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Torben Knudsen

Abstract— **This paper studies the data driven update of a model for a system where the number of inputs or outputs increased. Often existing control systems are equipped with an additional sensor or actuator to improve performance. If a good model for the present system is available it is advantageous to only estimate the additional part while keeping the present model, compared to estimating the whole model from scratch. The capabilities with convex methods are investigated. It is shown that model updating for static sensor/actuators can be done consistently for the deterministic part. The stochastic part is far more complicated and here convex methods gives a approximate solution. The total solution is demonstrated by simulation to improve state prediction and control performance.**

I. INTRODUCTION

To start with an example: Consider temperature control in e.g. a livestock stable. According to Skov A/S (a Danish company supplying climate control systems for stables), sometimes the climate in typically smaller parts of a stable are not acceptable after the commissioning of a standard control system. This can be due to special constructions in the specific stable. A remedy will often be to install an extra temperature sensor, air heater or ventilation device. However this requires new time consuming manual tuning. In the research project "Plug and Play Process Control" ($\mathrm{P}^3\mathrm{C}$) [1] the main idea is to develop general methods for automating this task. A subtask will then be to update the model with the additional device.

Assuming model based control design there will be a model for the present system. Estimating the model from scratch has disadvantages. For example there would be a risk of getting an inferior model for the present system compared to the already available one. Also if a new model is estimated from scratch by standard methods, e.g. *pem* from the matlab toolbox ident, the state space basis will change. This is undesirable as for example a present state feedback controller then can not be used. Consequently, the aim is to only estimate the parameters necessary to augments the present model with the new device.

To avoid problems with poor models due to local minima and problems with long execution time the choice is to see what results can be obtained using robust convex methods as e.g. least squares (LS) methods.

Incremental modelling in the sense that the system is fixed but the model is improved in a incrementally fashion is discussed in the literature [2], [3]. However, the problem here is different because the system is not fixed but increases in the number of inputs or outputs. This case is not found in the literature. Another related but different problem is in fault tolerant control where sensors or actuators fails i.e. disappears.

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The paper starts with presenting the model setup and notation. Then the LS methods for a additional input is developed. This is followed by LS methods for an additional output. This includes a development of a convex method to estimate an approximate stochastic model. A simulation example is then given followed by a conclusion.

The main contribution is the data driven incremental modelling by LS/convex methods especially the complicated stochastic part. This is to the authors knowledge all new. A preliminary version is included in an application paper [4]. However this preliminary version does not include consistency proofs and estimation for the stochastic part neither does it include the analysis presented in this paper explaining the complications regarding the Kalman gain update.

II. MODEL SETUP

As this is intended for multiple input multiple output (MIMO) systems a state space (SS) model seems the best choice. In this parametrization the additional parameters will appear as new rows or columns augmented to the present matrix parameters. The present parameters is left unchanged except for the Kalman gain if a additional output is added.

For the analysis two forms of the SS model are needed. First in basic form (1) where x_p is assumed to be "physical" states in the sense that w_p only includes unmeasured inputs/disturbances and v_p only includes the measurement noise. In this case it is reasonable to assume that the measurement noise v_p will be uncorrelated with the process noise w_p i.e. $R_{wv} = \underline{\underline{0}}$. Both noise processes are assumed white. Subscript p is for present model.

$$x_p(t+1) = A_p x_p(t) + B_p u_p(t) + w_p(t) \; , \tag{1a}$$

$$y_p(t) = C_p x_p(t) + D_p u_p(t) + v_p(t) \; , \tag{1b}$$

$$\mathrm{Cov}\begin{pmatrix} w_p \\ v_p \end{pmatrix} = \begin{pmatrix} R_w & R_{wv} \\ R_{vw} & R_v \end{pmatrix} \tag{1c}$$

$$u_p \in \mathcal{R}^m \; , \; x_p, w_p \in \mathcal{R}^n \; , \; y_p, v_p \in \mathcal{R}^l \tag{1d}$$

The other version needed is the innovation model (IM) (2).

$$\hat{x}_p(t+1) = A_p \hat{x}_p(t) + B_p u_p(t) + K_p e_p(t) \; , \tag{2a}$$

$$\hat{x}_p(t) \triangleq \mathrm{E}(x_p(t)|Y_p^{t-1}) \; , \tag{2b}$$

$$Y_p^t \triangleq y_p(t), y_p(t-1), \dots \; , \tag{2c}$$

$$y_p(t) = C_p \hat{x}_p(t) + D_p u_p(t) + e_p(t) \; , \tag{2d}$$

$$\mathrm{Cov}(e_p) = R_e \; , \; \mathrm{E}(e_p(t) e_p(s)^{\mathrm{T}}) = \underline{\underline{0}} \; , \; t \neq s \tag{2e}$$

$$\hat{y}_p(t) = C_p \hat{x}_p(t) + D_p u_p(t) \tag{2f}$$

$$e_p \in \mathcal{R}^l \tag{2g}$$

In this work (stochastic) stationarity is assumed. This gives the following basic properties which are used in the below development. The state prediction error is uncorrelated with previous measurement but the state prediction error is non white.

$$\tilde{x}_p(t) \triangleq x_p(t) - \hat{x}_p(t) \Rightarrow \tilde{x}_p(t), Y_p^{t-1} \text{ uncorrelated} \quad (3)$$

The output prediction error is uncorrelated with previous measurement and it is white noise.

$$\tilde{y}_p(t) \triangleq y_p(t) - \hat{y}_p(t) = e_p(t) \Rightarrow \quad (4)$$

$$\tilde{y}_p(t), Y_p^{t-1} \text{ uncorr.} \Rightarrow \tilde{y}_p(t), \tilde{y}_p(t-1), \ldots \text{ uncorr.} \quad (5)$$

It is assumed that the present model is known in its innovation form. Further the measurements available is assumed to be output only not states. The experimental conditions can be open loop (OL) or closed loop (CL) both with sufficient excitation.

III. ADDITIONAL INPUT

In the standard SS model, input is not assumed to be noise corrupted. This means that the new input part just has to be added to the otherwise unchanged IM (6). Notice especially that K_p is unchanged. The SS IM can be divided as (6a)–(6b) where subscript p and a means present and additional respectively.

$$x_p(t+1) = A_p x_p(t) + \begin{pmatrix} B_p & B_a \end{pmatrix} \begin{pmatrix} u_p(t) \\ u_a(t) \end{pmatrix} + K_p e_p(t) \quad (6a)$$

$$y_p(t) = C_p x_p(t) + \begin{pmatrix} D_p & D_a \end{pmatrix} \begin{pmatrix} u_p(t) \\ u_a(t) \end{pmatrix} + e_p(t) \, , \quad (6b)$$

$$R_p = \mathrm{Cov}(e_p)$$

$$u_p \in \mathcal{R}^m \, , \; u_a \in \mathcal{R} \, , \; x_p \in \mathcal{R}^n \, , \; y_p, e_p \in \mathcal{R}^l \quad (6c)$$

Then it is only necessary to estimate B_a, D_a. Notice that the predicted output are linear in these parameters as the Kalman filter predictor can be written as (7).

$$\begin{aligned} \hat{x}_p(t+1) &= (A_p - K_p C_p)\hat{x}_p(t) + K_p y_p(t) \\ &+ \left[\begin{pmatrix} B_p & B_a \end{pmatrix} - K_p \begin{pmatrix} D_p & D_a \end{pmatrix} \right] \begin{pmatrix} u_p(t) \\ u_a(t) \end{pmatrix} \end{aligned} \quad (7a)$$

$$\hat{y}_p(t) = C_p \hat{x}_p(t) + \begin{pmatrix} D_p & D_a \end{pmatrix} \begin{pmatrix} u_p(t) \\ u_a(t) \end{pmatrix} \quad (7b)$$

This means the output can be separated in a part from the present system and a linear combination of parts assuming that each new parameter θ_i (8) is one while the rest are zero.

$$\begin{pmatrix} \theta_1 & \ldots & \theta_n \end{pmatrix}^{\mathrm{T}} \triangleq B_a \, , \; \begin{pmatrix} \theta_{n+1} & \ldots & \theta_{n+l} \end{pmatrix}^{\mathrm{T}} \triangleq D_a \quad (8)$$

Define $\hat{y}_0$ as the predicted output from the present system i.e. where all additional parameters i.e B_a, D_a are zero and $\hat{y}_i$ as the predicted output where the present parameters B_p, D_p are zero and all additional parameters (8) are zero except number i which is one. Then the predicted output is a linear combination of these signals (9). Consequently, the measured output is given by (10) where e_p is the innovation.

$$\hat{y}_p(t) = \hat{y}_0(t) + \sum_{i=1}^{n+l} \theta_i \hat{y}_i(t) \Rightarrow \quad (9)$$

$$y_p(t) = \hat{y}_0(t) + \sum_{i=1}^{n+l} \theta_i \hat{y}_i(t) + e_p(t) \quad (10)$$

If signals for the whole measurement sequence are stacked into vectors and some more notation is introduced the following results can be obtained.

Theorem 1: (LS estimator for additional input) Assume a innovation model for the present system is known and the separation of the output predictor in (9) is used then the prediction error method reduces to a LS estimate (11).

$$\widehat{\Theta} = (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} Z \, , \quad (11)$$

$$\begin{aligned} &X \triangleq \begin{pmatrix} \widehat{Y}_1 & \cdots & \widehat{Y}_{n+l} \end{pmatrix} \, , \; Z \triangleq Y_p - \widehat{Y}_0 \, , \\ &Y_p \triangleq \begin{pmatrix} y_p(1) \\ \vdots \\ y_p(N) \end{pmatrix} \, , \; \widehat{Y}_i \triangleq \begin{pmatrix} \hat{y}_i(1) \\ \vdots \\ \hat{y}_i(N) \end{pmatrix} \, , \\ &\Theta \triangleq \begin{pmatrix} \theta_1 & \ldots & \theta_{n+l} \end{pmatrix}^{\mathrm{T}} \end{aligned} \quad (12)$$

Proof: Using the vector notation (12) the stacked output can be written as (13) which follows from (10). The definition of E_p is similar to (12). Further (13) can be turned into the multiple linear regression form (14) where it is well known [5, App. II.1] that the estimator minimizing the sum of squared prediction errors is given by (11).

$$Y_p = \widehat{Y}_0 + \widehat{Y}_1 \theta_1 + \cdots + \theta_{n+l} \widehat{Y}_{n+l} + E_p \Rightarrow \quad (13)$$

$$Z = Y_p - \widehat{Y}_0 = X\Theta + E_p \quad (14)$$

■

Theorem 2: (Consistent LS estimator for additional input) The LS estimator in theorem 1 is consistent in open as well as closed loop operation provided there is at least one time delay from output to input in the latter case.

Proof: Using the model equation (14) the estimator can be related to the parameters as seen in (15a). The limit value w.p.1 for $N \to \infty$ is (15b). The step from (15a) to (15b) follows from ergodicity which again follows from the stationarity assumption.

$$\begin{aligned} \widehat{\Theta} &= (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} Z \\ &= (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} (X\Theta + E_p) \\ &= \Theta + (X^{\mathrm{T}} X)^{-1} X^{\mathrm{T}} E_p \\ &= \Theta + \left(\frac{1}{Nl} X^{\mathrm{T}} X \right)^{-1} \frac{1}{Nl} X^{\mathrm{T}} E_p \quad (15a) \\ &\to \Theta + \left[\mathrm{E}\left(\frac{1}{Nl} X^{\mathrm{T}} X \right) \right]^{-1} \mathrm{E}\left(\frac{1}{Nl} X^{\mathrm{T}} E_p \right) \quad (15b) \\ &\quad \text{for } N \to \infty \qquad \text{(wp1)} \end{aligned}$$

The part $\mathrm{E}\left(\frac{1}{Nl} X^{\mathrm{T}} X\right)$ is invertible due to sufficient excitation (see also [5, App. II.2]). If the rows in X and the rows in E_p are uncorrelated the last term in (15) will go to zero. This term is a $n+l$ vector with element i given by (16).

$$\frac{1}{Nl} X^{\mathrm{T}} E_p = \frac{1}{Nl} \begin{pmatrix} \widehat{Y}_1 & \cdots & \widehat{Y}_{n+l} \end{pmatrix}^{\mathrm{T}} E_p = \frac{1}{Nl} \begin{pmatrix} \widehat{Y}_1^{\mathrm{T}} E_p \\ \vdots \\ \widehat{Y}_{n+l}^{\mathrm{T}} E_p \end{pmatrix} \, ,$$

$$\begin{aligned} &\frac{1}{Nl} \widehat{Y}_i^{\mathrm{T}} E_p = \frac{1}{Nl} \sum_{t=1}^{N} \sum_{j=1}^{l} \hat{y}_{ij}(t) e_{p,j}(t) \, , \\ &i = 1, \ldots, n+l \end{aligned} \quad (16)$$

Now, $\hat{y}_{ij}(t)$ is predictor part i output channel j at time t which is generated from inputs and outputs until and including time $t-1$ plus $u(t)$ for $D \neq \underline{\underline{0}}$ and these are uncorrelated with the innovation $e_{p,j}(t)$ even in closed loop as at least one time delay from output to input is assumed. Therefore the last vector $\mathrm{E}\left(\frac{1}{Nl}X^{\mathrm{T}}E_p\right)$ goes to zero w.p.1 and consequently so does the last term in (15) ∎

IV. ADDITIONAL OUTPUT

In contrast to additional input additional output is corrupted with measurement noise. The necessary augmentation to the model is then (17).

$$y_a(t) = C_a x_p(t) + D_a u_p(t) + v_a(t)\ , \tag{17a}$$

$$\mathrm{Cov}\begin{pmatrix} w_p \\ v_p \\ v_a \end{pmatrix} = \begin{pmatrix} R_w & R_{wv} & R_{wa} \\ R_{vw} & R_v & R_{va} \\ R_{aw} & R_{av} & R_a \end{pmatrix} \tag{17b}$$

Notice the important detail that y_a is related to the *physical* state x_p not the predicted state $\hat{x}_p$. As already mentioned this also means that it is reasonable to assume the measurement noise for the additional output v_a to be uncorrelated with the other noise sources i.e. $R_{a\bullet} = R_{\bullet a}^{\mathrm{T}} = \underline{\underline{0}}$.

A. LS Estimates of the Deterministic Part

If the state x_p is measured all parameters including the covariances (17b) can be estimated by the LS method based on (1a), (1b) and (17a).

As x_p is not assumed measured the prediction $\hat{x}_p$ must be used. This gives the regression equations below.

$$\begin{aligned} y_a(t) &= C_a x_p(t) + D_a u_p(t) + v_a(t)\ , \\ &= C_a \hat{x}_p(t) + D_a u_p(t) + C_a(x_p(t) - \hat{x}_p(t)) + v_a(t) \end{aligned} \tag{18}$$

The following results can now be obtained.

Theorem 3: (LS estimator for additional output, deterministic part) Assume a known innovation model for the present system then a LS estimator for the deterministic part is (19).

$$\hat{\theta} = \left(\sum_{t=1}^{N} \phi(t)\phi(t)^{\mathrm{T}}\right)^{-1} \sum_{t=1}^{N} \phi(t) y_a(t)\ , \tag{19}$$

$$\phi(t) = \begin{pmatrix} \hat{x}_p(t) \\ u_p(t) \end{pmatrix}\ ,\ \theta = \begin{pmatrix} C_a & D_a \end{pmatrix}^{\mathrm{T}} \tag{20}$$

Proof: The additional output can be written as (22) where it is well known that (19) minimizes the sum of squares (23).

$$r(t) \triangleq C_a \tilde{x}_p(t) + v_a(t)\ , \tag{21}$$

$$y_a(t) = \phi(t)^{\mathrm{T}}\theta + r(t) \Rightarrow \tag{22}$$

$$\hat{\theta} = \arg\min_{\theta} \sum_{t=1}^{N} (y_a(t) - \phi(t)^{\mathrm{T}}\theta)^2 \tag{23}$$

∎

Remark 3.1: Notice that (19) is not a PEM estimator as $\phi(t)^{\mathrm{T}}\theta$ is not the optimal predictor as it does not use y_a and accordingly the residuals is non white.

Theorem 4: (Consistent LS estimator for additional output, deterministic part) The LS estimator in theorem 3 is consistent under the same assumption as in theorem 2.

Proof: This proof is omitted to save space as it build on similar principles as the proof for the additional input theorem 2. ∎

B. Estimates of the Stochastic Part

The stochastic part can be specified in two ways. It can be based on the physical model then the parameter to estimate is the covariance (17b) for all noise involved. Or the innovation model can be used then a new K and R_e parameter including the additional output must be estimated.

It is crucial to understand that these two representations are only equivalent regarding the stochastic for the output. From the physical model the covariance for both state x_p and output can be calculated and the innovation model can be found. From the innovation model only the output covariance can be calculated and there is no general transformation back to the physical model. Consequently, having only the innovation model for the present system, stochastic specifications for the physical state are lacking.

1) Know Present "Physical" Noise Model: Consider first the case with know "physical" noise model i.e. the covariance (1c) known. Assume also the additional measurement noise to be uncorrelated which the present noise then only R_a is missing in the stochastic specification (17b). This variance can be estimated by (26) as $\tilde{x}_p$ and v_a are uncorrelated. The state prediction error covariance $R_{\tilde{x}_p}$ is found by the Kalman filter Riccati equation for the present system and $\widehat{R}_r$ is the estimated variance for the residuals r from the LS step.

$$r(t) = C_a \tilde{x}_p(t) + v_a(t) \Rightarrow \tag{24}$$

$$R_r = C_a R_{\tilde{x}_p} C_a^{\mathrm{T}} + R_a \tag{25}$$

$$\widehat{R}_a = \widehat{R}_r - C_a R_{\tilde{x}_p} C_a^{\mathrm{T}} \tag{26}$$

2) Only Present Innovation Model Known: When only the innovation model is known a first idea could be to improve the state estimate by including the additional output. This is however not possible. The reason is that the innovation model uses the one step predictor for the physical state as the state and consequently the state prediction error is zero. Therefore no additional outputs will be able to improve this state prediction and the Kalman gains from such additional outputs will be zero. As the output equation would be (18) the output noise will be $r(t)$ (21) which is non white due to $\tilde{x}_p(t)$. The best estimate for the variance on the output noise would then be $\widehat{R}_r$.

Using such a updated model it is possible to control the additional output which was not controlled before. However, it is not possible to improve control of present state or output as there is no "connection" in the updated model between the additional output and the present state or output.

All this of cause does not mean that additional output can not improve the prediction of the physical state x_p. It rather means that the improvement is not in $\hat{x}_p$ but rather in $\tilde{x}_p$ as formulated in (28). Here superscript a indicates that the predictor is also based on the additional output.

$$\hat{x}_p^a(t) \triangleq \mathrm{E}(x_p(t) | Y_p^{t-1}, Y_a^{t-1}) \Rightarrow \tag{27}$$

$$\begin{aligned} \hat{x}_p^a(t) &= \mathrm{E}(x_p(t) | Y_p^{t-1}, Y_a^{t-1}) \\ &= \mathrm{E}(\hat{x}_p(t) + \tilde{x}_p(t) | Y_p^{t-1}, Y_a^{t-1}) \\ &= \mathrm{E}(\hat{x}_p(t) | Y_p^{t-1}, Y_a^{t-1}) + \mathrm{E}(\tilde{x}_p(t) | Y_p^{t-1}, Y_a^{t-1}) \\ &= \hat{x}_p(t) + \mathrm{E}(\tilde{x}_p(t) | Y_p^{t-1}, Y_a^{t-1}) \\ &= \hat{x}_p(t) + \widehat{\tilde{x}}_p^a(t) \end{aligned} \tag{28}$$

Now, the IM is just one representation of the stochastic properties of the output. It has some good properties e.g. few parameters but it is unsuitable for incremental modelling of the stochastic part. A idea is then to try and find a representation which is better.

When the physical state is unknown only the stochastic properties of the output is given by the known IM. A SS model does only specify the second order stochastic properties therefore an alternative representation of the stochastic must give the same auto covariance function for the output. The method is then to calculate the output auto covariance from the IM and then to find another representation that gives the same output auto covariance. As this other representation is not at all unique it must be further specified. Here a physical system like (1) is assumed with a state which everything else depends on. Some of the inputs are measured the other inputs are included as process noise. With this setting it is natural to assume that all the measured outputs are a function of the physical state plus some measurement noise which is uncorrelated with the process noise. Consequently a representation with $R_{wv} = \underline{\underline{0}}$ is selected. Normally this is still not unique and then a representation with minimal maximal singular value for the state covariance $R_x(0)$ is selected.

Below the output auto covariance for the SS model (1) is calculated. Only the stationary properties are needed. Therefore the deterministic/mean part is left out and the mean state is assumed to be zero. Also the p index is left out. The following state relation can be the starting point.

$$\begin{aligned} x(t+k) &= Ax(t+k-1) + w(t+k-1) \\ &= A^k x(t) + \sum_{s=t}^{t+k-1} A^{t+k-1-s} w(s) \quad \Rightarrow \end{aligned} \tag{29}$$

$$R_x(k) \triangleq \mathrm{E}(x(t+k)x(t)^{\mathrm{T}}) = A^k R_x(0)\ ,\ k \geq 0\ , \tag{30}$$

$$R_x(k) = (R_x(-k))^{\mathrm{T}} = R_x(0)(A^{|k|})^{\mathrm{T}}\ ,\ k < 0 \tag{31}$$

If the system is stable $R_x(0)$ is the solution to the linear Lyapunov equation:

$$R_x(0) = AR_x(0)A^{\mathrm{T}} + R_w \tag{32}$$

The cross covariance from output to state is:

$$\begin{aligned} R_{yx}(k) &\triangleq \mathrm{E}(y(t+k)x(t)^{\mathrm{T}}) \\ &= \mathrm{E}([Cx(t+k) + v(t+k)]x(t)^{\mathrm{T}}) \\ &= CR_x(k) + \begin{cases} 0 & ,\ k \geq 0 \\ R_{vw}(A^{|k|-1})^{\mathrm{T}} & ,\ k < 0 \end{cases} \end{aligned} \tag{33}$$

All this finally gives the auto covariance for the output:

$$\begin{aligned} R_y(k) &\triangleq \mathrm{E}(y(t+k)y(t)^{\mathrm{T}}) \\ &= \mathrm{E}(y(t+k)[Cx(t) + v(t)]^{\mathrm{T}}) \\ &= R_{yx}(k)C^{\mathrm{T}} + \begin{cases} CA^{|k|-1}R_{wv} & ,\ k \geq 1 \\ R_v & ,\ k = 0 \\ 0 & ,\ k \leq -1 \end{cases} \\ &= CR_x(k)C^{\mathrm{T}} + \begin{cases} CA^{|k|-1}R_{wv} & ,\ k \geq 1 \\ R_v & ,\ k = 0 \\ R_{vw}(A^{|k|-1})^{\mathrm{T}}C^{\mathrm{T}} & ,\ k \leq -1 \end{cases} \end{aligned} \tag{34}$$

Based on the above the output auto covariance for any SS model can be calculated as long as all the parameters are known. To calculate it for a known IM model for the present system the IM model (2) parameters K_p and R_e must be transformed into the standard form (1) as follows.

$$\begin{aligned} \begin{pmatrix} R_w & R_{wv} \\ R_{vw} & R_v \end{pmatrix} &= \mathrm{Cov}\begin{pmatrix} w_p \\ v_p \end{pmatrix} \\ &= \mathrm{Cov}\begin{pmatrix} K_p e_p \\ e_p \end{pmatrix} = \begin{pmatrix} K_p R_e K_p^{\mathrm{T}} & K_p R_e \\ R_e K_p^{\mathrm{T}} & R_e \end{pmatrix} \end{aligned} \tag{35}$$

With these parameters $R_y(k)\ ,\ k = 0, \ldots, n$ are calculated.

To find the alternative representation with $R_{wv} = \underline{\underline{0}}$ the above relations (30), (32) and (34) are used again. First some definitions are needed.

$$\Gamma_i^j \triangleq \begin{pmatrix} CA^i \\ CA^{i+1} \\ \vdots \\ CA^j \end{pmatrix} \tag{36}$$

$$R_{y}{}_i^j \triangleq \begin{pmatrix} R_y(i) \\ R_y(i+1) \\ \vdots \\ R_y(j) \end{pmatrix} \tag{37}$$

Then $R_x(0)C^{\mathrm{T}}$ can be found from (34) by (38) because Γ_1^n has full column rank when the system is observable.

$$\Gamma_1^n R_x(0)C^{\mathrm{T}} = R_{y}{}_1^n \Leftrightarrow R_x(0)C^{\mathrm{T}} = \Gamma_1^{n\,\dagger} R_{y}{}_1^n \tag{38}$$

R_v is now found by the above and (34).

$$R_y(0) = CR_x(0)C^{\mathrm{T}} + R_v \Leftrightarrow \tag{39}$$

$$\begin{aligned} R_v &= R_y(0) - CR_x(0)C^{\mathrm{T}} \\ &= R_y(0) - C\Gamma_1^{n\,\dagger} R_{y}{}_1^n \end{aligned} \tag{40}$$

Now only R_w is missing. It is given by the Lyapunov equation (32) if $R_x(0)$ is given. $R_x(0)$ must therefore be found. The necessary conditions for $R_x(0)$ are the following, symmetric, positive definite, given positive definite R_w and given right $R_y(k)$.

$$R_x(0) = R_x(0)^{\mathrm{T}} \tag{41a}$$

$$R_x(0) \succ 0 \tag{41b}$$

$$R_x(0) - AR_x(0)A^{\mathrm{T}} \succ 0 \tag{41c}$$

$$R_x(0)C^{\mathrm{T}} = \Gamma_1^{n\,\dagger} R_{y}{}_1^n \tag{41d}$$

The necessary conditions (41) sometimes gives a unique solution for $R_x(0)$. This is the case if C^{T} in (41d) has full column rank i.e. if $l \geq n$ and C has full rank then $R_x(0)$ is uniquely given by (41d). However, in general (41) does not give a unique solution. From simulation experiments it seems to be and advantage to chose a "small" $R_x(0)$. This can be done by LMI using the Schur compliment. The maximal singular value of $R_x(0)$ is minimized by (42).

$$\min_{R_x(0)} \gamma \quad \text{s.t.} \quad \begin{pmatrix} \gamma I & R_x(0) \\ R_x(0)^{\mathrm{T}} & \gamma I \end{pmatrix} \succ 0 \tag{42}$$

Collecting the above a $R_x(0)$ can be found from the LMI consisting of the LMI optimization problem (42), the LMI constraints (41b), (41c) and the linear constraint (41d).

Having obtained a representation with R_w, R_v and $R_{wv} = \underline{\underline{0}}$ which is consistent with the output auto covariance the model can be updated with the method in section IV-B.1. It is however uncertain how close this representation is to the one generating x_p and y_a. Notice that the system must be stable to achieve the assumed (stochastic) stationarity.

The rationale behind using LMI methods for the solution is that they boil down to a convex optimization problem and there exists robust numerical methods for it exactly as for LS methods. This then still avoids non convex iterative optimization methods.

V. SIMULATION EXAMPLE

As there are no problems in estimating parameters for an additional input the methods for additional output will be exemplified.

The method above has been tested by simulation on a number of random systems of various order and number of input and outputs. This proved that the method in general works, as it improves the state estimate or leaves it unchanged. Also the parameter estimates seems to be consistent in both open and closed loop at least for the deterministic part.

To show an example where there is a expected effect of a additional output measurement a system has been constructed. The deterministic part is a zero order hold sampled version of the 2 order continuous time system (43).

$$\dot{x}(t) = Ax(t) + Bu(t) \,, \tag{43a}$$

$$y(t) = Cx(t) + Du(t) \,, \tag{43b}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -0.1 & -0.01 & 0.01 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \tag{43c}$$

Using the notation from (1) this gives the discrete time system (44) where the stochastic part also is chosen.

$$\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] = \left[\begin{array}{cc|c} 0.9002 & -0.0095 & 0.0095 \\ 0.95 & 0.9952 & 0.004833 \\ \hline 1 & 0 & 0 \end{array}\right] \tag{44a}$$

$$R_w = \begin{pmatrix} 0.0100 & 0.0050 \\ 0.0050 & 0.0100 \end{pmatrix} \,,\ R_v = 0.1 \,,\ R_{wv} = \underline{\underline{0}} \tag{44b}$$

$$K_p = \begin{pmatrix} 0.5414 \\ 0.4392 \end{pmatrix} \,,\ R_e = 0.0250 \tag{44c}$$

This system is constructed such that the second state is difficult to observe. The condition number for the observability matrix is 191. This improves to 3.65 when introducing the additional output (45) because the second state now is included in the output.

$$\left[\begin{array}{c|c} C_a & D_a \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0.1 \end{array}\right] \tag{45a}$$

$$R_a = 0.1 \,,\ R_{a\bullet} = R_{\bullet a} = \underline{\underline{0}} \tag{45b}$$

Using sufficient excitation on the input, simulation of 1000 samples from the system are shown in figure 1.

The result of using the above estimation methods is seen in (46). Recall that the "physical" noise model (44b) is not assumed known. Only the corresponding IM (44a), (44c) is known. The parameters estimated solely from data $\widehat{C}_a, \widehat{D}_a$ and partly from data $\widehat{R}_a$ does not match exactly the system values which is due to uncertainty as there is only 1000 samples. The parameters derived solely from the present model $\widehat{R}_w, \widehat{R}_v$ are different for the process noise compared

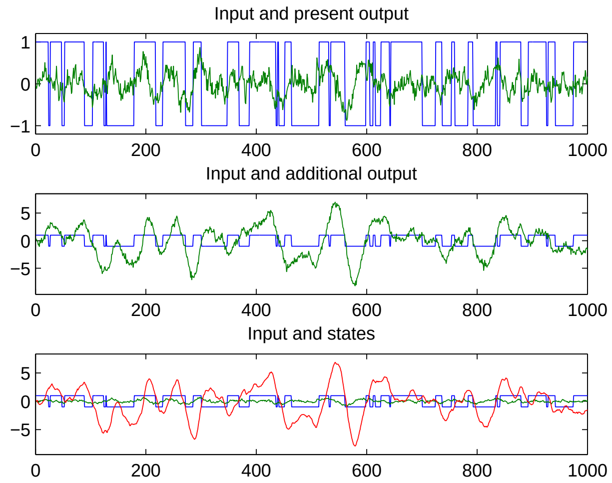

Fig. 1. Time plot for the simulation example. The input is blue and the first and second state in the bottom plot are green and red respectively.

to the system value because the method is not unique with 2 states and 1 output. Only $\widehat{R}_v$ is correct because the method gives a unique solution (40) for this parameter. Recall that the zero correlations between a measurement noise and other measurements and states noise are part of the assumptions, they are not estimated.

$$\left[\begin{array}{c|c} \widehat{C}_a & \widehat{D}_a \end{array}\right] = \left[\begin{array}{cc|c} 0.800 & 1.028 & 0.276 \end{array}\right] \tag{46a}$$

$$\widehat{R}_w = \begin{pmatrix} 0.0101 & -0.0036 \\ -0.0036 & 0.0013 \end{pmatrix} \,,\ \widehat{R}_v = 0.100 \,,\ \widehat{R}_{wv} = \underline{\underline{0}} \tag{46b}$$

$$\widehat{R}_a = 0.715 \,,\ \widehat{R}_{a\bullet} = \widehat{R}_{\bullet a} = \underline{\underline{0}} \tag{46c}$$

$$\widehat{K}_a = \begin{pmatrix} 0.5359 & 0.0051 \\ 0.5563 & 0.0919 \end{pmatrix} \widehat{R}_{ea} = \begin{pmatrix} 0.0249 & 0.0128 \\ 0.0128 & 0.7952 \end{pmatrix} \tag{46d}$$

The present model (44a) augmented with the estimated parameters (46) can now be turned into a updated IM. The estimated stochastic part is not in general consistent but a better approximation compared to zero gains in the Kalman filter part relating the additional output to the state predictions. This is also seen in the residual plots in figure 2. Especially the residual for the additional output is clearly non white. However, the maximum correlation for non zero lags is 0.5 which is quit small.

The above shows that the method is not perfect which is in accordance with the theoretical observations in section IV-B.2. The crucial question is then: can it give any improvement. This can be measured in terms of state prediction and control performance based on the present system and the system updated with the estimated model for the additional output.

For state prediction the results are seen in table I. For reference the table also includes the results from the ideal situation where both outputs are used and all the correct parameters (44), (45) are assumed known.

The improvement for the first state is minor because it is already well observe from the present output. For the second state the prediction error is reduced to approximately 1/3 for the estimated model compared to 1/5 for the ideal

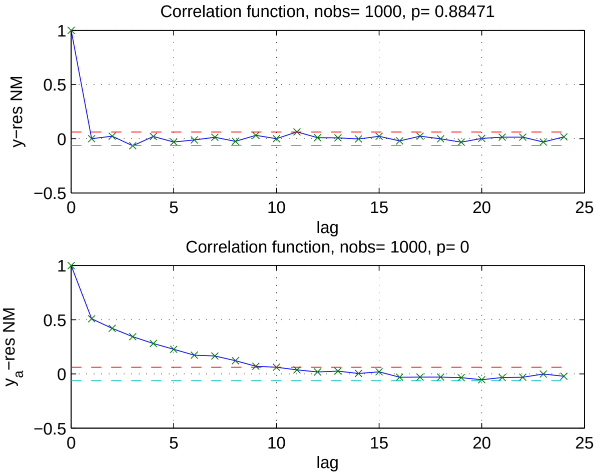

Fig. 2. Residual plot for additional output in the new model.

TABLE I
IMPROVEMENT IN STATE PREDICTION RMS ERRORS WHEN INCLUDING THE ADDITIONAL OUTPUT.

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{y}_p$	$\tilde{y}_a$
	rms values			
Present Model	0.117	1.029	0.150	2.833*
	rms values normalized with above row			
Present model	1.000	1.000	1.000	1.000
Updated model	0.986	0.377	0.991	0.180
Ideal updated model	0.956	0.208	0.974	0.147

model. Clearly the estimation method improves the state estimate when including an additional output. Rms on output prediction errors are also included. The present output is well predicted in all cases with a minor improvement using the additional output. The predicted additional output comes a bit odd into this comparison as the present model can not predict this output. Instead the value in the table marked with a * is the variance corresponding to the best prediction without a model. The result then shows that the improvement from the estimated model is close to the improvement obtained with the ideal model.

To evaluate the method in a control setting the state feedback controller design (47) has been chosen sufficiently fast to show differences in the quality for the state estimate.

$$F_o = \min_F \lim_{N\to\infty} \sum_{t=1}^{N} x(t)^{\mathrm{T}} Q_x x(t) + u(t)^{\mathrm{T}} Q_u u(t) \ , \qquad (47)$$
$$u(t) = -Fx(t) \ , \ Q_x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ , \ Q_u = 0.01$$

$$\begin{aligned} u(t) = -F_o\hat{x}(t) = -F_o \, \mathrm{E}(x(t)|Y^{t-1}) \ , \\ F_o = (30.8739 \quad 7.4953) \end{aligned} \qquad (48)$$

The result of using the three different state estimators in (48) are seen in table II. Notice that there is one time delay in the controller (48). According to the separation theorem [6, Theo. 5.1] the performance function has a part due to control and a part due to state prediction errors. As the control gain is fixed the differences in performance are due to the state predictors. In table II it is clearly seen that the updated model improves performance significantly and it is almost as good as the ideal model.

TABLE II
IMPROVEMENT IN STATE PREDICTION RMS ERRORS WHEN INCLUDING THE ADDITIONAL OUTPUT.

	$x^{\mathrm{T}}Q_x x$	$u^{\mathrm{T}}Q_u u$	Total
	Average values		
Present Model	1.188	0.002	1.190
	Average values normalized with above row		
Present model	1.000	1.000	1.000
Updated model	0.396	1.691	0.398
Ideal updated model	0.366	1.499	0.368

VI. CONCLUSION

Consider the situation where a additional sensor or actuator is needed in an existing system to improve estimation or control and where a model of the present system is known. Assume data driven system identification methods must be used to update the present model. It can be advantageous for control or estimation to keep the present state space and model and only estimate the necessary additional parameters.

In this work it is investigated what can be achieved with reliable, robust and convex numerical methods. The methods used are least squares and linear matrix inequalities.

The basic principle is to use the state prediction from the know present innovation model. The known state predictor can then be used in a regression type model to estimate the deterministic part of the model.

In case of a additional output there is also a stochastic part to be estimated. The Kalman gain has to be updated and the measurement noise for the additional output must be incorporated.

Using only the innovation form of the present model this is not possible. The solution develop is based on a transformation of the innovation model to a "physical" model in the sense that process and measurement noise are uncorrelated. This solution is to the authors knowledge new. Moreover, the problem of incremental modelling and the methods in this paper seems new.

The estimated parameters for the new device are consistent even in closed loop mode except for the stochastic part in the case with additional output. Here the estimate will in general be biased.

However, a simulation experiment shows that the developed methods successfully improve the state prediction and control performance for a system where one state initially has a low observability.

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