf{d})]$ of *all* paths in X from $\mathbf{c}$ to $\mathbf{d}$ in the compact-open topology (= uniform convergence topology).

Reparametrization equivalent d-paths [4] in X have the same directed image (= *trace*) in X. Dividing out the action of the monoid of (weakly-increasing) reparametrizations of the parameter interval $\vec{I}$, we arrive at trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$ [4, 25] which is shown in [26]to be homotopy equivalent to path space $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ for a far wider class of directed spaces X; in the latter paper, it is also shown that trace spaces enjoy nice properties; e.g., they are metrizable, locally compact, locally compact, and they have the homotopy type of a CW-complex.

It is the aim of the present paper to describe and analyze a combinatorial/topological model of both spaces of d-paths and traces in a model space X (up to homotopy equivalence) in order to make calculations of their algebraic topological invariants feasible.

2.2. **Subspaces of the model space.** We will now describe certain subspaces of X and then prove that associated spaces of d-paths *within* these subspaces are *either empty or contractible*.

Notation: For a real interval I and a real number x we write $x < I$ if and only if $x < t$ for all $t \in J$. Likewise we understand $>, \leq, \geq$. $\mathbf{a}^i = [a^i_1, \ldots, a^i_n], \mathbf{b}^i = [a^i_1, \ldots, a^i_n] \in I^n$ denote the lowest, resp. upmost vertices of hyperrectangle R^i. $\downarrow \mathbf{c} := \{\mathbf{x} \in X |\ \mathbf{x} \leq \mathbf{c}\}$ denotes the set of elements "below" $\mathbf{c} \in X$; not all of them can necessarily reach $\mathbf{c}$ by a d-path; likewise $\uparrow \mathbf{c} = \{\mathbf{x} \in X |\ \mathbf{c} \leq \mathbf{x}\}$ denotes the set of elements above $\mathbf{c}$. $\partial_+ \downarrow \mathbf{c}$ denotes the intersection of the upper boundary $\{\mathbf{x} \in \downarrow \mathbf{c} |\ \exists 1 \leq i \leq n : x_i = c_i\}$ of the hyperrectangle with upmost vertex in $\mathbf{c}$ with X.

Definition 2.1. (1) For $1 \leq i \leq l, 1 \leq j_i \leq n$, let

$$X_{j_1,\cdots,j_l} := \{\mathbf{x} \in I^n \mid \forall i :\ x_{j_i} < I^i_{j_i} \text{ or } \exists k : x_k > I^i_k\} \cap X.$$

(2) For non-empty subsets $J_i \subseteq [1:n], 1 \leq i \leq l$, let

$$X_{J_1,\cdots,J_l} := \{\mathbf{x} \in I^n \mid \forall i :\ x^i_{j_i} < I^i_{j_i}, j_i \in J_i, \text{ or } \exists k : x_k > I^i_k\} \cap X.$$

For later use, we note an equivalent formulation of these conditions:

$$\mathbf{x} \in X, \forall i :\ \mathbf{x} \leq \mathbf{b}^i \Rightarrow x_{j_i} < I^i_{j_i}; \tag{1}$$

$$\mathbf{x} \in X, \forall i :\ \mathbf{x} \leq \mathbf{b}^i \Rightarrow x_{j_i} < I^i_{j_i} (\text{for all } j \in I_j.) \tag{2}$$

Remark 2.2. An execution path in $X_{j_1,\cdots,j_l}$ has the following characterization: Processor j_i has not yet reached the "conflict" interval $J^i_{j_i}$ when one of the others, say k_i, has left "its" corresponding conflict interval $J^i_{k_i}$.

Example 2.3. In both rows in Figure 1, $X = \vec{I}^2 \setminus F$ is the complement of the black squares. The shaded areas show, in both cases, the subspaces X_{11}, X_{12}, X_{21}, resp. X_{22}. Remark that $\vec{P}(X_{ij})(\mathbf{0}, \mathbf{1}) = \emptyset$ occurs only in the second row – and only for $i = 1, j = 2$.

Example 2.4. In Figure 2 above, $X = \vec{I}^3 \setminus \vec{J}^3$ with $\vec{J} \subset \vec{I}$ an interior open interval. Apart from the forbidden region "black box" $\vec{J}^3$ with upper corner $\mathbf{b}$, you see the shaded areas $X_j \cap \partial_+ \downarrow \mathbf{b}, 1 \leq j \leq 3$. Remark that every pair of these areas intersect, whereas the intersection of all three is empty. As a consequence, $\vec{P}(X_J)(\mathbf{0}, \mathbf{1}) = \emptyset$ for $\emptyset \neq J \subseteq [1:3]$ if and only if $J = [1:3]$.

Lemma 2.5. (1) $X_{j_1,\cdots,j_l} \subset X$ *for all* $1 \leq i \leq l, 1 \leq j_i \leq n$.
(2) $X_{J_1,\cdots,J_l} = \bigcap_{j_i \in J_i} X_{j_1,\cdots,j_l}$.

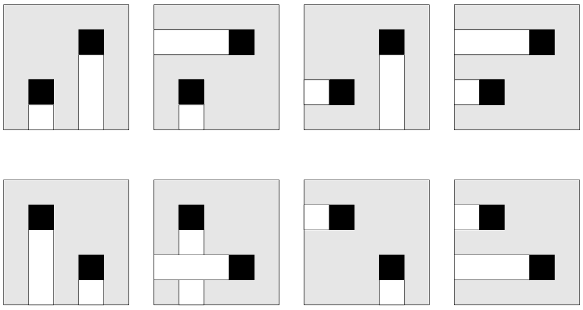


FIGURE 1. X is the complement of the black boxes. $X_{ij}, 1 \leq i, j \leq 2$, are represented by the shaded areas.

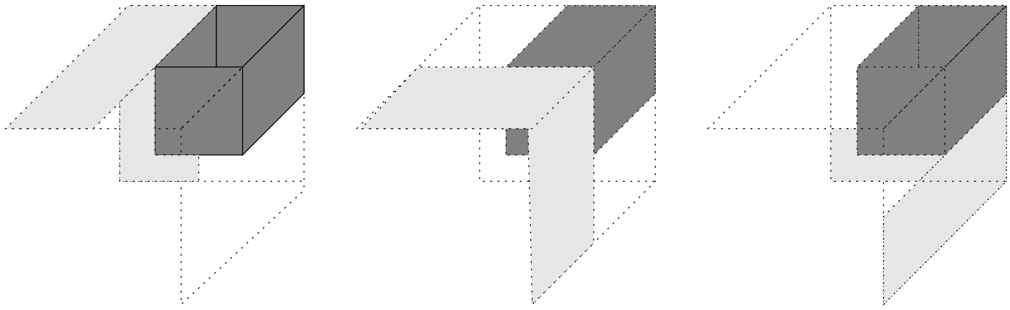


FIGURE 2. Intersections of X_i with the upper boundary $\partial_+ \downarrow \mathbf{b}$ of the box with upper corner $\mathbf{b}$.

(3) *If $a^i_j = 0$, then $\mathbf{0} \notin X_{j_1,\cdots,j_l}$ for $j_i = j$; in particular, there is then no d-path (trace) in $X_{j_1,\cdots,j_l}$ starting at $\mathbf{0}$.*

□

2.3. **Restricted path spaces are empty or contractible.** We consider the binary (maximum) operation $\vee$ on $\mathbf{R}^n$ given by $\mathbf{a} \vee \mathbf{b} = [\max(a_1, b_1), \ldots, \max(a_n, b_n)]$ and observe as a consequence of Definition 2.1:

Lemma 2.6. (1) *$X_{j_1,\ldots,j_l}$ is closed under $\vee$ for every choice $j_i \in [1:n], i \in [1:l]$.*
(2) *Intersections of $\vee$-closed sets are $\vee$-closed.*
(3) *$X_{J_1,\ldots,J_l}$ is closed under $\vee$ for every collection of* non-empty *subsets $J_i \subseteq [1:n]$, $1 \leq i \leq l$.*

□

Lemma 2.6 is no longer true if one of the sets J_i may be empty! A similar result holds for the binary minimum operation $\wedge$ given by $\mathbf{a} \wedge \mathbf{b} = [\min(a_1, b_1), \ldots, \min(a_n, b_n)]$.

The next observation is essential for our purposes:

Proposition 2.7. (1) *Let $A_1 \subseteq A \subseteq X, \mathbf{a} \in A$, and $A \times A \subseteq Y \subseteq X \times X$.*
Let $* : Y \to X$ denote a (continuous) d-map, (i.e., $\mathbf{x}_1 \leq \mathbf{x}_2, \mathbf{y}_1 \leq \mathbf{y}_2, (\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in$

$Y \Rightarrow \mathbf{x}_1 * \mathbf{y}_1 \leq \mathbf{x}_2 * \mathbf{y}_2)$ *satisfying* $\mathbf{a} * \mathbf{a} = \mathbf{a}, A * A \subseteq A$, *and* $(\uparrow \mathbf{a} \cap \downarrow A_1) * \downarrow A_1 \subseteq A_1$. *Then trace space* $\vec{T}(A)(\mathbf{a}, A_1)$ *is either empty or contractible.*

(2) *Let* $J_i \subseteq [1:n], 1 \leq i \leq l, \mathbf{c}, \mathbf{d} \in X_{J_1, \ldots J_l}$. *Then the trace spaces* $\vec{T}(X_{J_1, \ldots J_l})(\mathbf{c}, \mathbf{d})$ *and* $\vec{T}(X_{J_1, \ldots J_l})(\mathbf{c}, \partial_+ \downarrow \mathbf{d}) \cap X_{J_1, \ldots, J_l})$ *are either empty or contractible.*

Likewise, it can be shown that $\vec{T}(X_{J_1, \ldots J_l})(\partial_- \uparrow \mathbf{c} \cap X_{J_1, \ldots, J_l}, \mathbf{a})$ is empty or contractible using the operation $\wedge$.

Proof. (2) follows from (1) and Lemma 2.6 with $* = \vee$.
To prove (1), we show first that $\vec{P}(A)(\mathbf{a}, A_1)$ is either contractible or empty. If $\vec{P}(A)(\mathbf{a}, A_1)$ is non-empty, then, for any pair $p, q \in \vec{P}(A)(\mathbf{a}, A_1)$, let $H(p,q) : \vec{P}(A)(\mathbf{a}, A_1) \times I \to \vec{P}(A)(\mathbf{a}, A_1)$ be given by $H_t(p,q)(s) := q(s) * p(ts),\ t \in I$. Remark that $H_0(p,q)(s) = q(s) * \mathbf{a} = q(s), H_t(p,q)(0) = \mathbf{a} * \mathbf{a} = \mathbf{a}, H_t(p,q)(1) = q(1) * p(t) \in A_1$ and that $H_1(p,q)(s) = q(s) * p(s)$. Thus $H(p,q)$ defines an increasing d-homotopy [11] $q \mapsto p \vee q$ between d-paths in $\vec{P}(A)(\mathbf{a}, A_1)$. Likewise, $H(q,p)$ is an increasing d-homotopy $p \mapsto p \vee q$. Their concatenation $G(q,p) = H^-(q,p) * H(p,q)$ (orientations are reversed for the first d-homotopy) is a "zig-zag" d-homotopy from q to p. The map $G(-,-)$ defines a continuous section of the end point map $(\vec{P}(\mathbf{a}, A_1))^I \to \vec{P}(\mathbf{a}, A_1) \times \vec{P}(\mathbf{a}, A_1)$ associating to a pair (q,p) the d-homotopy $G(p,q)$.

Given an arbitrary $p \in \vec{P}(A)(\mathbf{a}, A_1)$, the map $G(-,p) : \vec{P}(\mathbf{a}, A_1) \times I \to \vec{P}(\mathbf{a}, A_1)$ is a contraction of $\vec{P}(\mathbf{a}, A_1)$ to p. By [26], Proposition 2.16, the trace space $\vec{T}(\mathbf{a}, A_1)$ is homotopy equivalent to the space of d-paths $\vec{P}(\mathbf{a}, A_1)$ and thus also contractible. $\square$

Remark 2.8. If $J_i = [1:n]$ for at least one i, then $\vec{T}(X_{J_1, \cdots, J_l})(\mathbf{0}, \mathbf{1})$ is always empty; in this case, condition (2) from Definition 2.1 amounts to $\mathbf{x} \leq \mathbf{b}^i \Rightarrow \mathbf{x} \leq \mathbf{a}^i$. But every d-path from $\mathbf{0}$ to $\mathbf{1}$ needs to pass through the region $\downarrow \mathbf{b}^i \setminus (\downarrow \mathbf{a}^i)$ inbetween.

For other end point conditions, this is no longer true in general: for example, if $\mathbf{y}$ is reachable from $\mathbf{0}$ and if $\mathbf{y} \leq \mathbf{a}^i$, then obstruction R^i does not play any role for d-paths ending at $\mathbf{y}$, and $J_i = [1:n]$ may occur as index set for a non-empty space $\vec{T}(X_{J_1, \ldots J_l})(\mathbf{0}, \mathbf{y})$.

The trace spaces considered above *cover* the total trace space: With notation as in Proposition 2.7, we obtain:

Lemma 2.9. *For any* $\mathbf{c}, \mathbf{d} \in X$,

- $\vec{T}(X)(\mathbf{c}, \mathbf{d}) = \bigcup_{[1:n]^l} \vec{T}(X_{j_1, \cdots, j_l})(\mathbf{c}, \mathbf{d})$.
- $\vec{T}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d})) = \bigcup_{[1:n]^l} \vec{T}(X_{j_1, \cdots, j_l})(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$.

Proof. We give the proof for the first statement: For a given d-path $p = [p_1, \ldots, p_n] \in \vec{P}(X)(\mathbf{c}, \mathbf{d}) \subseteq \vec{P}(\vec{I}^n)(\mathbf{c}, \mathbf{d})$ and $1 \leq i \leq l$, choose a minimal t_i such that there exists $k_i \in [1:n]$ with $p_{k_i}(t_i) = \tilde{b}^i_{k_i} := \min(b^i_{k_i}, d_i)$. If $\tilde{b}^i_{k_i} \leq a^i_{k_i}$, then $j_i \in [1:n]$ can be chosen arbitrarily; otherwise choose $s_i < t_i$ such that $p_{k_i}(]s_i, t_i[) =]a^i_{k_i}, \tilde{b}^i_{k_i}[$. Since $p(t) \notin R^i$ for every t and $p_j(t) < b_j^j$ for all (j,t) with $t < t_i$, there exists j_i such that $p_{j_i}(t) \leq a^i_{j_i}$ for $s_i < t < t_i$ and hence for $t < t_i$. In conclusion, $p \in \vec{P}(X_{j_1,\cdots,j_l})(\mathbf{c},\mathbf{d})$. □

Remark 2.10. In view of Lemma 2.5, it is enough to consider all index sets $(j_1,\dots,j_l)$ such that $a^i_{j_i} > 0$; likewise $(J_1,\dots,J_l)$ such that $a^i_{j_i} > 0$ for all $j_i \in J_i$.

In the following Section 3, we need a cover of trace space by *open* subsets. Therefore, we carefully augment the spaces $X_{J_1,\cdots,J_l}$: For every $1 \leq i \leq l, 1 \leq j \leq n$, choose $0 < \varepsilon^i_j$ such that $a^i_j < a^k_j, b^k_j \Rightarrow a^i_j + 2\varepsilon^i_j < a^k_j, b^k_j$.

Definition 2.11.
(1) $Y_{j_1,\dots,j_l} := \{\mathbf{x} \in I^n \mid \forall i:\ x_{j_i} < a^i_{j_i} + 2\varepsilon^i_{j_i} \text{ or } \exists k:\ x_k > b^i_k\} \cap X \supset X_{j_1,\dots,j_l}$.
(2) $Y_{J_1,\cdots,J_l} = \bigcap_{j_i \in J_i} Y_{j_1,\cdots,j_l} \supset X_{J_1,\cdots,J_l}$.

Proposition 2.12. *Suppose that, for every $1 \leq j \leq n$, no upper boundary b^i_j is a lower boundary a^k_j; i.e., that $\{a^i_j\}_i \cap \{b^i_j\}_i = \emptyset$ for every j.*

(1) *There exists a d-map $\varphi : X \to X$ (continuous and order preserving) and a d-homotopy* [11] *$\Phi = (\Phi_t) : X \times \vec{I} \to X, \varphi \to id_X$ keeping $X_{j_1,\cdots,j_l}$ pointwise fix that satisfy $\varphi(Y_{J_1,\dots,J_l}) \subseteq X_{J_1,\dots,J_l}$ and even $\Phi(Y_{J_1,\dots,J_l} \times I) \subseteq X_{J_1,\dots,J_l}$ for all $(J_1,\dots,J_l) \in [1:n]^l$.*
(2) *$X_{j_1,\cdots,j_l}$ is a deformation retract of $Y_{j_1,\cdots,j_l}$.*
(3) *$X_{J_1,\cdots,J_l}$ is a deformation retract of $Y_{J_1,\cdots,J_l}$.*

Proof. Choose piecewise linear and weakly increasing reparametrizations $\varphi_j : \vec{I} \to \vec{I}$, $1 \leq j \leq n$, of the unit interval I that are the identity outside the intervals $]a^i_j, a^i_j + 2\varepsilon^i_j[$ and that map $[a^i_j, a^i_j + \varepsilon^i_{j_i}]$ constantly to a^j_i. The product $\varphi = \prod_{j=1}^n \varphi_j : \vec{I}^n \to \vec{I}^n$ restricts to a map $\varphi : X \to X$ such that $\varphi(Y_{j_1,\dots,j_l}) \subseteq X_{j_1,\dots,j_l}$ and $\varphi(Y_{J_1,\dots,J_l}) \subseteq X_{J_1,\dots,J_l}$.

The linear homotopy $\Phi : \varphi \to id_{I^n}$ that connects φ and the identity map is a d-homotopy that restricts to d-homotopies on the spaces $Y_{j_1,\dots,j_l}$ and $Y_{J_1,\dots,J_l}$; it induces homotopies between the identity map and the maps induced by the restrictions of φ on associated trace spaces. □

Corollary 2.13.
(1) *$\vec{T}(Y_{J_1,\cdots,J_l})(\mathbf{c},\mathbf{d})$ is contractible, resp. empty, if and only if $\vec{T}(X_{J_1,\cdots,J_l})(\mathbf{c},\mathbf{d})$ is contractible, resp. empty;*
(2) *$\vec{T}(Y_{J_1,\cdots,J_l})(\mathbf{c},\partial_+ \downarrow \mathbf{d})$ is contractible, resp. empty if and only if $\vec{T}(X_{J_1,\cdots,J_l})(\mathbf{c},\partial_+ \downarrow \mathbf{d})$ is contractible, resp. empty.*

Proof. Immediate from Lemma 2.7 and Proposition 2.12. □

Remark 2.14. It should also be possible to exploit the max-operation $\vee$ for the definition and analysis of (future) components [5, 9] as follows: $\mathbf{x},\mathbf{y} \in X$ are elementarily future related if, for every $\mathbf{z}_1,\mathbf{z}_2$ with $\vec{P}(X)(\mathbf{x},\mathbf{z}_i) \neq \emptyset \neq \vec{P}(X)(\mathbf{z}_1,\mathbf{y}) : \mathbf{z}_1 \vee \mathbf{z}_2 \in X$; consider the equivalence relation *future equivalent* generated by symmetric and transitive closure.

3. (PROD)SIMPLICIAL MODELS FOR TRACE SPACES

In this and the next Section 4, we concentrate on an investigation of trace spaces $\vec{T}(X)(\mathbf{0},\mathbf{1})$ from the bottom vertex $\mathbf{0}$ to the top vertex $\mathbf{1}$ of $X \subseteq \vec{I}^n$ under the further simplifying restiction that all forbidden hyperrectangles $R^i \subset F$ are contained in the *interior* of I^n. The necessary modifications arising for more general state spaces and for trace spaces of type $\vec{T}(X)(\mathbf{c},\mathbf{d})$, resp. $\vec{T}(X)(\mathbf{c},\partial_+(\downarrow \mathbf{d}))$ will be discussed in Section 5.

3.1. **The index category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.**

3.1.1. *A matrix representation of a power poset.* The index multisets $(J_1,\cdots,J_l)$ with $J_i \subseteq [1:n]$ considered in the previous Section 2 may be viewed as elements of the power set $\mathcal{P}([1:l]\times[1:n]) \cong (\mathcal{P}([1:n]))^l$. Elements of that power set can be encoded by their characteristic functions which can be viewed as binary $l\times n$-matrices:

Let $M_{l,n} = M_{l,n}(\mathbf{Z}/2)$ denote the set of all binary $l\times n$-matrices – with 2^{ln} elements. Componentwise logical or ($\vee$), resp. logical and ($\wedge$) define addition and multiplication on $M_{l,n}$. The total order on $\mathbf{Z}/2$ given by $a \le b$ unless ($a = 1$ and $b = 0$) extends to a componentwise given partial order $\le$ on $M_{l,n}$. With this partial order defining the morphisms, $M_{l,n}$ will be viewed as a *poset category*.

There is a natural order-preserving bijection between the subsets of $[1:l]\times[1:n]$ (elements of the power set $\mathcal{P}([1:l]\times[1:n])$ with partial order given by inclusion) and elements in $M_{l,n}$ given by

$$J=(J_1,\dots,J_l)\mapsto M^J=(m^J_{ij}),\ m^J_{ij}=1 \Leftrightarrow j\in J_i \tag{3}$$

with inverse $M=(M_{ij})\mapsto J^M,\, j\in J^M_i \Leftrightarrow m_{ij}=1$.

Under this bijection, the relevant multisets $J=(J_1,\dots,J_l)$ with $J_i\neq\emptyset, 1\le i\le l$, correspond to matrices in the subset $M^R_{l,n}\subset M_{l,n}$ consisting of the $(2^n-1)^l$ matrices such that *no row vector is a zero vector*. We view $M^R_{l,n}$ as a *full subposet category* within $M_{l,n}$.

3.1.2. *Subcategories and pasting functors.* To ease notation, we will in the following write $\vec{T}(X_M)(\mathbf{0},\mathbf{1})$ instead of $\vec{T}(X_{J^M})(\mathbf{0},\mathbf{1})$. The relevant index category to consider here is the full subposet category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})\subset M^R_{l,n}$ consisting of all matrices M such that

- $\vec{T}(X_M)(\mathbf{0},\mathbf{1})$ is *non-empty*.

This index category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ gives rise to both a contravariant functor $\mathcal{D}$ and a covariant functor $\mathcal{E}$ into **Top**: The functor $\mathcal{D}:\mathcal{C}(X)(\mathbf{0},\mathbf{1})\to\mathbf{Top}$ associates $\vec{T}(X_M)(\mathbf{0},\mathbf{1})$ to the matrix M; inclusions in $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ correspond to reverse inclusions in **Top**. The functor $\mathcal{E}:\mathcal{C}(X)(\mathbf{0},\mathbf{1})\to\mathbf{Top}$ restricts from a functor $\mathcal{E}^l_n: M_{l,n}\to\mathbf{Top}$; it associates to $M=^J$ with $(J_1,\dots,J_l)$ – all $J_i\neq\emptyset$! – the standard simplex $\Delta^{|J_1|-1}\times\cdots\times\Delta^{|J_l|-1}\subset\mathbf{R}^{|J_1|}\times\cdots\times\mathbf{R}^{|J_l|}\subset(\mathbf{R}^n)^l$; $\mathbf{R}^{|J_i|}$ is included in $\mathbf{R}^n$ as the subspace given by the equations $x_j=0, j\notin J_i$. The functor $\mathcal{E}^l_n$ should be considered as a pasting scheme for the product of simplices $(\Delta^{n-1})^l$; the functor $\mathcal{E}$ becomes then a pasting scheme for a sub-prodsimplicial complex [18] $X_M \subseteq (\Delta^{n-1})^l$ to be explained below.

Remark 3.1. We prefer $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ as indexing category to the nerve of the covering given by the spaces $X_{j_1,\cdots,j_l}$, since an intersection X_M – in view of Lemma 2.5(2) – can arise in many ways as intersection of the basic spaces $X_{j_1,\cdots,j_l}$ corresponding to matrices in which every row is a *unit* vector; even as intersection of a varying number of the basic covering sets. The nerve of that latter covering is bigger than necessary, it carries redundant information since it does not take care of the product structure giving rise to automatically commuting morphims. That nerve is in fact the barycentric subdivision of $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$; cf. below.

3.2. **Trace spaces and prodsimplicial complexes as colimits.** Regarding these functors as pasting schemes, we consider their colimits, which yield:

- $\mathbf{colim}(\mathcal{D}) = \vec{T}(X)(\mathbf{0},\mathbf{1})$ – by Lemma 2.9;
- $\mathbf{colim}(\mathcal{E}_n^l) = (\Delta^{n-1})^l$;
- $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) := \mathbf{colim}(\mathcal{E}) \subset \mathbf{colim}(\mathcal{E}_n^l) = (\Delta^{n-1})^l$ is a prodsimplicial complex (in the terminology of [18]) consisting of those products of simplices $\Delta^{|J_1|-1} \times \cdots \times \Delta^{|J_l|-1}$ that correspond to tuples $(J_1,\dots,J_l)$ with $M^J \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$; in other words, the functor $\mathcal{E}$ is regarded as a pasting scheme for products of simplices, one product for each non-empty space $\vec{T}(X_M)(\mathbf{0},\mathbf{1})$.

Remark 3.2. This prodsimplicial complex is *not* a general complex of morphisms in the sense of [18], 9.2.4. Whether $\vec{T}(X_{J_1,\cdots J_l})(\mathbf{0},\mathbf{1})$ is non-empty cannot be decided by investigating whether all $\vec{T}(X_{j_1,\cdots j_l})(\mathbf{0},\mathbf{1}), j_i \in J_i$ are non-empty. The topology of the complex does not only depend on its 1-skeleton ([18], Proposition 18.1).

Comparing with $\mathbf{colim}(\mathcal{E}_n^l) = (\Delta^{n-1})^l$, we obtain at once:

Lemma 3.3. *The prodsimplicial complex* $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ *is a subcomplex of* $(\partial\Delta^{n-1})^l \cong (S^{n-2})^l$*; in particular, it has at most* n^l *vertices and* $\dim(\mathbf{T}(X)(\mathbf{0},\mathbf{1})) \leq (n-2)l$.

Proof. From Remark 2.8, it follows that $\vec{T}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$ as soon as M has a row vector consisting of digits one only; in particular, no product can have a (full) factor Δ^{n-1}. The complex $(\partial\Delta^{n-1})^l$ has the number of vertices and the dimension given in the lemma. $\square$

Example 3.4. Assume that the obstruction hyperrectangles $R^i =]\mathbf{a}^i, \mathbf{b}^i[$ have the property $\mathbf{b}^i < \mathbf{a}^{i+1}, 1 \leq i < l$; i.e., the holes are ordered with respect to the partial order in $\mathbf{R}^n$. This is the case in the first row of Example 2.3. It is not difficult to see in that case that $\vec{T}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$ if and only if M has a row consisting of all 1 digits. As a consequence, $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = (\partial\Delta^{n-1})^l \cong (S^{n-2})^l$ in this case. Hence the bounds given in Lemma 3.3 are sharp!

3.3. **Homotopy equivalences.**

Theorem 3.5. *Assume that, for every $1 \leq i \leq n$, no upper boundary coordinate b_i^j is equal to a lower boundary coordinate a_i^k. Then trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subset (\partial\Delta^{n-1})^l$ and to the nerve of the category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$; the latter simplicial complex arises as a barycentric subdivision of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.*

Proof. First, we determine the *homotopy* colimits of the functors defining the pasting schemes above. We apply the homotopy lemma (Theorem 15.12 in [18]) to the natural transformation $\Psi : \mathcal{D} \Rightarrow \mathcal{T}$ from $\mathcal{D}$ to the trivial functor $\mathcal{T} : \mathcal{C}(X) \to \mathbf{Top}$ which sends every object into the same one-point space. Since the maps corresponding to Ψ are homotopy equivalences at any object M in $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ (from a contractible space $\vec{T}(X_M)(\mathbf{0},\mathbf{1})$ – by Proposition 2.7(2) – to a point), the map $\mathbf{hocolim}\mathcal{D} \to \mathbf{hocolim}\mathcal{T}$ induced by Ψ is a homotopy equivalence by the homotopy lemma. By definition, $\mathbf{hocolim}\mathcal{T}$ is the nerve $\Delta(\mathcal{C}(X)(\mathbf{0},\mathbf{1}))$ of the indexing category.

The same argument applies also to the trivial natural transformation from $\mathcal{E}$ to $\mathcal{T}$ and shows that $\mathbf{hocolim}\mathcal{E}$ is also homotopy equivalent to the nerve $\Delta(\mathcal{C}(X)(\mathbf{0},\mathbf{1}))$ – which is thus a barycentric subdivision of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.

Next, we wish to apply the projection lemma (Theorem 15.19 in [18]) – with two twists – to the fiber projection maps $\mathbf{hocolim}\mathcal{D} \to \mathbf{colim}\mathcal{D}$ and $\mathbf{hocolim}\mathcal{E} \to \mathbf{colim}\mathcal{E}$. If applicable, that lemma ensures that these projection maps are homotopy equivalences. Altogether, the maps discussed above fit to yield a homotopy equivalence

$$\vec{T}(X)(\mathbf{0},\mathbf{1}) = \mathbf{colim}(\mathcal{D}) \leftarrow \mathbf{hocolim}(\mathcal{D}) \to \mathbf{hocolim}(\mathcal{T}) \leftarrow \mathbf{hocolim}(\mathcal{E}) \to \mathbf{colim}(\mathcal{E}) = \mathbf{T}(X)(\mathbf{0},\mathbf{1}).$$

The first twist alluded to above consists in using, instead of the nerve diagram of the covering given by the spaces $X_{j_1,\cdots,j_l}$, the functors $\mathcal{D}$, resp. $\mathcal{E}$ with indexing category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$, cf. Remark 3.1. For the functor $\mathcal{E}$, we use moreover, that $\mathbf{T}(X)(\mathbf{c},\mathbf{d})$ has a prodsimplicial and thus a CW-structure and Remark 15.20 in [18].

As to the functor $\mathcal{D}$, we need to verify the conditions of the projection lemma: It was shown in [26], that $\vec{T}(X)(\mathbf{0},\mathbf{1})$ is paracompact – even under much weaker assumptions to X. Furthermore, we may replace the cover given by the subspaces $\vec{T}(X_{j_1,\ldots,j_l})(\mathbf{0},\mathbf{1})$ to that given by the homotopy equivalent *open* subspaces $\vec{T}(Y_{j_1,\ldots,j_l})(\mathbf{0},\mathbf{1})$ from Definition 2.11, with the same colimit and a homotopy equivalent homotopy colimit. □

Remark 3.6. A modified version of Theorem 3.5 holds without assuming that the obstruction hyperrectangles are contained in the interior of I^n; also for trace spaces of type $\vec{T}(X)(\mathbf{c},\mathbf{d})$ and $\vec{T}(X)(\mathbf{c},\partial_+ \downarrow \mathbf{d})$ described in Section 2. The only necessary change is a different description of the corresponding index category; this will be explained in Section 5.

4. Determination of the index category

To determine the prodsimplicial model $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ of trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ (Theorem 3.5), we need to describe the indexing category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ and hence to determine which of the subspaces $\vec{T}(X_M)(\mathbf{0},\mathbf{1})$ are empty and which not.

It turns out that (non-)emptyness can be investigated by a method that was originally designed for the detection of deadlocks and associated unsafe regions in models for the simple Higher Dimensional Automata from Section 2.1 described in [6]. A *deadlock* in X is an element $\mathbf{x} \in X$ that admits only the constant path as d-path with source $\mathbf{x}$. The *unsafe region* corresponding to $\mathbf{x}$ consists of all $\mathbf{y} \in X$ such that no d-path in X with source $\mathbf{y}$ can leave the hyperrectangle spanned by $\mathbf{y}$ and $\mathbf{x}$.

It will be shown that $\vec{T}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$ is equivalent to the existence of a deadlock ($\neq$ $\mathbf{1}$ within X_M. This in turn depends on whether a certain set of *inequalities* – determined by M – between coordinates of the obstruction hyperrectangles holds.

A simple-minded version of the procedure described here restricted to the case $n = 2$ was described earlier in [22].

4.1. **Empty path spaces and deadlocks.** Remember the notation convention: $1 \leq i \leq l$ denumerates the obstruction hyperrectangles R^i; $1 \leq j \leq n$ denumerates the n coordinate directions in $\mathbf{R}^n$.

We begin with a "dual" look at the spaces $X_{J_1,\dots,J_l}$ from Definition 2.1, resp. X_M from (3) in Section 3.1. For each of the original forbidden hyperrectangles $R^i = \prod_{j=1}^n I_j^i$ (cf. Section 2), we define n *extended* hyperrectangles

$$R_j^i = \prod_{k=1}^{j-1} \bar{I}_k^i \times I_j^i \times \prod_{k=j+1}^{n} \bar{I}_k^i,\ 1 \leq i \leq l, 1 \leq j \leq n, \tag{4}$$

and $\bar{I}_k^i = [0, a_k^i] \cup I_k^i = [0, b_k^i[$, an interval with 0 as its lower boundary.

Remark 4.1. Each of the hyperrectangles R_j^i has the property that all apart at most one of the coordinates of the lowest vertex are 0.

By negating (2) from Definition 2.1, one obtains immediately for every matrix $M = (m_{ij}) \in M_{l,n}$:

Lemma 4.2. $X_M = \vec{I}^n \setminus \bigcup_{m_{ij}=1} R_j^i$. □

The following result shows that (non)-emptyness of the relevant trace spaces can be established by checking a bunch of inequalities. For this, we have to find deadlocks in the subspaces X_M by identifying *non-empty intersections* of n extended hyperrectangles among the R_j^i, $m_{ij} = 1$, and the associated unsafe regions.

Proposition 4.3. *For $M \in M_{l,n}^R$, the following are equivalent:*

(1) *M is* not *an object in $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.*
(2) *$\vec{T}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$.*

(3) *There is a map* $i : [1:n] \to [1:l]$ *such that*

$$m_{i(j),j} = 1 \text{ for all } 1 \le j \le n \text{ and } \bigcap_{1\le j\le n} R_j^{i(j)} \neq \emptyset.$$

(4) *There is a such a map* i *with*

$$a_j^{i(j)} < b_j^{i(k)} \text{ for all } j,k \in [1:n].$$

Proof. The equivalence of (1) and (2) follows from the definition of $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$. To establish equivalence of (3) and (4), note that an intersection of (homothetic) hyperrectangles is non-empty if and only if each lower coordinate of one of the participating rectangles is smaller than all corresponding upper coordinates. From Remark 4.1 we know that all but one of the lower coordinates of the $R_j^{i(j)}$ are *zero*; the requirement has only to be checked for $a_j^{i(j)}$; exactly what is required in (4).

Assuming (2), i.e., $\vec{T}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$, then $\mathbf{0}$ must be contained in the unsafe region associated to a deadlock ($\neq \mathbf{1}$) for some configuration of n forbidden hyperrectangles chosen among the $R_j^i, j \in J_i$. (If a deadlock making $\mathbf{0}$ unsafe arises by a configuration containing one or several of the *original* hyperrectangles R^i, extending it to some $R_j^i, m_{ij} = 1$, will enlarge the compound obstruction and certainly again give rise to a configuration with the same property. Hence, we may restrict attention to configurations consisting of *extended* hyperrectangles only. It is important that the matrix $M \in M_{l,n}^R$ has no zero row vector for this argument; cf. also Remark 4.4 below.) The existence of a deadlock in X_M is equivalent to the existence of a non-empty intersection $\bigcap_{1\le j\le n} R_j^{i(j)}$[6], i.e., of a map as given in (3).

On the other hand, granted (3), if $\bigcap_{1\le j\le n} R_j^{i(j)} \neq \emptyset$, the intersection gives rise to a deadlock $\mathbf{e} = [e_1, \ldots, e_n] \neq \mathbf{1}$ in X_M; in fact the coordinates e_j of $\mathbf{e}$ are maximal among the j-th coordinates of the $R_j^{i(j)}$; in our case $e_j = a_j^{i(j)}$, cf. [6]. The associated unsafe region has as its *lowest* vertex the point in X the n coordinates of which are *next to* maximal among these lower coordinates of the $R_j^{i(j)}$ [6]. Now we use that the extended hyperrectangles are special (cf. Remark 4.1), in the sense that all these coordinates (next to maximal among the lower coordinates) are 0. Hence $\mathbf{0}$ is automatically in the unsafe region associated to the deadlock $\mathbf{e}$ in X_M! In particular, there is no d-path with source $\mathbf{0}$ leaving $\downarrow \mathbf{c}$; this proves (2): $\vec{T}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$. □

Remark 4.4. In proving (2) implies (3) above, it is crucial that all index sets J_i are non-empty. Otherwise, a number of extended hyperrectangles R_j^i might, together with some of the original R^i, generate a deadlock with $\mathbf{0}$ in the unsafe region that does not arise from a non-empty intersection of *extended* hyperrectangles. Below, you find an illustration for that phenomenon in dimension two:

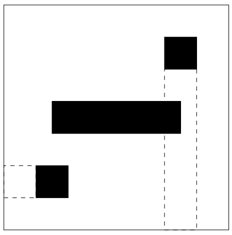


FIGURE 3. Deadlock arising from combination of extended and non-extended rectangles.

4.2. **Algorithmic Determination of $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.**

4.2.1. *The map Ψ and its properties.* We will also consider the following subset of binary matrices:

- $M^C_{l,n} \subset M_{l,n}$ consists of the matrices such that *every column vector is a unit vector* – with l^n elements. Every such matrix $M = M(i)$ represents the characteristic function of the graph of some map $i : [1:l] \to [1:n]$, cf. Proposition 4.3(3).

Define the map $\Psi : M_{l,n} \to \mathbf{Z}/2$ by $\Psi(M) = 1 \Leftrightarrow \vec{T}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset$; equivalently, $\Psi(M) = 0 \Leftrightarrow M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ for matrices $M \in M^R_{l,n}$.

Proposition 4.5. (1) *Ψ is order-preserving; for $M \in M_{l,n}$, we have:*
(2) *$\Psi(M) = 0$ if M has a zero vector among its column vectors.*
(3) *$\Psi(M) = 1 \Leftrightarrow$ there exists $N \in M^C_{l,n}$ with $\Psi(N) = 1$ and $N \leq M$.*

Proof. (1) If $M \leq M' \in M_{l,n}$, then $\vec{T}(X_{M'})(\mathbf{0},\mathbf{1}) \subseteq \vec{T}(X_M)(\mathbf{0},\mathbf{1})$. If the latter set is empty, the first set is empty, as well.
(2) Assume that the j-th column in M is the zero vector. Then no obstruction hyperrectangle is extended in direction j. Hence, all j-th lower coordinates chosen from the extended hyperrectangles corresponding to M are *strictly positive*. In particular, $\mathbf{0}$ is *not* contained in the unsafe region of any deadlock occuring in X_M; in particular, there exists a d-path from $\mathbf{0}$ to $\mathbf{1}$ since neither the face $x_j = 0$ nor the upper boundary $\partial_+ \downarrow \mathbf{1}$ intersect any of the extended hyperrectangles.
(3) $\Leftarrow$ is an immediate consequence of (1). In view of (2), we may assume that M has no zero vector among its column vectors. The graph of a map $i : [1:n] \to [1:l]$ satisfying (3) in Proposition 4.3 has as its characteristic function the matrix $M(i) \in M^C_{l,n}$ with $\Psi(M(i)) = 1$ and $M(i) \leq M$.

□

The determination of Ψ can thus be performed in two steps. First, we determine the restriction of Ψ to the subset $M^C_{l,n}$ corresponding to maps $i : [1:n] \to [1:l]$. In particular, we determine the set of matrices (D for "dead")

$$D(X)(\mathbf{0},\mathbf{1}) := \{M \in M^C_{l,n} \mid \Psi(M) = 1\}. \tag{5}$$

Using this set $D(X)(\mathbf{0},\mathbf{1})$, we will then determine the set of matrices

$$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) := \{M \in M^R_{l,n} \mid \Psi(M) = 0\}. \tag{6}$$

describing the objects of the relevant index category.

4.2.2. *Determination of* $D(X)(\mathbf{0},\mathbf{1})$. As described in Section 4.1, the graph of a map $i : [1:n] \to [1:l]$ has a matrix $M(i) \in M^C_{l,n}$ as its characteristic function.

To a matrix $M \in M^C_{l,n}$, we associate its *row set* $R(M) := \{1 \le i \le l \mid \mathbf{m}^i \neq \mathbf{0}\} \subseteq [1:l]$ – indexing the *non-zero* rows $\mathbf{m}^i$ of M. The row set $R(M(i))$ is equal to the image $i([1:n]) \subseteq [1:l]$.

The condition from Proposition 4.3(4) leads to consider the *same* upper bounds $b_j^{i(k)}$ for matrices M with the *same* row set $R(M) = R \subseteq [1:l]$: To every of the $\sum_{k=1}^{\min(n,l)} \binom{n}{k} \le 2^l - 1$ non-empty subsets $R \subseteq [1:l]$ of cardinality at most $\min(n,l)$ corresponds an upper bound $\mathbf{b}^R = [b_1^{r_1}, \dots, b_n^{r_n}] \in [0,1]^n$ with $b_j^{r_j} = \min_{i \in R} b_j^i$.

Ordering the j-th coordinates a_j^i, resp. b_j^i of subinterval boundaries for $\vec{I}_j^i \subset \vec{I}_j$ (e.g. by a quicksort algorithm) gives rise to $2n$ (not necessarily well-determined) permutations $\pi_j^0, \pi_j^1 \in \Sigma_l$ such that $a_j^{\pi_j^0(1)} \le \dots \le a_j^{\pi_j^0(n)}$ and $b_j^{\pi_j^1(1)} \le \dots \le b_j^{\pi_j^1(n)}$. Furthermore, let $C_j : [1:l] \to [1:l]$ be given by $C_j(k) := \max\{r \mid 1 \le r \le l, a_j^{\pi_j^0(r)} < b_j^{\pi_j^0(k)}\}, 1 \le j \le n$. Note that $C_j(k) \ge k$ for all j and that C_j is monotone; the maps C_j arise from sorting the union $\{a_j^i\} \cup \{b_j^i\}$.

For a subset $B \subseteq [1:l]$, the bound $\mathbf{b}^B = [b_1^B, \dots, b_n^B] = [\min b_1^{i_1}, \dots, \min b_n^{i_n}], i_j \in B$ corresponds to a multiindex $(k_1^B, \dots, k_n^B) \in [1:l]^n$ with $\pi_j^1(k_j^B) = b_j$. For $\emptyset \neq B \subseteq [1:l]$, let $\tilde{R}_j(B) := \pi_j^0(C_j(k_j^B)) = \{i \in [1:l] | a_j^i < b_j^B\}$ and $R_j(B) = \tilde{R}_j(B) \cap B$. From condition (4) in Proposition 4.3, we conclude:

Lemma 4.6. *A map* $i : [1:n] \to [1:l]$ *gives rise to a matrix* $M = M(i) \in D(X)(\mathbf{0},\mathbf{1})$ *if and only if*

$$i(j) \in R_j(i([1:n])) \text{ for every } 1 \le j \le n.$$

□

What is left is a method to determine the sets $R_j(B)$ for every subset $B \subseteq [1:l]$ of cardinality at most n: Starting from one-elements sets B, work your way incrementally through all non-empty subsets $B \subset [1:l]$ with at most n elements. For a one element set $B = \{i\}$, $\mathbf{b}^B = \mathbf{b}^i$. In general, determine $\mathbf{b}^B$ using that $\mathbf{b}^{S \cup T} = \min(\mathbf{b}^S, \mathbf{b}^T)$ for subsets $S, T \subseteq [1:l]$ – this corresponds to taking the minima of the corresponding multi-indices. For the determination of the sets $R_j(B)$, the following properties – in particular (4) – are helpful:

Lemma 4.7. (1) *If* $B = \{i\}$ *is a one-element set, then* $R_j(B) = B$ *for* $j \in [1:n]$. *Hence* $\Psi(M) = 1$ *for each of the* l *matrices* $M \in M^C_{l,n}$ *with a one-element row set* $R(M)$.

(2) $\emptyset \neq B \subseteq C \subseteq [1:l] \Rightarrow \tilde{R}_j(B) \supseteq \tilde{R}_j(C), 1 \leq j \leq n$.
(3) $R_j(B \cup C) = (R_j(B) \cap \tilde{R}_j(C)) \cup (R_j(B) \cap \tilde{R}_j(C))$.
(4) *For $k \notin B$, we have*

$$R_j(B \cup \{k\}) = \begin{cases} R_j(B), & b_j^B < a_j^k \Leftrightarrow a_j^k \notin C_j(b_j^B) \\ R_j(B) \cup \{k\}, & a_j^k < b_j^B < b_j^k \Leftrightarrow a_j^k \in C_j(b_j^B), \pi_j^1(k) > \pi_j^1(k_j^B) \\ C_j(b_j^k) \cap R_j(B) \cup \{k\}, & b_j^k < b_j^S \Leftrightarrow \pi_j^1(k) < \pi_j^1(k_j^B) \end{cases}.$$

Proof. (1) follows from (the proof of) Lemma 3.3; (2) is obvious. For (3), note that $R_j(B \cup C) = (\tilde{R}_j(B) \cap \tilde{R}_j(C)) \cap (B \cup C)$ and use distributivity. (4) is an easy consequence. □

4.2.3. *Determination of $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$.* From Proposition 4.5 and Lemma 4.6, we can conclude immediately:

Proposition 4.8. *Let $M \in M_{l,n}^R$.*
(1) $\Psi(M) = 1$ *if and only if there is a matrix $N \in D(X)(\mathbf{0}, \mathbf{1})$ (cf. Lemma 4.6) such that $n_{ij} \leq m_{ij}$ for all $1 \leq i \leq l, 1 \leq j \leq n$;*
(2) $\Psi(M) = 0 \Leftrightarrow M \in \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ *if and only if, for every matrix $N \in D(X)(\mathbf{0}, \mathbf{1})$, there is a pair $(i, j) \in [1:l] \times [1:n]$ such that $m_{ij} = 0, n_{ij} = 1$.*

□

Matrices that are *maximal* with respect to the partial order $\leq$ on binary matrices within $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ constitute the subset $\mathcal{C}_{max}(X)(\mathbf{0}, \mathbf{1}) \subseteq \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$.

To determine the matrices contained in these two sets, we consider (choice) subsets $C \subseteq [1:l] \times [1:n]$ characterized by the property:
For every matrix $N \in D(X)(\mathbf{0}, \mathbf{1})$ there exists $(i, j) \in C$ with $n_{ij} = 1$. Remark that one index (i, j) can count for several matrices N.
Functions $m_C = 1 - \chi(C)$ for such choices are exactly the characteristic functions for matrices $M_C = (m_{ij}) \in \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$.

A choice C is *minimal*, if for every $C' \subset C$ there is a matrix $N \in D(X)(\mathbf{0}, \mathbf{1})$ with $n_{ij} = 0$ for every $(i, j) \in C'$. The function $m_C = 1 - \chi(C)$ for a *minimal* choice function is a *maximal* matrix $M_C \in \mathcal{C}_{max}(X)(\mathbf{0}, \mathbf{1})$.

We describe a simple-minded algorithm constructing $\mathcal{C}_{max}(X)(\mathbf{0}, \mathbf{1})$ step by step given $D(X)(\mathbf{0}, \mathbf{1}) = \{D_1, \dots, D_p\}$ starting with $A_{max}^0(X)(\mathbf{0}, \mathbf{1})$ with the matrix $\mathbf{1}$ consisting of only 1s as the only element. Assume $A_{max}^{i-1}(X)(\mathbf{0}, \mathbf{1}) = \{M_1, \dots, M_{q_{i-1}}\}$ to consist of the maximal binary matrices M such that $N_k \not\leq M$ for $1 \leq k \leq i-1 < p$.

Compare the matrices $M_l \in A_{max}^{i-1}(X)(\mathbf{0}, \mathbf{1})$ to N_i. If $N_i \not\leq M_l$, then keep M_l unchanged as an element of $A_{max}^i(X)(\mathbf{0}, \mathbf{1})$; if $N_i \leq M_l$, then replace M_l by the n matrices $M_l^1, \dots M_l^n \in A_{max}^i(X)(\mathbf{0}, \mathbf{1})$: M_l^j arises from M_l by replacing the j-th entry 1 in N_l (which is also 1 in M_l) by 0.

Assessing whether $N \leq M$ is easy, given N: form the binary product $\bigwedge_{n_{pq}=1} m_{pq}$; this product is always over n entries.

The maximal matrices in $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})$ correspond to the *maximal simplex products* that are patched together in $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subset (\Delta^{n-1})^l$ while the matrices in $D(X)(\mathbf{0},\mathbf{1})$ correspond to *minimal non-faces* in $(\Delta^{n-1})^l$. The construction above reminds of a similar construction of a simplicial complex $K(\mathcal{F})$ associated to a set system $\mathcal{F}$ used for topological investigations of colouring problems, cf. e.g. [19]; in our case, we have the product structure in the underlying category $M_{l,n}$ as an additional feature.

Corollary 4.9. (1) *The simplex product in* $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ *corresponding to* $M_c \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ *considered as an object in* $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ *(cf. Section 3.2) has dimension* $(n-1)l - |C|$.
(2) $\dim \mathbf{T}(X)(\mathbf{0},\mathbf{1}) = (n-1)l - \min|C|$ *where C ranges over all choice subsets in* $[1 : l] \times [1 : n]$.

Proof. The simplex product corresponding to $\mathcal{R}^{M_C}$ has type $\prod_{1\le i\le l} \Delta^{n-1-c_i}$ with $c_i = \{j|\ (i,j) \in C\}$. □

Corollary 4.10. *The Lusternik-Schnirelmann category of trace space* $T(X)(\mathbf{0},\mathbf{1})$ *satisfies the inequality* $cat(T(X)(\mathbf{0},\mathbf{1})) \le |\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})|$.

Proof. The prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ homotopy-equivalent to $\vec{T}(X)(\mathbf{0},\mathbf{1})$ is covered by maximal products of simplices; there are $|\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})|$ of those. As products of simplices, they are contractible; they are deformation retracts of contractible open neighbourhoods in $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$. □

Example 4.11. (1) X a square with two square holes as in Example 2.3, upper row: $D(X)(\mathbf{0},\mathbf{1})$ consists of the two matrices $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\1&1\end{bmatrix}$ (deadlock only if one extends in both directions from the same obstruction). There are four choices $C \subset [1 : 2] \times [1 : 2]$; and any of these is minimal and of cardinality 2. Hence $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) = \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ consists of the four matrices $\begin{bmatrix}0&1\\0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\begin{bmatrix}1&0\\0&1\end{bmatrix}$, and $\begin{bmatrix}1&0\\1&0\end{bmatrix}$ with extensions that allow exactly one d-homotopy class around the (extended) holes. The corresponding prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ consists of four points of type $(\Delta^0)^2$.
(2) X a square with two square holes as in Example 2.4, lower row: This time, $D(X)(\mathbf{0},\mathbf{1})$ consists of the three matrices $\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\1&1\end{bmatrix}$, and $\begin{bmatrix}0&1\\1&0\end{bmatrix}$: one additional deadlock configuration comes up with one extension for every hole. There are only three choices C, all minimal and of cardinality 2 corresponding to the matrices $\begin{bmatrix}0&1\\0&1\end{bmatrix}$, $\begin{bmatrix}1&0\\0&1\end{bmatrix}$, and $\begin{bmatrix}1&0\\1&0\end{bmatrix}$. $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ consists thus of three points.
(3) X a square with three holes as in Figure 3: In this case,

$$D(X)(\mathbf{0},\mathbf{1}) = \{ \begin{bmatrix}1&1\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\1&1\end{bmatrix} \}.$$

There are four minimal choices giving rise to the matrices in

$$\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) = \{ \begin{bmatrix}1 & 0\\ 1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\\ 0 & 1\end{bmatrix} \}.$$

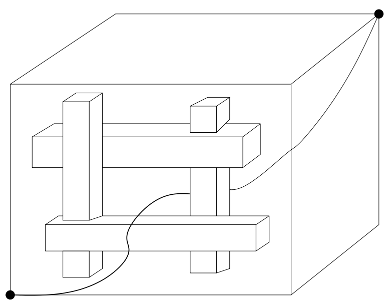
The complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ consists of four points.

(4) $X = \vec{I}^n \setminus \vec{J}^n$ as in Example 2.4: In this case, $D(X)(\mathbf{0},\mathbf{1}) = \{[1,1,\ldots,1]\}$. The n minimal choices correspond to the one row matrices with exactly one entry 0 in $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})$. These matrices correspond to the maximal simplices in $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1}$.

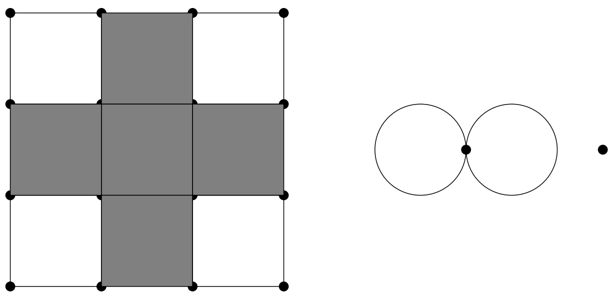
Corollary 4.12. $\vec{T}(\vec{I}^n \setminus \vec{J}^n)(\mathbf{0},\mathbf{1}) \simeq \partial\Delta^{n-1}$. □

Previous attempts to prove Corollary 4.12 directly were much more complicated.

Example 4.13. The space X in the figure below shows a cube from which two wedges, each of them composed of two rectangular boxes are removed. Remark that the two wedges do not touch each other. The trace in that drawing from bottom to top is homotopic but not dihomotopic (homotopic through a 1-parameter deformation of d-paths) to a trace on the boundary of the cube. A simple-minded analysis of this model in [23] showed by a quite intricate argument that the trace space for this d-space (from bottom to top) is not connected.



State space X



Associated models for trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$

The general method described in this article yields a model for trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ as a subspace of the 4-torus $(\partial\Delta^2)^4 \cong (S^1)^4$. It turns out by inspection that one can handle the two wedges as one obstruction and therefore that trace space can be seen as the union of five squares and a disjoint extra ("corner") point in the two-torus $(\partial\Delta^2)^2 = (S^1)^2$. This subspace is of course homotopy equivalent to the disjoint union of a wedge of circles and of an extra point: $\vec{T}(X)(\mathbf{0},\mathbf{1}) \simeq (S^1 \vee S^1) \sqcup *$.

It would be interesting to find more general methods for dimension reduction as the one described above.

4.2.4. *A reformulation: Minimal transversals in hypergraphs.* The search for minimal choices in $D(X)(\mathbf{0},\mathbf{1})$ can be translated into a well-known and well-investigated problem in combinatorics.[1] The set $D(X)(\mathbf{0},\mathbf{1})$ may be considered as a *hypergraph* (with hyperedges = simplices connecting a number of vertices; every matrix in $D(X)(\mathbf{0},\mathbf{1})$ defines a hyperedge) on the vertex set $[1:l]\times[1:n]$. A minimal choice is then a *minimal transversal* (or *hitting set*) of that hypergraph, i.e., it has nonempty intersection with every hyperedge and it is minimal with this property. Computing minmal transversals has many applications (e.g., machine learning, indexing of databases, data mining and optimization). There are several articles about algorithms for finding minimal transversals and their complexity in the literature; cf. e.g. [17].

The hypergraph given by the matrices in $D(X)(\mathbf{0},\mathbf{1})$ has special properties: All hyperedges have the same cardinality n; even more so, they are graphs of functions from $[1:n]$ to $[1:l]$. This ought to simplify the setting.

4.3. **Homology of the trace space.** By Theorem 3.5, the homology of the trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ may be calculated as the homology of the associated prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$. Given the poset category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$, this is the homology of a particular chain complex $C(X)(\mathbf{0},\mathbf{1})$ with one generator for every product of simplices.

More precisely, let $C_k(X)(\mathbf{0},\mathbf{1})$ denote the free R-module generated by all matrices in $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ with $(k+l)$ entries 1; R denotes the chosen ring. For a matrix $M\in M_{l,n}$ with $m_{pq}=1$, let M_{pq} be given by $(m_{pq})_{ij}=\begin{cases} m_{ij} & (i,j)\neq(p,q)\\ 0 & (i,j)=(p,q)\end{cases}$.

The boundary operator ∂ on $C(X)(\mathbf{0},\mathbf{1})$ is then given by $\partial(M)=\sum_{m_{pq}=1}(-1)^{|(p,q)|}M_{pq}$ with alternating sign: $|(p,q)|=\sum_{i=1}^{p-1}\sum_{j=1}^{n}m_{ij}+\sum_{j=1}^{q}m_{pj}-1$ takes account of the ones in M preceeding $m_{pq}=1$.

It should be interesting to perform actual homology calculations in "real life" examples that give rise to huge chain complexes. The algorithms for the calculation of homology in [16] by reduction of chain complexes (with field coeffectients) might be helpful. Likewise a modification of the algorithms in [15] for the homology of cubical complexes.

5. Obstructions on the boundary

5.1. **Introduction.** In Sections 3 and 4, we had assumed that all obstruction hyperrectangles R^i are contained in the *interior* of I^n. This assumption is not valid in many applications; we will now briefly describe how to modify the index category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ in the general case; in fact, we will describe the index categories

- $\mathcal{C}(X)(\mathbf{c},\mathbf{d})$ with set of objects $\{M\in M^R_{l,n}\mid \vec{T}(X_M)(\mathbf{c},\mathbf{d})\neq\emptyset\}$ corresponding to trace space $\vec{T}(X)(\mathbf{c},\mathbf{d})$ with traces starting at $\mathbf{c}$ and ending at $\mathbf{d}$; and

[1] I would like to thank my colleague Leif Kjær Jørgensen, Aalborg University, for mentioning hypergraphs and their transversals to me.

- $\mathcal{C}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$ with set of objects $\{M \in M^R_{l,n} | \vec{T}(X_M)(\mathbf{c}, \partial_+ \downarrow \mathbf{d}) \neq \emptyset\}$ corresponding to traces ending on the *upper boundary* of the box with $\mathbf{d}$ as upper corner.

Remark 5.1. (1) Models for Higher Dimensional Automata have often obstructions intersecting the boundary ∂I^n; those arrise always as soon as semaphores of an arity $r < n-1$ (at most r processors can proceed concurrently) are involved.
(2) The trace space $\vec{T}(X)(\mathbf{0}, \partial_+ \downarrow \mathbf{1})$ is interesting in the analysis of algorithms for wait-free protocols (cf. e.g., [14]) in which all processors with at least one exception are allowed to "die", i.e., cease to communicate. In this case, the accepting states correspond to the points contained in $\partial_+ \downarrow \mathbf{1}$.

In the same way as described in Section 3.1, the matrix poset categories $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ and $\mathcal{C}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$ serve as pasting schemes that give rise to prodsimplicial complexes $\mathbf{T}(X)(\mathbf{c}, \mathbf{d})$ and $\mathbf{T}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$. Under the general conditions of Theorem 3.5, but allowing obstruction hyperrectangles to intersect the boundary of $[\mathbf{c}, \mathbf{d}]$, we obtain using Proposition 2.7:

Theorem 5.2. (1) *Trace space $\vec{T}(X)(\mathbf{c}, \mathbf{d})$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{c}, \mathbf{d})$ and to the nerve of the category $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$.*
(2) *Trace space $\vec{T}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$ is homotopy equivalent to the prodsimplicial complex $\mathbf{T}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$ and to the nerve of the category $\mathcal{C}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$.*

□

For an algorithmic determination of these index categories (as in Section 4), we need to describe several modifications of the matrix subsets $D(X)(-,-)$ and $\mathcal{C}(X)(-,-)$ with respective boundaries.

5.2. **Which trace spaces are (non-)empty?** In both cases, rectangles R^i that do not intersect the box $[\mathbf{c}, \mathbf{d}]$ become irrelevant. This can be handled by reducing the number of rows in the matrices constituting the index categories: We separate
$[1:l] = [1:l]^{in} \sqcup [1:l]^{out}$ with $i \in [1:l]^{in} \Leftrightarrow (1 \leq j \leq n \Rightarrow a^i_j < d_j, c_j < b^i_j)$ and let $l' := |[1:l]^{in}|$.

Remark 5.3. Comparing trace spaces with varying end points, it may be necessary to take account of these irrelevant rectangles nevertheless. On the prodsimplicial side this will result in taking a product with one or several simplices Δ^{n-1}; cf. Section 5.4.

Lemma 5.4. *Suppose $a^i_j \leq c_j$. Then $M \notin \mathcal{C}(X)(\mathbf{c}, -)$ for every matrix $M \in M^R_{l',n}$ with $m_{ij} = 1$.*

Proof. Supposing $\mathbf{x} \in X_M$ and $\mathbf{x} \leq \mathbf{b}^i$ implies $x_j < a^i_j \leq c_j$, i.e., $\mathbf{x} \notin [\mathbf{c}, \mathbf{1}]$; in particular, $\vec{T}(X_M)(\mathbf{c}, -) = \emptyset$. □

Under these circumstances, we will thus only have to investigate matrices

$$M \in \bar{M}^R_{l',n} \text{ with } a^i_j \leq c_j \Rightarrow m_{ij} = 0. \tag{7}$$

We will deal with the easier case of the index category $\mathcal{C}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$ first. Proposition 4.3 has the following immediate modification:

Proposition 5.5. *For $M \in \bar{M}^R_{l',n}$, the following are equivalent:*

(1) *M is* not *an object in $\mathcal{C}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$.*
(2) *$\vec{T}(X_M)(\mathbf{c}, \partial_+(\downarrow \mathbf{d})) = \emptyset$.*
(3) *There is a map $i : [1:n] \to [1:l']$ such that $m_{i(j),j} = 1$ and $\bigcap_{1 \le j \le n} R_j^{i(j)} \neq \emptyset$.*
(4) *There is a map $i : [1:n] \to [1:l']$ with $a_j^{i(j)} < b_j^{i(k)}$ for all $j,k \in [1:n]$.*

Proof. The proof is an easy modification of the one given for Proposition 4.3. Note that a deadlock on the boundary $\partial_+(\downarrow \mathbf{d})$ is irrelevant for paths/traces in $\vec{T}(X)(\mathbf{c}, \partial_+ \downarrow \mathbf{d})$. □

For the analysis of $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$ in general, we have to deal with obstruction hyperrectangles *intersecting* $\partial_+ \downarrow \mathbf{d}$; this may also be the case for $\mathbf{d} = \mathbf{1}$ for hyperrectangles intersecting the boundary of I^n. For every such "intersection direction" $1 \le j \le n$ with a hyperrectangle intersecting the j-th face $x_j = d_j$ of $\partial_+ \downarrow \mathbf{d}$, (i.e., such that there exists an i with $b_j^i > d_j$ or $b_j^i = d_j = 1$), we introduce *new* obstruction hyperrectangles $R_j^0 = [0,1]^{i-1} \times [d_j, 1] \times [0,1]^{n-i}$ – no longer open; degenerate for $d_j = 1$. In particular, $a_j^0 = d_j$ for these "intersecting directions".

Proposition 4.3 can then be modified as follows:

Proposition 5.6. *For $M \in \bar{M}^R_{l',n}$, the following are equivalent:*

(1) *M is* not *an object in $\mathcal{C}(X)(\mathbf{c}, \mathbf{d})$.*
(2) *$\vec{T}(X_{\tilde{\mathcal{R}}})(\mathbf{c}, \mathbf{d}) = \emptyset$.*
(3) *There is a map $i : [1:n] \to [0:l']$ such that $i(j) \neq 0 \Rightarrow m_{i(j),j} = 1$ and such that $\bigcap_{1 \le j \le n} R_j^{i(j)} \neq \emptyset$.*
(4) *There is a map $i : [1:n] \to [0:l']$ such that $i(j) \neq 0 \Rightarrow m_{i(j),j} = 1$ and such that*
$$\begin{cases} a_j^{i(j)} < b_j^{i(k)} \text{ for } j,k \in [1:n], i(j) > 0 \text{ or } i(j) = 0, a_j^0 = d_j < 1; \\ b_j^{i(k)} = 1 \text{ for } j,k \in [1:n], i(j) = 0, a_j^0 = d_j = 1. \end{cases}$$

□

Compared to Proposition 4.3, remark that further intersections involving hyperrectangles R_j^0 need to be considered.

5.3. **Modified Algorithms.** The matrix representation from Section 4.2.1 needs a few minor modifications. First of all, in both cases, only obstructions intersecting $[\mathbf{c}, \mathbf{d}]$ need to be taken care of, and this may reduce the number of rows from l to l' in the matrices to be considered. For the category $\mathcal{C}(X)(\mathbf{c}, \partial_+(\downarrow \mathbf{d}))$, one may then proceed as in Section 4.2 – with the simplification that only matrices in $\bar{M}^R_{l',n}$ (cf. (7)) need to be considered.

5.3.1. *Determination of* $D(X)(\mathbf{c},\mathbf{d})$. For a category of type $\mathcal{C}(X)(\mathbf{c},\mathbf{d})$, we replace the matrix set $M^C_{l,n}$ by the set $\bar{M}^C_{l',n}$. A matrix $M\in\bar{M}^C_{l',n}$ has the following properties:

- $a^i_j<c_j\Rightarrow m_{ij}=0$;
- every column vector $\mathbf{m}_j$ is either a unit vector or the zero vector $\mathbf{0}$;
- if $\mathbf{m}_j=\mathbf{0}$, then j is an intersection direction.

A matrix $M\in\bar{M}^C_{l',n}$ codes the map $i_M:[1:n]\to[0:l'], i_M(j)=\begin{cases} i(j) & \mathbf{m}_j=\mathbf{e}_{i(j)} \\ 0 & \mathbf{m}_j=\mathbf{0}\end{cases}$. Vice versa, a (relevant) map $i:[1:n]\to[0:l']$ comes with a characteristic matrix $M(i)=(m_{ij})\in\bar{M}^C_{l',n}, m_{ij}=1\Leftrightarrow 0<i(j)=i$. With $\Psi:\bar{M}_{l',n}\to\mathbf{Z}/2$ defined as in Section 4.2.1, we obtain the following analogue to Proposition 4.5:

Proposition 5.7. *A matrix* $M\in\bar{M}^R_{l',n}$ *satisfies* $\Psi(M)=1$ *if and only if there exists* $N\in\bar{M}^C_{l',n}$ *with* $\Psi(N)=1$ *and* $N\le M$.

We wish to determine $\Psi(M)$ for $M\in\tilde{M}^C_{l',n}$ and, in particular, $D(X)(\mathbf{c},\mathbf{d}):=\{M\in\tilde{M}^C_{l',n}|\Psi(M)=1\}$. For that purpose, one has to consider both the row set $R(M)\subset[1:l']$ (cf. Section 4.2.2) and the *column set* $C(M)\subset[1:n]$ indexing the *non-zero* rows, resp. columns of M. An analogue of Lemma 4.6 (again with $\mathbf{b}^B=[b^B_1,\dots,b^B_n]$ and $b_j=\min_{i\in B}b^i_j$) for a row set $B\subset[1:l']$ reads:

Lemma 5.8. *A map* $i:[1:n]\to[0:l']$ *gives rise to a matrix* $M=M(i)\in D(X)(\mathbf{c},\mathbf{d})$ *if and only if*

(1) $i(j)=0(\Leftrightarrow\mathbf{m}_j=\mathbf{0})\Rightarrow d_j<b^{r_j}_j$ *or* $d_j=b^{r_j}_j=1$ *and*
(2) $i(j)\in R_j(i([1:n])\setminus\{0\})$.

The method described in Lemma 4.7 has to be extended: for a given non-empty (row) subset $B\subset[1:l']$, one determines a *maximal* column set $C(B)\subset[1:n]$ consisting of those $j\in[1:n]$ that satisfy (1) above; this requires checking n (in)equalities. Next, for every subset of $C\subseteq C(B)$the sets $\tilde{R}_j(B;C):=\{i\in[1:l']|j\notin C\Rightarrow a^i_j<b^B_j\}$ have to be determined – decrementally – as in Lemma 4.7; a zero column corresponds to every $j\in C$. As in Proposition 4.6, we end up determining the set of matrices $M\in D(X)(\mathbf{c},\mathbf{d}):=\{M\in\bar{M}^C_{l',n}|\ \Psi(M)=1\}$.

Having found $D(X)(\mathbf{c},\mathbf{d})$, we can determine the matrices in $\mathcal{C}(X)(\mathbf{c},\mathbf{d}):=\{M\in\bar{M}^R_{l,n}|\ \Psi(M)=0\}$ in the same way as described in Proposition 4.8; again, only matrices in $\bar{M}^R_{l',n}$ need to be checked.

5.4. **Varying end points.** By concatenation, traces $\sigma\in\vec{T}(X)(\mathbf{d},\mathbf{d}'),\tau\in\vec{T}(X)(\mathbf{c}',\mathbf{c})$ induce continuous maps $\sigma_\sharp:\vec{T}(X)(\mathbf{c},\mathbf{d})\to\vec{T}(X)(\mathbf{c},\mathbf{d}')$ and $\tau^\sharp:\vec{T}(X)(\mathbf{c},\mathbf{d})\to\vec{T}(X)(\mathbf{c}',\mathbf{d})$.

In order to find out "what happens" between $\mathbf{d}$ and $\mathbf{d}'$, one has to study the effect of these induced maps; it suffices to look at d-homotopy classes $[\sigma]\in\vec{\pi}_1(X)(\mathbf{d},\mathbf{d}')$ of paths/traces $\sigma \in \vec{T}(X)(\mathbf{d}, \mathbf{d}')$, cf. [24] and the discussion of the functor $\vec{T}^X : \vec{D}(X) \to Ho - Top$ from the double category $\vec{D}(X)$ associated to X there. Similarly, one may analyse what happens between $\partial_+ \downarrow \mathbf{d}$ and $\partial_+ \downarrow \mathbf{d}'$; moreover, there are similar contravariant versions "between $\mathbf{c}'$ and $\mathbf{c}$. For a discussion/determination of o-called components in X ([5, 9, 24]), one would like to know which traces σ induce homotopy equivalences (or at least bijections on sets of path components).

For brevity, we restrict to the first case, the concatenation map $\sigma_\sharp : \vec{T}(X)(\mathbf{c}, \mathbf{d}) \to \vec{T}(X)(\mathbf{c}, \mathbf{d}')$. Assume that l' of the hyperrectangles intersect $[\mathbf{c}, \mathbf{d}]$ whereas $l \geq l'$ intersect $[\mathbf{c}, \mathbf{d}']$. In order to compare, we will use the *same* larger index set $[1 : l]$ in both cases. A hyperrectangle R^i not intersecting $[\mathbf{c}, \mathbf{d}]$ does not pose any conditions to the question $\vec{T}(X_M)(\mathbf{c}, \mathbf{d}) \neq \emptyset$ defining index categories; the corresponding i-th row in the matrix M is irrelevant. As a result, the index category $\mathcal{C}(X)(\mathbf{c}, \mathbf{d}) \subset M^R_{l',n}$ will be replaced by the pullback

$$\begin{array}{ccc} \tilde{\mathcal{C}}(X)(\mathbf{c}, \mathbf{d}) & \xrightarrow{\pi} & \mathcal{C}(X)(\mathbf{c}, \mathbf{d}) \\ \downarrow \subseteq & & \downarrow \subseteq \\ M^R_{l,n} & \xrightarrow{\pi} & M^R_{l',n} \end{array}$$

with $\pi : M^R_{l,n} \to M^R_{l',n}$ leaving out superfluous rows.

The pasting scheme corresponding to $\tilde{\mathcal{C}}(X)(\mathbf{c}, \mathbf{d})$ gives rise to the prodsimplicial complex $\tilde{\mathbf{T}}(X)(\mathbf{c}, \mathbf{d}) = \mathbf{T}(X)(\mathbf{c}, \mathbf{d}) \times (\Delta^{n-1})^{l-l'}$ homotopy equivalent to $\mathbf{T}(X)(\mathbf{c}, \mathbf{d})$.

The index category $\mathcal{C}(X)(\mathbf{c}, \mathbf{d}')$ becomes then a *sub*category of $\tilde{\mathcal{C}}(X)(\mathbf{c}, \mathbf{d})$ with certain matrices eliminated; one needs to analyse the effect of the associated *inclusion* of prodsimplicial complexes $\mathbf{T}(X)(\mathbf{c}, \mathbf{d}') \hookrightarrow \tilde{\mathbf{T}}(X)(\mathbf{c}, \mathbf{d})$. This and the consequences for components will be analysed in future work.

It is also relevant to ask is what happens if one digs an extra forbidden hyperrectangle out of the state space $X \subset I^n$; interesting in particular for an inductive determination of index categories and associated prodsimplicial models of trace spaces. Again, the associated map between prodsimplicial models is a combination of a homotopy equivalence (taking the product with a simplex) and an inclusion map reflecting the additional obstruction. The effect of this map (and the map induced on homology) has still to be investigated more closely.

5.5. **A particular case: Semaphores of arity one.** Matters get simplified for an HDA model in which every semaphore allows only a *single* process to proceed. In this case the forbidden region F is a union of hyperrectangles $R^i = \prod I^i_j$ such that $I^i_j = [0,1]$ for every $1 \leq i \leq l$, except for two choices $1 \leq j_1(i), j_2(i)$: $I^i_{j_k(i)} = [a^i_{j_r(i)}, b^i_{j_r(i)}], r = 1, 2$; $0 < a^i_{j_r(i)} < b^i_{j_r(i)} < 1$. Note that every such semaphore gives rise to $\binom{n}{2}$ forbidden hyperrectangles, and that k such semaphores hence produce $l = k\binom{n}{2}$ hyperrectangles. In this case, we have:

Proposition 5.9. (1) *If $\vec{T}(X_{J_1,\cdots,J_l})(\mathbf{0},\mathbf{1})$ is non-empty, then every index set J_k has exactly* one *element j_k.*

(2) *$\vec{T}(X)(\mathbf{0},\mathbf{1})$ is homotopy equivalent to a* finite discrete *space; its (contractible) connected components are the non-empty ones among the spaces $\vec{T}(X_{j_1,\cdots j_l})(\mathbf{0},\mathbf{1})$.*

Proof. (1) Suppose J_i has at least two elements $1 \leq j_1 < j_2 \leq n$ for some $1 \leq i \leq l$. If one of them, say $j_1 \notin \{j_1(i), j_2(i)\}$, then $a^i_{j_1} = 0$ and hence trace space is empty by Proposition 4.3(1).

If $\{j_1, j_2\} = \{j_1(i), j_2(i)\}$, we define a map $i : [1 : n] \to [0 : l]$ with $i(j_1) = i(j_2) = i$ and $i(j) = 0$ for all other j. We check that condition (4) in Proposition 5.6 is satisfied: $a^i_{j_1} < b^i_{j_1}, a^i_{j_2} < b^i_{j_2}$, and $b^i_j = 1$ for $j_1 \neq j \neq j_2$.

(2) It follows from (1), that the subposet category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ has no non-trivial morphisms, and that the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ has dimension 0.

□

It remains thus to determine *which* of the spaces $\vec{T}(X_{j_1,\cdots j_l})(\mathbf{0},\mathbf{1})$ are *empty*. Remark that there are $2^l, l = k\binom{n}{2}$ such spaces – for every i, one may choose either $j_1(i)$ or $j_2(i)$. State spaces corresponding to semaphores of arity one possess additional structure (apart from giving rise to obstruction hyperrectangles of the type studied in Proposition 5.9) that we are now going to exploit:

Let us first consider a *single* semaphore of arity one given by intervals $]a_j, b_j[\subset [0,1]$, $1 \leq j \leq n$. The associated forbidden region is the union $F = \bigcup_{1\leq j_1,j_2\leq n, j_1\neq j_2} R(j_1, j_2)$ of hyperrectangles $R(j_1, j_2) = \{\mathbf{x} \in I^n |\ a_{j_i} < x_{j_i} < b_{j_i}, i = 1, 2\}$. Remark that $R(j_1, j_2) = R(j_2, j_1)$; there are $\binom{n}{2}$ such hyperrectangles. As usual, let $X = I^n \setminus F$.

In the proof of the next result, we will also need the extended hyperrectangles $R_{j_1}(j_1, j_2) = \{\mathbf{x} \in I^n |\ x_{j_2} < b_{j_2}, a_{j_1} < x_{j_1} < b_{j_1}\}$ and likewise $R_{j_2}(j_1, j_2)$; moreover, as in Section 5.2, the degenerate hyperrectangles $R^0_j = [0,1]^{j-1} \times \{1\} \times [0,1]^{n-j}$, $1 \leq j \leq n$.

Proposition 5.10. *The map $\vec{x} : \Sigma_n \to \vec{T}(X)(\mathbf{0},\mathbf{1})$ from a discrete space Σ_n with $n!$ elements parametrized by permutations $\pi : [1 : n] \to [1 : n]$ given by $\vec{x}(\pi)(t) = [x_1(t), \ldots, x_n(t)]$ with* $x_{\pi(k)}(t) = \begin{cases} 0 & t \leq \frac{k-1}{n} \\ (nt-(k-1)) & \frac{k-1}{n} \leq t \leq \frac{k}{n} \\ 1 & \frac{k}{n} \leq t \leq n \end{cases}$ *is a homotopy equivalence.*

Proof. Note that $\vec{x}(\pi)$ describes a d-path on the 1-skeleton of $\vec{I}^n$ – which does not intersect F.

Let $\mathcal{P}_2(n)$ denote the set of all 2-element subsets of $[1 : n]$ (with $\frac{n(n-1)}{2}$ elements), and let $c : \mathcal{P}_2(n) \to [1 : n]$ denote a *choice* function with the property $c(\{j_1, j_2\}) \in \{j_1, j_2\}$. For such a choice function c – determining in which order to pass the obstructions $R(j_1, j_2)$ – let $F_c = \bigcup_{1\leq j_1,j_2\leq n, j_1\neq j_2} R_{c(j_1,j_2)}(j_1, j_2)$ and $X_c = I^n \setminus F_c$. By Theorem 5.2 and Proposition 5.9, the (contractible) components of $\vec{T}(X)(\mathbf{0},\mathbf{1})$ correspond to those choice functions c giving rise to *non-empty* trace spaces $\vec{T}(X_c)(\mathbf{0},\mathbf{1})$.

A choice function c gives rise to a relation on $[1:n]$ defined by $j_1 \leq_c j_2$ if $c(j_1, j_2) = j_1$ and its reflexive and transitive closure $\preceq_c$. If $\preceq_c$ defines a *total order* on $[1:n]$, then this total order is given by a permutation $\pi \in \Sigma_n$: $\pi(1) \preceq_c \pi(2) \preceq_c \cdots \preceq_c \pi(n)$. We claim that $\vec{T}(X_c)(\mathbf{0},\mathbf{1}) \neq \emptyset$ if and only if $\preceq_c$ is a total order.

If $\preceq_c$ is *not* a total order, then there is a chain $j_1 \preceq_c \cdots \preceq_c j_k \preceq_c j_1$ with $k < n$; let $j_{k+1}, \ldots, j_n$ denote the remaining elements of $[1:n]$. The extended, resp. degenerate hyperrectangles $R_{j_1}(j_1, j_2), \ldots R_{j_k}(j_k, j_1), R^0_{j_{k+1}}, \ldots, R^0_{j_n}$ with non-empty intersection $\{\mathbf{x} \in I^n \mid a_{j_i} < x_{j_i} < b_{j_i}, i \leq k; x_{j_i} = 1, i > k\}$ give then rise to a deadlock $\mathbf{x} = [x_1, \ldots, x_n]$, i.e., $x_{j_i} = a_{j_i}, i \leq k; x_{j_i} = 1, i > k$.

Suppose now that $\preceq_c$ is a total order. For every choice of n among the extended and degenerate hyperrectangles $R_{j_p}(j_p, j_q), p \preceq_c q$ and $R^0_{j_r}$, their intersection will be empty: since there is no $\preceq_c$ loop, the of union of all $\{j_p, j_q\}$ corresponding to extended hyperrectangles has at least one element j in common with the set $\{j_r\}$ corresponding to degenerate hyperrectangles. An element $\mathbf{x} \in \bigcap R_{j_p}(j_p, j_q) \cap \bigcap R^0_{j_r}$ would have to satisfy both $x_j < b_j$ and $x_j = 1$. Hence, all these intersections are empty, there are no deadlocks in X_c. □

Let us now consider a state space corresponding to a collection of k semaphores of arity one, i.e, $X = I^n \setminus \bigcup_{1 \leq i \leq k} F^i, F^i = \bigcup_{1 \leq j_1 < j_2 \leq n} R^i(j_1, j_2)$. To every collection $\pi = (\pi^1, \ldots, \pi^k) \in (\Sigma^n)^k$ of k permutations, we associate extended forbidden regions $\bar{F}^i = \bigcup_{1 \leq j_1 < j_2 \leq n} R^i_{\pi_i(j_2)}(\pi_i(j_1), \pi_i(j_2))$ with $R^i_{\pi_i(j_2)}(\pi_i(j_1), \pi_i(j_2)) =$ $\{\mathbf{x} \in I^n \mid x_{\pi_i(j_1)} < b^i_{\pi_i(j_1)}, a^i_{\pi_i(j_2)} < x_{\pi_i(j_2)} < b^i_{\pi_i(j_2)}\} \supset R^i(\pi_i(j_1), \pi_i(j_2))$, and the state space $X_\pi = I^n \setminus \bigcup_{1 \leq i \leq k} \bar{F}^i$. Then, with $l = k\binom{n}{2}$ and l choice functions $c_1, \ldots, c_l$, we get:

$$\vec{T}(X)(\mathbf{0},\mathbf{1}) \simeq \{(c_1, \ldots, c_l) \mid \vec{T}(X_{c_1,\ldots,c_l})(\mathbf{0},\mathbf{1}) \neq \emptyset\} \simeq \{\pi = (\pi_1, \ldots, \pi_k) \mid \vec{T}(X_\pi)(\mathbf{0},\mathbf{1}) \neq \emptyset\}.$$

The first homotopy equivalence above is, as in the proof of the previous Proposition 5.10, a consequence of Theorem 5.2 and Proposition 5.9. The second homotopy equivalence follows from Proposition 5.10: only those tuples of choice functions arising from permutations can give rise to non-empty trace spaces.

Consider the set of all $a^i_j, b^i_j \in I, i \leq k, 1 \leq n$ boundary coordinates of contributing semaphores. For every collection $\pi = (\pi_1, \ldots, \pi_k) \in (\Sigma^n)^k$, we consider several order relations on subsets of these real numbers:

- The natural order $\leq$, inherited from the reals, on numbers a^i_j, b^i_j with the *same* subscript (direction) j;
- $b^i_{\pi_i(j_1)} \preceq a^i_{\pi_i(j_2)}$ for $1 \leq i \leq k, 1 \leq j_1 < j_2 \leq n$.

We call the collection π *compatible* if the transitive hull of these relations is a *partial order*.

Proposition 5.11. *Let $X = I^n \setminus F$ denote the state space corresponding to a collection of l semaphores of arity one. Then $\vec{T}(X)(\mathbf{0},\mathbf{1})$ is homotopy equivalent to the discrete space $\{\pi = (\pi_1, \ldots, \pi_k) \in (\Sigma^n)^k \mid \pi$ compatible$\}$.*

Proof. We need to show: $\vec{T}(X_\pi)(\mathbf{0},\mathbf{1}) \neq \emptyset$ if and only if π is compatible.

Assume first that $\vec{T}(X_\pi)(\mathbf{0},\mathbf{1}) \neq \emptyset$. Every d-path $p : \vec{I} \to X_\pi$ from $\mathbf{0}$ to $\mathbf{1}$ yields the order relation $t_1 \leq t_2, x_j(t_1) = c^{i_1}_{j_1}, x_k(t_2) = d^{i_2}_{j_2}, ; c, d = a, b \Rightarrow c^{i_1}_{j_1} \leq d^{i_2}_{j_2}$ compatible with the relations defined above. In particular, the transitive hull is an order relation.

Now assume that $\vec{T}(X_\pi)(\mathbf{0},\mathbf{1}) = \emptyset$, i.e., the forbidden regions $\bar{F}^i$ give rise to a deadlock. A deadlock arises as lower corner of a non-empty intersection of n hyperrectangles among the $R^i_{\pi_i(j^i_2)}(\pi_i(j^i_1), \pi_i(j^i_2)), 1 \leq i \leq k, 1 \leq j^i_1 < j^i_2 \leq n$ and the degenerate hyperrectangles $R^0_j, 1 \leq j \leq n$.

A non-empty intersection gives rise to at least one chain of coordinates

$$a^{i_1}_{\pi_{i_1}(j^{i_1}_2)} < b^{i_2}_{\pi_{i_2}(j^{i_2}_1)} \preceq a^{i_2}_{\pi_{i_2}(j^{i_2}_2)} < b^{i_3}_{\pi_{i_3}(j^{i_3}_1)} \preceq \cdot < \cdots < \cdot \preceq a^{i_r}_{\pi_{i_{r-1}}(j^{i_{r-1}}_2)}$$

with $\pi_{i_{s+1}}(j^{i_{s+1}}_1) = \pi_{j_s}(j^{i_s}_2)$ – since, at a deadlock, every a_j coordinate is less ($<$) than every b_j-coordinate; and every contributing b-coordinate is $\preceq$ some a-coordinate corresponding to the same obstruction – and *equal ends* since one cannot continue an infinite number of times. This contradicts the partial order condition. $\square$

It does not seem easy to check which of the k-tuples $\pi = (\pi_1, \ldots, \pi_k)$ are compatible: The relation generated by $\leq$ and by $\preceq$ defines a digraph G_π on the boundary coordinates a^i_j, b^i_j; the k-tuple is compatible if and only if G_π does not contain a directed cycle.

6. MODELS FOR MORE GENERAL TRACE SPACES

6.1. **Trace spaces in products of digraphs corresponding to non-linear programs.** So far, we have only looked at model spaces corresponding to concurrent *linear* programs, without branchings, mergings and loops. More realistic models can be investigated using more or less the same tools: Let $\Gamma = \prod_{j=1}^n \Gamma_j$ denote a product of *directed* graphs (brachings, mergings and loops allowed); each Γ_j represents a program run by a single processor. The Γ_j are regarded as d-spaces (realizations of pre-cubical sets of dimension one), and Γ is given the product structure: as an n-dimensional pre-cubical complex with d-space structure[11].

A *directed interval* J_j from a_j to b_j in the geometric realization of a component Γ_j is uniquely given by the image $p_j(I)$ of a trace $p_j \in \vec{T}(\Gamma_j)(a_j, b_j)$. A *(generalized) hyperrectangle* in Γ is a product $R = \prod_j J_j \subseteq \prod_j \Gamma_j = \Gamma$ of such directed intervals. A forbidden region $F = \bigcup_i R^i$ is the union of such generalized hyperrectangles, and the state space $X = \Gamma \setminus F$ is its complement.

The aim is to analyse the space of d-paths $\vec{P}(X)(\mathbf{x},\mathbf{y}) \subseteq \vec{P}(\Gamma)(\mathbf{x},\mathbf{y})$ (or the space of traces $\vec{T}(X)(\mathbf{x},\mathbf{y})$ homotopy equivalent to it) between two points $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n)$ in the space X. First, we have a look at $\vec{T}(\Gamma)(\mathbf{x},\mathbf{y})$ and then, we will use the map induced on traces by the inclusion map $i_X : X \hookrightarrow \Gamma$.

For a directed graph – no cubes of higher dimension supporting homotopies available – dihomotopy of d-paths (with fixed end points) is equivalent to reparametrization equivalence, cf. [4]. In particular, each factor $\vec{T}(\Gamma_j)(x_j, y_j)$ is *discrete*; every component is represented by a (reparametrization equivalence class of) a particular directed path from x_j to y_j. The product structure of Γ yields:

Lemma 6.1. $\vec{T}(\Gamma)(\mathbf{x}, \mathbf{y}) \simeq \prod \vec{T}(\Gamma_j)(x_j, y_j)$*; in particular,* $\vec{T}(\Gamma)(\mathbf{x}, \mathbf{y})$ *is* discrete.

Proof. The d-space structure and the dihomotopy relations factor:
$\vec{T}(\Gamma) \simeq \vec{P}(\Gamma) \cong \prod \vec{P}(\Gamma_j) \simeq \prod \vec{T}(\Gamma_j)$. □

Remark 6.2. To enumerate the components (=traces) of the space of d-paths in a directed graph Γ, one should first reduce it to normal form $N(\Gamma)$: Vertices with valency $(1, -1)$ – one ingoing and one outgoing arrow, different from each other – are suppressed; the two arrows are concatenated to one. The normal form $N(\Gamma)$ does no longer have such vertices.

Attach a unique label to each arrow in a directed graph Γ *in normal form* and form words in these labels along concatenable arrows. Then $\vec{T}(\Gamma)$ corresponds to the discrete set of such words; $\vec{T}(\Gamma)(x, y)$ to the words starting and ending with one or several specific labels, depending on whether x, y correspond to vertices or to points on a directed edge. There is no need to distinguish between points on the same edge.

Each component $C \in \vec{T}(\Gamma)(\mathbf{x}, \mathbf{y})$ can thus be represented by an n-tuple of (traces of) specific d-paths $c_j \in \vec{P}(\Gamma_j)(x_j, y_j)$. As representatives, we choose $c_j \in \vec{R}(\Gamma_j)(x_j, y_j) \subset \vec{P}(\Gamma_j)(x_j, y_j)$ to be *regular* (i.e., locally injective, cf. [4])); every other d-path in $\vec{P}(\Gamma_j)(x_j, y_j)$ dihomotopic to c_j is then a reparametrization $c_j \circ \varphi_j$ of c_j with $\varphi_j \in \vec{P}(\vec{I})(0, 1)$ an (increasing) d-path in the standard ordered unit interval $\vec{I}$ ([4], Theorem 3.6 and Proposition 3.8). The d-paths c_j altogether define a d-map $c : \vec{I}^n \to \Gamma$ by $c(t_1, \ldots, t_n) = [c_1(t_1), \ldots, c_n(t_n)]$, and

Lemma 6.3. *The d-map* $c : \vec{I}^n \to \Gamma$ *induces a homeomorphism*
$c\circ : \vec{T}(\vec{I}^n)(\mathbf{0}, \mathbf{1}) \to C \subset \vec{T}(\Gamma)(\mathbf{x}, \mathbf{y}),\ p \mapsto c \circ p.$ □

Given such a component $C \in \pi_0(\vec{T}(\Gamma)) \cong \prod \pi_0(\vec{T}(\Gamma_j))$, the following two questions arise naturally:

(1) Does C lift to X (i.e., can it be represented by an – interleaving – d-path in X rather than in Γ)?
(2) Determine the topology of $i_X^{-1}(C)$, i.e., of the space of all d-paths in X whose projections to the Γ_j are (reparametrizations of) these specified paths ("interleavings of these execution paths").

Every directed interval $J =]a_j, b_j[\subset \Gamma_j$ (in the sense above) pulls back to the standard interval $c_j^{-1}(]a_j, b_j[) \subset I$ – which is an open subinterval of I in the subspace topology, possibly empty. To each generalized hyperrectangle $R^i = \prod J_j^i \subset \Gamma$ corresponds thus an (honest) hyperrectangle $\bar{R}^i = c^{-1}(R^i) = \prod c_j^{-1}(J_j^i)$, possibly empty. The forbidden region $F \subset \Gamma$ corresponds to a forbidden region $\bar{F} = c^{-1}(F) = \bigcup_i \bar{R}^i \subset I^n$, leaving $\bar{X} = I^n \setminus \bar{F} \subset I^n$ as state space (with the d-structure inherited from $\vec{I}^n$). By restricting the homeomorphism $c\circ$ from Lemma 6.3, we obtain

Corollary 6.4. *The d-map $c : \bar{X} \to X$ induces a homeomorphism* $c\circ : \vec{T}(\bar{X})(\mathbf{0},\mathbf{1}) \to i_X^{-1}(C) \subset \vec{T}(X)(\mathbf{x},\mathbf{y})$. □

Example 6.5. Let $\Gamma_1 = \vec{S}^1$ denote a circular digraph and $\Gamma_2 = \vec{I}$ a linear one. Let $X = (\Gamma_1 \times \Gamma_2) \setminus J^2$ with $J^2 \subset (\Gamma_1 \times \Gamma_2)$ an open rectangular hole. The component C_r in $\vec{T}(\Gamma_1 \times \Gamma_2)(\mathbf{0},\mathbf{1})$ corresponding to $r + \frac{1}{2}$ spiral tours leads to a state space $\bar{X}_r$ with r rectangular holes with an exponential covering map back to X:

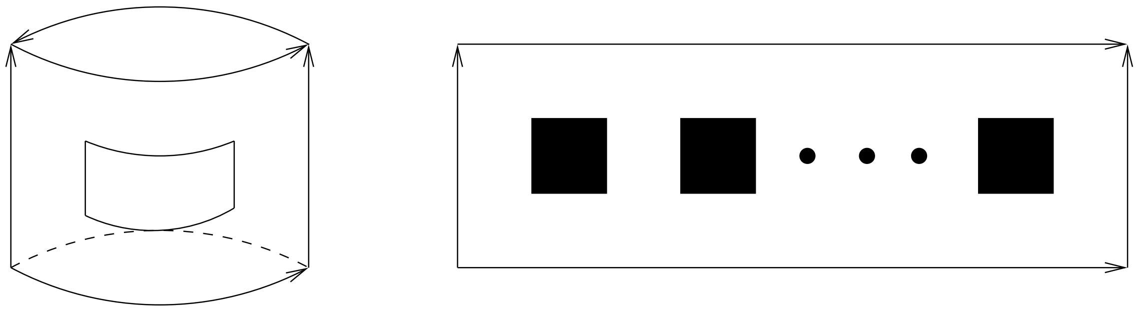


FIGURE 4. State spaces X (directed cylinder with hole) and $\bar{X}_r$.

Corollary 6.4 allows us to attack the questions above:

(1) is equivalent to: Is $\vec{T}(\bar{X})(\mathbf{0},\mathbf{1})$ non-empty? This is the case if $\mathbf{0}$ is *not* contained in the unsafe region corresponding to any deadlock in $\bar{X}$ – this can be settled using the techniques described in [6]; compare also [8].
(2) The topology of a nonempty space $i_X^{-1}(C) \cong \vec{T}(\bar{X})(\mathbf{0},\mathbf{1})$ can be analysed as that of the prodsimplicial complex $\mathbf{T}(\bar{X})(\mathbf{0},\mathbf{1})$ as described in Sections 4 and 5.

Remark 6.6. The components $C \subset \vec{T}(\Gamma)(-,-)$ form the morphisms of the fundamental category $\vec{\pi}_1(\Gamma)$ (composition induced by concatenation of paths) with the elements of Γ as objects. In particular, loop components act on (the left and on the right) on components with matching end points. In [8], the authors have shown that unsafe areas corresponding to a specific deadlock point can look quite different for components with the same end points. It should be interesting to investigate how the topology of the spaces $i_X^{-1}(C)$ behaves under composition with loops. For applications, it is essential to find out whether there is an algorithm determining them in an "inductive" fashion.

6.2. **Simplicial models for trace spaces in pre-cubical complexes.** The methods used in this paper can certainly be applied more generally. In [26], we investigated spaces of d-paths in a non-self-linked pre-cubical complex X (with a compatible d-structure defined at first for every cube) and showed that, for all $x_0, x_1 \in X$, the path spaces $\vec{P}(X)(x_0, x_1)$ are ELCX (equi locally convex) in the sense of Milnor's [20]; hence that they are locally contractible and possess the homotopy type of a CW-complex.

The main ingredient in the proof is the construction of a locally defined average map $\mu : U \to X$ defined on a neighbourhood $U = \bigcup_\beta V_\beta \times V_\beta$ of the diagonal with V_β the open star neighbourhood of a vertex β in X. This average map plays a role very similar to that of max in Section 2.

In particular, a directed sequence of adjacent vertices and their open star neighbourhoods gives rise to the *contractible* space of d-paths (or traces) progressing through that sequence of neighbourhoods. The spaces of d-paths in all possible such sequences give rise to a covering of the space of *all* d-paths (with given end points) by contractible subspaces ([26], Proposition 3.16). Using the same method (and restrictions of the map μ above), it can be shown that intersections of such subspaces (through intersections of open stars of certain vertices) are empty or also contractible.

The nerve lemma ([18]) shows then, that spaces of d-paths (and thus of traces) in a pre-cubical complex (with given end points) are homotopy equivalent to the *nerve* of the covering described above. In particular, $\vec{T}(X)(x_0, x_1)$ has an explicit structure of a simplicial complex. To describe it explicitly, one needs to know which sets of sequences of adjacent vertices give rise to intersections of open star neighbourhoods containing a d-path.

Remark 6.7. More abstractly, one may describe a category of *contractible cube paths in semi-cubical complexes* (with contractible trace spaces and such that all sub-cube paths are contractible, as well) and then consider the induced category *over* X (objects = semi-cubical maps from a contractible cube path into X respecting given end points). The nerve of that category (or of any subcategory that covers all d-paths in X with given end points) is then homotopy equivalent to $\vec{T}(X)(x_0, x_1)$.

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