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Frier, Christian; Damkilde, Lars

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# **Lower Bound Limit State Analysis using the Interior-Point Method with Spatial Varying Barrier Function**

**Christian Frier\* and Lars Damkilde**

Department of Civil Engineering  
Aalborg University, Aalborg, Denmark  
e-mail: cf@civil.aau.dk

**Summary** A method of conducting lower bound Limit State analysis is to apply the interior-point method. The aim of the paper is to refine the method by reducing the number of optimization variables considerably by eliminating the equilibrium equations a priori. Another new idea is to adapt a spatially varying barrier function. Significant gains are made regarding computational speed and robustness of the algorithm.

## **Introduction**

Limit State analysis has been used in design for decades e.g. the yield line theory for concrete slabs, [1]. The Limit State analysis is very well suited for manual methods especially the upper-bound methods and is therefore used in practical engineering design calculation. Analysis of elastic structures was around 1960 revolutionized by the introduction of computers and the Finite Element concept. Soon after the first attempts to solve Limit State problems by computers were implemented, see [2]. However, the methods did not penetrate into practice in the same impressive way as the linear Finite Element analysis did. The field of Computerized Limit State analysis did grow and extended the applications from frame and slabs also to include geotechnical problems, see e.g. [3] and reinforced plates, see e.g. [4]. In the last decade the main developments have been in the optimization procedure, where the interior point method in various formulations has increased the performance considerably, see e.g. [5].

The lower-bound formulation results in a non-linear convex optimization problem. The variables consist of the stress state in the elements and a load parameter. The object function will in this context be the load carrying capacity. The restrictions are linear equilibrium equations and non-linear convex yield criteria. The most effective solution methods are based on variants of Karmarkar's interior point method. In order to have a more efficient implementation two remedies can be used. The first is to eliminate the equality constraints a priori. This gives a considerable reduction in the number of variables. The method has in previous studies shown its capability, see [6]. The second is to deal with the non-linear yield criteria directly and in this respect avoiding the large number of linear inequalities, see e.g. [7]. Recently, both aspects have been implemented with success in [8]. In the present work the method is improved in terms of computational efficiency and improvements on the optimization algorithm. In the interior-point method a barrier function is used to ensure that the optimization variables stay feasible during the iterative solution process. It is suggested to use a spatial varying barrier function for which the barrier is different for each stress point. More details and further improvements on the optimization algorithm is to appear in [9]. In the paper the method is illustrated by a single example used by other researchers. However, the method is fully general and can be used for all types of limit state problems. The method is illustrated on a plane strain problem, but it is fully general.

## **Computational aspects**

A lower bound solution is a stress state where equilibrium is satisfied and the yield criteria are not violated. The problem is discretized by the traditional Finite Element concept with stress-based

elements, and in this context only plane strain problems are considered. A triangular element with 9 stress parameters first formulated by Sloan is used, see [3], with the formulation from [10] adapted. The lower bound optimization problem can be formulated as:

$$\begin{aligned} & \text{maximize} && \alpha \\ & \text{subject to} && \mathbf{H}\boldsymbol{\beta}_s = \alpha\mathbf{R} + \mathbf{R}_0 \\ & && f_j(\boldsymbol{\beta}_j) \leq 0, \quad j = 1, 2, \dots, p \end{aligned} \quad (1)$$

where  $\mathbf{H}$  is the global flexibility matrix,  $\boldsymbol{\beta}_s$  are stress parameters for the whole system,  $\mathbf{R}$  are global nodal forces scalable by the load parameter,  $\alpha$ , and  $\mathbf{R}_0$  are constant global nodal forces independent of the load parameter.  $f_j$  are non-linear yield criteria evaluated in the  $j$ th stress point with stress parameters,  $\boldsymbol{\beta}_j$ , of  $p$  in total.

In order to reduce the problem size and improve the numerical stability the equilibrium equations can be eliminated a priori. The elimination is a standard Gauss elimination which reduces the number of independent stress parameters from  $\boldsymbol{\beta}_s$  to  $\boldsymbol{\beta}_f$ , the so-called free stress parameters. The relation between the stress variables can be written:

$$\boldsymbol{\beta}_s = \mathbf{B}\boldsymbol{\beta} + \mathbf{c} \quad (2)$$

where  $\mathbf{B}$  is a matrix and  $\mathbf{c}$  is a vector of constant elements relating the free stress variables,  $\boldsymbol{\beta} = [\boldsymbol{\beta}_f \ \alpha]^T$ , to the entire set of stress variables,  $\boldsymbol{\beta}_s$ , which are obtained during the Gauss elimination process. Note that the load multiplier for the sake of convenience has been included in the set of stress variables.

The optimization problem, can be solved by the interior-point method. A barrier function,  $\mu_j$ , is added to the objective function in (1), see e.g. [11]. Furthermore, non-negative Slack variables,  $\mathbf{s}$ , are added to transform the non-linear inequality constrains into equality constrains. The Lagrangian of the augmented optimization problem can then be formulated:

$$\mathcal{L}(\boldsymbol{\beta}, \mathbf{s}, \boldsymbol{\lambda}) = \mathbf{b}^T \boldsymbol{\beta} + \sum_{j=1}^p \mu_j \log s_j - \boldsymbol{\lambda}^T (\mathbf{f}(\boldsymbol{\beta}) + \mathbf{s}) \quad (3)$$

where  $\mathbf{b} = [\mathbf{0} \ 1]^T$  and  $\mathbf{f}$  is the vector of the yield criteria, evaluated in all material points,  $\mathbf{0}$  is a vector of zeros and  $\boldsymbol{\lambda}$  is a vector of non-negative Lagrange multipliers.

The idea behind the barrier function is to prevent the gradient search process to end too close to the boundary. A new idea in the present work is to use a barrier function which differs between the stress points, thus hopefully increasing the convergence rate of the algorithm. The barrier functions are chosen as either of the following:

$$\mu_j = c\delta^k, \quad \mu_j = c [\max(s_j - s_{max}, 0.1)\delta]^k \quad (4)$$

where  $c$  is a scaling factor,  $\delta$  is a constant controlling the speed by which the barrier is reduced, and  $k$  is the iteration number. The constant  $c$  is chosen such that the initial barrier parameter is just below one, in this work  $c = 0.95$  and  $\delta = 0.7$  is chosen.  $s_{max}$  is the largest slack variable.

The Kuhn-Tucker conditions states that the gradient of the Lagrangian must vanish at the optimum. By differentiation of (3), a non-linear equation system is to be solved for variables  $\beta$ ,  $s$  and  $\lambda$ . This can be done by Newton's method, where increments on the variables are found iteratively. During the iterations, the barrier function, (4), is reduced and the iterations are started from an initial feasible point, i.e.  $\beta = \mathbf{0}$ ,  $s = \mathbf{e}$  and  $\lambda = \mathbf{e}$ . Here,  $\mathbf{e}$  is a vector of ones. Line search is conducted in order for the increments to be feasible, i.e. non-negative values of  $s$  and  $\lambda$  and the stress state,  $\beta$ , must be within the yield criteria. After calculating the increments, they are multiplied by a factor below one, in this work 0.8 is used. The iterations are stopped when the duality gap between the slack variables,  $s$ , and the lagrange multipliers,  $\lambda$ , becomes sufficiently small.

### Numerical example

As a test example the slotted block in plane strain, shown in Figure 1 is considered.

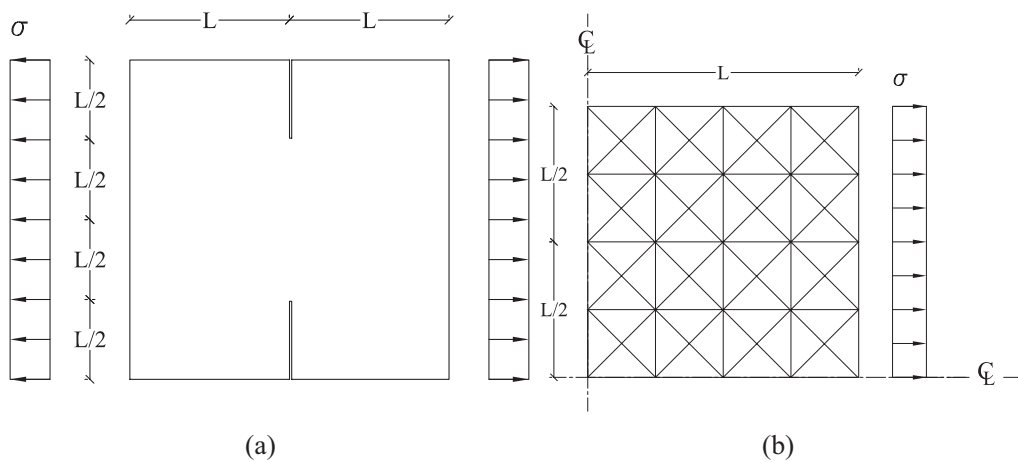


Figure 1: Slotted block problem (a) and element discretization,  $N = 4$  (b).

The example has been treated by Andersen and Christiansen [12] and by Krabbenhoft and Damkilde [7]. The square block has two notches as shown in Figure 1.(a). The material is governed by the von Mises yield criterion in plane strain, with a yield stress  $f_0 = \sqrt{3}$ . In Figure 2 is shown the result of the optimization process in terms of the convergence of the load multiplier  $\alpha$  as a function of the iteration number. Results are shown for both  $N = 4$ , as shown in Figure 1.(b), and for  $N = 12$ . The optimization process has been conducted with both the conventional constant barrier function and the new spatial varying barrier function in (4). It can be observed, that the load multiplier converges in all cases towards a value that does not differ much, suggesting that the  $N = 12$  discretization is adequate in the present case. However this might not be a general conclusion for other structures. An interesting conclusion is, that the convergence is faster when using the spatial varying barrier function, suggesting that it is favorable.

### Concluding remarks

In this paper the interior-point method is used to conduct Limit State analysis with the lower bound method for structural problems in plane strain. Focus is on improvements on the optimization algorithm in two different aspects. First, the equality constraints are eliminated prior to the optimization, reducing the number of optimization variables and constraints. Secondly, a spatially varying barrier function is suggested in order to speed up the convergence of the algorithm. Both suggestions improve the convergence of the optimization algorithm.

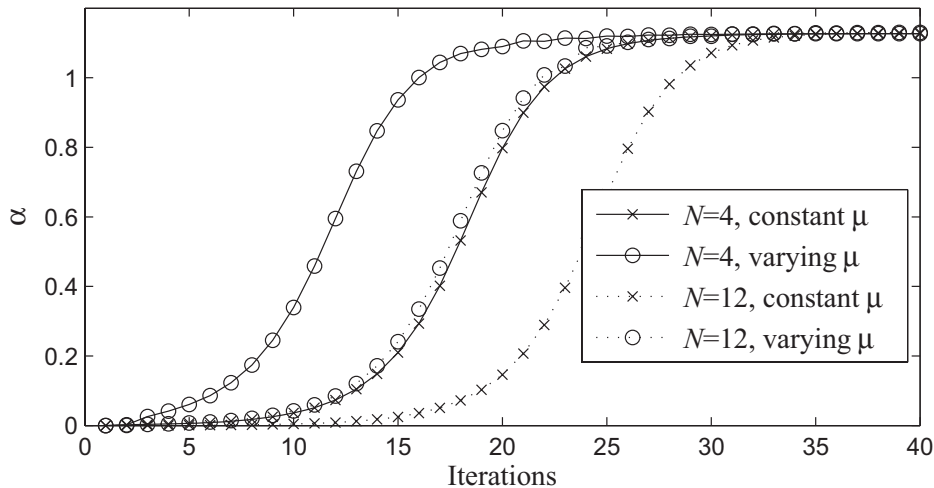


Figure 2: Convergence of load multiplier, computed for  $N = 4$  and  $N = 12$  as shown in Figure 1.b. Both a constant and a spatial varying barrier function is considered.

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