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NON-LINEAR TRIPLE PRODUCT $A^*B - B^*A$ DERIVATIONS ON $*$-ALGEBRAS

Mohammad Shavandi and Ali Taghavi

Abstract. Let $\mathcal{A}$ be a unital prime $*$-algebra that possesses a nontrivial projection, and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a non-linear map which satisfies

$$\Phi(A \diamond B \diamond C) = \Phi(A) \diamond B \diamond C + A \diamond \Phi(B) \diamond C + A \diamond B \diamond \Phi(C)$$

for all $A, B, C \in \mathcal{A}$, where $A \diamond B = A^*B - B^*A$. Then, if $\Phi(\alpha \frac{I}{2})$ is self-adjoint map for $\alpha \in \{1, i\}$, we show that Φ is additive $*$-derivation.

1 Introduction

Let $\mathcal{R}$ be a $*$-algebra. For $A, B \in \mathcal{R}$, we indite $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$ for $*$-Jordan product and $*$-Lie product, respectively. These products are playing significant roles in some research topics, so the study of their unique features has recently attracted many author's attention, in particular during the recent decade. (for example, see [3, 8, 11, 16]).

Recall that a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if

$$\Phi(A + B) = \Phi(A) + \Phi(B)$$

and

$$\Phi(AB) = \Phi(A)B + A\Phi(B)$$

for all $A, B \in \mathcal{R}$. A map Φ is additive $*$-derivation if it is an additive derivation as well as $\Phi(A^*) = \Phi(A)^*$. The notion of (non-linear) derivations is very important both in theory and applications, and is being studied continuously by enthusiasts ([2, 12, 13, 14]).

The λ-Jordan $*$-product has been defined as $A \bullet_\lambda B = AB + \lambda BA^*$. We say that a map $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ is a λ-Jordan $*$-derivation provided that $\Phi(A \bullet_\lambda B) = \Phi(A) \bullet_\lambda B + A \bullet_\lambda \Phi(B)$ for all $A, B \in \mathcal{A}$. It is clear that for $\lambda = -1$ and $\lambda = 1$, a λ-Jordan $*$-derivation map is a $*$-Lie derivation and $*$-Jordan derivation, respectively [1].

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A Von Neumann algebra $\mathcal{A}$ is a self-adjoint subalgebra of some $B(H)$, the algebra of bounded linear operators acting on some complex Hilbert space, which satisfies the double commutant property: $\mathcal{A}^{''} = \mathcal{A}$ where $\mathcal{A}^{'} = \{T \in B(H), TA = AT, \forall A \in \mathcal{A}\}$ and $\mathcal{A}^{''} = \{\mathcal{A}^{'}\}^{'}$. The center of a (von Neumann) algebra $\mathcal{A}$ is usually specified by $\mathcal{Z}(\mathcal{A}) = \mathcal{A}^{'} \cap \mathcal{A}$. A Von Neumann algebra $\mathcal{A}$ is called factor if it has trivial center, i.e., $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$. For $A \in \mathcal{A}$, the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P such that $PA = A$. It is not difficult to see that $\overline{A}$ is the projection onto the closed subspace spanned by $\{BAx : B \in \mathcal{A}, x \in H\}$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{A}) : S = S^*, S \leq A\}$. If $A = P$ is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. A projection P is said to be core-free if $\underline{P} = 0$ (see [10]). It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$, [5, 6].

Recently, Yu and Zhang in [18] have shown that every non-linear $*$-Lie derivation from a factor Von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang in [7] have investigated a non-linear λ-Jordan $*$-derivation. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a Von Neumann algebra without central abelian projections and λ is a non-zero scalar, then $\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a non-linear λ-Jordan $*$-derivation if and only if Φ is an additive $*$-derivation.

On the other hand, many mathematicians devoted themselves to study the $*$-Jordan product $A \bullet B = AB + BA^*$. In [19], Zhang proved that every non-linear $*$-Jordan derivation map $\Phi : \mathcal{A} \to \mathcal{A}$ on a factor Von Neumann algebra is an additive $*$-derivation. In [17], second auther and his colleagues showed that $*$-Jordan derivation map on every factor von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is additive $*$-derivation.

Quite recently, the authors of [4] discussed some bijective maps preserving the new product $A^*B + B^*A$ between von Neumann algebras with no central abelian projections. To more precisely, if the map Φ preserves the condition

$$\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A)$$

for all $A, B \in \mathcal{A}$, then ϕ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. Recently, we have characterized that maps which preserve the new n-tuple product $A^*B - B^*A$ between factor von Neumann algebras are additive $*$-derivation under some certain assumptions[15].

The authors of [9] introduced the concept of $*$-Lie triple derivations. They have considered a nonlinear $*$-Lie triple derivation as map $\Phi : \mathcal{A} \to \mathcal{A}$ such that

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

hold for all $A, B, C \in \mathcal{A}$, where $[A, B]_* = AB - BA^*$. They also showed that if Φ preserves the above characterizations on factor von Neumann algebras then Φ is additive $*$-derivation.

Motivated by the above results, we are following the problems related to derivations on general algebraic structures, algebras with less properties, which lead to the generalization of previous results. In the first step, we need to show the existence of such non-linear derivations. For this purpose, we introduce the triple product $A \diamond B \diamond C := (A \diamond B) \diamond C$, where $A \diamond B = A^*B - B^*A$. Thus, in this note, we shall prove the following sentence:

Let $\mathcal{A}$ be a prime $*$-algebra, and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map which fulfill the following condition

$$\Phi(A \diamond B \diamond C) = \Phi(A) \diamond B \diamond C + A \diamond \Phi(B) \diamond C + A \diamond B \diamond \Phi(C)$$

for all $A, B, C \in \mathcal{A}$. Provided that $\Phi(\alpha \frac{I}{2})$ is self-adjoint map for $\alpha \in \{1, i\}$,then ϕ is a additive $*$-derivation.

We note that $\mathcal{A}$ is prime, that is, if $A\mathcal{A}B = \{0\}$, for all $A, B \in \mathcal{A}$, then $A = 0$ or $B = 0$.

2 Main Results

Our main theorem is as follows:

Theorem 1. *Let $\mathcal{A}$ be a prime $*$-algebra, and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map which satisfies in*

$$\Phi(A \diamond B \diamond C) = \Phi(A) \diamond B \diamond C + A \diamond \Phi(B) \diamond C + A \diamond B \diamond \Phi(C) \qquad (2.1)$$

for all $A, B, C \in \mathcal{A}$. If $\Phi(\alpha \frac{I}{2})$ is self-adjoint map for $\alpha \in \{1, i\}$, then Φ is additive $*$-derivation.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$. So, it is easy to check that $P_2 = I_{\mathcal{A}} - P_1$ is also a nontrivial projection in $\mathcal{A}$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$, for distinct values of $i, j = 1, 2$; then it is straightforward to verify that $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$, we are able to write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. In order to show additivity of Φ on $\mathcal{A}$, we apply above decomposition of $\mathcal{A}$ to demonstrate some claims that prove Φ is additive on each component $\mathcal{A}_{ij}$ for $i, j = 1, 2$.

We prove the above theorem through several claims as following:

Claim 2. *An initial calculus shows $\Phi(0) = 0$ which is left to readers.*

Claim 3. *We show that $\Phi(\frac{I}{2}) = 0$, $\Phi(-\frac{I}{2}) = 0$ and $\Phi(i\frac{I}{2}) = 0$.*

Indeed, by 2.1, it is clear that

$\Phi(i\frac{I}{2} \diamond \frac{I}{2} \diamond \frac{I}{2}) = \Phi(i\frac{I}{2}) \diamond \frac{I}{2} \diamond \frac{I}{2} + i\frac{I}{2} \diamond \Phi(\frac{I}{2}) \diamond \frac{I}{2} + \frac{I}{2} \diamond i\frac{I}{2} \diamond \Phi(\frac{I}{2})$.

A straightforward computation shows

$\Phi(i\frac{I}{2}) = \frac{1}{4}\Phi(i\frac{I}{2}) - \frac{1}{4}\Phi(i\frac{I}{2})^* - \frac{1}{4}\Phi(i\frac{I}{2})^* + \frac{1}{4}\phi(i\frac{I}{2}) + \frac{i}{4}\Phi(\frac{I}{2})^* + \frac{i}{4}\Phi(\frac{I}{2}) + \frac{i}{4}\Phi(\frac{I}{2}) + \frac{i}{4}\Phi(\frac{I}{2})^* + \frac{i}{2}\Phi(\frac{I}{2}) + \frac{i}{2}\Phi(\frac{I}{2})^*$

which implies that

$$-\frac{1}{2}\Phi(\frac{iI}{2})^{*}-\frac{1}{2}\Phi(\frac{iI}{2})+i\Phi(\frac{I}{2})+i\Phi(\frac{I}{2})^{*}=0. \tag{2.2}$$

By taking adjoint from both sides of equ.2.2, we possess that

$$-\frac{1}{2}\Phi(\frac{iI}{2})-\frac{1}{2}\Phi(\frac{iI}{2})^{*}-i\Phi(\frac{I}{2})-i\Phi(\frac{I}{2})^{*}=0. \tag{2.3}$$

Hence, the summation of the equations (2.2)and (2.3) give us $\frac{1}{2}\Phi(\frac{iI}{2})+\frac{1}{2}\Phi(\frac{iI}{2})^{*}=0$. *Since* $\Phi(\frac{iI}{2})$ *is self-adjoint,* $\Phi(\frac{iI}{2})=0$. *By applying the equ.2.2. and insert the zero value of* $\Phi(\frac{iI}{2})$ *in it, we deduce* $\Phi(\frac{I}{2})+\Phi(\frac{I}{2})^{*}=0$. *But initial assumption declared* $\Phi(\frac{I}{2})^{*}$ *is self-adjoint, so* $\Phi(\frac{I}{2})^{*}=0$.

Similarly, we can show that $\Phi(-\frac{I}{2})^{*}=0$. *It is enough to set zero values in extention of* $\Phi(i\frac{I}{2}\diamond\frac{I}{2}\diamond\frac{I}{2})$ *and use the self-adjoint assumption.*

Claim 4. *For each* $A\in\mathcal{A}$, *we have*

1. $\Phi(-iA)=-i\Phi(A)$.
2. $\Phi(iA)=i\Phi(A)$.

It is easy to check that

$$\Phi\left(-iA\diamond\frac{I}{2}\diamond\frac{I}{2}\right)=\Phi\left(A\diamond i\frac{I}{2}\diamond\frac{I}{2}\right).$$

So, we have

$$\Phi(-iA)\diamond\frac{I}{2}\diamond\frac{I}{2}=\Phi(A)\diamond i\frac{I}{2}\diamond\frac{I}{2}.$$

It follows that

$$-\Phi(-iA)^{*}+\Phi(-iA)=-i\Phi(A)^{*}-i\Phi(A). \tag{2.4}$$

On the other hand, one can easily check that

$$\Phi\left(-iA\diamond i\frac{I}{2}\diamond\frac{I}{2}\right)=\Phi\left(A\diamond-\frac{I}{2}\diamond\frac{I}{2}\right),$$

which implies that

$$\Phi(-iA)\diamond i\frac{I}{2}\diamond\frac{I}{2}=\Phi(A)\diamond-\frac{I}{2}\diamond\frac{I}{2}.$$

Consequentely, we obtain

$$-i\Phi(-iA)^* - i\Phi(-iA) = -\Phi(A) + \Phi(A)^*. \tag{2.5}$$

Equivalently, we can establish

$$-\Phi(-iA)^* + \Phi(-iA) = -i\Phi(A) + i\Phi(A)^*. \tag{2.6}$$

By summing the equations (2.4) and (2.6), we achieve first desired result, i.e.,

$$\Phi(-iA) = -i\Phi(A).$$

Similarly, we can show that $\Phi(iA) = i\Phi(A)$.

Claim 5. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$ we have*

$$\Phi(A_{11} + A_{12}) = \Phi(A_{11}) + \Phi(A_{12}).$$

Let $T = \Phi(A_{11} + A_{12}) - \Phi(A_{11}) - \Phi(A_{12})$. We have to show that $T = 0$.

$$\begin{aligned}
&\Phi(A_{11} + A_{12}) \diamond C_{21} \diamond I + (A_{11} + A_{12}) \diamond \Phi(C_{21}) \diamond I \\
&+(A_{11} + A_{12}) \diamond C_{21} \diamond \Phi(I) \\
&= \Phi(A_{11} + A_{12} \diamond C_{21} \diamond I) \\
&= \Phi(A_{11} \diamond C_{21} \diamond I) + \Phi(A_{12} \diamond C_{21} \diamond I) = \Phi(A_{11}) \diamond C_{21} \diamond I \\
&+A_{11} \diamond \Phi(C_{21}) \diamond I + A_{11} \diamond C_{21} \diamond \Phi(I) + \Phi(A_{12}) \diamond C_{21} \diamond I \\
&+A_{12} \diamond \Phi(C_{21}) \diamond I + A_{12} \diamond C_{21} \diamond \Phi(I) = (\Phi(A_{11}) + \Phi(A_{12})) \diamond C_{21} \diamond I \\
&+(A_{11} + A_{12}) \diamond \Phi(C_{21}) \diamond I + (A_{11} + A_{12}) \diamond C_{21} \diamond \Phi(I).
\end{aligned}$$

Since $T = T_{11} + T_{12} + T_{21} + T_{22}$, then

$$-T_{22}^* C_{21} - T_{21}^* C_{21} + C_{21}^* T_{22} + C_{21}^* T_{21} = 0.$$

Thus $T_{22} = T_{21} = 0$. In a similar vein, we have

$$\begin{aligned}
&\Phi(A_{11} + A_{12}) \diamond C_{12} \diamond P_1 + (A_{11} + A_{12}) \diamond \Phi(C_{12}) \diamond P_1 \\
&+(A_{11} + A_{12}) \diamond C_{12} \diamond \Phi(P_1) = \Phi((A_{11} + A_{12}) \diamond C_{12} \diamond P_1) \\
&= \Phi(A_{11} \diamond C_{12} \diamond P_1) + \Phi(A_{12} \diamond C_{12} \diamond P_1) \\
&= (\Phi(A_{11}) + \Phi(A_{12})) \diamond C_{12} \diamond P_1 + (A_{11} + A_{12}) \diamond \Phi(C_{12}) \diamond P_1 \\
&+(A_{11} + A_{12}) \diamond C_{12} \diamond \Phi(P_1).
\end{aligned}$$

Thus $T \diamond C_{12} \diamond P_1 = 0$ which implies that $-T_{11}^* C_{12} + C_{12}^* T_{11} = 0$. So, we get $T_{11}^* C_{12} = 0$. Therefore, for all $C \in \mathcal{A}$, we have $T_{11}^* C P_2 = 0$. As respects to the prime assumption of $\mathcal{A}$, we possess that $T_{11} = 0$. In a complete similar way, by applying P_2 instead of P_1 in above, we can show that $T_{12} = 0$.

Claim 6. *For each* $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}$ *and* $A_{22} \in \mathcal{A}_{22}$*, the following equalities hold:*

1. $\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.
2. $\Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})$.

We show that

$$T = \Phi(A_{11} + A_{12} + A_{21}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) = 0.$$

From Claim 5, we obtain

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond C_{21} \diamond I + (A_{11} + A_{12} + A_{21}) \diamond \Phi(C_{21}) \diamond I \\
&+(A_{11} + A_{12} + A_{21}) \diamond C_{21} \diamond \Phi(I) = \Phi(A_{11} + A_{21} + A_{12} \diamond C_{21} \diamond I) \\
&= \Phi(A_{11} \diamond C_{21} \diamond I) + \Phi(A_{21} \diamond C_{21} \diamond I) + \Phi(A_{12} \diamond C_{21} \diamond I) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond C_{21} \diamond I \\
&+(A_{11} + A_{12} + A_{21}) \diamond \Phi(C_{21}) \diamond I + (A_{11} + A_{12} + A_{21}) \diamond C_{21} \diamond \Phi(I).
\end{aligned}$$

It follows that $T \diamond C_{21} \diamond I = 0$. Since $T = T_{11} + T_{12} + T_{21} + T_{22}$, we have

$$-T_{22}^* C_{21} - T_{21}^* C_{21} + C_{21}^* T_{22} + C_{21}^* T_{21} = 0.$$

Therefor, $T_{22} = T_{21} = 0$.
Another application of previous claim, claim5, indicates

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond P_1 + (A_{11} + A_{12} + A_{21}) \diamond \Phi(P_1) \diamond P_1 \\
&+(A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond \Phi(P_1) = \Phi((A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond P_1) \\
&= \Phi(A_{11} \diamond P_1 \diamond P_1) + \Phi(A_{12} \diamond P_1 \diamond P_1) + \Phi(A_{21} \diamond P_1 \diamond P_1) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond P_1 \diamond P_1 + (A_{11} + A_{12} + A_{21}) \diamond \Phi(P_1) \diamond P_1 \\
&+(A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond \Phi(P_1).
\end{aligned}$$

which implies that $T \diamond P_1 \diamond P_1 = 0$. So $2T_{11} - 2T_{11}^* + T_{12} - T_{12}^* = 0$. Thus, we have

$$T_{12} = 0, T_{11} - T_{11}^* = 0. \tag{2.7}$$

In addition, from Claim 4 and Claim 5, we have

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond iP_1 \diamond I + (A_{11} + A_{12} + A_{21}) \diamond \Phi(iP_1) \diamond I \\
&+(A_{11} + A_{12} + A_{21}) \diamond iP_1 \diamond \Phi(I) = \Phi(A_{11} + A_{12} \diamond iP_1 \diamond I) \\
&+\Phi(A_{21} \diamond iP_1 \diamond I) = \Phi(A_{11} \diamond iP_1 \diamond I) + \Phi(A_{12} \diamond iP_1 \diamond I) \\
&+\Phi(A_{21} \diamond iP_1 \diamond I) = \Phi(A_{11}) \diamond iP_1 \diamond I + A_{11} \diamond \Phi(iP_1) \diamond I + A_{11} \diamond iP_1 \diamond \Phi(I) \\
&+\Phi(A_{12}) \diamond iP_1 \diamond I + A_{12} \diamond \Phi(iP_1) \diamond I + A_{12} \diamond iP_1 \diamond \Phi(I) \\
&+\Phi(A_{21}) \diamond iP_1 \diamond I + A_{21} \diamond \Phi(iP_1) \diamond I + A_{21} \diamond iP_1 \diamond \Phi(I) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond iP_1 \diamond I \\
&+(A_{11} + A_{12} + A_{21}) \diamond \Phi(iP_1) \diamond I + (A_{11} + A_{12} + A_{21}) \diamond iP_1 \diamond \Phi(I)
\end{aligned}$$

which shows that $T \diamond iP_1 \diamond I = 0$. Therefore,

$$-T_{11} - T_{11}^* = 0. \tag{2.8}$$

The equations (2.7) and (2.8) reveal $T_{11} = 0$.
Similarly, we can show that

$$\Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

Claim 7. *For each $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$ we have*

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

We need to show that

$$T = \Phi(A_{11} + A_{12} + A_{21} + A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) - \Phi(A_{22}) = 0.$$

By previous claim, i.e. claim 6, it is obvious that

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond I + (A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(C_{12}) \diamond I \\
&+(A_{11} + A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond \Phi(I) = \Phi((A_{11} + A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond I) \\
&= \Phi((A_{11} + A_{12} + A_{21}) \diamond C_{12} \diamond I) + \Phi(A_{22} \diamond C_{12} \diamond I) \\
&= \Phi(A_{11} \diamond C_{12} \diamond I) + \Phi(A_{12} \diamond C_{12} \diamond I) + \Phi(A_{21} \diamond C_{12} \diamond I) + \Phi(A_{22} \diamond C_{12} \diamond I) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})) \diamond C_{12} \diamond I \\
&+(A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(C_{12}) \diamond I + (A_{11} + A_{12} + A_{21} + A_{22}) \diamond C_{12} \diamond \Phi(I).
\end{aligned}$$

So, $T \diamond C_{12} \diamond I = 0$ which follows that

$$C_{12}^* T_{11} + C_{12}^* T_{12} - T_{11}^* C_{12} - T_{12}^* C_{12} = 0.$$

Thus $T_{11} = T_{12} = 0$.

Similarly, by applying C_{21} instead of C_{12} in recent computation, we obtain

$$T_{21} = T_{22} = 0.$$

Claim 8. *For each $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $i \neq j$, we have*

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

It is easy to show that

$$(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} = -A_{ij} - B_{ij} + A_{ij}^* + B_{ij}^*.$$

So, we can write

$$\Phi(-A_{ij} - B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) = \Phi((P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2})$$
$$= \Phi(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} + (P_i + A_{ij}^*) \diamond \Phi(P_j + B_{ij}) \diamond \frac{I}{2}$$
$$+(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \Phi(\frac{I}{2}) = (\Phi(P_i) + \Phi(A_{ij}^*)) \diamond (P_j + B_{ij}) \diamond \frac{I}{2}$$
$$+(P_i + A_{ij}^*) \diamond (\Phi(P_j) + \Phi(B_{ij})) \diamond \frac{I}{2} + (P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \Phi(\frac{I}{2})$$
$$= \Phi(P_i \diamond B_{ij} \diamond \frac{I}{2}) + \Phi(A_{ij}^* \diamond P_j \diamond \frac{I}{2})$$
$$= \Phi(-B_{ij}) + \Phi(B_{ij}^*) + \Phi(-A_{ij}) + \Phi(A_{ij}^*).$$

Therefore, we show that

$$\Phi(-A_{ij} - B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) = \Phi(-A_{ij}) + \Phi(-B_{ij}) + \Phi(A_{ij}^*) + \Phi(B_{ij}^*). \qquad (2.9)$$

A initial and easy computation explains that

$$(P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} = iA_{ij} + iB_{ij} + iA_{ij}^* + iB_{ij}^*.$$

Hereupon, we can compute that

$$\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi((P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2})$$
$$= \Phi(P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} + (P_i + A_{ij}^*) \diamond \Phi(iP_j + iB_{ij}) \diamond \frac{I}{2}$$
$$+(P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \Phi(\frac{I}{2}) = (\Phi(P_i) + \Phi(A_{ij}^*)) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2}$$
$$+(P_i + A_{ij}^*) \diamond (\Phi(iP_j) + \Phi(iB_{ij})) \diamond \frac{I}{2}$$
$$+(P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \Phi(\frac{I}{2})$$
$$= \Phi(P_i \diamond iB_{ij} \diamond \frac{I}{2}) + \Phi(A_{ij}^* \diamond iP_j \diamond \frac{I}{2}) = \Phi(iB_{ij}) + \Phi(iB_{ij}^*)$$
$$+\Phi(iA_{ij}) + \Phi(iA_{ij}^*).$$

Thus we conclude the following equality:

$$\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi(iA_{ij}) + \Phi(iB_{ij}) + \Phi(iA_{ij}^*) + \Phi(iB_{ij}^*).$$

On the other hand, claim 4 and the above equation implies that

$$-\Phi(A_{ij} + B_{ij}) - \Phi(A_{ij}^* + B_{ij}^*) = -\Phi(B_{ij}) - \Phi(B_{ij}^*) - \Phi(A_{ij}) - \Phi(A_{ij}^*). \qquad (2.10)$$

By collecting equations (2.9) and (2.10), we obtain

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

Claim 9. *For each $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $1 \leq i \leq 2$, we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Similar to the previous claim, we have to show that

$$T = \Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}) = 0.$$

An easy computation disclose that

$$\begin{aligned}
&\Phi(A_{ii} + B_{ii}) \diamond P_j \diamond I + (A_{ii} + B_{ii}) \diamond \Phi(P_j) \diamond I + (A_{ii} + B_{ii}) \diamond P_j \diamond \Phi(I) \\
&= \Phi((A_{ii} + B_{ii}) \diamond P_j \diamond I) = \Phi(A_{ii} \diamond P_j \diamond I) + \Phi(B_{ii} \diamond P_j \diamond I) \\
&= \Phi(A_{ii}) \diamond P_j \diamond I + A_{ii} \diamond \Phi(P_j) \diamond I + A_{ii} \diamond P_j \diamond \Phi(I) + \Phi(B_{ii}) \diamond P_j \diamond I \\
&+ B_{ii} \diamond \Phi(P_j) \diamond I + B_{ii} \diamond P_j \diamond \Phi(I) \\
&= (\Phi(A_{ii}) + \Phi(B_{ii})) \diamond P_j \diamond I + (A_{ii} + B_{ii}) \diamond \Phi(P_j) \diamond I + (A_{ii} + B_{ii}) \diamond P_j \diamond \Phi(I).
\end{aligned}$$

So, we have

$$T \diamond P_j \diamond I = 0.$$

Therefore, we obtain $T_{ij} = T_{ji} = T_{jj} = 0$.
On the other hand, for every $C_{ij} \in \mathcal{A}_{ij}$, we have

$$\begin{aligned}
&\Phi(A_{ii} + B_{ii}) \diamond C_{ij} \diamond I + (A_{ii} + B_{ii}) \diamond \Phi(C_{ij}) \diamond I \\
&+ (A_{ii} + B_{ii}) \diamond C_{ij} \diamond \Phi(I) \\
&= \Phi((A_{ii} + B_{ii}) \diamond C_{ij} \diamond I) = \Phi(A_{ii} \diamond C_{ij} \diamond I) \\
&+ \Phi(B_{ii} \diamond C_{ij} \diamond I) = (\Phi(A_{ii}) + \Phi(B_{ii})) \diamond C_{ij} \diamond I \\
&+ (A_{ii} + B_{ii}) \diamond \Phi(C_{ij}) \diamond I + (A_{ii} + B_{ii}) \diamond C_{ij} \diamond \Phi(I).
\end{aligned}$$

Recent equation shows that $T \diamond C_{ij} \diamond I = 0$ which implies $T_{ii} \diamond C_{ij} \diamond I = 0$. Thus, we have $-T_{ii}^* C_{ij} + C_{ij}^* T_{ii} = 0$. By recent equality and the fact that $\mathcal{A}$ is prime, we possess $T_{ii} = 0$.
Therefore, the additivity of Φ derives from the above claims.

In the rest of the paper, we show that Φ is $*$-derivation.

Claim 10. Φ *preserves star.*

Since $\Phi(\frac{I}{2}) = 0$ then we can write

$$\Phi\left(A \diamond \frac{I}{2} \diamond \frac{I}{2}\right) = \Phi(A) \diamond \frac{I}{2} \diamond \frac{I}{2}.$$

Then

$$\Phi(A - A^*) = \Phi(A) - \Phi(A)^*.$$

So, we showed that Φ preserves star.

Claim 11. Φ *is derivation.*

For every $A, B \in \mathcal{A}$, it is easy to check that

$$\begin{aligned}\Phi(-AB + B^*A^*) &= \Phi(A^* \diamond B \diamond \frac{I}{2}) = \Phi(A^*) \diamond B \diamond \frac{I}{2} + A^* \diamond \Phi(B) \diamond \frac{I}{2} \\ &= B^*\Phi(A)^* - \Phi(A)B - A\Phi(B) + \Phi(B)^*A^*\end{aligned}$$

So we have

$$\Phi(B^*A^* - AB) = \Phi(B)^*A^* + B^*\Phi(A)^* - \Phi(A)B - A\Phi(B) \tag{2.11}$$

An standard application of the Claims 4 and 10 shows that

$$\begin{aligned}&i\Phi(-B^*A^* - AB) = \Phi(-iA^* \diamond B \diamond \frac{I}{2}) = (-iA)^* \diamond B \diamond \frac{I}{2} + (-iA^*) \diamond \Phi(B) \diamond \frac{I}{2} \\ &= i(-\Phi(B)^*A^* - B^*\Phi(A)^* - \Phi(A)B - A\Phi(B))\end{aligned}$$

which implies

$$\Phi(-B^*A^* - AB) = -\Phi(B)^*A^* - B^*\Phi(A)^* - \Phi(A)B - A\Phi(B). \tag{2.12}$$

Thus, by (2.11) and (2.12), we conclude that

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

This completes the proof.

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Mohammad Shavandi
University of Mazandaran,
Faculty of Mathematical Sciences, Department of Mathematics,
P. O. Box 47416-1468, Babolsar, Iran.
e-mail: mo.shavandi@gmail.com

Ali Taghavi
Faculty of Mathematical Sciences, Department of Mathematics,
P. O. Box 47416-1468, Babolsar, Iran.
e-mail: taghavi@umz.ac.ir

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