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Publication date:
2009

Document Version
Publisher's PDF, also known as Version of record

Link to publication from Aalborg University

Citation for published version (APA):
Frendrup, A., Vestergaard, P. D., & Yeo, A. (2009). *Total domination in partitioned graphs*. Department of Mathematical Sciences, Aalborg University. Research Report Series No. R-2009-06

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R-2009-06 April 2009

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ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Total domination in partitioned graphs

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To appear in Graph Theory and Combinatorics

Abstract. We present results on total domination in a partitioned graph $G = (V, E)$. Let $\gamma_t(G)$ denote the total dominating number of G. For a partition $V_1, V_2, \ldots, V_k$, $k \geq 2$, of V, let $\gamma_t(G; V_i)$ be the cardinality of a smallest subset of V such that every vertex of V_i has a neighbour in it and define the following
$f_t(G; V_1, V_2, \ldots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)$
$f_t(G; k) = \max\{f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$
$g_t(G; k) = \max\{\Sigma_{i=1}^{k} \gamma_t(G; V_i) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$

We summarize known bounds on $\gamma_t(G)$ and for graphs with all degrees at least δ we derive the following bounds for $f_t(G; k)$ and $g_t(G; k)$.

(i) For $\delta \geq 2$ and $k \geq 3$ we prove $f_t(G; k) \leq 11|V|/7$ and this inequality is best possible.

(ii) for $\delta \geq 3$ we prove that $f_t(G; 2) \leq (5/4 - 1/372)|V|$. That inequality may not be best possible, but we conjecture that $f_t(G; 2) \leq 7|V|/6$ is.

(iii) for $\delta \geq 3$ we prove $f_t(G; k) \leq 3|V|/2$ and this inequality is best possible.

(iv) for $\delta \geq 3$ the inequality $g_t(G; k) \leq 3|V|/4$ holds and is best possible.

Key words. Total domination, Partitions and Hypergraphs.

1. Notation

By $G = (V, E)$ we denote a *graph* G with *vertex* set $V = V(G)$ and *edge* set $E = E(G)$. The *order* of G is $|V(G)| = n$. For $x \in V(G)$ we denote by $N_G(x)$ the set of *neighbours* to x and $N_G[x] = \{x\} \cup N_G(x)$. Indices may be omitted if clear from context. The *degree* of x is $d_G(x) = |N_G(x)|$, the number of neighbours to x. We let $\delta(G) = \delta$ denote the *minimum degree* in G and $\Delta(G) = \Delta$ the *maximum* degree. A *hypergraph* $H = (V, E)$ has *vertex set* $V = V(H)$ and its set of *hyperedges*, or *edges* for short, is $E = E(H)$. Each hyperedge e is a subset of V, $e \subseteq V(H)$. A vertex v is *incident* with an edge e if $v \in e$, the *degree* of

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v is the number of hyperedges in H containing v. We let $\delta(H) = \delta$ denote the *minimum degree* in H and $\Delta(H) = \Delta$ the *maximum degree.* H is *r-regular* if each vertex has degree r, i.e. $d_H(x) = r$, or equivalently, x is contained in precisely r edges. H is *k-uniform* if each hyperedge contains exactly k vertices. Two edges e_1 and e_2 are said to be *overlapping* if $|V(e_1) \cap V(e_2)| \geq 2$. Let $Y \subseteq V(H)$ then $E(Y)$ denotes all hyperedges, e, contained in Y (i.e. $V(e) \subseteq Y$).

For a hypergraph H a *hitting set* or a *transversal* $\mathcal{T}$ is a set of vertices $\mathcal{T} \subseteq V(H)$ such that $e \cap \mathcal{T} \neq \emptyset$ for each hyperedge e in $E(H)$, i.e. each edge e contains at least one vertex from $\mathcal{T}$. $\mathcal{T}(H)$ denotes the minimum cardinality of a transversal for the hypergraph H. For sets $S, T \subseteq V$, in a graph G the set S *totally dominates* T if every vertex in T is adjacent to some vertex of S. The minimum number of vertices needed to totally dominate V is the *total domination number* $\gamma_t(G)$. For a subset S of V we let $\gamma_t(G; S)$ denote the smallest number of vertices in G which totally dominates S. A *partition* $V = (V_1, V_2, \ldots, V_k)$ of $V(G)$ into k disjoint sets, $k \geq 2$, has $V = \bigcup_{i=1}^{k} V_i$, $V_i \cap V_j = \emptyset$, $1 \leq i < j \leq k$. For a partition $(V_1, V_2, \ldots, V_k)$ of V, we define the following.

$$f_t(G; V_1, V_2, \ldots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)$$
$$g_t(G; V_1, V_2, \ldots, V_k) = \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)$$

We furthermore define $f_t(G; k)$ and $g_t(G; k)$ as follows.

$$f_t(G; k) = \max\{f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$$
$$g_t(G; k) = \max\{g_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}$$

For further notation we refer to Chartrand and Lesniak [1].

2. Introduction

The theory of domination is outlined in two books by Haynes, Hedetniemi and Slater [5, 6]. A combination of domination and partitions is treated by Hartnell and Vestergaard [7], Seager [14], Tuza and Vestergaard [17], Henning and Vestergaard [11]. There has been an upsurge in the study of total domination. New results on total domination are given by Henning, Kang, Shan, Thomassé and Yeo in [10,12,15,18]. In [9] Henning surveys recent results on total domination. Here we shall study total domination in partitioned graphs.

3. Bounds on γ_t

We summarize in Theorem 1 results found by Henning, Thomassé and Yeo. If C_{10} : $v_1, v_2, \ldots, v_{10}, v_1$ is the circuit with 10 vertices then let G_{10} denote the graph obtained from C_{10} by addition of the edge v_1v_6 and let H_{10} denote the graph obtained from C_{10} by addition of the edges v_1v_6 and v_2v_7.

Theorem 1. *Let G be a connected graph with n vertices and minimum degree $\delta(G) = \delta$. Then*

$\delta \geq 2$ implies $\gamma_t(G) \leq 4n/7$ for $G \notin \{C_3, C_5, C_6, C_{10}, G_{10}, H_{10}\}$ ([8, Corollary 6], [9, Theorem 27]).
$\delta \geq 3$ implies $\gamma_t(G) \leq n/2$. ([15]).

$\delta \geq 4$ implies $\gamma_t(G) \leq 3n/7$ ([15]) and there exists some $\epsilon > 0$ such that $\gamma_t(G) \leq (3/7-\epsilon)n$ for $G \neq G_{14}$, where G_{14} is an incidence bipartite graph of order 14 derived from the Fano plane ([19]).

It is a conjecture that $\delta \geq 5$ implies $\gamma_t(G) \leq 4n/11$.
Theorem 2 and Theorem 3 below, give conditions for equality in Theorem 1.

Theorem 2. ([9, Theorem 29]) *Let G be a connected graph of order $n > 14$ with $\delta \geq 2$. Then $\gamma_t(G) = 4n/7$ if and only if G can be obtained from a connected graph F of order at least three by adding $|V(F)|$ disjoint copies of C_6, one corresponding to each $v \in V(F)$, such that either v is joined by a new edge to a vertex in its corresponding C_6 or by two new edges to two vertices at distance two apart in its corresponding C_6.*

The family $\mathcal{G} \cup \mathcal{H}$ is constructed in [3] as follows. Take two copies $a_1b_1a_2b_2 \dots a_kb_k$ and $c_1d_1c_2d_2 \dots c_kd_k$, of the path $P_{2k}, k \geq 2$, and add edges a_id_i, b_ic_i for $i = 1, 2, \dots, k$. ¿From this the graph of order $4k$ belonging to the infinite family $\mathcal{G}$ is obtained by adding a_1c_1 and b_kd_k, while the graph of order $4k$ in $\mathcal{H}$ is obtained by adding a_1b_k and c_1d_k, The generalized Petersen graph GP_{16} is obtained from two circuits $u_1u_2u_3 \dots u_7u_8$ and $v_1v_2v_3 \dots v_7v_8$ by addition of edges u_1v_1, u_2v_4, u_3v_7, u_4v_2, u_5v_5, u_6v_8, u_7v_3, u_8v_6.

Theorem 3. ([12, Theorem 5]) *Let G be a connected graph with $\delta(G) \geq 3$. Then $\gamma_t(G) = n/2$ if and only if $G \in \mathcal{G} \cup \mathcal{H}$ or $G = GP_{16}$.*

4. f_t for k-partitioned graphs with $\delta \geq 2$

We have that f_t increases with the number of partition classes, i.e.,
$f_t(G;k) \leq f_t(G;k+1)$. Therefore we get a weaker inequality if we partition V into more than two classes. That is demonstrated in Theorem 4 below.

Theorem 4. *Let G be a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$. If $k \geq 2$ then $f_t(G;k) \leq 11n/7$.*

If $k = 2$ then $f_t(G;k) \leq 3n/2$. Equality holds if and only if G is a circuit of length zero modulo four, $G = C_{4t}, t \geq 1$.

If $k = 3$ then $f_t(G;k) \leq 11n/7$. For $n > 14$ equality holds if and only if G can be obtained from a circuit or a path of order at least three by joining each of its vertices by one edge to disjoint copies of C_6.

If $k \geq 4$ then $f_t(G;k) \leq 11n/7$ and for $n > 14$ equality holds if and only if $\Delta(G) \leq k$ and G can be obtained from a connected graph F having order at least three and $g_t(F;k) = |V(F)|$ by adding disjoint copies of C_6, one corresponding to each $v \in V(F)$, such that either v is joined by a new edge to one vertex in its corresponding C_6 or by two new edges to two vertices at distance two apart in its corresponding C_6.

Proof. By Theorem 1 we have $\gamma_t(G) \leq 4n/7$ and assigning to each vertex its own class dominator we have $g_t(G;k) \leq n$. Therefore $f_t(G;k) = \gamma_t(G) + g_t(G;k) \leq 11n/7$. The result for $k = 2$ is proven by Frendrup, Henning and Vestergaard in [4, Theorem 2]. For $k \geq 3$ the equality $f_t(G;k) = 11n/7$ implies $\gamma_t(G) = 4n/7$ and $g_t(G;k) = n$ and therefore G has the structure described in Theorem 2. Since $g_t(G;k) = n$ each subgraph H of G must satisfy $g_t(H;k) = |V(H)|$ and further $\Delta(G) \leq k$. Let H_1 be the graph obtained from a circuit $C_6 : v_1v_2 \dots v_6$ by adding a new vertex x and the edge xv_1 and let $H_2 := H_1 + xv_3$. Observe for $k = 3$ that $g_t(H_1; k) = |V(H_1)|$ (obtainable from partitioning $x, v_1, v_2 \dots, v_6$ into classes indexed 1122133 or 1221133) while $g_t(H_2; k) < |V(H_2)|$. For $k \geq 4$ we can easily show that $g_t(H_i; k) = |V(H_i)|$, $i = 1, 2$. This proves for $k \geq 3$ that $f_t(G; k) = 11n/7$ implies G has the structure described in this theorem. Conversely, assume first that $k = 3$ and that G is obtainable as a disjoint union of H_1's with edges added between the vertices named x, so they span F, where F is a path or circuit. We must exhibit a partition of $V(G)$ proving that $f_t(G; k) = 11n/7$, i.e. that $g_t(G; k) = |V(G)|$. It is easy to find a partition V_1', V_2', V_3' of $V(F)$ such that $g_t(F; k) = |V(F)|$. If $k = 3$ we can extend this partition to all the H_1's such that the following holds, which proves that $g_t(G; V_1', V_2', V_3') = n$.

– $N(x) = N_F(x) \cup \{v_1\}$ contains at most one vertex from each V_1', V_2', V_3' (just put v_1 in the partition set which doesn't contain any of the two vertices in $N_F(x)$).
– $N(v_1) = \{x, v_2, v_6\}$ contains one vertex from each V_1', V_2', V_3' (just put v_2 and v_6 in the partition sets such that this holds).
– $N(v_3), N(v_5) \subset \{v_2, v_4, v_6\}$, which contains one vertex from each V_1', V_2', V_3' (just put v_4 in the same set as x).
– $N(v_2), N(v_4), N(v_6) \subset \{v_1, v_3, v_5\}$, which contains one vertex from each V_1', V_2', V_3' (just put v_3 and v_5 in the partition sets such that this holds).

Assume next that $k \geq 4$. Then a vertex $x \in F$ may belong to a unit H_1 or H_2. Again there is a partition $V_1', V_2', \dots, V_k'$ of $V(F)$ such that $g_t(F; k) = |V(F)|$ and similarly to above we can extend this partition to all of G, such that the neighbourhood of every vertex in G contains at most one vertex from any partition set. The details are left to the reader. This proves that $g_t(G; k) = n$. □

5. g_t for two-partitioned graphs with $\delta \geq 3$

Chvátal and McDiarmid [2] and Tuza [16] independently established the following result about transversals in hypergraphs (see also Thomassé and Yeo [15] for a short proof of this result).

Theorem 5. ([2,16,15]) *If H is a hypergraph with all edges of size at least three, then $\mathcal{T}(H) \leq (|V(H)| + |E(H)|)/4$.*

Theorem 6. *Let G be a graph of order n with $\delta \geq 3$. Then $g_t(G; 2) \leq 3n/4$.*

Proof. ¿From the two-partitioned graph G, we define for $i = 1, 2$, H_i to be the hypergraph on n vertices and m_i edges where $V(H_i) = V(G)$ and the hyperedges of H_i are the sets of neighbourhoods of class i vertices. In other words, $e \in E(H_i)$ precisely if, for some vertex v in V_i, $e = N_G(v)$. Each edge in H_i has at least three vertices because $\delta(G) \geq 3$. In G we see that a set $\mathcal{T}_i$ of vertices totally dominates V_i if and only if $\mathcal{T}_i$ is a transversal of H_i. Applying Theorem 5 to H_1 and H_2 separately we obtain transversals $\mathcal{T}_i$ of H_i, $i = 1, 2$, satisfying

$$|\mathcal{T}_1| \leq \tfrac{m_1+n}{4} \qquad |\mathcal{T}_2| \leq \tfrac{m_2+n}{4}.$$

Since $m_1 + m_2 = n$ we obtain $|\mathcal{T}_1| + |\mathcal{T}_2| \leq \frac{m_1+n}{4} + \frac{m_2+n}{4} = \frac{3n}{4}$. This proves Theorem 6. □

An example of graphs with equality $g_t(G; 2) = 3n/4$ is given in the next section.

6. An infinite family of graphs extremal for Theorem 6

We have the following theorem.

Theorem 7. *For each integer $r \geq 1$ there exists a connected bipartite graph G_r of order $n = 16r$ with $\delta(G_r) = 3$ such that $g_t(G_r; 2) = 3|V(G_r)|/4$ and $f_t(G_r; 2) \geq 9|V(G_r)|/8$.*

Proof. We define the graph G_r as follows. Define the vertex set of G_r to be $V(G_r) = W_r \cup A_r \cup B_r$, where

$$\begin{aligned} W_r &= \{w_0, w_1, w_2, \ldots, w_{8r-1}\} \\ A_r &= \{a_0, a_1, a_2, \ldots, a_{4r-1}\} \\ B_r &= \{b_0, b_1, b_2, \ldots, b_{4r-1}\} \end{aligned}$$

We define the edge set of G_r such that the following holds, for all $i \in \{0, 1, 2, \ldots, r-1\}$ (where $b_{-1} = b_{4r-1}$ by definition):

$$\begin{array}{ll} N(w_{8i}) = \{a_{4i}, a_{4i+1}, b_{4i}\} & N(w_{8i+1}) = \{a_{4i}, a_{4i+1}, b_{4i}\} \\ N(w_{8i+2}) = \{a_{4i}, a_{4i+2}, b_{4i}\} & N(w_{8i+3}) = \{a_{4i+1}, a_{4i+2}, b_{4i-1}\} \\ N(w_{8i+4}) = \{a_{4i+2}, b_{4i+1}, b_{4i+2}\} & N(w_{8i+5}) = \{a_{4i+3}, b_{4i+1}, b_{4i+2}\} \\ N(w_{8i+6}) = \{a_{4i+3}, b_{4i+1}, b_{4i+3}\} & N(w_{8i+7}) = \{a_{4i+3}, b_{4i+2}, b_{4i+3}\} \end{array}$$

We now assume $r \geq 1$ is fixed, and therefore omit the subscripts of the above sets and graph. Define V_1 and V_2 as follows.

$$V_1 = A \cup \cup_{i=0}^{r-1} \{w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+5}\}$$
$$V_2 = B \cup \cup_{i=0}^{r-1} \{w_{8i}, w_{8i+4}, w_{8i+6}, w_{8i+7}\}$$

We will now show that if S_i is a set such that every vertex in V_i has a neighbour in S_i, then $|S_i| \geq 3|V(G)|/8$, for $i = 1, 2$. This would imply that $f_t(G; 2) \geq 9|V(G)|/8$ and $g_t(G) \geq 6|V(G)|/8$ when $k = 2$ (as clearly the above would also imply that $\gamma_t(G) \geq 3|V(G)|/8$). From Theorem 6 follows that $g_t(G) = 3|V(G)|/4$.

Let S_1 be a set that totally dominates V_1 (i.e. every vertex in V_1 has a neighbour in S_1). As w_{8i+5} has a neighbour in S_1 we note that $|S_1 \cap \{a_{4i+3}, b_{4i+1}, b_{4i+2}\}| \geq 1$, for all $i = 0, 1, 2, \ldots, r-1$. As w_{8i+1}, w_{8i+2} and w_{8i+3} all have a neighbour in S_1 we note that $|S_1 \cap \{a_{4i}, a_{4i+1}, a_{4i+2}, b_{4i}, b_{4i-1}\}| \geq 2$, for all $i = 0, 1, 2, \ldots, r-1$ (recall that $b_{-1} = b_{4r-1}$). As the above sets are all disjoint we note that $|S_1 \cap (A \cup B)| \geq 3|A \cup B|/8$.

As a_{4i+3} has a neighbour in S_1 we note that $|S_1 \cap \{w_{8i+5}, w_{8i+6}, w_{8i+7}\}| \geq 1$, for all $i = 0, 1, 2, \ldots, r-1$. As a_{4i}, a_{4i+1} and a_{4i+2} all have a neighbour in S_1 we note that $|S_1 \cap \{w_{8i}, w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+4}\}| \geq 2$, for all $i = 0, 1, 2, \ldots, r-1$. As the above sets are all disjoint we note that $|S_1 \cap W| \geq 3|W|/8$. This implies the desired result for S_1.

The fact that if S_2 totally dominates V_2, then $|S_2| \geq 3|V(G)|/8$ is proved analogously to above. We now just need to show that G is connected. Let $P_i = \{w_{8i}, w_{8i+1}, \ldots, w_{8i+7}\}$ and let $Q_i = \{a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, b_{4i}, b_{4i+1}, b_{4i+2}, b_{4i+3}\}$ for all $i = 0, 1, 2, \ldots, r-1$. Note that $G[P_i \cup Q_i]$ is connected. As the edges $w_{8i+3}b_{4i-1}$, for all $i = 0, 1, 2, \ldots, r-1$ connects P_i with Q_{i-1} ($Q_{-1} = Q_{r-1}$) we are done. □

7. $f_t(G)$ for two-partitioned graphs with $\delta \geq 3$

Let G be a graph of order n with $\delta(G) \geq 3$.

From Theorems 1 and 6 it follows immediately that $f_t(G;2) = \gamma_t(G) + g_t(G;k) \leq n/2 + 3n/4 = 5n/4$ when $\delta(G) \geq 3$. We shall in Theorem 8 below prove a slightly stronger result and later pose an even stronger conjecture.

The following result is known (see for example [13]).

Lemma 1. *([13]) If G is a 3-regular graph, then there exists a matching M in G, such that $|M| \geq \frac{7}{16}|V(G)|$.*

Lemma 2. *Let H be a 2-regular 3-uniform hypergraph with no two edges overlapping. Then $\mathcal{T}(H) \leq \frac{|V(H)|+|E(H)|}{4} - \frac{|V(H)|}{24}$.*

Proof. Let H be a 2-regular 3-uniform hypergraph with no overlapping edges. Define the graph G_H as follows $V(G_H) = E(H)$ and $E(G_H) = \{e_1e_2 : |V(e_1) \cap V(e_2)| = 1\}$. As there are no overlapping edges and H is 2-regular and 3-uniform, we note that G_H is a 3-regular graph. By Lemma 1, there exists a matching M in G_H, such that $|M| \geq \frac{7}{16}|V(G_H)|$.

If $e_1e_2 \in M$, then by the definition of G_H we note that $V(e_1) \cap V(e_2) = \{x_{e_1e_2}\}$ for some $x_{e_1e_2} \in V(H)$. Let $X = \{x_f \mid f \in M\}$ and note that $2|M|$ edges in H contain a vertex from X (as M was a matching). Let X' be a set of vertices of order $|E(H)| - 2|M|$ containing a vertex from every edge in H, which does not contain a vertex from X. Note that $X \cup X'$ is a transversal of H of order $|M| + (|E(H)| - 2|M|)$. By the above bound on $|M|$ we get the following, as $3|E(H)| = \sum_{x \in V(H)} d(x) = 2|V(H)|$.

$$\begin{aligned}\mathcal{T}(H) &\leq |E(H)| - |M| \leq |E(H)| - \tfrac{7}{16}|E(H)| \\ &= \tfrac{|E(H)|}{4} + \tfrac{5|E(H)|}{16} = \tfrac{|E(H)|}{4} + \tfrac{5}{16} \times \tfrac{2|V(H)|}{3} \\ &= \tfrac{|V(H)|+|E(H)|}{4} - \tfrac{|V(H)|}{24}\end{aligned}$$

□

Lemma 3. *Let H be a 3-uniform hypergraph, where multiple edges are allowed. For each edge and vertex in H we assign a non-empty subset of $\{0,1,2\}$. Let this subset be denoted by $L(q)$ for all $q \in V(H) \cup E(H)$. Let H_i be the 3-uniform hypergraph containing vertex-set $V_i = \{v : i \in L(v)$ and $v \in V(H)\}$ and edge-set $E_i = \{e : i \in L(v)$ and $e \in E(H)\}$, for $i = 0,1,2$. Let $Y \subseteq V(H)$ be arbitrary and assume that the following holds.*

(a): $\Delta(H_1), \Delta(H_2) \leq 2$
(b): $\Delta(H - E(Y)) \leq 4$.
(c): There are no overlapping edges in H_i, $i \in \{1,2\}$.
(d): If $e \in E(H) - E(Y)$, then $0 \in L(e)$ and $|L(e)| \geq 2$.

This implies that the following holds.

$$\sum_{i=0}^{2} \mathcal{T}(H_i) \leq \left(\sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4}\right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

Remark. We assume here in Lemma 3 that the assignment of a set $L(q)$ to each q is done such that H_0, H_1, H_2 really are hypergraphs, i.e., such that each hyperedge in E_i consists of vertices from V_i, $i = 0,1,2$. This requirement will be satisfied in the proof of Theorem 8 where the lemma is applied.

Proof. Assume that the lemma is false, and that H is a counterexample with minimum $|E_0| + |E_1| + |E_2|$. Clearly $|E_0| + |E_1| + |E_2| > 0$, as otherwise $\sum_{i=0}^{2} \mathcal{T}(H_i) = 0$. For simplicity we will use the following notation:

$T^* = \sum_{i=0}^{2} \mathcal{T}(H_i)$
$S^* = \sum_{i=0}^{2} \frac{|V_i|+|E_i|}{4}$
$V^* = V(H_0) \cap V(H_1) \cap V(H_2)$

We recall that H was assumed to be a "minimal" counterexample to $T^* \leq S^* - (|V^* \setminus N_H[Y]|)/372$. We will now prove a few claims, which end in a contradiction, thereby proving the lemma. For H the left hand side of the inequality, ℓ, and the right hand side of the inequality, r, in Lemma 3 satisfies $\ell > r$. We shall construct smaller H' which also satisfies (a)-(d) and which therefore has $\ell' \leq r'$ by the minimality of H. H' is to be constructed such that there exist $\alpha \leq \beta$ for which $\ell - \alpha \leq \ell'$ and $r' \leq r - \beta$. Those inequalities combine to give the desired contradiction $\ell \leq r$.

Claim A: If we add a vertex to Y, then $N[Y]$ does not increase by more than 9 vertices.

Proof of Claim A: This follows from the fact that H is 3-uniform and $\Delta(H - E(Y)) \leq 4$, by (b) in the statement of the lemma.

Claim B: There is no $e = \{v_1, v_2, x\} \in E_i$, such that $d_{H_i}(v_1) = d_{H_i}(v_2) = 1$ and $d_{H_i}(x) = 2$, for $i = 0, 1, 2$.

Proof of Claim B: Assume that there is such an edge $e = \{v_1, v_2, x\} \in E_i$. Let $e' = \{w_1, w_2, x\}$ be the other edge in H_i containing x. Now delete v_1, v_2, x, e and e' from H_i and add $\{v_1, v_2, x, w_1, w_2\}$ to Y. Note that (a)-(d) still hold and that T^* decreases by 1 as we simply add x to any transversal in the new H_i in order to get a transversal in the old H_i. By Claim A the set $N[Y]$ does not increase by more than 45 vertices. As V^* does not decrease by more than 3 vertices and S^* decreases by 5/4, we are done by the "minimality" of H (as $\alpha = 1 \leq 5/4 - 48/372 = \beta$ in the argument above Claim A).

Claim C: There is no $e = \{x, v_1, v_2\} \in E_i$, such that $d_{H_i}(v_1) = d_{H_i}(v_2) = 2$ and $d_{H_i}(x) = 1$, for $i = 1, 2$.

Proof of Claim C: Assume that there is such an edge $e = \{x, v_1, v_2\} \in E_i$. Let $e_1 = \{w_1, w_2, v_1\}$ be the other edge in H_i containing v_1 and let $e_2 = \{u_1, u_2, v_2\}$ be the other edge in H_i containing v_2. As there are no overlapping edges in H_i (by (c) in the statement of the lemma) we note that $e_1 \neq e_2$ and $|\{w_1, w_2, u_1, u_2\}| \geq 3$. Let S be any subset of $\{w_1, w_2, u_1, u_2\}$ such that $|S| = 3$. We now separately consider the cases when addition of S as a new hyperedge to H_i causes overlapping edges in H_i, and when it doesn't.

Assume that adding S to E_i does not cause overlapping edges in $H_i - e_1 - e_2$. Now delete x, v_1, v_2, e, e_1 and e_2 from H_i and add the edge S to H_i (and H). Furthermore add $\{x, v_1, v_2, w_1, w_2, u_1, u_2\}$ to Y. Note that (a)-(d) still hold. If T' is a transversal in the new H_i then due to the edge S we either have $\{u_1, u_2\} \cap T' \neq \emptyset$, in which case $T' \cup \{v_1\}$ is a transversal in the old H_i or $\{w_1, w_2\} \cap T' \neq \emptyset$, in which case $T' \cup \{v_2\}$ is a transversal in the old H_i. Therefore T^* decreases by at most one. By Claim A we have that $N[Y]$ does not increase by more than 63 vertices. As V^* does not decrease by more than 3 and S^* decreases by 5/4, we are done by the "minimality" of H (as $1 \leq 5/4 - 66/372$).

So now assume that the above addition of S would cause overlapping edges in $H_i - e_1 - e_2$. This can only happen if there is an edge $e' \in E_i$ such that $|S \cap V(e')| \geq 2$. Note that by (a) the degree in H_i is two for all vertices in $S \cap V(e')$ (they only lie in S and e'). Now delete the vertices $\{x, v_1, v_2\} \cup (S \cap V(e'))$ from H_i and delete the edges e, e_1, e_2 and e' from H_i (do not add the edge S to H_i). Furthermore add $\{x, v_1, v_2, w_1, w_2, u_1, u_2\} \cup (V(e') - S)$ to Y. Note that (a)-(d) still hold. By a similar argument to above we note that T^* decreases by at most two. By Claim A we see that $N[Y]$ does not increase by more than 72 vertices. As V^* does not decrease by more than 6 and S^* decreases by at least $9/4$, we are done by the "minimality" of H (as $2 \leq 9/4 - 78/372$).

Claim D: There is no $e = \{x, v_1, v_2\} \in E_0$, such that $d_{H_0}(v_1) = d_{H_0}(v_2) = 2$ and $d_{H_0}(x) = 1$ and $|N_{H_0}[V(e)]| \geq 6$.

Proof of Claim D: Assume that there is such an edge $e = \{x, v_1, v_2\} \in E_0$. Let $e_1 = \{w_1, w_2, v_1\}$ be the other edge in H_0 containing v_1 and let $e_2 = \{u_1, u_2, v_2\}$ be the other edge in H_0 containing v_2. If $e_1 = e_2$, then $|N_{H_0}[V(e)]| \leq 4$, a contradiction. So assume that $e_1 \neq e_2$. As $|N_{H_0}[V(e)]| \geq 6$ we note that $|\{w_1, w_2, u_1, u_2\}| \geq 3$. We are now done analogously to Claim C.

Claim E: $\Delta(H_1), \Delta(H_2) \leq 1$.

Proof of Claim E: Assume that $\Delta(H_1) \geq 2$. By (a) we have $\Delta(H_1) = 2$. By Claim B and Claim C we note that there is a 2-regular component, R, in H_1. There are no overlapping edges in R by (c). By Lemma 2 there is a transversal T_R in R of order at most $(|V(R)| + |E(R)|)/4 - |V(R)|/24$. So delete all edges and vertices in R and add all vertices in R to Y. By Claim A we have that $N[Y]$ increases by at most $9|V(R)|$ vertices. We now have a contradiction to the "minimality" of H, as $|V(R)|/24 \geq 9|V(R)|/372$. Analogously we can show that $\Delta(H_2) \leq 1$.

Claim F: Assume $e_1, e_2 \in E(H_0)$ overlap and $e_i = (x_1, x_2, u_i)$ for $i = 1, 2$, where $u_1 \neq u_2$. If $d_{H_0}(x_1) = d_{H_0}(x_2) = 2$, then there is an edge $e' \in E(H_0)$ such that $\{u_1, u_2\} \subseteq V(e')$.

Proof of Claim F: Let e_1 and e_2 be defined as in the Claim, and assume that there is no edge $e' \in E(H_0)$ such that $\{u_1, u_2\} \subseteq V(e')$. Delete e_1, e_2, x_1, x_2 and u_1 from H_0. For every edge, e'', in H_0 that contains u_1, delete e'' and add the edge $(e'' - \{u_1\}) \cup \{u_2\}$ instead. Furthermore add $\{x_1, x_2, u_1, u_2\}$ and $V(e'')$ from all transformed edges, to Y. As there is at most 4 edges containing u_1 in $H_0 - E(Y)$ we note that Y increases by at most 10 (the neighbours of u_1 in $H_0 - E(Y)$ and $\{u_1, u_2\}$). Therefore $V^* - N[Y]$ decreases by at most $3 + 90$, by Claim A. We also note that S^* decreases by $5/4$.

We now show that T^* decreases by at most one. If $u_2 \in T'$ then $T' \cup \{u_1\}$ is a transversal in the old H_0. If $u_2 \notin T'$ then $T' \cup \{x_1\}$ is a transversal in the old H_0. As (a)-(d) still holds after the above operations, we have a contradiction to the "minimality" of H, as $1 \leq 5/4 - 93/372$.

Definition G: Let $x \in V^* - N[Y]$ be arbitrary. The vertex x exists since otherwise we would be done by Theorem 5.

Claim H: $d_{H_1}(u) = d_{H_2}(u) = 1$ for all $u \in N_{H_0}[x]$, where x is defined in Definition G.

Proof of Claim H: Assume that $u \in N_{H_0}[x]$ has $d_{H_2}(u) = 0$ or $u \notin V(H_2)$, which are the only possibilities for u, if $d_{H_2}(u) \neq 1$ (by Claim E). If $u \in V(H_2)$ and $d_{H_2}(u) = 0$, then delete u from $V(H_2)$. We are now done as T^* is unchanged, S^* decreases by $1/4$ and $V^* - N[Y]$ does not decrease by more than one. So we may assume that $u \notin V(H_2)$. Since $x \in V^*$ we note that $x \in V(H_1)$ and $x \in V(H_2)$, which by the above argument implies that $d_{H_1}(x) = d_{H_2}(x) = 1$ and $u \neq x$. Let $e_1 = \{x, u, q\}$ be the edge in H_1 (and H_0) containing u and x. Let e_2 be the edge in H_2 (and H_0) that contains x. Note that $d_{H_0}(x) = 2$ and $d_{H_0}(u) = 1$. If $d_{H_0}(q) = 1$ then we are done by Claim B. So $d_{H_0}(q) \geq 2$. However as any edge containing q must also lie in H_1 or H_2, as $q \notin Y$, we note that

$d_{H_0}(q) = 2$. Let e_q be the edge in H_2 that contains q. Note that $e_q \neq e_2$, by Claim F. As e_q and e_2 do not intersect we note that $|N_{H_0}[V(e)]| = 7 \geq 6$, so we are done by Claim D.

Claim I: Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing x (defined in Definition G). They exist by Claim H. Then $V(e_1) \cap V(e_2) = \{x\}$.

Proof of Claim I: Assume for the sake of contradiction that $|V(e_1) \cap V(e_2)| \geq 2$. If $|V(e_1) \cap V(e_2)| = 3$, then we delete e_1 from H_0 and add $V(e_1)$ to Y. This contradicts the "minimality" of H, as T^* remains unchanged, S^* decreases by 1/4 and $N[Y]$ increases from Claim A by at most 27. Therefore assume that $|V(e_1) \cap V(e_2)| = 2$. Let $e_1 = \{x, v, w\}$ and let $e_2 = \{x, v, y\}$ where $w \neq y$. As $d_{H_0}(x) = d_{H_0}(v) = 2$, there is an edge, e', in H_0 such that $\{w, y\} \subseteq V(e')$, by Claim F. However $e' \not\in E(H_1)$ and $e' \not\in E(H_2)$ by Claim E. This is however a contradiction to (d), as $w, y \not\in Y$.

Claim J: We now obtain a contradiction.

Proof of Claim J: : Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing x (defined in Definition G). They exist by Claim H and $V(e_1) \cap V(e_2) = \{x\}$, by Claim I. Let $e_1 = \{x, v_1, v_2\}$ and let $e_2 = \{x, w_1, w_2\}$. Let e_1' be the edge in H_1 containing w_1 and let e_1'' be the edge in H_1 containing w_2 (they exist by Claim H). Let e_2' be the edge in H_2 containing v_1 and let e_2'' be the edge in H_2 containing v_2 (they exist by Claim H).

If $e_1' = e_1''$, then $V(e_1') \cap V(e_2) = \{w_1, w_2\}$ and $e_1' = \{w_1, w_2, r\}$ for some $r \in V(H_0)$. By Claim F, there is an edge in H_0 that contains x and r. But this is a contradiction, as neither e_1 or e_2 contain r, by Claim H. Therefore $e_1' \neq e_1''$. Analogously we can show that $e_2' \neq e_2''$.

We now delete e_1, e_1', e_1'' from H, H_0 and H_1. Delete e_2, e_2', e_2'' from H, H_0 and H_2. Delete $V(e_1) \cup V(e_1') \cup V(e_1'')$ from $V(H_1)$ and delete $V(e_2) \cup V(e_2') \cup V(e_2'')$ from $V(H_2)$. Delete $V(e_1) \cup V(e_2)$ from H and H_0. Let S_1 be any subset of size three in $V(e_1') \cup V(e_1'') - \{w_1, w_2\}$ and let S_2 be any subset of size three in $V(e_2') \cup V(e_2'') - \{v_1, v_2\}$. Add the edges S_1 and S_2 to H and H_0. Finally add all vertices in $V(e_1') \cup V(e_1'') \cup V(e_2') \cup V(e_2'') - \{w_1, w_2, v_1, v_2, x\}$ to Y.

We first show that T^* decreases by at most 8. It is clear that the transversal size drops by three in both H_1 and H_2. So assume that T' is a transversal of the new H_0. As in the proof of Claim C we note that one of the three edges e_1, e_2', e_2'' are already covered by a vertex in T' (due to S_2) and the other two edges can be covered by one additional vertex. Similarly by adding one more vertex to T' we can make sure that e_2, e_1', e_1'' are all covered. Therefore the transversal size drops by at most two in H_0.

Note that S^* drops by 33/4 as we delete 9 vertices in each of H_1 and H_2 and we delete 5 vertices in H_0. We also delete three edges in each of H_1 and H_2 and six edges in H_0. But we also add two edges in H_0.

$N[Y]$ increases by at most 72 vertices by Claim A, as $|V(e_1') \cup V(e_1'') \cup V(e_2') \cup V(e_2'') - \{w_1, w_2, v_1, v_2, x\}| \leq 8$. As V^* decreases by at most 13, we note that $V^* - N[Y]$ decreases by at most 85. We note that (a)-(d) still holds after the above operations. We therefore have a contradiction to the "minimality" of H, as $8 \leq 33/4 - 85/372$.

Theorem 8. *If G is a graph with $\delta(G) \geq 3$ then $f_t(G; 2) \leq (\frac{5}{4} - \frac{1}{372})|V(G)|$.*

Proof. Let G be any graph with $\delta(G) \geq 3$ and let (W_1, W_2) be a partition of $V(G)$. Define the hypergraph H_G, such that $V(H_G) = V(G)$ and $E(H_G)$ is obtained by selecting for each $v \in V(G)$ one set of three vertices from $N_G(v)$ to form a hyperedge. $E(H_G) =$

$\{e_v : v \in V(G)\}$, $e_v = \{x_v, y_v, z_v\} \subseteq N_G(v)$. Furthermore for every hyperedge, $e \in E(H_G)$ let $L(e)$ be the set $\{0, i\}$ if $v \in W_i$. For reasons which will be clear later we let $L(v) = \{0, 1, 2\}$ for every $v \in V(H_G)$. Let H_i be the 3-uniform hypergraph containing vertex-set $V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}$ and edge-set $E_i = \{e : i \in L(e) \text{ and } e \in E(H)\}$, for $i = 0, 1, 2$. Note that a transversal of H_0 corresponds to a total dominating set in G and a transversal of H_i ($i \in \{1, 2\}$) corresponds to a total dominating set in G of the set W_i. Therefore we would be done if we could show that $\mathcal{T}(H_0) + \mathcal{T}(H_1) + \mathcal{T}(H_2) \leq (\frac{5}{4} - \frac{1}{372})|V(G)|$. Let Y be an empty set. We note that $|E_1| + |E_2| = |E_0| = |V_0| = |V_1| = |V_2| = |V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]| = |V(G)|$ and therefore the inequality above is equivalent to

$$(*) \qquad \sum_{i=0}^{2} \mathcal{T}(H_i) \leq \Big(\sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4}\Big) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

For simplicity we will use the following notation:
$T^* = \sum_{i=0}^{2} \mathcal{T}(H_i)$
$S^* = \sum_{i=0}^{2} \frac{|V_i|+|E_i|}{4}$
$V^* = V(H_0) \cap V(H_1) \cap V(H_2)$
We will now do a few transformations on H, H_0, H_1, H_2.

Transformation 1: While there is some vertex $x \in V(H)$ with $d_{H_0}(x) \geq 5$ (or equivalently $d_H(x) \geq 5$), delete x and all edges incident with x from H (and therefore also from H_0, H_1 and H_2).

Claim A: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim A: We note that T^* drops by at most three, as we may place x in the transversal of the new H_i's in order to get transversals in the old H_i's. We note that S^* decreases by at least $13/4$, as we delete x from H_0, H_1, H_2 and 5 edges from H_0 plus a total of 5 edges from H_1 and H_2. As V^* decreases by one and $N_H[Y] = \emptyset$ remains unchanged, we are done.

Transformation 2: While there is a vertex $x \in V(H)$ with $d_{H_1}(x) \geq 3$, delete x and all edges incident to x from H_0 and H_1. Also delete these edges from H (but do not delete x or any edges incident to x in H_2). If $d_{H_2}(x) = 0$ then delete x from H_2 (i.e. delete 2 from $L(x)$). If $d_{H_2}(x) > 0$ then note that $d_{H_2}(x) = 1$ (as we have performed transformation 1 as long as we could) and put $N_{H_2}[x]$ in Y.

Claim B: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim B: We note that T^* drops by at most two, as we may place x in the transversal of the new H_0 and H_1 in order to get transversals in the old H_0 and H_1. We note that S^* decreases by at least $9/4$, as we delete 3 edges and 1 vertex from H_0 and H_1 and we either delete a vertex in H_2 or 4 edges from H_0. As V^* decreases by one and $N_H[Y]$ increases by at most 21 (as $\Delta(H) \leq 4$, after Transformation 1), we are done.

Transformation 3: While there is a vertex $x \in V(H)$ with $d_{H_2}(x) \geq 3$, then do the following. Delete x and all edges incident to x from H_0 and H_2. Also delete these edges from H (but do not delete x or any edges incident to x in H_1). Furthermore delete any vertices in H_2, which get degree zero by the above transformation. If $d_{H_1}(x) = 0$ then delete x from H_1. If $d_{H_1}(x) > 0$, then we put $N_{H_1}[x]$ in Y.

Claim C: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim C: We note that T^* drops by at most two, as we may place x in the transversal of the new H_0 and H_2 in order to get transversals in the old H_0 and H_2. Lets count any edge, e, in H_1, which does not lie in H_0 as contributing $1 + |V(e) \cap V(H_0)|/3$ to the sum S^*. We note that there are no such edges when we start the transformation 3's.

We note that S^* now decreases by at least 25/12, because of the following. For every edge containing x in H_2, which does not lie in H_0 there is a vertex of degree one in the edge, due to the above transformations. Therefore we either delete an edge in H_0 or a vertex in H_2 for each of the edges containing x in H_2. As we also delete the edges in H_2 and the vertex x in H_0 and H_2 we note that S^* drops by at least 8/4. So if $d_{H_1}(x) = 0$ then S^* decreases by at least 9/4 as claimed. If $d_{H_1}(x) > 0$ and the edge, e, containing x in H_1 also lies in H_0, then we are done as we delete an extra edge in H_0 and the edge left in H_1 is counted as at most $1 + 2/3$. If $d_{H_1}(x) > 0$ and the edge, e, containing x in H_1 does not lie in H_0, then we decrease the value of e by 1/3 as $1 + |V(e) \cap V(H_0)|/3$ decreases. This shows that S^* decreases by at least 25/12.

As V^* decreases by one and $N[Y]$ increases by at most 21 (as $\Delta(H) \leq 4$, after Transformation 1), we are done.

Transformation 4: If $e_1, e_2 \in E(H_i)$ and $|V(e_1) \cap V(e_2)| \geq 2$ for some $i \in \{1, 2\}$, then we do the following.

If $|V(e_1) \cap V(e_2)| = 3$, then if $e_1, e_2 \in E_0$ we delete e_2 from both H_0 and H_i. If $e_j \notin E_0$ ($j \in \{1, 2\}$) then we delete e_j from H_i (in this case $V(e_j) \subseteq Y$). So now assume that $|V(e_1) \cap V(e_2)| = 2$ and $e_1 = (u_1, x, y)$ and $e_2 = (u_2, x, y)$, where $u_1 \neq u_2$,

If $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$, then by the above transformations we note that $e_1, e_2 \in E_0$. We now add a new vertex q to H, H_0 and H_i. We delete e_1 and e_2 from H, H_i and H_0 and add the edges $\{q, x, y\}$ to H, H_i and H_0.

If $d_{H_i}(u_j) = 1$, for some $j \in \{1, 2\}$, then do the following. Delete e_1, e_2 and the vertices $\{u_j, x, y\}$ from H_i. Add the vertices $\{u_1, u_2, x, y\}$ to Y.

Claim D: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim D: In the case when $|V(e_1) \cap V(e_2)| = 3$ we note that T^* remains unchanged, S^* decreases by 1/4 and $V^* - N[Y]$ remains unchanged. We are now done with this case.

In the case when $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$, we note that T^*, S^* and V^* remain unchanged and $N[Y]$ can only grow by adding q to it, but $q \notin V^*$. We also note that the above transformation decreases the number of edges in H_i, so it cannot continue indefinitely. We are now done with this case.

In the case when $d_{H_i}(u_j) = 1$, we note that T^* decreases by at most one, S^* decreases by 5/4, V^* decreases by at most three and $N[Y]$ increases by at most 24 (In $H - e_1 - e - 2$ we note that u_1 and u_2 have degree at most 3 while x and y have degree at most 2). As $1/4 \geq 27/372$ we are done with this case.

Claim E: $\Delta(H_1), \Delta(H_2) \leq 2$ and $\Delta(H - E(Y)) \leq 4$ and there are no overlapping edges in H_i, $i \in \{1, 2\}$.

Proof of Claim E: The fact that $\Delta(H_1), \Delta(H_2) \leq 2$ follow from Transformations 2 and 3. As $\Delta(H) \leq 4$ after Transformation 1 and no other transformation increases $\Delta(H)$, we note that $\Delta(H - E(Y)) \leq \Delta(H) \leq 4$. There are no overlapping edges in H_i, $i \in \{1, 2\}$ due to Transformation 4.

Claim F: If $e \in E(H) - E(Y)$, then $0 \in L(e)$ and $|L(e)| \geq 2$.

Proof of Claim F: This was true before Transformation 1 as it was true for all edges. Transformation 1 clearly does not change this property. In Transformation 2, we only keep an edge, e, in H_i, where $i \in \{1, 2\}$ but delete it in H_0 if we put $V(e)$ in Y. So the above still holds after Transformation 2. Analogously it also holds after Transformation 3. It is not difficult to check that it also holds after Transformation 4 (note that the above property holds for the edge we might add to H in Transformation 4).

We now see that (*) holds due to Lemma 3. That implies the theorem.

8. Possible strengthening of Theorem 8

No graph extremal for Theorem 8 is known and probably an inequality $f_t(G; 2) \leq \alpha|V(G)|$ can be obtained for some α smaller than $\frac{5}{4} - \frac{1}{372}$. Certainly α must be at least 9/8, that is demonstrated by the graphs of section 6.

There is a graph of order 12 having $f_t(H_{12}; 2) = 7n/6$, namely H_{12} from the family $\mathcal{H}$ defined after Theorem 2, with the two P_6's as its partition classes. Unless we, e.g., demand that the order of the graphs be large, H_{12} shows that we cannot get a better inequality than the following conjecture.

Conjecture 1. Let G be a graph of order n with $\delta \geq 3$ then $f_t(G; k) \leq 7n/6$.

9. Three partition classes

Theorem 9. *Let G be a graph of order n with $\delta \geq 3$ then $f_t(G; 3) \leq 3n/2$.*

For arbitrarily large n, $n \equiv 0$ (mod 6), there exist graphs G_n with $g_t(G_n; 3) = n$, $\gamma_t(G_n) = n/3$, $f_t(G; 3) = 4n/3$.

Proof. By Theorem 1 we have that $\gamma_t(G) \leq n/2$, and $g_t(G; 3) \leq n$ holds trivially, so by addition we get $f_t(G; 3) \leq 3n/2$ as desired.

Assume a graph G has $g_t(G; 3) = n$. Then $\Delta(G) \leq 3$ and as $\delta(G) \geq 3$, G is cubic. Since each vertex has three neighbours, one in each partition class, we see for each $i = 1, 2, 3$, that vertices in class V_i span a matching in G.

Listing the 3 neighbours to each V_i-vertex we count each vertex of G once, so $3|V_i| = n$ giving $|V_1| = |V_2| = |V_3| = n/3$.

Each V_1-vertex is adjacent to precisely one V_2-vertex and that has no other V_1-neighbour, so there is a perfect matching of V_1V_2-edges and analogously G contains perfect matchings of V_1V_3- and V_2V_3-edges.

One partition class V_i totally dominates G so $\gamma_t(G) \leq n/3$. In fact, $\gamma_t(G) = n/3$ because each vertex in G can totally dominate at most its three neighbours.

Following the steps above, it is now easy for $n \equiv 0 \pmod 3$ to construct a graph G_n with $g_t(G_n; 3) = n$. This graph has $f_t(G_n; 3) = \gamma_t(G_n) + g_t(G_n; 3) = 4n/3$.

We do not know if there, for $\delta \geq 3$, are graphs G with $4n/3 < f_t(G; 3) \leq 3n/2$, but we pose the following conjecture.

Conjecture 2. There exists some positive ϵ such that the following holds. If G is a graph with $\delta(G) \geq 3$, then $f_t(G; 3) \leq (3/2 - \epsilon)|V(G)|$.

Theorem 10. *Let G be a graph of order n with $\delta \geq 3$ and let $k \geq 4$. $f_t(G; k) \leq 3n/2$ and there exists an infinite family of graphs with $f_t(G; k) = 3n/2$.*

Proof. The inequality is proven as in Theorem 9. For a graph with $f_t(H; k) = 3n/2$ take $H \in \mathcal{H}$ ($\mathcal{H}$ is defined after Theorem 2). Let $v_1, v_2, \ldots, v_{n/2}$ and $u_1, u_2, \ldots, u_{n/2}$ be two disjoint paths in H such that $\{v_1u_2, v_2u_1, v_1v_{n/2}, u_1u_{n/2}\} \subseteq E(H)$. Let V_1, V_2, V_3, V_4 be a partition of H such that $l(v_1), l(v_2), \ldots, l(v_{n/2}) .... = 1, 2, 3, 4, 1, 2, 3, 4, ...$ and
$l(u_1), l(u_2), \ldots, l(u_{n/2}) .... = 4, 3, 2, 1, 4, 3, 2, 1, ...$ where $l(x) = i$ if $x \in V_i$,
then $f_t(H; V_1, V_2, V_3, V_4) = 3n/2$.

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Received: December, 2008