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A dissertation submitted to the Faculty of Technology and Science, Aalborg University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Reconstruction Methods for Inverse Problems

In danish:
Rekonstruktionsmetoder for inverse problemer

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Preface

This dissertation is the result of a Ph.D. study at The Department of Mathematics, Aalborg University, Denmark, conducted in the period from September 1st, 1998 to August 31st 2002. The theme of this dissertation is Reconstruction Methods for Inverse Problems as they apply to the Inverse Medium Problem, Inverse Scattering Problem and Inverse Conductivity Problem.

First an introduction to the physical and intuitive rational behind Inverse Problems, as they apply to non-destructive testing, is made. In Chapter 2 and Chapter 5 general remarks on the Inverse Medium Problem, and the Inverse Conductivity Problem are included. These chapters are included to give an understanding of the mathematical problems and to put results in their contexts. For more exhastive treatments I refer to the numerous references included in the bibliography.

In Chapter 3, the results in Berntsen, Cornean, and Moeller (2001), are given [**BCM01**]. Chapter 4 includes the work in Berntsen and Moeller (2002), [**BM02**]. Chapters 6 and 7 contain results obtained for the The Single Measurement Conductivity Problem, and numerical results hereon.

Early on in my Ph.D. studies, I was introduced to regularization techniques by Per Christian Hansen and Arnold Neumaier at the “Second Interdisciplinary Inversion Summer School”. A great deal of my research has been spent on utilizing regularization of different integral equations, as well as back projection methods in Computerized Tomography. I was introduced to the latter topic in 1999, during studies with Professor G. Uhlmann, at the University of Washington, Seattle, USA.

Acknowledgements

I would like to thank G. Uhlmann of the University of Washington, for kindly inviting me to participate in a semester-long workshop on Inverse Problems, at the Mathematical Sciences Research Institute “MSRI” in Berkeley, CA, USA. During said workshop, I had the opportunity to confer with Dr. R. Kress, of the University of Gottingen, and Dr. H. Kang, of Seoul National University, about my research, and thank them both for fruitful discussions. I am also grateful to the Knud Højgaard Foundation, and the Direktør Ib Henriksen Foundation for partially funding my studies at MSRI.

I especially thank my supervisor, Associate Professor Svend Berntsen, for his time and valuable guidance during my work on this dissertation. I extend gratitude to the Department of Mathematical Sciences at Aalborg University, for hosting me during my Ph.D. studies, and the faculty, who were always open for discussions. Special recognition goes to Assistant Professor Horia Cornean, and my colleagues Lasse Borup, and Kim Knudsen for their contributions in creating a highly stimulating environment in which to work, and for reading preliminary versions of this dissertation.

Finally, I would like to thank my family, and especially, my wife, Connie, for invaluable support and encouragement.

Aalborg, August, 2002 Steen Møller

This dissertation was defended on the 11^{th} of December, 2002, at Aalborg University. The opponents where Professor Arne Jensen (Aalborg University) (chair), Privat Docent Roland Potthast (University of Göttingen) and Michael Pedersen (Technical University of Denmark). I extend my gratitude to the opponents for their carefull reading off and comments to this dissertation. Any errors that remains are my responsibility

Minneapolis, May, 2003 Steen Moeller

Summary

This dissertation explorers problems of uniqueness, construction, and stability for three different Inverse Problems. These can be the physical problems where internal properties of different medias are sought from boundary measurement. Although the question of existence has not been answered, the notation of admissibility, data for which the reconstruction methods work that is not too restrictive, are introduced for two of the problems. The range space for Inverse Problems are normally not closed, so a good and not too restrictive characterization of admissibility can normally not be expected. The three Inverse Problems that have been explored are:

- The Inverse Medium Problem - time-harmonic fields.
- The Inverse Scattering Problem - one time dependent field.
- The Inverse Conductivity Problem - one current experiment.

In chapter 3, we considered the Inverse Medium Problem in $\mathbb{R}^3$ for a fixed frequency, when the unknown permittivity only depends on two variables and has compact support. A new reconstruction method was established, by reducing the problem to solving a Second-kind Fredholm problem. This enabled the definition of admissible data as a number of condition, such that the Second-kind Fredholm integral equation had one and only one solution. The conditions for admissible data could also be proved to hold for almost all frequencies, making it near optimal. A stability result followed for data satisfying the admissibility condition.

In chapter 4, we established a theory, Generalized Fourier Transforms, for the inversion of a large class of First-kind Integral Equations. The theory describes classes, such that the inverse operator for each class, will have a certain explicit simple structure. These classes may in some cases be characterized explicitly. This was done for one class related to the Fourier Transform, and for another class a subset hereof was characterized explicitly. Spaces, such that the Generalized Fourier Transforms tool was a homeomorphic mapping, was found for both of these classes. As an application of the theory, an Inverse Scattering Problem using one time dependent experiment was considered for an isotropic medium without absorption. Using the theory a reconstruction method was derived, and an associated uniqueness result was established in the Born approximation. A stability result hereof, related to reconstructing Fourier coefficients of the unknown permittivity, was obtained.

In chapter 6, the Inverse Conductivity Problem with one experiment was investigated. The conductivity was considered to be piecewise constant with known location of the jumps. An algebraic equation for the conductivity constants with at most two solutions was found. The solutions of this algebraic expressions depends continuously on the induced current. If the conductivity consisted of any finite number of disjoint piecewise constant inhomogeneities, then these could also be reconstructed uniquely from one boundary experiment. Numerical verification of the latter was done in chapter 7, where numerical stability, and dependence on limited aperture data, where tested.

Dansk Resumé
(Summary in Danish)

Denne afhandling afdækker spørgsmål om entydighed, konstruktion og stabilitet for tre forskellige Inverse Problemer. Disse problemstillinger kan være de fysiske problemer hvor objekters indre egenskaber ønskes bestemt fra målinger foretaget på randen af sådanne objekter. Eksistensproblematikken er ikke løst men begrebet tilladeligedata, som er mindst muligt restriktive krav for hvilke rekonstruktionsmetoderne virker, er introduceret for to af problemerne.

Billedrummet for mange Inverse Problemer er ofte ikke lukket, så en god og ikke for restriktiv karaterisering af tilladeligedata er normalt ikke forventelig. De tre Inverse problemer der er blevet behandlet er

- Det Inverse Medium Problem - tidsharmoniske felter.
- Det Inverse Sprednings Problem - et tidsafhængigt felt.
- Det Inverse Ledningsevne Problem - et strøm eksperiment.

I kapitel 3 betragtes det Inverse Medium Problem i $\mathbb{R}^3$ for fastholdt frekvens, når den ukendte permittivitet kun afhænger af to variabler og har kompakt støtte. En ny rekonstruktionsmetode blev etableret ved at reducere problemet til et Second-kind Fredholm integrallignings problem. Dette gjorde det muligt at definere tilladelige data som en række betingelser således at Second-kind Fredholm integralligningen havde en og højst en løsning. Betingelserne for tilladelige data kunne vises at holde for næsten alle frekvenser, hvilket er næsten optimalt. Et stabilitets resultat fulgte for tilladelige data.

I kapitel 4 blev teorien Generaliserede Fourier Transformationer etableret til at løse en stor klasse af First-kind Integralligninger. Teorien beskriver klasserne således at den inverse operator for hver klasse vil have en bestemt eksplicit struktur. Selve klasserne kan i visse tilfælde karakteriseres eksplicit. Dette er gjort for en klasse relateret til Fourier Transformationen og for en anden klasse er en delmængde heraf karakteriseret eksplicit. Rum, således at den Generaliserede Fourier Transformation er en homeomorf afbildning er fundet for disse klasser. Som en anvendelse af teorien er et inverst spredningsproblem for et isotropisk legeme uden absorption betragtet for en indkommende bølge. Ud fra teorien er en rekonstruktionsmetode udledt og i Born Approximationen er et tilhørende entydighedsresultat fundet. Et stabilitetsresultat hørende til Fourierkoefficienter for den ukendte permittivitet er opnået.

I kapitel 6 er det Inverse Ledningsevne problem med et eksperiment undersøgt. Ledningsevnen er antaget at være stykkevis konstant og med kendt lokalisering af springene. En algebraisk ligning med højst to løsninger er fundet for de ukendte ledningsevnekonstanter. Løsningerne til denne algebraiske ligning afhænger kontinuert af den påtrykte strøm. Hvis ledningsevnen består af et endeligt antal disjunkte stykkevis konstante inhomogeniteter så kan disse også rekonstrueres entydigt fra et randeksperiment. Numerisk verifikation af det sidste er gjort i kapitel 7, hvor numerisk stabilitet såvel som afhængighed af begrænset apparatur er undersøgt.

Contents

Chapter 1

Introduction

The theme of this dissertation is Reconstruction Methods for Inverse Problems related to non-destructive testing. Results are established on the Inverse Scattering Problem, Inverse Medium Problem, and Inverse Conductivity Problem. All are Inverse Boundary Value Problem for a body Ω embedded in $\mathbb{R}^n$, $n = 2, 3$. In Chapters 2 and 5 the mathematical introductions of the Inverse Medium Problem, and the Inverse Conductivity Problem, are made.

The problems of non-destructive testing are, from knowledge of measured data on the boundary of $\partial\Omega$, to reconstruct what is inside Ω. For the Inverse Conductivity Problem, the boundary data consists of pairs of data. The pairs are a current and the measured potential related to this current, or by symmetry, the current related to the applied potential. For the Inverse Scattering Problem, and Inverse Medium Problem the data are acoustic fi elds at $\partial\Omega$ or any boundary encircling Ω. These fi elds are scattered waves from time-dependent and time-independent waves respectively. The time-dependent scattered wave is known for all times and the time-independent wave is known for all incident waves with a fi xed frequency.

The mathematical question of uniqueness for the Inverse Boundary Value Problem is, initially, to determine what type of data can be obtained, and then, if all possible experiments are suffi cient, for a unique determination of what is inside Ω.

In full generality, for anisotropic mediums, the answers for uniqueness have been either negative or are still unknown. However, there are many important problems where a-priori knowledge leads to Inverse Boundary Value Problems, where uniqueness may indeed be established.

For the above mentioned Boundary Value Problems, extensive literature is available. These include, the study of existence, uniqueness, and stability. The actual numerical computations of solutions to forward problem may often be very extensive calculations, and are treated through Finite Element Methods, Finite Difference Methods, Integral Equation Methods, and combinations hereof.

For the two elliptic Inverse Boundary Value Problems with isotropic mediums, the Inverse Medium Problem and the Inverse Conductivity Problem, the most complete treatment has been undertaken using the Lippmann-Schwinger-Fadeev scattering solutions. In $\mathbb{R}^3$ uniqueness of the Inverse Medium Problem has been found, using the scattered fields related to incident plane-waves from all directions [**Nac88**]. In $\mathbb{R}^3$ uniqueness has been established for the Inverse Conductivity Problem when the conductivity σ has smoothness $\sigma \in C^{1,1}$, i.e. almost two derivatives, [**Nac88**]. For $\mathbb{R}^2$ uniqueness has not been established for the Inverse Medium Problem, but it has been established for regularity of the class $W^{2,p}$, $p > 1$ [**BU97**] for the Inverse Conductivity Problem. For anisotropic medias, the question of uniqueness has been answered negative for both the Inverse Medium Problem and the Inverse Conductivity Problem.

When the medium has some a-priori regularity, or less information about Ω is sought, many uniqueness results have been established. This has led to the question of how little information is needed in order to achieve some of these uniqueness results. Some of these are what can be obtained from one experiment and what can be obtained from partial aperture? Answers to questions of this type will be addressed.

From the viewpoint of applications, the problems of uniqueness are less relevant. Instead, questions of reconstruction and stability is at heart. The pitfall for many inverse problems are that they suffer from being ill-posed. This means that large changes in the subject may only induce small differences in the observations and visa versa, making it difficult to distinguish the internal properties. Stability estimates are therefore sparse.

The concept of Inverse Boundary Value Problems, may nicely be understood by an analogy, that Christopher R. Johnson made in the introduction for an IEEE theme, on "Computational Inverse Problems in Medicine" [**Joh95**], where he compared inverse methods with imitating the great detective Sherlock Holmes. Johnsen writes:

> " Methods for solving Inverse Problems are the opposite of prediction? Basically, you carefully study the evidence of the scene and from it try to infer who was there and what happened. To do this you might simply observe, but more likely, you will poke or tap at the scene - send various signals through it, for instance, and take sensitive detector readings of what happened to them. Then fitting the evidence to your knowledge of how the world works? you essentially throw out all scenarios that are impossible and whatever is left, however improbable, must be the truth."

The Aim of this Dissertation

The aim of this dissertation is to consider the following Inverse Boundary Value Problems, and under some a-priori assumptions address the questions of reconstruction and stability. The problems and reconstruction to be considered are:

1. For The Inverse Medium Problem for $\mathbb{R}^3$ using one frequency and fields from all incident directions, to reconstruct the refractive index.

2. For The Inverse Scattering Problem using one time dependent incident field to reconstruct the refractive index.

3. For The Inverse Conductivity Problem using one current experiment to determine a piecewise constant conductivities with fixed location.

Numerical implementation of all these reconstruction methods have been undertaken, but after some effort, two of these have been abandoned, since results could not be expected within the project time.

The Result of this Dissertation

For the Inverse Medium Problem in $\mathbb{R}^3$, if the scatter has some smoothness, and depends on only two variables, we found:

- An explicit reconstruction method;
- Verifiable conditions (admissible data) that almost always are satisfied for the scattered data from scatters, which satisfy the assumptions;
- Stability of the method, within the class of admissible data.

For the Inverse Conductivity Problem, if the conductivity is piecewise constant, and the location in the jumps in the conductivity is known, we found:

- That one pair of a current and the associated potential are sufficient for the unique determination of N disjoint constants;
- That there exist currents such that, for two nested piecewise constant inhomogeneities, there exists at most two different sets of constants.

For the Inverse Scattering Problem:

- A theory for finding simple inverse integral operators to a large number of classes of integral equations. The theory has been made explicit for some classes, and mapping properties of the operators are found. Spaces for which the inverse integral operators are homeomorphic mappings are found;
- The use of the integral operators in the Born Approximation of the Inverse Scattering Problem for acoustic time-dependent waves was done. Fourier Coefficients related to the unknown wavenumber is expressed explicitly as an algebraic expression in terms of the measured scattered wave.

The Structure of this Dissertation

Chapter 2, considers the physical problem of Acoustic Scattering, and discusses the results of Chapters 3 and 4.

Chapter 3 is the work in S. Berntsen, H. Cornean, and S. Moeller (2001), *Wavenumber Reconstruction from Boundary Measurements*, Technical Report R-01-2005.

Respectively, Chapter 4 includes the work in S. Berntsen and S. Moeller (2002), *Generalized Fourier Transform Classes, Integral Transforms and Special Functions*, Volume 13, Issue 5, 2002, pages 447-459. Also, an application hereof to the Inverse Scattering Problem is included.

Chapter 5 review the Conductivity problem, and Chapters 6 and 7 are new theoretical and numerical results for The Single Measurement Inverse Conductivity Problem.

Each chapter may be accessed independently of each other except for chapter 7 which builds on results of chapter 6.

Chapter 2

Acoustic Scattering

In this chapter, the Acoustic Scattering Problem is considered and the results on Inverse Acoustic Problem obtained in [**BCM01**] and [**BM02**] are discussed. These results are presented in Chapter 3 and 4. The aim of this chapter is to motivate some physical applications, define the mathematical problems and highlight some key results. Reconstruction results will be emphasized with focus on the Dual Space Method, and the reduction to and motivation of the Linear Sampling Method. The reconstruction results from Chapter 3 and 4 concludes this chapter.

Acoustic scattering, is the scattering of a sound wave by an obstacle embedded in a (normally) homogeneous background. An obstacle can be either acoustically penetrable or impenetrable, depending on whether the wave can travel through the media or not. If a penetrable obstacle has a variation in the density, it is an inhomogeneous obstacle. Different types of obstacles, will be characterized by different boundary conditions on the obstacle, depending on how the wave gets scattered and/or penetrate the obstacle.

The direct problem of acoustic scattering is, given the knowledge of what type of scatter and what the incident wave is, to find the scattered wave. The inverse problem takes this answer to the direct scattering problem as its' starting point, and asks what is the nature of the inhomogeneity that gave rise to a particular scattered field.

If the acoustic wave is time harmonic, then the field amplitude is described as the solution of Helmholtz equation with a radiation condition. Whereas, if it is a time dependent field, then the wave is described through the wave equation.

What is referred to as the forward problem, is discussed and Far Field Patterns and the Born Approximation are introduced. Subsequently, general questions of uniqueness, existence, and reconstruction will be discussed for the Inverse Problem. Here, it is beneficiary to remark that there are only very few existence results for inverse problems. This is due to the non-trivial task of specifying appropriate conditions on data.

2.1. The Forward Problem

Inspired by the theory of Electromagnetic Propagation denote $n(\mathbf{x})$ the refractive index and assume that $n_0 - n(\mathbf{x})$ is compactly supported in a bounded set D with C^2 boundary and that $n \in C^1(D)$. An imaginary part of n is used for modeling absorption. The wave vector is defined as $k = \omega/c_0$ where c_0^2 is the sound speed for the background medium.

The mathematical formulation for the scattering of a Time Harmonic Acoustic Wave u^{in} in $\mathbb{R}^3$ is for fixed k to find the total field $u \in C^2(\mathbb{R}^3)$ solving

$$\begin{aligned} (\nabla^2 + k^2 n(\mathbf{x}))u(\mathbf{x},\omega) &= 0 && \mathbf{x} \in \mathbb{R}^3 && (a) \\ u(\mathbf{x},\omega) &= u^{in}(\mathbf{x},\omega) + u^{sc}(\mathbf{x},\omega) && && (b) \\ (\nabla^2 + k^2 n_0)u^{in}(\mathbf{x},\omega) &= 0 && \mathbf{x} \in \mathbb{R}^3 && (c) \\ \lim_{r\to\infty} r\left(\frac{\partial u^{sc}}{\partial r} - iku^{sc}\right) &= 0 && r = |\mathbf{x}| && (d) \end{aligned} \tag{2.1}$$

where the background permittivity is n_0 and $\mathrm{Im}(n) \geq 0$.

The most common tool for studying existence, uniqueness and continuous dependence of u on u^{in} of (2.1a)-(2.1d) is the Lippmann-Schwinger equation. This equation is defined as

$$u(\mathbf{x},\omega) = u^{in}(\mathbf{x},\omega) - k^2 \int_D \Phi(\mathbf{x},\mathbf{y})(n_0 - n(\mathbf{x}))u(\mathbf{y},\omega)d\mathbf{y} \qquad \mathbf{x} \in \mathbb{R}^n, \tag{2.2}$$

where $\Phi(\mathbf{x},\mathbf{y})$ is the free space fundamental solution for the Helmholtz equation defined as

$$\Phi(\mathbf{x},\mathbf{y}) = \begin{cases} (e^{ik\sqrt{n_0}|\mathbf{x}-\mathbf{y}|})/(|\mathbf{x}-\mathbf{y}|) & \mathbf{x} \in \mathbb{R}^3 \\ \frac{i}{4} H_0^1(k\sqrt{n_0}|\mathbf{x}-\mathbf{y}|) & \mathbf{x} \in \mathbb{R}^2 \end{cases}$$

where H_0^1 is a Hankel function of the first type and order zero. The acoustic problem (2.1a)-(2.1d) and (2.2) are equivalent for functions in $C^2(\mathbb{R}^n)$.

Since (2.2) is a Second Kind Fredholm Integral Equation, existence, uniqueness, and continuous dependence of u on u^{in} are readily discussed. From a unique continuation argument follows that (2.1) has at most one solution [**CK98**]. From the Riesz-Fredholm theory then follows that (2.2) can be solved uniquely for $u \in C^2(\mathbb{R}^3)$ and that $u \in C^2(\mathbb{R}^3)$ depends continuously on u^{in}. The forward problem of acoustic scattering is therefore well-posed in the setting of Hadamard [**Had23**]. That is, it satisfies: existence, uniqueness and stability.

The concept of Far Field Patterns that is to be introduced is a functions essentially containing all possible information about the scatter. And, even though it will not be addressed further we will in passing mention that the Far Field Pattern are for the Acoustic Scattering Problem what the Dirichlet-Neumann map is for the Conductivity Problem from Chapter 5.

Every radiating solution u to the Helmholtz equation in $\mathbb{R}^3$ has the asymptotic behavior of an outgoing spherical wave

$$u^{sc}(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\Big\{u_\infty(\hat{\mathbf{x}}) + O\big(\frac{1}{|\mathbf{x}|}\big)\Big\}, \quad |\mathbf{x}| \to \infty \tag{2.3}$$

uniformly in all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, where the function $u_\infty(\hat{x}, d)$ defined on the unit sphere S^2 is known as the Far Field of u. Furthermore any incident field may be expressed as the combination of incident plane waves. The Far Field Pattern is the Far Field from all incident plane waves with directions $\hat{d} \in S^2$, and is denoted $u_\infty(\hat{\mathbf{x}}, \hat{d})$. Therefore if

$$u^{in}(\mathbf{x}) = \int e^{ik\mathbf{x}\cdot\mathbf{d}} g_{pq}\, d\, s(d) \quad \mathbf{x} \in \mathbb{R}^3 \quad g_{pq} \in \mathscr{L}^2(\Omega)$$

(which is also the definition of a Herglotz Wave Function) then the Far Field for u^{in} is

$$u_\infty(\hat{\mathbf{x}}) = \int_{S^2} u_\infty(\hat{\mathbf{x}}, d) g(d)\, ds(d).$$

The Far Field operator $\mathcal{F} : \mathscr{L}^2(S^2) \to \mathscr{L}^2(S^2)$ is defined as

$$(\mathcal{F}g)(\hat{\mathbf{x}}) = \int_{S^2} u_\infty(\hat{\mathbf{x}}, d) g(d)\, ds(d)$$

and has dense range for $\hat{\mathbf{x}} \in S^2$ if Im$(n) > 0$, [**CK98**]. From (2.2) and (2.1c) is seen that

$$u^{sc}(\mathbf{x}) = -k^2 \int_{\mathbb{R}^3} \Phi(\mathbf{x}, \mathbf{y})(1 - n(\mathbf{y}))u(\mathbf{y})d\,\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.4}$$

Since for fixed k,

$$\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\Big(e^{-ik\hat{x}\cdot y} + O\big(\frac{1}{|\mathbf{x}|}\big)\Big)$$

it follows from (2.3) that

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}}(1 - n(\mathbf{y}))u(\mathbf{y})d\mathbf{y}.$$

If $k\|n\|_\infty \ll 1$ then $u(\mathbf{y}) \approx u^{in}(\mathbf{y})$, and the Born Approximation is obtained as

$$u^b_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int e^{ik\hat{x}\cdot\mathbf{y}}(1 - n(\mathbf{y}))u^{in}(\mathbf{y})d\,\mathbf{y} \tag{2.5}$$

Some methods for solving (2.1a)-(2.1d) are discussed in [**CK98**] and general methods for solving the Second Kind Fredholm Integral Equation (2.2) can be found in [**Atk76**], [**Kre98**]. Recently, Vainikko proposed a new numerical method for solving the Lippmann-Schwinger equation [**Vai00**]. His method has recently been used by Hohage [**Hoh01**] for the Acoustic Scattering Problem and by Siltanen [**Sil99**], in his implementation of Nachman's reconstruction method for the Inverse Conductivity Problem.

2.2. The Inverse Problem

Considering the Inverse Problem for recovering the permittivity that caused an observed scattered field, two distinct methods for discussing a solution appear. One is to find the support of n, which amounts to the question "where is it?" and the other, is the actual reconstruction of n, which is the answer to "what is it?"

The onset for studying the Inverse Medium Problem is the determination of sufficient physical information for uniqueness. Uniqueness for obstacle scattering was from knowledge of the Far Field Pattern on the units sphere S^2 and one frequency initially proved by Schiffer (1960), [**CK83**]. The question whether the knowledge of the Far Field Patterns for $\hat{x}, d \in \Omega$ and fixed k also uniquely determines the index of refraction in $\mathbb{R}^3$ was answered in the affirmative by Nachmann [**Nac88**] and Novikov [**Nov88**], independently. Their ideas were motivated by the paper of Sylvester and Uhlmann [**SU87**] on Complex Geometrical Optics Solutions for the Inverse Conductivity Problem. The proof has subsequently been simplified by Hähner [**Häh96**] using Fourier series techniques.

For $\mathbb{R}^2$ uniqueness is yet not known, but partial results exist. If two Far Field Patterns for two different permittivities and same frequency agree then the difference between these refractive index is $C^{0,\alpha}$ [**SU93**]. Also, if the Far Field Pattern agree for an interval of frequencies uniqueness can be proved.

The uniqueness proof of Nachmann follow as a consequence of a unique reconstruction algorithm. The reconstruction method though has so far not been implemented for the Acoustic Problem since it is rather difficult to follow up all the steps from which the reconstruction algorithm is made of. Stability for the Inverse Medium Problem was, however, proved by Hähner using a method of Stefanov [**Ste90**], that does not rely on a reconstruction method. If two permittivities n_1 and n_2 are close Hähner found that

$$\|n_1(\mathbf{x}) - n_2(\mathbf{x})\|_\infty \leq c(-\ln(\|F_{n_1} - F_{n_2}\|))^{-1/15}$$

where a complicated norm was used on the Far Field operators F_{n_i} [**Häh96**]. Recently, an estimate with the L^2 norm on the Far Field Operators and the factor $(-1/15)$ expressed in terms of a Sobolev regularity index s have been found by Hohage and Hähner, [**HH01**]. There the assumption of n_1 and n_2 being close was also removed. Furthermore, they found that better stability estimates can be obtained for point sources than for plane waves.

From applicational and theoretical aspects, it is of interest to state under what a-priori conditions simple reconstruction algorithm can be derived, and to see what can be recovered for incomplete data.

2.2.1. Reconstruction Methods. The Inverse Medium Problem that may be proposed from the Lippmann-Schwinger equation (2.4) is to find a function $n(\mathbf{x})$ such that

$$A_u n = u^{sc} \tag{2.6}$$

where $A_u : \mathscr{L}^2(D) \to \mathscr{L}^2(S^2 \times \partial B(0,R))$ depends on the total field itself and when $\partial B(0,R)$ is a sphere containing D. This equation is nonlinear in u and linear in n. The data are normally the Far Field Patterns

$$\mathcal{F} := \{u_\infty(\cdot, d_n) | n = 1, 2, \cdots\}$$

where $\{d_n | n = 1, 2, \cdots\}$ denote a countable dense set of vectors on the unit sphere indicating incident directions of plane waves $u^{in}(\mathbf{x}, \omega, d)$. For recovering the index of refraction n, various Born Approximations have been suggested linearizing (2.6) to essentially (2.5). In the survey article [**CR88**] some different Born Approximations are discussed, along with error estimates, when such can be established.

In some engineering papers, the Lippmann-Schwinger equation is referred to as the Contrast Type Integral Equation and the Born Approximation of (2.6) is the Distorted-wave Born Approximation [**TBLdH01**].

Instead of working with the Lippmann-Schwinger equation, Gutman and Klibanov [**GK94**] in a series of papers, considered the Helmholtz equation directly. They developed a regularized quasi-reversible method for determining a finite set of Fourier Components of $n(\mathbf{x})$. Their method applies to slowly varying permittivities which is, for instance, the case in underwater acoustics. Some medical applications also satisfy this assumption [**Ish78**].

In many applications, slowly varying permittivities and Born Approximations are not justified, and methods that deal with the full nonlinearity of the problem are needed.

Monk and Colton have developed the Dual Space Method, to be explained in section 2.2.2, for reconstructing $n(\mathbf{x})$ which does not directly use (2.5) but leads to a least squares optimization method involving the Lippmann-Schwinger equation [**CK98**]. Their method is somewhere between solving the nonlinear equation (2.6) and the linearized equation (2.5). Their investigations have led to the suggestion of the Linear Sampling Method, to be explained in section 2.2.3, for determining the support of inhomogeneous scatters. The first version of the Linear Sampling Method was proposed by Kirsch and Colton [**CK96**], and subsequently improved by Colton, Piana, Potthast [**CPP97**], and Kirsch [**Kir98**] and [**Kir99**].

2.2.2. The Dual Space Method. The Dual Space Method of Colton and Monk is essentially an extension to the Inverse Medium Problem of a method developed for the Inverse Obstacle Problem. It is based on properties of Herglotz wave functions which are entire solutions to the Helmholtz equation $(\Delta + k^2)v = 0$, defined as

$$v(\mathbf{x}) = \int_{S^2} e^{ik\mathbf{x}\cdot\mathbf{d}} g(\mathbf{d}) ds(\mathbf{d}), \quad \mathbf{x} \in \mathbb{R}^3, \qquad g \in \mathscr{L}^2(S^2).$$

It can be shown [**CK98**] that the Far Field Pattern is proportional to $\frac{i^{p-1}}{k} Y_p^q(\hat{\mathbf{x}})$ if and only if the scattered field is $u_p^g = h_p^1(k|\mathbf{x}|) Y_p^q(\hat{\mathbf{x}})$. The Dual Space Method is to first determine g_{pq} (i.e. the incident field) from

$$Fg_{pq} := \int_{\Omega} u_{\infty}(\hat{\mathbf{x}}, \mathbf{d}) g_{pq}(\mathbf{d})\ ds(\mathbf{d}) = \frac{i^{p-1}}{k} Y_p^q(\hat{\mathbf{x}}). \tag{2.7}$$

Then from the differential equations

$$\Delta w + k^2 n(\mathbf{x}) w = 0, \qquad \Delta v + k^2 v = 0 \quad \mathbf{x} \in D, \tag{2.8}$$

and the boundary conditions

$$w = u_p^q + v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial u_p^q}{\partial \nu} + \frac{\partial v}{\partial \nu} \quad \mathbf{x} \in \partial D, \tag{2.9}$$

to determine w and $n(\mathbf{x})$. This method for finding w and n in turn leads to a large optimization problem.

If $\text{Im}(n) = 0$ the Dual Space Method have problems because there may exist nontrivial solutions of the homogeneous problem of (2.9) and (2.8). Different modifications to the Dual Space Method has been suggested to avoid this problem. A comparison between two methods can be seen in [**CK98**]. It can be seen that when $\text{Im}(n) \approx 0$ the solvability problems for the Dual Space Method becomes apparent.

2.2.3. Reconstructing the Support. The equations for the linear sampling method is essentially the same as for the Dual Space Method except that it does not solve a large optimization problem as in the Dual Space Method.

The price paid for avoiding solving a large nonlinear optimization problem is that only the support of anomalies in the index of refraction against a background (not necessarily constant, see [**CCM00**] for discussion hereof) are obtained rather than the actual values of the index of refraction itself. Their method also works for inhomogeneous background mediums, which is not the case for most other methods.

Instead of using $\frac{i^{p-1}}{k} Y_p^q$ in (2.7) the Far Field Pattern $\Phi_\infty = (4\pi)^{-1} e^{ik\hat{x}\cdot\mathbf{y}}$ for the fundamental solution $\Phi(\mathbf{x}, \mathbf{y})$ is used. The Far Field Operator has dense range if $\text{Im}(n) > 0$ and hence for every $\varepsilon > 0$ there exist a function $g(\cdot, \mathbf{z}) \in \mathscr{L}^2(S^2)$ such that $\|Fg_{pq} - \Phi_\infty\|_{\mathscr{L}^2(S^2)} < \varepsilon$. For $\mathbf{z}$ approaching ∂D it can be shown that the Herglotz wave Function becomes unbounded in $\mathscr{L}^\infty$ which is only possible if g_{pq} becomes unbounded. The support of n can be found from solving (2.7) for different values of $\mathbf{y}$ and, observing where $g_{pq}(\cdot, \mathbf{y})$ becomes unbounded.

Kirsch proved that for a non-absorbing medium the function $\Phi_\infty(\cdot, \mathbf{y})$ is in the range of $(F^*F)^{1/4} g_{pq} = \Phi_\infty(\cdot, \mathbf{y})$ if and only if $\mathbf{y} \in D$. From the operator equation $(F^*F)^{1/4}$ it is therefore possible to conclude that $\|g(\cdot, z)\|$ becomes unbounded both when $\mathbf{z}$ approached ∂D from inside and outside D. A property that has not been proven for the original linear sampling method, but was observed.

Numerical experiments with the Linear sampling method shows so far to be better for finding the support of impenetrable scatters than inhomogeneous obstacles [**TCP02**], even though the method does not a-priori assume that the boundary conditions on ∂D are known.

Another method for finding the boundary of D is the Point Source Method of Potthast [**Pot97**], [**Pot01**] which also is based on the existence of a Herglotz Wave Function. In light of the improved stability estimate for point sources by Hähner and Hohage point sources seem favorable over say the Linear Sampling Method that uses plane waves. But since here both methods are only used for finding the support of D, and not the refractive index which is what the stability estimate is about, the validity of such a statement has not been proved. However an advantage of the Point Sources Method is that a minima is sought, whereas a maximum tending to infinity is sought in the Linear Sampling Method. This means that the regularization for the Linear Sampling Method is used differently from normal regularization where g's norm usually is penalized in order that is does not get to large.

For the Inverse Medium Problem both (2.7) and (2.5) needs to be regularized in order to get satisfactory reconstructions of g_{pq} and n respectively. For the linear Sampling Method a comparison between the four different regularization strategies TSVD, Tikhonov, Landweber Iteration, and Conjugated Gradient, has been made in [**TCP02**].

2.3. Wave Number Reconstruction

In [**BCM01**] the reconstruction of the refractive index (or the permittivity) $n(\mathbf{x})$ for dimensional simple scatters was considered. The refractive index was assumed to be cylindric, depending only on base variables, i.e. $n(\mathbf{x}) = \Lambda \times [0, a]$, where Λ depends only on base variables x_1, x_2 and has compact support. The background medium was assumed to have a positive absorption, which is the physical realistic case, i.e. $\text{Im}(n_0) > 0$. For the Inverse Medium Problem it is possible to derive a simple reconstruction algorithm. The reconstruction method is based on the Helmholtz Equation. Because of the simplicity of the reconstruction method, it is then possible to list explicitly necessary demands that the scattered field must satisfy in order to come from a permittivity of the form $n(\mathbf{x}) = \Lambda \times [0, a]$. These conditions are not minimal in the sense that they are neccesary, but scattered fields satisfying these assumptions can be proved to exist. The reconstruction method is proved to be stable and unique for small frequencies and bounded permittivities in $C^1(D)$. For higher frequencies, there exists at most a discrete set of frequencies for which the reconstruction method does not work, and they do not accumulate at zero.

When comparing with solving (2.7) and (2.5), finding conditions that the scattered field must satisfy is appealing. For (2.7) and (2.5) regularization techniques much be applied, and appropriate conditions on the data are not known.

The reconstruction method in [**BCM01**] is based on the commutation of the Laplacian in rectangular coordinates with the directional derivative in the direction of independence; $[\partial_{x_3}, \Delta_{x_1,x_2,x_3}] = 0$. Consider the simpler model of Helmholtz equation on the bounded convex Lipschitz domain $D = supp(n)$. Assume that $\phi_1, \phi_2 \in H^{1/2}(\partial D)$ and let u_i, $i = 1, 2$ be solutions of the boundary value problem

$$(\Delta + n(\mathbf{x}))u_i = 0 \qquad \mathbf{x} \in D \tag{2.10}$$

$$u_i = \phi_i \qquad \mathbf{x} \in \partial D \tag{2.11}$$

Since D is convex with Lipschitz boundary, (2.10) and (2.11) has a unique solution, [**Gri85**] [**Isa98**]. From Green's theorem follows that

$$\begin{aligned}\int_{supp(\Lambda)|_0^a} u_1 u_2 n(\mathbf{x}) dx_1 dx_2 &= \int_D \partial_{x_3}(u_1 u_2 n(\mathbf{x})) d\mathbf{x} \\ &= \int_{supp(\Lambda)} (\partial_{x_3} u_1)(\partial_{x_3} u_2) - \nabla_{x_1,x_2} u_1 \cdot \nabla_{x_1,x_2} u_2 \Big|_0^a dx_1 dx_2 + \\ &\int_0^a \int_{supp(\partial\Lambda)} \partial_{x_3} u_1 \nu_2 \cdot \nabla_{x_1 x_2} u_2 + \partial_{x_3} u_2 \nu_2 \cdot \nabla_{x_1 x_2} u_1 d\mathbf{x} =: \eta \end{aligned} \tag{2.12}$$

where ν_2 is the outward normal to $\partial\Lambda \subset \mathbb{R}^2$. The boundary integral η only contains known boundary contributions.

Hence, for this dimensional simple domain, it is easy to fi nd $n(\mathbf{x})$ from knowledge of the total fi eld on the boundary. In [**BCM01**], the formula is essentially (2.12), except for being over a ball, see (3.15) page 22. Becausse $\partial_\nu u$ is not continous across $\partial\Omega$, the extension of η in (2.12) to an integral over a sphere is nontrivial.

By assuming that u^{in} is a plane-wave with a fi xed frequency and complex wave number $k^2 n_0$ and that $u_1 = u_2$, the kernel $u^2\big|_0^a$ decomposes as

$$\begin{aligned} u^2(a, \underline{p}) - u^2(0, \underline{p}) &= (e^{i\underline{p}\cdot\underline{x} + ip_3 a} + u^{sc}(\underline{x}, a, \underline{p}))^2 - (e^{i\underline{p}\cdot\underline{x}} + u^{sc}(\underline{x}, 0, \underline{p}))^2 \\ &= (e^{ip_3 a} - 1) e^{i2\underline{p}\underline{x}} + \mathcal{L}_1 \end{aligned}$$

and hence

$$\int e^{i\underline{p}\underline{x}} n(\mathbf{x}) + \int \mathcal{L}_1 n(\mathbf{x}) = \eta \tag{2.13}$$

If $\mathcal{L}_1 \in \mathscr{L}^2(\Lambda \times \mathbb{R}^2)$ and $\eta \in \mathscr{L}^2(\mathbb{R}^2)$ then the Fourier transform turns (2.13) into a second Fredholm integral problem. In [**BCM01**] conditions necessary for (2.13) to be solvable was identifi ed, see defi nition 3.5, and scattered fi elds satisfying this list of conditions was denoted as admissible experiments;$\mathcal{A}_\omega$. The list of conditions for admissible experiments was then for a fi xed $n(\mathbf{x})$ shown to hold for all experiments, except for at a discrete set of frequencies.

We restate (see theorem 3.9 page 24)

Theorem 2.1. *Let $\omega_0 > 0$ and $N > 0$ be arbitrarily large, but fixed.*
i. Assume that $k \in C^1(x_1, x_2)$ and ω is allowed to take values in $(0, \omega_0)$. Denote by $u \in H^2_{\mathrm{loc}}(\mathbf{R}^3)$ the solution to the forward problem corresponding to k^2 (see Definition 3.1 and formula (3.12)). Then there exists a finite set $M \subset (0, \omega_0)$ such that for any frequency $\omega \in (0, \omega_0) \setminus M$ one has $u^{sc}|_{\partial\Lambda'} \in \mathcal{A}_\omega$.

ii. There exists $\omega_N > 0$ such that for any $\omega \in (0, \omega_N)$ and any $k^2 n(\mathbf{x}) \in C^1(x_1, x_2)$ with $\max(\|Re(k\sqrt{n})\|_{C^1}, \|Im(k\sqrt{n})\|_{C^1}) \leq N$ (see Definition 3.3 page 21) one has $u^{sc}|_{\partial\Lambda'} \in \mathcal{A}_\omega$. Therefore, in this case $k^2 n(\mathbf{x})$ can be uniquely reconstructed by (2.13).

The uniqueness results of theorem 2.1 for solving (2.13) means knowing the scattered field related to incident plane waves for all directions. For a finite number of incident plane waves, uniqueness is not established but reconstruction of $n(\mathbf{x})$ in some finite dimensional space is possible by solving the First Kind Fredholm Integral equation (2.12). This is similar in spirit to Gutman and Klibanov trying to obtain a finite number of Fourier coefficients, except here the full nonlinearity is used.

2.3.1. Ill-posed Integral Equations. Assume that the total fields from a finite number of incident fields are known on a sphere with radius R containing the cylinder $supp(n)$, i.e. $\left\{u^{sc}(\mathbf{x}, \underline{p}_n) \middle| \underline{p}_n \in \mathbb{R}^2 \times \mathbb{R}^2, n = 1, \cdots N\right\}$. Firstly, the field has to be backprojected to the hyper-surfaces $(\Lambda, 0)$ and (Λ, a). Since u is known on the surface of $B(0, R)$, u can be found in the complement of $\overline{B(0,R)}$ from solving the exterior Dirichlets problem. Denote by ϕ a vector in the x_3 direction, and let Γ^- and Γ^+ denote hyper-surfaces perpendicular to ϕ at $(0, 0, -R)$ and $(0, 0, R)$ respectively. The scattering problem (2.1) is then an initial value problem

$$\begin{aligned} (\Delta u + k^2(n_0 + n(\mathbf{x}))u &= 0 \qquad && x_3 > -R \\ u &= u^- && x_3 = -R \\ \partial_\nu u &= \partial_\nu u^- && x_3 = -R \end{aligned}$$

and a similar problem can be written for Γ^+. This formulation is identical to the one of Natterer [**Nat97**] where he uses this formulation to propagate $u|_{\Gamma^-}$ to $u|_{\Gamma^+}$ for known $n(x)$. He proves that the instability in recovering $u|_{\Gamma^+}$ is a pure high-frequency phenomenon, hence frequency regularized $u|_{x_3=0}$ and $u|_{x_3=a}$ can be found stably for sufficiently large k^2.

Since $\mathcal{L}_1$ in (2.13) is needed for all $\underline{p} \in \mathbb{R}^2$, (2.12) can be used for reconstructing $n(\mathbf{x})$, except that the boundary data must be taken on the sphere or any other geometry that contains $supp(n)$, unless Λ is convex, in which case the boundary data can be propagated to $supp(n)$. The first kind integral equation (2.12) therefore reads

$$\int_\Lambda u^2(\underline{x}, \underline{p}) n(\underline{x}) dx = \eta(\underline{p})$$

and must be solved using regularization methods.

Remark: The boundary integral η is numerically unstable since it contains derivatives of $u\big|_{\partial B(0,R)}$. Also for finite number of incident directions, $u\big|_\Lambda$ is also regularized, making (2.12) more ill-conditioned.

2.4. Generalized Fourier Transforms

A second reconstruction method that has been established for non-absorbing media (i.e. $\text{Im}(k) = 0$) was based on the data from one time dependent experiment. For this the inversion tool "Generalized Fourier Transforms" for inverting First Kind Integral Equations with special classes of integral kernels, has been established, [**BM02**]. This theory complements the theory of $\mathbf{H}$-transforms established by [**SSK98**] for product kernels. An application hereof to low-contrast mediums of small size have been indicated in [**Ber00**], and is here extended to general low-contrast mediums. An explanation of the "Generalized Fourier Transforms" theory is made in section 2.4.1 and the application to low-contrast mediums is explained in section 2.4.2

2.4.1. The Generalized Inverse. For some First Kind Integral Equations analytic inversion tools exist that are better than the regularization methods discussed in [**Han98**] and [**EHN96**]. If the kernel is a convolution kernel $k(s,t) = k(s-t)$ and $k(s,t) \in \mathscr{L}^2(\mathbb{R} \times \mathbb{R})$ or if $k(s,t) = e^{ist}$ then the Fourier identities for convolution operators apply to deconvolve, and if the problem is not ill-posed then this is favorable. Also, using regularization methods, uniqueness and continous dependence gets lost for the original problem.

In [**BM02**] product kernels $k(s,t) = k(st)$ where considered and a new result within the theory of $\mathcal{M}$-transformations, studied in [**SSK98**], was established. Consider the integral equation $\mathcal{B}f : \mathbb{R}_+ \to \mathbb{R}$

$$F(\omega) = \mathcal{B}f(\omega) = \int_0^\infty B(\omega r) f(r) dr \quad \omega \in \mathbb{R} \tag{2.14}$$

If B satisfy a number of conditions, then it was proved that there exists an explicit operator $\mathcal{P}$ such that the inversion is given as

$$f(r) = \int_{-\infty}^{\infty} \mathcal{P}(B(-\omega r)) F(\omega) d\omega \quad r \in \mathbb{R}_+ \tag{2.15}$$

There are different types of conditions B can satisfy, and these in turn define the operator $\mathcal{P}$. The conditions on B are easiest expressed in the Mellin transformed space (see the definition of $\mathcal{A}_0$ page 49 and $\mathcal{A}_1$ page 53). It was shown that all operators defining the same operator $\mathcal{P}$ form an Abelian group.

The space T was defined as the smallest Hilbert space, with the inner product

$$\langle f, g \rangle_T = \langle f, g r^{-2} \rangle_{\mathscr{L}^2(\mathbb{R}_+)},$$

which contain the space of test functions $\mathcal{D}(\mathbb{R}_+)$, and the space Ω was defined as $\Omega = \mathcal{B}_0 T$, where $\mathcal{B}_0$ is the canonical element for the Abelian group. The operator $\mathcal{B}$ was found to be a homeomorphism between these spaces. Hence existence, uniqueness, construction, and continuity of f solving (2.14) was proved.

2.4.2. The Reconstruction Method from One Time Dependent Experiment. This method for reconstructing the index of refraction using the transformation theory is again essentially based on the Helmholtz equation under the assumption that $\mathrm{Im}(n) = 0$.

Assume that the scattered field of a time-dependent scattering experiment is known on the boundary of $\partial B(0,a)$ encircling the inhomogeneity. This is equivalent to knowing the solution of the time-independent scattering experiment on $\partial B(0,a)$ for all frequencies. Here let $n_0 = 1$.

If v is a free space solution of $(\Delta + k^2)u = 0$ for $\mathbf{x} \in \mathbb{R}^3$, then from Green's Formula applied to Helmholtz Equation follow that

$$\int_{\partial B(0,a)} v\frac{\partial u^{sc}}{\partial \nu} + u^{sc}\frac{\partial v}{\partial \nu} d\mathbf{x} = \int_{B(0,a)} k^2(1 - n(x))u \ v \ d\mathbf{x}.$$

A general free space solution v can be expressed as $v = \sum c_m j_n(k|\mathbf{x}|)Y_n^m(\hat{\mathbf{x}})$ where j_n are spherical Bessel functions and Y_n^m are spherical harmonics. From the Born Approximation $u \approx u^{in}$ follows

$$\int_{\partial B(0,a)} v\frac{\partial u^{sc}}{\partial \nu} + u^{sc}\frac{\partial v}{\partial \nu} d\mathbf{x} = \int_{B(0,a)} k^2(1 - n(x))u^{in} \ v \ d\mathbf{x} := F(\omega)$$

The Fourier Coefficients $f_{nm}(r)$, with $r = |\mathbf{x}|$, of $(1 - n(\mathbf{x}))$ expanded in Spherical Harmonics are determined by

$$\begin{aligned} F_n^m(\omega) &:= \frac{1}{k}\int_{\partial B(0,a)} Y_n^m\Big(j_n(kr)\frac{d}{dr}u^{sc} + u^{sc}\frac{d}{dr}j_n(kr)\Big)d\mathbf{x} \\ &= \int_0^a kr \ u^{in} \, j_n(kr) r f_{nm}(r) \ dr = \mathcal{B}_n(r f_{nm}). \end{aligned} \tag{2.16}$$

If u^{in} is chosen as either h_0^1 or j_n the product of $r j_n \, u^{in}$ can be shown to be a kernel for which the Generalized Fourier Transformation theory applies.

Hence, if $F_n^m(\omega)$ is known for all ω and satisfies that the Fourier transformation of $F_n^m(\omega)$, $\mathcal{F}^{-1}F_n^m$, has compact support on $\mathbb{R}$ and $\mathcal{F}^{-1}(\omega F)(t), \mathcal{F}^{-1}F(t), t^{-1}(\mathcal{F}^{-1}F)(t) \in \mathscr{L}^2(\mathbb{R})$, then $r f_{nm}(r)$ is readily calculated by (2.15).

The reconstruction of $n(\mathbf{x})$ is formally

$$(1 - n(\mathbf{x})) = \sum_{nm} f_{nm}(r) Y_n^m(\hat{\mathbf{x}})$$

where $r f_{nm}(r)$ depends continuously on F, for $F \in \Omega$.

Wavenumber Number Reconstruction for the Acoustic Problem

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Abstract. We formulate a three dimensional Inverse Medium Problem in which the inhomogeneity is bounded, has a cylindrical shape and only depends on base's variables. We prove that the wave number can be uniquely reconstructed as soon as we know the scattered field on the cylinder's boundary. The reconstruction algorithm is explicitly given and proved to be stable.

Key Words and Phrases Inverse medium Problem, Fredholm integral equation

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3.1. Introduction

This paper considers the inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium, which is usually referred to as the Acoustic Inverse Medium Problem (see [**CK98**] and references therein).

As is very well known, solving such an inverse problem would mean to determine the wave number of a (compactly supported) scatterer immersed in a homogeneous host medium. The typical experiments consist in scattering some particularly chosen incident fields, and then their corresponding scattered fields are measured somewhere outside the scatterer (or equivalently, we measure their far field patterns).

There are at least two very interesting questions regarding the full three dimensional Inverse Medium Problem. First, the uniqueness of its' solution under some a priori assumptions on the scatterer's properties and second, the availability of a stable, as explicit and simple as possible reconstruction procedure.

The uniqueness aspects are well understood by now and many elegant ways of proving such results may be found in the literature. Among these, probably the most complete results were obtained employing the powerful method of Lippmann-Schwinger-Fadeev scattering solutions. Let us cite here only the works of Nachman [**Nac88**], Novikov [**Nov88**] and Ramm [**Ram86**, **Ram88**].

Even though the above mentioned method also allows one to reconstruct the wave number (actually, in this case the uniqueness is a simple consequence of the unique reconstruction), it is rather difficult to follow up all the steps from which the reconstruction algorithm is made of. It is even more difficult to formulate stability results, especially when reconstruction procedures are not available. For the conductivity problem this goal was achieved by Alessandrini [**Ale88**] who inspired Stefanov [**Ste90**] in the inverse problem of potential scattering; Stefanov's proof was subsequently carried over to the electromagnetic case by Hähner [**Häh00**].

In order to be able to formulate inverse medium problems with more easily solvable uniqueness, reconstruction and stability issues, one has to impose some more restrictive conditions on the scatterer's assumed properties. In [**Ber02**] this was established for piecewise constant wavenumbers with fixed location using one experiment. The present paper generalizes the method to scatters with more general geometry. What we actually do in this paper: we formulate a three dimensional inverse medium problem in which the scatterer is bounded, has a cylindrical shape and the wave number only depends on base variables.

We then show that the wave number can be uniquely reconstructed as soon as we know the scattered field on the cylinder's boundary. The reconstruction algorithm is explicitly given and proved to be stable.

Let us now briefly describe the structure of the paper:

Section 2 contains the rigorous description of our setting and the statements of our results. We start by describing the a-priori assumptions we make on the scatterer, and after a few definitions concerning various direct problems we arrive at our inverse problem (see Definition 3.4). The next step we make is to define a space of "admissible data"; a function belongs to this space if it fulfills a certain list of conditions. These conditions build in fact the algorithm one should follow for the reconstruction of the wave number.

Clearly, the difficult problem resides in proving that these conditions are also necessary for scattering fields coming from "almost all" forward problems. For more precision, see Theorem 3.6.

Theorem 3.8 and Corollary 3.9 reformulate the "almost" equivalence between the inverse problem and the space of admissible data. Corollary 3.10 states a uniqueness result from the knowledge of partial data.

Finally, Theorem 3.11 employs the reconstruction procedure in order to conclude that small changes in the measured boundary field lead to small changes in its corresponding wave number.

Section 3 contains the main technical core of our paper, being entirely dedicated to the proof of Theorem 3.6. For reader's convenience, we added a concluding overview intended to "put all the things together".

Section 4 gives the proofs for Theorem 3.8 and Corollaries 3.9 and 3.10.

Section 5 contains the proof of Theorem 3.11.

3.2. Preliminaries and the Results

3.2.1. General Notations. The acoustic scatterer will be modeled by a cylindrical domain $\Lambda \subset \mathbb{R}^3$. If $\Omega \subset \mathbb{R}^2$ is an open, bounded and connected C^∞ domain, then for $a > 0$ we define

$$\Lambda = \Omega \times [0, a] \ . \tag{3.1}$$

Throughout the paper, three dimensional vectors $\mathbf{x}$ will sometimes be represented as $\mathbf{x} = (\underline{x}, x_3)$, where $\underline{x} \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$ (or $\mathbb{C}$). The characteristic function of a set M will be denoted by $\mathbf{1}_M$.

From technical reasons, we introduce a smoother domain $\Lambda' \subset \{\mathbf{x} \in \mathbb{R}^3,\ x_3 \geq 0\}$, obeying

$$\Lambda \subset \Lambda', \quad (\Omega, 0), (\Omega, a) \subset \partial\Lambda',\ \partial\Lambda' \in C^\infty \ . \tag{3.2}$$

Let us now enumerate our a-priori assumptions on k^2; the set of all such wave numbers will be generically denoted by W:

Assumptions 1.

(1) Outside the scatterer, the square of the wave number is constant and equal to $k_0^2 = \frac{\omega^2}{c_0^2} + i\omega\sigma_0,\ \sigma_0 > 0,\ \omega > 0$, while inside it only depends on two variables: $k^2(\mathbf{x}) = \kappa_\omega(\underline{x})\mathbf{1}_{[0,a]}(x_3) + k_0^2$, $\mathrm{supp}(\kappa) \subseteq \Omega$;

(2) There exist two real valued functions $\kappa_1, \kappa_2 \in C_0^1(\Omega)$ such that for $\omega > 0$:

(a)

$$\kappa_\omega(\underline{x}) = \omega^2\kappa_1(\underline{x}) + \imath\ \omega\kappa_2(\underline{x}); \tag{3.3}$$

(b)

$$\begin{aligned} k^2(\mathbf{x}) &= \kappa_\omega(\underline{x})\mathbf{1}_{[0,a]}(x_3) + k_0^2 \\ &= \omega^2[\kappa_1(\underline{x})\mathbf{1}_{[0,a]}(x_3) + 1/c_0^2] + \imath\ \omega\ [\kappa_2(\underline{x})\mathbf{1}_{[0,a]}(x_3) + \sigma_0]\ ; \end{aligned} \tag{3.4}$$

(c)

$$\inf_{\mathbf{x}\in\mathbb{R}^3}[\kappa_2(\underline{x})\mathbf{1}_{[0,a]}(x_3) + \sigma_0] > 0. \tag{3.5}$$

Throughout the paper, by $\sqrt{z}$ we mean the principal branch of the complex square root, holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. Define

$$\mathbf{p} = (\underline{p}, p_3) = (\underline{p}, \sqrt{k_0^2 - |\underline{p}|^2}) \in \mathbb{C}^3,\ \underline{p} \in \mathbb{R}^2,\ p_3 = p_3(\underline{p}) \in \mathbb{C},\ \mathrm{Im}(p_3) > 0. \tag{3.6}$$

Notice that (see (3.6)) $e^{ip_3a} - 1 \neq 0$ for all $\underline{p}$ and moreover,

$$\lim_{|\underline{p}|\to\infty} \frac{1}{|e^{ip_3a} - 1|} = 1.$$

The incidents fields we work with are:

$$u^{in}(\mathbf{x};\mathbf{p}) = u^{in}(\underline{x}, x_3; \underline{p}) = e^{i\underline{x}\cdot\underline{p} + ip_3x_3}\ , \tag{3.7}$$

for all possible values of $\underline{p} \in \mathbb{R}^2$ at some frequency $\omega > 0$.

Definition 3.1. We say that $u(\mathbf{x})$ solves the forward acoustic scattering problem if:

(1) $(\Delta + k^2(\mathbf{x}))u = 0$, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$;
(2) $u(\mathbf{x}) = u^{in}(\mathbf{x}) + u^{sc}(\mathbf{x})$;
(3) $(\Delta + k_0^2)u^{in} = 0$ in $\mathbb{R}^3$, $k_0^2 = \frac{\omega^2}{c_0^2} + i\omega\sigma_0,\ \sigma_0 > 0,\ \omega > 0$;
(4) $\lim_{r\to\infty} r\left(\frac{\partial u^{sc}}{\partial r} - ik_0u^{sc}\right) = 0$.

Under our a-priori assumptions on k^2, the above forward problem has a unique solution (see Theorem 8.7 in [**CK98**]). We also know that the solution to the forward problem also solves the Lippmann-Schwinger equation in $H^2_{loc}(\mathbb{R}^3)$:

$$u(x) = u^{in}(\mathbf{x}) - \int_\Lambda G_0(\mathbf{x}, \mathbf{y}, \omega)(k^2(\mathbf{y}) - k_0^2)u(\mathbf{y})d\mathbf{y}, \tag{3.8}$$

where

$$G_0(\mathbf{x},\mathbf{y};\omega) = (4\pi\ |\mathbf{x}-\mathbf{y}|)^{-1}\exp\left[i\ k_0(\omega)|\mathbf{x}-\mathbf{y}|\right]. \tag{3.9}$$

Denote by $\mathcal{G}_\omega(k^2)$ the integral operator having the kernel $G_0(\mathbf{x},\mathbf{y};\omega)(k^2(\mathbf{y})-k_0^2)$. Then $\mathcal{G}_\omega(k^2)$ is a compact operator on $C^0(\Lambda)$. Due to the unique solvability of (3.8), it can be easily argued that the operator $\mathbf{1}+\mathcal{G}_\omega(k^2)$ is one-to-one, therefore invertible (the Fredholm alternative). Denote by $\xi(\mathbf{x},\underline{p},\omega) = \{\mathcal{G}_\omega(k^2)u^{in}(\cdot\ ;\underline{p},\omega)\}(\mathbf{x}) \in C^0(\Lambda)$; the same notation will be employed for its natural extension to $H^2(\mathbb{R}^3)$:

$$\xi(\mathbf{x},\underline{p},\omega) = \int_\Lambda G_0(\mathbf{x},\mathbf{y};\omega)(k^2(\mathbf{y})-k_0^2)u^{in}(\mathbf{y},\underline{p})d\mathbf{y}. \tag{3.10}$$

Then the restriction to Λ of the scattered field u^{sc} is represented as:

$$u^{sc}(\mathbf{x};\underline{p},\omega) = -\{[\mathbf{1}+\mathcal{G}_\omega(k^2)]^{-1}\xi(\cdot\ ,\underline{p},\omega)\}(\mathbf{x}) \tag{3.11}$$

The solution to (3.8) is:

$$\begin{aligned} u(\mathbf{x};\underline{p},\omega) &= u^{in}(\mathbf{x},\underline{p},\omega) - \xi(\mathbf{x},\underline{p},\omega) + \\ &\int_\Lambda G_0(\mathbf{x},\mathbf{y};\omega)(k^2(\mathbf{y})-k_0^2)\{[\mathbf{1}+\mathcal{G}_\omega(k^2)]^{-1}\xi(\cdot\ ,\underline{p},\omega)\}(\mathbf{y}). \end{aligned} \tag{3.12}$$

Definition 3.2. We say that $\tilde{u}(f)$ solves the exterior Dirichlet problem if:

(1) $(\Delta+k_0^2)\tilde{u}=0$ in $\mathbb{R}^3\setminus\overline{\Lambda'}$, $\tilde{u}\in H^2_{\text{loc}}(\mathbb{R}^3\setminus\overline{\Lambda'})$;
(2) $\tilde{u}|_{\partial\Lambda'} = f \in H^{3/2}(\partial\Lambda')\subset C^0(\partial\Lambda')$;
(3) $\tilde{u}$ satisfies the Sommerfeld radiation condition at infinity.

It is well known (see Theorem 3.21 in [**CK83**] or Theorem 3.9 in [**CK98**]) that the exterior Dirichlet problem has a unique solution.

3.2.2. Stating the Inverse Problem. As we have already outlined in the introduction, our main interest resides in reconstructing the wave number from the measured boundary data. The experiments we consider consist in the scattering of the particularly chosen incident fields $u^{in}(\cdot\ ;\underline{p},\omega)$ (see (3.7)) at a fixed frequency $\omega>0$ and all possible values of $\underline{p}\in\mathbb{R}^2$.

A particularly interesting set is composed from the values of the scattered field on the boundary of Λ', generated by all possible $k\in W$; this set will be denoted by F_ω:

$$F_\omega := \{u^{sc}(\mathbf{x};\underline{p},\omega)\mid \mathbf{x}\in\partial\Lambda',\ \underline{p}\in\mathbb{R}^2,\ k\in W\}\ . \tag{3.13}$$

Another important definition comes next:

Definition 3.3. Fix $N>0$. We denote by $W_N\subseteq W$ the set of wave numbers for which $\max\{||\kappa_1||_{C^1},||\kappa_2||_{C^1}\}\le N$. Then by $F_{N,\omega}\subset F_\omega$ we understand the subset of only those Φ_ω's which correspond (via the forward problem) to wave numbers in W_N.

The inverse problem we study can be stated as follows:

Definition 3.4. Fix a frequency $\omega > 0$ and perform the above experiments, for every possible $\underline{p} \in \mathbb{R}^2$. Denote by $\Phi_\omega(\cdot, \underline{p})$ the scattered field restricted to $\partial\Lambda$. Then

i. Existence: find sufficient conditions for a function $\Phi_\omega(\cdot, \underline{p}) \in H^{3/2}(\partial\Lambda')$ to be an element of F_ω.

ii. Unique reconstruction: if $\Phi_\omega \in F_\omega$, then construct a unique $k^2(\Phi_\omega) \in W$ such that by introducing it into the forward problem (see Definition 3.1), the scattered field such obtained coincides with Φ_ω on $\partial\Lambda'$.

Remark. In this paper we only answer the unique reconstruction question *ii.* The more difficult problem of giving sufficient conditions for a Φ_ω in order to be an element of F_ω, remains open.

In what follows, we are mainly interested in three things:

(1) First, to formulate a list of sufficient conditions on the measured data $\Phi_\omega \in F_\omega$ which should hold in order to permit the unique reconstruction of the wave number (or κ_ω); these conditions will also provide the reconstruction algorithm for κ_ω. The set of measured scattered fields having those properties will be called *the space of admissible data* and generically denoted by $\mathcal{A}_\omega$; clearly, $\mathcal{A}_\omega \subseteq F_\omega$.

(2) Second, to prove that $\mathcal{A}_\omega$ "is not empty" and sometimes equals F_ω (for more precision see Theorem 3.6).

(3) Third, to study some stability properties of the mapping $F_\omega \ni \Phi_\omega \mapsto k^2(\Phi_\omega) \in W$.

Let us now start the rigorous description of $\mathcal{A}_\omega$:

Definition 3.5. We say that $\Phi \in \mathcal{A}_\omega$ if

(1) $\Phi(\cdot, \underline{p}) \in H^{3/2}(\partial\Lambda')$, for all $\underline{p} \in \mathbb{R}^2$;

(2) Denote by $[\tilde{u}(\Phi)](\cdot, \underline{p})$ the solution to the exterior Dirichlet problem corresponding to the boundary value $\Phi(\cdot, \underline{p})$ and denote by

$$[u(\Phi)](\cdot;\underline{p}) = u^{in}(\cdot;\mathbf{p}) + [\tilde{u}(\Phi)](\cdot;\underline{p}) \ . \tag{3.14}$$

Take $\rho > 0$ so large that the ball centered at the origin with radius ρ (i.e. $B(0, \rho)$) includes $\overline{\Lambda'}$. Define

$$[\eta(\Phi)](\underline{p}) = \frac{1}{e^{ip_3 a} - 1} \int_{|\mathbf{y}|=\rho} \{-[\partial_3 u]\ [\partial_\nu u] + u[\partial_\nu \partial_3 u]\}\ d\sigma(\mathbf{y}), \tag{3.15}$$

where ν is the exterior normal; our second condition is

$$[\eta(\Phi)] \in \mathscr{L}^2(\mathbb{R}^2). \tag{3.16}$$

(3) If $\underline{x} \in \Omega$, define

$$\begin{aligned}\mathcal{L}(\underline{x},\underline{p}) &:= \frac{1}{e^{ip_3a}-1}\{[u(\Phi)]^2(\underline{x},a;\underline{p}) - [u(\Phi)]^2(\underline{x},0;\underline{p})\} = \\ &= e^{i2\underline{p}\cdot\underline{x}} + \mathcal{L}_1(\underline{x},\underline{p}),\end{aligned} \tag{3.17}$$

where

$$\mathcal{L}_1(\underline{x},\underline{p}) := \frac{1}{e^{ip_3a}-1}\big\{2[u^{in}\Phi](\underline{x},a;\underline{p}) - 2[u^{in}\Phi](\underline{x},0;\underline{p}) + \Phi^2(\underline{x},a;\underline{p}) - \Phi^2(\underline{x},0;\underline{p})\big\}. \tag{3.18}$$

Our third condition is:

$$\mathcal{L}_1 \in \mathscr{L}^2(\Omega\times\mathbb{R}^2), \text{ i.e. } \int_{\Omega\times\mathbb{R}^2} |\mathcal{L}_1|^2(\underline{x},\underline{p})d\underline{x}\; d\underline{p} < \infty\ ; \tag{3.19}$$

(4) If $\mathcal{F}_{\underline{p}}$ denotes the partial Fourier transform with respect to the "$\underline{p}$" variable, then define

$$\mathcal{K}(\underline{x},\underline{y}) = \left[\mathcal{F}_{\underline{p}}\left(\mathcal{L}_1(\underline{y},\underline{p}/2)\right)\right](\underline{x}). \tag{3.20}$$

Denote by K the Hilbert-Schmidt operator corresponding to $\mathcal{K}$ acting on $\mathscr{L}^2(\mathbb{R}^2\times\Omega)$ and with the same letter its natural restriction to $L^2(\Omega\times\Omega)$. Then our fourth condition is that the operator $\mathbf{1}+K$ is invertible in $B(\mathscr{L}^2(\Omega\times\Omega))$;

(5) Denote by

$$[\tilde{\eta}(\Phi)](\underline{y}) = \left[\mathcal{F}_{\underline{p}}\left(\eta(\Phi)(\underline{p}/2)\right)\right](\underline{y}), \tag{3.21}$$

and with the same letter its natural restriction to $\mathscr{L}^2(\Omega)$.

Denote by

$$\kappa(\Phi) = (\mathbf{1}+K)^{-1}\tilde{\eta}(\Phi) \in \mathscr{L}^2(\Omega), \tag{3.22}$$

extended by zero outside Ω. Our fifth condition is that

$$k^2(\Phi) := \kappa(\Phi)\mathbf{1}_{[0,a]} + k_0^2, \tag{3.23}$$

obeys the Assumptions 1;

(6) Introduce $k^2(\Phi)$ in (3.11) and denote the scattered field such obtained by $v^{sc}(\Phi) \in H^2_{\text{loc}}(\mathbb{R}^3)$. The last condition is

$$v^{sc}(\Phi)|_{\partial\Lambda'} = \Phi. \tag{3.24}$$

Remarks. 1. It may seem that the last two conditions are awkward and superfluous as soon as we assume that Φ_ω comes from a forward problem (i.e. belongs to F_ω). But they are justified if we reason from the point of view of a practical application. Indeed, the scattering experiments provide us with a Φ_ω which could come from a wave number which is not in W (this would mean that our a-priori assumptions for the scatterer are wrong). Hence even if the computation of $k^2(\Phi)$ in (3.23) is possible, we still have to check that (3.24) holds in order to conclude that our a-priori assumptions about the scatterer are correct.

2. Although equation (3.24) is highly nonlinear, it only involves Φ; a characterization of its solutions would automatically lead us to an affirmative answer to the "existence" part of our inverse problem.

3.2.3. The Results. We now are prepared to give our first result. It essentially says that the space of admissible data $\mathcal{A}_\omega$ is not empty:

Theorem 3.6. *Let $\omega_0 > 0$ and $N > 0$ be arbitrarily large, but fixed.*

i. Assume that $k \in W$ and ω is allowed to take values in $(0, \omega_0)$. Denote by $u \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ the solution to the forward problem corresponding to k^2 (see Definition 3.1 and formula (3.12)). Then there exists a finite set $M \subset (0, \omega_0)$ such that for any frequency $\omega \in (0, \omega_0) \setminus M$ one has $u^{sc}|_{\partial\Lambda'} \in \mathcal{A}_\omega$.

ii. There exists $\omega_N > 0$ such that for any $\omega \in (0, \omega_N)$ and any $k \in W_N$ (see Definition 3.3) one has $u^{sc}|_{\partial\Lambda'} \in \mathcal{A}_\omega$. Therefore, in this case $\mathcal{A}_\omega = F_{N,\omega}$.

Remark. During the proof of Theorem 3.6, we will show that $u^{sc}|_{\partial\Lambda'}$ always obeys the first three conditions in Definition 3.5. The only problem which could appear in the reconstruction process is that the operator $\mathbf{1} + K$ introduced in the fourth condition might not be invertible. That is why the following definition is justified:

Definition 3.7. Fix $\omega > 0$ and take $k^2 = k^2(\omega)$ as in (3.4). Construct the integral kernel $\mathcal{K}(\underline{x}, \underline{y}; \omega)$ as in (3.20). We say that ω is regular with respect to κ_1 and κ_2 if the operator $\mathbf{1} + K(\omega)$ is invertible.

The next theorem couples Theorem 3.6, Definition 3.5 and Definition 3.7, stating the conditions we need such that the reconstruction part of our inverse problem to have a unique solution:

Theorem 3.8. *Fix $\omega_0 > 0$ and choose some ω in $(0, \omega_0)$. Then the following two statements are equivalent:*

i. The scattered field restricted to the boundary belongs to the space of admissible data;

ii. The inverse problem (see Definition 3.4) has a unique solution $k \in W$ given by some $\kappa_1, \kappa_2 \in C^1_0(\Omega)$ (see (3.4)), and the frequency is regular with respect to κ_1 and κ_2.

The next corollary is a natural consequence of the above theorems, saying that if the wave numbers are restricted to some W_N, we can uniquely reconstruct them from the knowledge of the scattered field on the boundary, for some sufficiently small frequency:

Corollary 3.9. *Fix $N > 0$ and assume that the wave numbers are only allowed to belong to W_N (see Definition 3.3). Then there exists $\omega_N > 0$ such that for any frequency $\omega \in (0, \omega_N)$, the inverse problem has a unique solution in W_N, given by (3.23) and (3.22).*

We can also formulate a uniqueness result claiming that if we have two wave numbers in W for which we know that their corresponding scattered fields are equal at the boundary for some frequencies and some (not all) $\underline{p}$'s, then those wave numbers must be equal:

Corollary 3.10. *Let $\omega > 0$ and let $k_1, k_2 \in W$. Denote by $u_1(\mathbf{x};\underline{p},\omega)$ ($u_2(\mathbf{x};\underline{p},\omega)$) the total field obtained by introducing k_1 (k_2) in (3.12), i.e. the solution to the forward problem.*

Assume that $u_1|_{\partial\Lambda'}(\cdot\,;\underline{p},1/n) = u_2|_{\partial\Lambda'}(\cdot\,;\underline{p},1/n)$ for all $\underline{p}$ in an open subset of $\mathbb{R}^2$ and $n > N_0$. Then $k_1 = k_2$.

Finally, let us state our stability result for the inverse problem. We will prove (see (3.35)) that if $\Phi \in F_\omega$, then $\sup_{\mathbf{y}\in\partial\Lambda'} |\Phi(\mathbf{y};\cdot)| \in \mathscr{L}^2(\mathbb{R}^2)$. Then introduce the following norm on F_ω:

$$|||\Phi|||^2 := \int_{\mathbb{R}^2} \left(\sup_{\mathbf{y}\in\partial\Lambda'} |\Phi(\mathbf{y};\underline{p})| \right)^2 d\underline{p}. \tag{3.25}$$

It is not diffi cult to see that

$$||\cdot||^2_{\mathscr{L}^2(\partial\Lambda'\times\mathbb{R}^2)} \leq |\partial\Lambda'| \quad |||\cdot|||^2. \tag{3.26}$$

We thus justifi ed the use of $|||\cdot|||$-norm for measuring the distance between two boundary data:

Theorem 3.11. *Fix $N > 0$, choose an $\omega \in (0,\omega_N)$ and fix an arbitrary $\Phi \in F_{N,\omega}$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$||k^2(\Psi) - k^2(\Phi)||_{\mathscr{L}^2(\Omega)} < \epsilon \quad whenever \quad \Psi \in F_{N,\omega} \quad and \quad |||\Psi - \Phi||| < \delta\ .$$

3.3. Proof of Theorem 3.6

The strategy will consists in showing that if $k \in W$, all the conditions in Defi nition 3.5 can be verifi ed except for the case in which the frequency ω belongs to a fi nite subset M of $(0,\omega_0)$.

3.3.1. An Equation for κ. Consider $\rho > 0$ large enough such that the ball $B(0,\rho)$ includes $\overline{\Lambda'}$. Then:

Lemma 3.12. *Let $k \in W$ (see Assumptions 1) and let $u(\cdot,\underline{p}) \in H^2_{loc}(\mathbb{R}^3)$ be the corresponding solution to the forward problem (see Definition 3.1). Denote by $\Phi(\cdot\,;\underline{p}) \in H^{3/2}(\partial\Lambda')$ the restriction of $u^{sc}(\cdot\,;\underline{p})$ to $\partial\Lambda'$. Then κ satisfies the following integral equation (see (3.15) and (3.17)):*

$$\int_\Omega \mathcal{L}(\underline{x},\underline{p})\kappa(\underline{x})d\underline{x} = [\mathcal{F}^{-1}(\kappa)](2\underline{p}) + \int_\Omega \mathcal{L}_1(\underline{x},\underline{p})\kappa(\underline{x})d\underline{x} = [\eta(\Phi)](\underline{p})\ . \tag{3.27}$$

Proof. Due to the uniqueness properties for both exterior and forward problems (see Definitions 3.2 and 3.1), one can easily see that (3.27) is equivalent to

$$\int_\Omega [u^2(\underline{y},0) - u^2(\underline{y},a)]\kappa(\underline{y})d\underline{y} = \int_{|\mathbf{y}|=\rho} \{(\partial_3 u)(\partial_\nu u) - u(\partial_\nu \partial_3 u)\}d\sigma(\mathbf{y}), \qquad (3.28)$$

where ν is the exterior normal.

We will first give the formal derivation of (3.28) and then we will argue why the formal computations are justified. First, write the Helmholtz equation $(\Delta + k^2)u = 0$ and differentiate it with respect to x_3:

$$\begin{aligned} 0 = \partial_{x_3}(\nabla^2 u + k^2 u) &= \nabla^2(\partial_{x_3} u) + k^2(\partial_{x_3} u) + u\kappa\{\delta(x_3) - \delta(x_3 - a)\} \\ &= (\nabla^2 + k^2)(\partial_{x_3} u) + u[\partial_{x_3}(k^2 - k_0^2)], \end{aligned} \qquad (3.29)$$

where the Dirac distribution acts on $\mathcal{S}(\mathbb{R})$.

Then

$$\int_{|\mathbf{x}|<\rho} \{(\partial_{x_3} u)[(\nabla^2 + k^2)u] - u[(\nabla^2 + k^2)(\partial_{x_3} u)]\}d\mathbf{x} = \int_{|\mathbf{x}|<\rho} u^2[\partial_{x_3}(k^2 - k_0^2)]d\mathbf{x}. \qquad (3.30)$$

Apply Green's formula in the left hand side, taking into account the fact that the distribution $\partial_{x_3}(k^2 - k_0^2)$ is supported on $(\Omega, 0)$ and (Ω, a):

$$\int_{|\mathbf{x}|=\rho} \{-(\partial_{x_3} u)(\partial_\nu u) + u(\partial_{x_3}\partial_\nu u)\}d\sigma(\mathbf{x}) = \int_\Omega \kappa(\underline{y})[u^2(\underline{y},a) - u^2(\underline{y},0)]d\underline{y}. \quad (3.31)$$

Notice that the above integrals are well defined, since u is a C^∞ function outside Λ and continuous on $\partial\Lambda'$ (in fact, the restriction of u to $\partial\Lambda'$ belongs to $H^{3/2}(\partial\Lambda')$).

In order to justify these formal computations, consider $\{\alpha_n\}_{n\geq 1} \subset \mathcal{S}(\mathbb{R}^3)$ an approximation of the Dirac distribution, where

$$\operatorname{supp}(\alpha_n) \subset B(0, 1/n) \text{ and } \alpha_n(\mathbf{x}) = \alpha_n(|\mathbf{x}|). \qquad (3.32)$$

Define $u_n = \alpha_n * u$; they are C^∞ functions and for any $r > 0$:

$$\lim_{n\to\infty} \Big\{ ||u_n - u||_{H^2(B(0,r))} + ||[\partial_{x_3} u_n]\alpha_n * [(k^2 - k_0^2)u] - u_n\alpha_n * [(k^2 - k_0^2)\partial_{x_3} u]||_{L^1(\mathbb{R}^3)} \Big\} = 0. \quad (3.33)$$

Then due to (3.32) the following identity holds (n sufficiently large):

$$\begin{aligned} &\int_{|\mathbf{x}|<\rho} \big\{[\partial_{x_3} u_n][(\Delta + k^2)u_n + \alpha_n * ((k^2 - k_0^2)u)] - \\ - \;& u_n[(\Delta + k^2)\partial_{x_3} u_n + \alpha_n * ((k^2 - k_0^2)\partial_{x_3} u)]\big\}\, d\mathbf{x} = \\ = \;& \int_\Omega \kappa(\underline{y}) \int_{|\mathbf{x}|<\rho} \big[u_n(\mathbf{x})\alpha_n(\mathbf{x} - (\underline{y},0))u(\underline{y},0) - u_n(\mathbf{x})\alpha_n(\mathbf{x} - (\underline{y},a))u(\underline{y},a)\big]\, d\mathbf{x} = \\ = \;& \int_\Omega \kappa(\underline{y})\{u(\underline{y},0)[\alpha_n * u_n](\underline{y},0) - u(\underline{y},a)[\alpha_n * u_n](\underline{y},a)\}d\underline{y}\, . \end{aligned} \qquad (3.34)$$

Notice that $\alpha_n * u_n = \beta_n * u$, where $\beta_n = \alpha_n * \alpha_n$ is another approximation of the Dirac distribution and

$$\lim_{n\to\infty} ||\beta_n * u - u||_{H^2(B(0,\rho))} = 0.$$

Taking n to the limit, employing Green's formula and the continuity of the trace operator between $H^2(B(0,\rho))$ and $\mathscr{L}^2(\partial\Lambda')$, (3.28) follows. □

3.3.2. Checking (3.19). This subsection will prove that the scattered field $u^{sc}(\mathbf{x};\underline{p})$ corresponding to a wave number $k \in W$ is sufficiently well localized in $\underline{p}$ in order to insure (3.19). Looking at the definition of $\mathcal{L}_1$, one sees that it would be enough proving for $u^{sc}(\mathbf{x};\underline{p})$ an estimate of the form

$$\sup_{\mathbf{x}\in\Lambda} |u^{sc}(\mathbf{x};\underline{p})| \leq \frac{C}{(1+\underline{p}^2)^{\frac{1+\delta}{2}}}, \tag{3.35}$$

where C is some constant and $\delta > 0$. The next lemmas will make this precise.

Remember that $k \in W$ is determined by $\kappa_\omega(\underline{x}) = \omega^2\kappa_1(\underline{x}) + i\omega\kappa_2(\underline{x})$, where $\omega > 0$ and $\kappa_{1,2} \in C_0^1(\Omega)$. In particular, this implies:

$$||\kappa_{1,2}||_{C^1} := \max_{|\alpha|\leq 1} \sup_{\underline{x}\in\overline{\Omega}} |D^\alpha \kappa_{1,2}(\underline{x})| < \infty. \tag{3.36}$$

Recall first that $k_0^2 = \omega^2/c_0^2 + i\sigma_0\omega$, $\sigma_0 > 0$, $\omega > 0$, $p_3 = \sqrt{k_0^2 - \underline{p}^2}$ and $\text{Im}(p_3) > 0$. For further purposes, we introduce $0 < \omega_0$ and

$$\mathcal{S}_0 := \{z \in \mathbb{C};\ 0 < \text{Re}(z) < \omega_0,\ |\text{Im}(z)| < \sigma_0 c_0/4\}. \tag{3.37}$$

If $\omega \in \mathcal{S}_0$, then $\text{Im}(p_3) > 0$. Clearly, there exists a constant $A > 1$, only depending on $\mathcal{S}_0$ such that if $|\underline{p}| \geq A$ then $p_3 \sim i\ |\underline{p}|$ i.e.

$$\text{Im}(p_3) \geq \frac{|\underline{p}|}{2}, \quad |\underline{p}| \geq A > 1. \tag{3.38}$$

Lemma 3.13. *Fix $\mathbf{x} \in \overline{\Lambda}$, $\underline{p} \in \mathbb{R}^2$ and $\omega \in \mathcal{S}_0$. Consider (see also (3.10) and (3.7))*

$$\xi(\mathbf{x},\underline{p},\omega) = \int_\Lambda G_0(\mathbf{x},\mathbf{y};\omega)\kappa_\omega(\underline{y})\mathbf{1}_{[0,a]}(y_3)e^{i\underline{p}\cdot\underline{y}+ip_3y_3}d\mathbf{y}\ . \tag{3.39}$$

Fix $0 < \delta < 1$. Then there exists a positive constant $C(\delta,\Lambda,\mathcal{S}_0)$ such that for any $\omega \in \mathcal{S}_0$ and $\underline{p} \in \mathbb{R}^2$:

$$\sup_{\mathbf{x}\in\overline{\Lambda}} |\xi(\mathbf{x},\underline{p},\omega)| \leq \frac{C(\delta,\Lambda,\mathcal{S}_0)}{(1+|\underline{p}|)^{1+\delta}} ||\kappa_\omega||_{C^1}\left[1 + \frac{1}{\mathit{Im}(k_0)^{2\delta}} + \frac{|k_0|}{\mathit{Im}(k_0)}\right]. \tag{3.40}$$

Proof. Fix $0 < \delta < 1$. Define $\gamma(\underline{p}) = (1 + |\underline{p}|)^{-\delta}$. One of the key ingredients used in the proof of (3.40) is the following estimate, uniform in $\omega \in \mathcal{S}_0$ (see also (3.38)):

$$\exp\left[-a\gamma(\underline{p})\mathrm{Im}(p_3)\right] \leq \begin{cases} 1 & \text{if} \quad |\underline{p}| \leq A(\delta, \mathcal{S}_0) \\ \exp\left[-a|\underline{p}|^{1-\delta}/2\right] & \text{if} \quad |\underline{p}| \geq A(\delta, \mathcal{S}_0) \end{cases} \tag{3.41}$$

First, rewrite ξ as

$$\xi(\mathbf{x}, \underline{p}, \omega) = \int_\Omega \int_0^a G_0(\mathbf{x}, \mathbf{y}; \omega)\kappa_\omega(\underline{y}) e^{i\underline{p}\cdot\underline{y} + ip_3y_3} d\underline{y}\, dy_3 := \xi_1(\mathbf{x}, \underline{p}, \omega) + \xi_2(\mathbf{x}, \underline{p}, \omega), \tag{3.42}$$

where

$$\xi_1(\mathbf{x}, \underline{p}, \omega) = \int_\Omega \int_{a\gamma(\underline{p})}^a G_0(\mathbf{x}, \mathbf{y}; \omega)\kappa_\omega(\underline{y}) e^{i\underline{p}\cdot\underline{y} + ip_3y_3} d\underline{y}\, dy_3 \tag{3.43}$$

and

$$\xi_2(\mathbf{x}, \underline{p}, \omega) = \int_\Omega \int_0^{a\gamma(\underline{p})} G_0(\mathbf{x}, \mathbf{y}; \omega)\kappa_\omega(\underline{y}) e^{i\underline{p}\cdot\underline{y} + ip_3y_3} d\underline{y}\, dy_3. \tag{3.44}$$

Let us first treat ξ_1. Since $a\gamma(\underline{p}) < y_3 < a$,

$$|e^{ip_3y_3}| \leq e^{-\mathrm{Im}(p_3)y_3} \leq \exp\left[-a\gamma(\underline{p})\mathrm{Im}(p_3)\right],$$

it follows from (3.41) that

$$\begin{aligned} (1 + |\underline{p}|)^{1+\delta}|\xi_1(\mathbf{x}, \underline{p}, \omega)| \quad &\leq \quad C(\delta, \mathcal{S}_0)\ ||\kappa_\omega||_{C^1} \int_\Lambda \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \leq \\ &\leq \quad C(\delta, \mathcal{S}_0)||\kappa_\omega||_{C^1}\ \mathrm{const}(\Lambda). \end{aligned} \tag{3.45}$$

Therefore, ξ_1 obeys (3.40).

Secondly, let us study ξ_2. If $|\underline{p}| \leq A(\delta, \mathcal{S}_0)$, then ξ_2 obeys an estimate similar to (3.45), therefore the “nontrivial” region is $|\underline{p}| \geq A(\delta, \mathcal{S}_0)$ (where we also have (3.38)).

For technical reasons, we introduce $\xi_{2,\varepsilon}$ by

$$\xi_{2,\varepsilon}(\mathbf{x}, \underline{p}, \omega) = \int_0^{a\gamma(\underline{p})} e^{ip_3y_3}\psi_\varepsilon(\mathbf{x}, y_3; \underline{p}, \omega) dy_3 \tag{3.46}$$

where

$$\psi_\varepsilon(\mathbf{x}, y_3; \underline{p}, \omega) = \int_{\Omega\cap\{|\underline{x}-\underline{y}|\geq\varepsilon\}} G_0(\mathbf{x}, \mathbf{y}; \omega)\kappa_\omega(\underline{y}) e^{i\underline{p}\cdot\underline{y}} d\underline{y}. \tag{3.47}$$

Since (see (3.9))

$$|G_0(\mathbf{x}, \mathbf{y}, \omega)| \leq \frac{1}{4\pi|\underline{x} - \underline{y}|}, \tag{3.48}$$

we have $\lim_{\varepsilon\to 0} \xi_{2,\varepsilon}(\mathbf{x}, \underline{p}, \omega) = \xi_2(\mathbf{x}, \underline{p}, \omega)$. Our goal now consists in proving (3.40) for $\xi_{2,\varepsilon}$ uniformly in ε, which would end the proof.

Let us first remark a useful identity:

$$p_3\xi_{2,\epsilon}(\mathbf{x},\underline{p},\omega)$$
$$-\, i\left[\psi_\varepsilon(\mathbf{x},y_3;\underline{p},\omega)e^{ip_3y_3}\right]_0^{a\gamma(\underline{p})} + i\int_0^{a\gamma(\underline{p})} e^{ip_3y_3}\frac{\partial\psi_\varepsilon}{\partial y_3}(\mathbf{x},y_3;\underline{p},\omega)dy_3 \;. \quad (3.49)$$

Taking the limit:

$$\begin{aligned} p_3\xi_2(\mathbf{x},\underline{p},\omega) &= -i\psi_0(\mathbf{x},a\gamma(\underline{p});\underline{p},\omega)e^{ip_3a\gamma(\underline{p})} + i\psi_0(\mathbf{x},0;\underline{p},\omega) + \\ &+ i\lim_{\varepsilon\to 0}\int_0^{a\gamma(\underline{p})} e^{ip_3y_3}\frac{\partial\psi_\varepsilon}{\partial y_3}(\mathbf{x},y_3;\underline{p},\omega)dy_3 \;. \end{aligned} \quad (3.50)$$

The first term in (3.50) is exponentially small, due to (3.41) and to the estimate (see (3.47))

$$|\psi_0(\mathbf{x},y_3;\underline{p},\omega)| \le \mathrm{const}(\Lambda)\ ||\kappa_\omega||_{C^1}. \quad (3.51)$$

We decompose ψ_0 into two terms $\psi_0^{(1)} + \psi_0^{(2)}$:

$$\psi_0^{(1)}(\mathbf{x},0;\underline{p},\omega) = \kappa_\omega(\underline{x})e^{i\underline{p}\cdot\underline{x}}\int_{\mathbb{R}^2} e^{i\underline{p}\cdot(\underline{y}-\underline{x})}G_0(\mathbf{x}-(\underline{y},0);\omega)d\underline{y} \quad (3.52)$$

$$\psi_0^{(2)}(\mathbf{x},0;\underline{p},\omega) = \int_{\mathbb{R}^2} e^{i\underline{p}\cdot\underline{y}}G_0(\mathbf{x}-(\underline{y},0);\omega)[\kappa_\omega(\underline{y})-\kappa_\omega(\underline{x})]d\underline{y} \quad (3.53)$$

In order to finish the proof of (3.40) for $|\underline{p}| \ge A > 1$, it would be enough having three more estimates:

$$\sup_{\mathbf{x}\in\Lambda}|\psi_0^{(1)}(\mathbf{x},0;\underline{p},\omega)| \le \mathrm{const}(\Lambda)\ ||\kappa_\omega||_{C^1}|\underline{p}|^{-1}, \quad (3.54)$$

$$\sup_{\mathbf{x}\in\Lambda}|\psi_0^{(2)}(\mathbf{x},0;\underline{p},\omega)| \le \mathrm{const}(\Lambda,\mathcal{S}_0)\ (1+|\underline{p}|)^{-\delta}\frac{||\kappa_\omega||_{C^1}}{\mathrm{Im}(k_0)^{2\delta}} \quad (3.55)$$

and

$$\sup_{\varepsilon>0}\left|\frac{\partial\psi_\varepsilon}{\partial y_3}(\mathbf{x},y_3;\underline{p},\omega)\right| \le \mathrm{const}(\Lambda,\mathcal{S}_0)\ ||\kappa_\omega||_{C^1}\left[1+\frac{|k_0|}{\mathrm{Im}(k_0)}\right]. \quad (3.56)$$

Indeed, if we replace them in (3.50), then (3.40) follows.

Let us now prove (3.54). We transform $\psi_0^{(1)}$ using a partial Fourier transform identity for G_0:

$$\frac{1}{2\pi}\int_{\mathbb{R}} dq\frac{e^{iqx_3}}{k_0^2-\underline{p}^2-q^2} = \int e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}G_0((\underline{x}-\underline{y},x_3);\omega)d\underline{y}, \quad (3.57)$$

Using that $\mathrm{Im}(k_0^2) > 0$, we split $k_0^2 - \underline{p}^2 - q^2 = -(q - \sqrt{k_0^2 - \underline{p}^2})(q + \sqrt{k_0^2 - \underline{p}^2})$ and apply the residue theorem to the left hand side of (3.57), yielding:

$$-i\frac{e^{i\sqrt{k_0^2-p^2}|x_3|}}{2\sqrt{k_0^2-p^2}},$$

hence (3.54) holds for $\psi_0^{(1)}$.

For $\psi_0^{(2)}$ we will finally need the C^1 regularity on $\kappa_{1,2}$:

$$\begin{aligned} p_{1,2}\psi_0^{(2)} &= -i\int_{\mathbb{R}^2}\left[\frac{\partial}{\partial y_{1,2}}e^{i\underline{p}\cdot\underline{y}}\right]G_0(\mathbf{x}-(\underline{y},0);\omega)(\kappa_\omega(\underline{y})-\kappa_\omega(\underline{x}))d\underline{y} \\ &= i\int_{\mathbb{R}^2}e^{i\underline{p}\cdot\underline{y}}\left\{\left[\frac{\partial}{\partial y_{1,2}}G_0(\mathbf{x}-(\underline{y},0);\omega)\right](\kappa_\omega(\underline{y})-\kappa_\omega(\underline{x}))\right. \\ &\qquad\left.+\frac{\partial\kappa_\omega}{\partial y_{1,2}}(\underline{y})G_0(\mathbf{x}-(\underline{y},0);\omega)\right\}. \end{aligned} \tag{3.58}$$

Since $\kappa_{1,2}(\underline{y}) \in C^1$, $|\kappa_{1,2}(\underline{y}) - \kappa_{1,2}(\underline{x})| \le \mathrm{const}|\underline{y} - \underline{x}|$. Also,

$$\left|\frac{\partial}{\partial y_{1,2}}G_0(\mathbf{x}-(\underline{y},0);\omega)\right| \le \mathrm{const}\ \left(1+\frac{1}{|\underline{x}-\underline{y}|^2}\right)e^{-\mathrm{Im}(k_0)\ |\underline{x}-\underline{y}|}\ . \tag{3.59}$$

Inserting the above estimate into (3.58) we get:

$$|\psi_0^{(2)}(\mathbf{x},0;\underline{p},\omega)| \le \mathrm{const}\ (1+|\underline{p}|)^{-1}||\kappa_\omega||_{C^1}\frac{1}{\mathrm{Im}(k_0)^2}. \tag{3.60}$$

Unfortunately, (3.60) does not have a sufficiently good behavior when ω is real and tends to zero. To improve it, we will use a small trick: combine (3.54) and (3.51) to obtain

$$|\psi_0^{(2)}(\mathbf{x},0;\underline{p},\omega)| \le \mathrm{const}(\Lambda,\mathcal{S}_0)||\kappa_\omega||_{C^1}. \tag{3.61}$$

Then write

$$|\psi_0^{(2)}| = |\psi_0^{(2)}|^\delta\ |\psi_0^{(2)}|^{1-\delta} \le \mathrm{const}(\Lambda,\mathcal{S}_0)\ (1+|\underline{p}|)^{-\delta}\frac{||\kappa_\omega||_{C^1}}{\mathrm{Im}(k_0)^{2\delta}}$$

which proves (3.55).

Finally, let us prove (3.56). Performing the derivative with respect to y_3 in (3.47) we get:

$$\begin{aligned} &\frac{\partial\psi_\varepsilon}{\partial y_3}(\mathbf{x},y_3;\underline{p},\omega) = \\ &\qquad\int_{\Omega\cap\{|\underline{x}-\underline{y}|\ge\varepsilon\}}\left\{-\frac{e^{ik_0|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^3}+ik_0\frac{e^{ik_0|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^2}\right\}(y_3-x_3)\kappa_\omega(\underline{y})e^{i\underline{p}\cdot\underline{y}}d\underline{y}. \end{aligned} \tag{3.62}$$

Assume $x_3 - y_3 \neq 0$ (otherwise there is nothing to prove). Using polar coordinates in (3.62) (i.e. $|\underline{y} - \underline{x}| = r > \varepsilon$) we get

$$\left|\frac{\partial \psi_\varepsilon}{\partial y_3}\right| \leq 2\pi ||\kappa_\omega||_{C^1} |x_3 - y_3| \int_\varepsilon^\infty \left\{ \frac{e^{-\mathrm{Im}(k_0) r}}{(r^2 + |x_3 - y_3|^2)^{3/2}} + |k_0| \frac{e^{-\mathrm{Im}(k_0) r}}{(r^2 + |x_3 - y_3|^2)} \right\} r\, dr. \tag{3.63}$$

Changing the variable in $r' = r/|x_3 - y_3|$ we get:

$$\left|\frac{\partial \psi_\varepsilon}{\partial y_3}\right| \leq 2\pi ||\kappa_\omega||_{C^1} \int_0^\infty \left\{ \frac{r'}{(r'^2 + 1)^{3/2}} + |k_0|\ |x_3 - y_3| e^{-\mathrm{Im}(k_0)|x_3 - y_3| r'} \right\} dr'. \tag{3.64}$$

While the first term in the above integrand is well behaved in terms of ω, the second one will generate a $k_0/\mathrm{Im}(k_0)$; hence, (3.56) is proved. □

Lemma 3.14. *The mapping* $\mathcal{S}_0 \ni z \mapsto \xi(\cdot\,, \underline{p}, z) \in C^0(\Lambda)$ *is analytic.*

Proof. Take $z_0 \in \mathcal{S}_0$. It is easy to see that for fixed $\mathbf{x} \in \overline{\Lambda}$ and $\underline{p} \in \mathbb{R}^2$, the function $\xi(\mathbf{x}, \underline{p}, \cdot) : \mathcal{S}_0 \mapsto \mathbb{C}$ is holomorphic (see (3.39)). Take a disk $B(z_0, r) \subset \mathcal{S}_0$ and apply the Cauchy integral formula ($z \in B(z_0, r)$):

$$\xi(\mathbf{x}, \underline{p}, z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{\xi(\mathbf{x}, \underline{p}, \zeta)}{\zeta - z} = \sum_{n \geq 0} (z - z_0)^n a_n(\mathbf{x}, \underline{p}), \tag{3.65}$$

where

$$a_n(\mathbf{x}, \underline{p}) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{\xi(\mathbf{x}, \underline{p}, \zeta)}{(\zeta - z_0)^{n+1}}.$$

Finally, employ (3.40) and obtain

$$||a_n(\cdot\,, \underline{p})||_\infty \leq \frac{\mathrm{const}(\mathrm{r}, \delta, \Lambda, \mathrm{z}_0)}{r^n} (1 + |\underline{p}|)^{-1-\delta} \tag{3.66}$$

which finishes the proof. □

Lemma 3.15. *Consider the natural extension of* $k^2 = k_\omega^2 \in W$ *for complex values of* ω *(see Assumptions 1).*

i. There exists an open strip $\mathcal{S} \subseteq \mathcal{S}_0$ *containing* $(0, \omega_0)$ *such that* $\inf_{\mathbf{x} \in \mathbb{R}^3} Im(k_\omega^2(\mathbf{x})) > 0$ *whenever* $\omega \in \mathcal{S}$*;*

ii. The forward problem corresponding to k_ω^2 *has a unique solution, whose scattered field* $u^{sc}(\mathbf{x}; \underline{p}, \omega)$ *is given by the natural extension of (3.11) to* $\mathcal{S}$.

Moreover, the mapping $\mathcal{S} \ni z \mapsto u^{sc}(\cdot\,; \underline{p}, z) \in C^0(\Lambda)$ *is analytic.*

Proof i. Relation (3.4) yields

$$\begin{aligned}\mathrm{Im}(k_\omega^2(\mathbf{x})) & \qquad (3.67)\\ =&\mathrm{Re}(\omega)\ \{2\mathrm{Im}(\omega)[\kappa_1(\underline{x})\mathbf{1}_{[0,a]}(x_3)+1/c_0^2]+[\kappa_2(\underline{x})\mathbf{1}_{[0,a]}(x_3)+\sigma_0]\}\\ \geq&\mathrm{Re}(\omega)\ \left\{\inf_{\mathbf{y}\in\mathbb{R}^3}[\kappa_2(\underline{y})\mathbf{1}_{[0,a]}(y_3)+\sigma_0]-|\mathrm{Im}(\omega)|\ [||\kappa_1||_\infty+1/c_0^2]\right\}\end{aligned}$$

Now use the condition (3.5); then $\mathcal{S}$ may be defined as the intersection (see also (3.37)):

$$\mathcal{S}_0\cap\left\{z\in\mathbb{C},\ |\mathrm{Im}(z)|<\frac{\inf_{\mathbf{y}\in\mathbb{R}^3}[\kappa_2(\underline{y})\mathbf{1}_{[0,a]}(y_3)+\sigma_0]}{2[||\kappa_1||_\infty+1/c_0^2]}\right\}. \qquad (3.68)$$

ii. Since *i* holds, the argument which led to (3.11) still works. It is now easy to establish that the mapping

$$\mathcal{S}\ni\omega\mapsto\mathcal{G}_\omega(k_\omega^2)\in B(C^0(\Lambda))$$

is analytic, and this also remains true for

$$\mathcal{S}\ni\omega\mapsto[\mathbf{1}+\mathcal{G}_\omega(k_\omega^2)]^{-1}\in B(C^0(\Lambda)).$$

Employing the analyticity of ξ (see the previous lemma), and by a similar argument with that one used for establishing (3.65) one gets

$$u^{sc}(\mathbf{x}\ ;\underline{p},z)=\sum_{n\geq 0}(z-z_0)^n u_n(\mathbf{x},\underline{p}) \qquad (3.69)$$

where

$$u_n(\mathbf{x},\underline{p})=-\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{1}{(\zeta-z_0)^{n+1}}\left\{[\mathbf{1}+\mathcal{G}_\zeta(k_\zeta^2)]^{-1}\xi(\cdot\ ,\underline{p},\zeta)\right\}(\mathbf{x}). \qquad (3.70)$$

Since

$$\sup_{\{|\zeta-z_0|=r\}}\left\|[\mathbf{1}+\mathcal{G}_\zeta(k_\zeta^2)]^{-1}\right\|<\infty,$$

(3.70) and (3.40) imply

$$||u_n(\cdot\ ,\underline{p})||_\infty\leq\frac{\mathrm{const}(r,\delta,\Lambda,z_0)}{r^n}(1+|\underline{p}|)^{-1-\delta}, \qquad (3.71)$$

therefore the sequence in (3.69) converges in $C^0(\Lambda)$ and the proof is completed. □

Denote by $\Phi(\cdot\ ,\underline{p},\omega)$ the restriction to $\partial\Lambda'$ of $u^{sc}(\cdot\ ;\underline{p},\omega)$, where $\omega\in\mathcal{S}$ and $\underline{p}\in\mathbb{R}^2$. Define (see also (3.18)):

$$\begin{aligned}\mathcal{L}_1(\underline{x},\underline{p};\omega)\ \ &:=\ \ \frac{1}{e^{ip_3(\omega)a}-1}\left\{2[u^{in}\Phi](\underline{x},a;\underline{p},\omega)-2[u^{in}\Phi](\underline{x},0;\underline{p},\omega)\ +\right.\\ &+\ \ \left.\Phi^2(\underline{x},a;\underline{p},\omega)-\Phi^2(\underline{x},0;\underline{p},\omega)\right\}. \qquad (3.72)\end{aligned}$$

Among other things, the next corollary proves (3.19).

Corollary 3.16. *Denote by* $L(\omega)$ *the integral operator corresponding to* $\mathcal{L}_1(\underline{x}, \underline{p}; \omega)$ *acting on* $\mathscr{L}^2(\Omega \times \mathbb{R}^2)$*. Then:*

i. $L(\omega)$ *is a Hilbert-Schmidt operator (i.e. (3.19) holds) and the following mapping is analytic:*

$$\mathcal{S} \ni \omega \mapsto L(\omega) \in \mathcal{B}(\mathscr{L}^2(\Omega \times \mathbb{R}^2)) \; .$$

ii. For real values of $\omega \in \mathcal{S}$*,*

$$\lim_{\omega \searrow 0} ||L(\omega)|| = 0 \; . \tag{3.73}$$

Proof i. First notice that (see (3.7)) $|u^{in}(\mathbf{x}, \underline{p}, \omega)| \leq 1$ if $\mathbf{x} \in \Lambda$, $\underline{p} \in \mathbb{R}^2$ and $\omega \in \mathcal{S}$. Introduce (with $n = 0$) (3.71) in (3.72) and (3.19) follows. Furthermore, reasoning as in the previous two lemmas, one can write a power series expansion for $\mathcal{L}_1(\underline{x}, \underline{p}; \omega)$ similar to that one in (3.69), where the coeffi cients obey an estimate as in (3.71). Therefore, the power expansion holds in $\mathscr{L}^2(\Omega \times \mathbb{R}^2)$, too, and the mapping $\mathscr{L}(\omega)$ is analytic even in the Hilbert-Schmidt norm.

ii. Let us begin by noticing that (see (3.11)):

$$\lim_{\omega \searrow 0} ||\mathcal{G}_\omega(k_\omega^2)||_{B(C^0(\Lambda))} = 0 \; .$$

Using (3.40) in estimating the sup-norm in the right side of (3.11) we obtain that for $0 < \omega < \epsilon(c_0, \sigma_0, \omega_0, \delta, \Lambda)$

$$||u^{sc}(\cdot\ , \underline{p}, \omega)||_{C^0(\Lambda)} \leq 2||\xi(\cdot\ , \underline{p}, \omega)||_{C^0(\Lambda)} \leq \frac{\text{const}}{(1 + |\underline{p}|)^{1+\delta}} \omega^{1-\delta}, \tag{3.74}$$

where the above constant depends on everything but ω and $\underline{p}$.

We have already seen that there exists a constant $A > 0$ such that if $|\underline{p}| > A$, one has $\text{Im}(p_3(\omega)) > |\underline{p}|/2$ uniformly in $\omega \in \mathcal{S}_0$.

If $|\underline{p}| > A$, then

$$|\mathcal{L}_1(\underline{x}, \underline{p}; \omega)| \leq \frac{\text{const}}{1 - e^{-Aa/2}} (1 + |\underline{p}|)^{-1-\delta} \omega^{1-\delta}.$$

If $|\underline{p}| < A$ and $\omega \in \mathcal{S}_0$, defi ne the (bounded and continuous) function

$$f(\underline{p}, \omega) = \frac{i p_3(\omega) a}{e^{i p_3(\omega) a} - 1}.$$

Since $|p_3(\omega)| \geq \sigma_0^{1/2} \omega^{1/2}$ for any $\underline{p}$, we have

$$|\mathcal{L}_1(\underline{x}, \underline{p}; \omega)| \leq \text{const}\ (\sup |f|)\ (1 + |\underline{p}|)^{-1-\delta} \omega^{1/2-\delta}.$$

Now choose some $0 < \delta < 1/2$; then the $\mathscr{L}^2$ norm of $\mathcal{L}_1(\cdot\ , \cdot\ ; \omega)$ yet the Hilbert-Schmidt norm of $L(\omega)$ tends to zero with ω; therefore (3.73) holds. □

3.3.3. Completing the Proof of Theorem 3.6. The next lemma verifies the remaining needed conditions for u^{sc} in order to be an element of $\mathcal{A}_\omega$:

Lemma 3.17. *i. The function* $[\eta(\Phi)](\cdot)$ *defined in (3.15) belongs to* $\mathscr{L}^2(\mathbb{R}^2)$ *(i.e. (3.16) holds);*

ii. Having in mind (3.20), redenote by $\mathcal{K}(\underline{x}, \underline{y}; \omega)$ *the partial Fourier transform of* $\mathcal{L}_1$*, and by* $K(\omega)$*,* $\omega \in \mathcal{S}$*, the Hilbert-Schmidt operator corresponding to* $\mathcal{K}$ *acting on* $\mathscr{L}^2(\Omega \times \Omega)$*. Then the operator* $\mathbf{1} + K(\omega)$ *is invertible in* $B(\mathscr{L}^2(\Omega \times \Omega))$ *for all frequencies except maybe for a discrete set* $\mathcal{M} \subset \mathcal{S}$*;*

iii. The intersection $M := \mathcal{M} \cap (0, \omega_0)$ *is finite.*

Proof i. Since both terms in the left hand side of (3.27) are $\mathscr{L}^2$ vectors, $\eta(\Phi)$ is modulus square integrable, too. Notice that this result is not at all obvious from the definition of $\eta(\Phi)$ in (3.15), since the incident field has an exponential increase in $|\underline{p}|$ when $y_3 < 0$.

ii. We intend to use the analytic Fredholm theorem (Theorem VI.14 in [**RS80**]). Since the partial Fourier transform is a unitary operator on $\mathscr{L}^2(\Omega \times \mathbb{R}^2)$, we can restate Corollary 3.16 with $L(\omega)$ replaced by $K(\omega) \in B(\mathscr{L}^2(\Omega \times \Omega))$. Next, notice that $\mathbf{1} + K(\omega)$ is invertible for sufficiently small and positive frequencies (see (3.73)), therefore the "right" Fredholm alternative holds.

iii. Firstly, we have just seen that 0 is not an accumulation point for the eventual positive singularities. Secondly, let us show that ω_0 cannot be an accumulation point for $\mathcal{M}$. Indeed, one can reconsider the first two statements of this lemma for strips containing the interval $(0, 2\omega_0)$, hence if ω_0 is not regular, it must be isolated from the other singularities. □

3.3.3.1. *Putting All the Things Together.* Since the proof of Theorem 3.6 required a lot of intermediary technical results, let us now give an overview of the argument. We started by choosing $k \in W$ of the form given in (3.3) and (3.4). We then considered the scattering field appearing in the forward problem (see (3.11), (3.12) and the argument leading to them).

Denote by $\Phi_\omega(\cdot, \underline{p})$ the restriction of the scattered field to $\partial\Lambda'$ and let us see that all six conditions listed in Definition 3.5 are fulfilled, i.e. k can be reconstructed from the knowledge of $\Phi_\omega(\cdot, \underline{p})$ except maybe for a finite number of frequencies in the interval $(0, \omega_0)$:

1. $\Phi_\omega(\cdot, \underline{p}) \in H^{3/2}(\partial\Lambda')$ by the trace theorem;

2. Since the scattered field $u^{sc}(\cdot; \underline{p}, \omega)$ coincides outside Λ' with the solution to the exterior problem $\tilde{u}(\Phi_\omega)$ (see Definition 3.2), we conclude that $u(\Phi_\omega)$ which enters in the definition of $[\eta(\Phi_\omega)](\underline{p})$ (see (3.15)) is in fact the total field which solves the forward problem. We then established the key equation (3.27) which allow us to conclude that $\eta(\Phi_\omega)(\cdot) \in \mathscr{L}^2(\mathbb{R}^2)$ as soon as (3.19) holds;

3. As we have already mentioned, equation (3.27) lies at the very foundation of our arguments. The whole reconstruction process depends on its solvability. Since we see $\mathcal{L}_1$ as an integral kernel of a certain operator $L(\omega)$ (see Corollary 3.16), we would very much like to have a property as (3.19) since it would automatically imply that $L(\omega)$ is compact, and moreover, even after the partial Fourier transform with respect to the "$\underline{p}$" variable, the newly obtained operator $K(\omega)$ (see Lemma 3.17) remains compact.

As one can see in (3.18) or (3.72), the proof of (3.19) can be reduced to the proof of an estimate as in (3.35). The main idea consists in using (3.11), where one can see that the "$\underline{p}$" dependence is only contained by the "free" term ξ. Hence we concentrated our efforts in proving (3.40), and then we used (3.11) in passing on the decay in $\underline{p}$ to the scattered field.

4. We also payed a lot of attention to the frequency dependence. Since we eventually want to invert the operator $\mathbf{1} + K(\omega)$ in the Hilbert space $\mathscr{L}^2(\Omega \times \Omega)$, we focused on verifying the conditions of the analytic Fredholm alternative. This has been ultimately achieved in Corollary 3.16 and Lemma 3.17. The main idea consisted in proving that the compact operators $L(\omega)$ and $K(\omega)$ admit analytic extensions to a strip $\mathcal{S}$ containing the interval $(0, \omega_0)$, and secondly, proving that there are points in this strip for which the inverse exist. This was the main reason for the careful study of the frequency behaviour of the constant appearing in (3.40) (since k_0^2 is ω dependent). In fact, we should emphasize that (see Lemma 3.17 *ii.* and *iii.*) for sufficiently small frequency, we can always invert $\mathbf{1} + K(\omega)$ and hence to solve (3.27) and reconstruct k.

5 and 6. These points are now automatically satisfied.

3.4. Proof of Theorem 3.8 and Corollaries 3.9 and 3.10

At this point, having explained and motivated the structure of the space of admissible data $\mathcal{A}_\omega$ at a given positive frequency ω, the proofs which follow are more or less straightforward.

3.4.1. Proof of Theorem 3.8. We prove the theorem by double implication.

Assume *i* holds. The existence in W of a solution for the inverse problem is guaranteed by (3.23) and (3.24). That ω is regular (see Definition 3.7) with respect to any such solution follows from the fourth condition in Definition 3.5.

As for the uniqueness of the solution to the inverse problem, assume that we have two solutions in W corresponding to the same $\Phi_\omega \in \mathcal{A}_\omega$. Then they both solve equation (3.27), and since ω is regular with respect to both solutions (the operator $K(\omega)$ is the same), the uniqueness follows since $\mathbf{1} + K(\omega)$ is one to one.

Assume *ii* holds. Then since $k \in W$, by following the proof of Theorem 3.6 we see that the first three conditions in Definition 3.5 are satisfied. Since ω is regular with respect to k, we conclude that the fourth condition holds, too. Therefore the measured data belongs to $\mathcal{A}_\omega$. □

3.4.2. Corollary 3.9. We know from Theorem 3.6 *ii* that there exists an $\omega_N > 0$ such that for any frequency $0 < \omega < \omega_N$, the restriction to $\partial\Lambda'$ of the scattered field corresponding to any wave number in W_N belongs to the space of admissible data $\mathcal{A}_\omega$. Therefore, the reconstruction process can take place (according to Theorem 3.8), yielding a unique solution to the inverse problem. □

3.4.3. Corollary 3.10. The proof will have two steps:

First, let us argue why the fields are in fact equal for all $\underline{p}$'s, not only for an open set as stated in the hypothesis. Without loss, assume that the fields coincide for all $|\underline{p}| < 1$ at some frequency $\omega > 0$. We know that both u_1 and u_2 can be expressed as in (3.12) and (3.10) where k is replaced by k_1 and k_2 respectively. Then we employ the next lemma:

Lemma 3.18. *Let $R > 1$ be arbitrary large. Fix $\omega > 0$, $k \in W$ and $\mathbf{x} \in \partial\Lambda'$. Then there exists an open strip $S_R \subset \mathbb{C}^2$ containing the "real" ball $B(0, R)$ such that the mapping (see (3.12) and (3.10))*

$$B(0, R) \ni \underline{p} \mapsto u(\mathbf{x}; \underline{p}, \omega) \in \mathbb{C}$$

admits an analytic extension to S_R.

Proof. The strip is chosen such that extending the function (see (3.6))

$$B(0, R) \ni \underline{p} \mapsto p_3(\underline{p}) = \sqrt{k_0^2 - \underline{p}^2} \in \mathbb{C}$$

to it, becomes analytic in $\underline{p}$. This construction is always possible since the imaginary part of k_0^2 is strictly positive. Then we "propagate" this extension from (3.7) to (3.10) and finally to (3.12) employing a similar approach to that one used in proving Lemma 3.14. □

An immediate consequence of this lemma is that the fields must now coincide on $B(0, R)$ and since R was arbitrary, they must coincide for all $\underline{p} \in \mathbb{R}^2$.

Second, since we now know that the fields are equal for all $\underline{p}$'s and for a sequence of decreasing frequencies ($\omega = 1/n,\ n > N_0$), there must exist $m > N_0$ such that $1/m$ is a regular frequency for both k_1 and k_2 (since the set M in Theorem 3.6 *i* is finite). Therefore the fields are in the space of admissible data and the reconstruction procedure assures the equality $k_1 = k_2$. □

3.5. Proof of Theorem 3.11

We know from Theorem 3.6 *ii*. that $F_{N,\omega} = \mathcal{A}_\omega$ if $\omega \in (0, \omega_N)$. Therefore, one can associate to any $\Psi \in F_{N,\omega}$ (via the reconstruction procedure) a unique $k(\Psi) \in W_N$. In fact (see (3.23), what we really need to compute is $\kappa(\Psi)$ given by (3.22). The rest of this section will prove that small changes in Ψ lead to small changes in $\kappa(\Psi)$.

Let us rewrite (3.22) as:

$$\begin{aligned}\kappa(\Psi)-\kappa(\Phi) & \qquad\qquad (3.75)\\ =&\{[\mathbf{1}+K(\Psi)]^{-1}-[\mathbf{1}+K(\Phi)]^{-1}\}\tilde\eta(\Phi)+[\mathbf{1}+K(\Psi)]^{-1}\{\tilde\eta(\Psi)-\tilde\eta(\Phi)\}\\ =&[\mathbf{1}+K(\Psi)]^{-1}\left\{[K(\Phi)-K(\Psi)][\mathbf{1}+K(\Phi)]^{-1}\tilde\eta(\Phi)+\tilde\eta(\Psi)-\tilde\eta(\Phi)\right\}.\end{aligned}$$

The theorem would be concluded if we can prove the following three technical lemmas (see also (3.25)):

Lemma 3.19. *Under the same conditions as in Theorem 3.11, one can find $\delta_1>0$ such that*

$$||K(\Psi)-K(\Phi)||_{B(\mathscr{L}^2(\Omega))}<\epsilon \quad whenever \quad \begin{array}{l}\Psi\in F_{N,\omega} \quad and\\ ||\Psi-\Phi||_{\mathscr{L}^2(\partial\Lambda'\times\mathbb{R}^2)}<\delta_1 .\end{array}$$

Lemma 3.20. *Under the same assumptions, given $\epsilon>0$ there exist $\delta_2>0$ and a constant $C>0$ such that*

$$||[\mathbf{1}+K(\Psi)]^{-1}||_{B(\mathscr{L}^2(\Omega))}<C \quad whenever \quad \begin{array}{l}\Psi\in F_{N,\omega} \quad and\\ ||\Psi-\Phi||_{\mathscr{L}^2(\partial\Lambda'\times\mathbb{R}^2)}<\delta_2 .\end{array}$$

Lemma 3.21. *Under the same assumptions, given $\epsilon>0$ there exists $\delta_3>0$ such that (see (3.25))*

$$||\tilde\eta(\Psi)-\tilde\eta(\Phi)||_{\mathscr{L}^2(\Omega)}<\epsilon \quad whenever \quad \begin{array}{l}\Psi\in F_{N,\omega} \quad and\\ |||\Psi-\Phi|||<\delta_3 .\end{array}$$

3.5.1. Proof of Lemmas 3.19 and 3.20. Instead of a "direct" study of the $B(\mathscr{L}^2(\Omega))$-norm for the operator $K(\Phi)-K(\Psi)$, we will estimate its Hilbert-Schmidt norm. We know that each operator K corresponds to an integral kernel $\mathcal{K}(\underline{x},\underline{y})$ (see (3.20) and (3.18)), obtained from $\mathcal{L}_1(\underline{x},\underline{p})$ via a partial Fourier transform over the "$\underline{p}$"-variable. Hence,

$$||K(\Phi)-K(\Psi)||_{B(\mathscr{L}^2(\Omega))}\le||\mathcal{L}_1(\Phi)-\mathcal{L}_1(\Psi)||_{\mathscr{L}^2(\Omega\times\mathbb{R}^2)},$$

where (see (3.18)) the kernel $\mathcal{L}_1(\Phi)-\mathcal{L}_1(\Psi)$ looks like

$$\begin{aligned}&\frac{1}{e^{ip_3(\omega)a}-1}\left\{2[u^{in}(\Phi-\Psi)](\underline{x},a;\underline{p},\omega)-2[u^{in}(\Phi-\Psi)](\underline{x},0;\underline{p},\omega)\right. +\\ +\quad&\left.\Phi^2(\underline{x},a;\underline{p},\omega)-\Psi^2(\underline{x},a;\underline{p},\omega)-\Phi^2(\underline{x},0;\underline{p},\omega)+\Psi^2(\underline{x},0;\underline{p},\omega)\right\}. \qquad (3.76)\end{aligned}$$

Remember that $\omega>0$ is fixed; then (see (3.6) and (3.7))

$$\sup_{\underline{p}\in\mathbb{R}^2}\left|\frac{1}{e^{ip_3(\underline{p},\omega)a}-1}\right|\le\text{const},\quad \sup_{\mathbf{x}\in\Lambda}|u^{in}(\mathbf{x};\underline{p})|\le 1.$$

Another important thing is to apply the estimate (3.74) for both Φ and Ψ. Before that, let us mention that the constant appearing in (3.74) also depends on the C^1-norm of its corresponding wave number (i.e. κ_1 and κ_2) which by assumption is bounded from above by N. Therefore,

$$\sup_{\mathbf{x}\in\Lambda}\sup_{\underline{p}\in\mathbb{R}^2}|\Psi|(\mathbf{x};\underline{p},\omega)\leq \mathrm{const}(N,\Lambda) \tag{3.77}$$

and a similar estimate can be written for Φ, too.

It follows that

$$||\mathcal{L}_1(\Phi)-\mathcal{L}_1(\Psi)||_{\mathscr{L}^2(\Omega\times\mathbb{R}^2)}\leq \mathrm{const}(N,\Lambda)||\Phi-\Psi||_{\mathscr{L}^2(\partial\Lambda'\times\mathbb{R}^2)}$$

which ends the proof of Lemma 3.19.

As for Lemma 3.20, we only remark that it is a straightforward consequence of Lemma 3.20 and of a well known identity:

$$[\mathbf{1}+K(\Psi)]^{-1}=[\mathbf{1}+K(\Phi)]^{-1}+[\mathbf{1}+K(\Psi)]^{-1}[K(\Phi)-K(\Psi)][\mathbf{1}+K(\Phi)]^{-1}.$$

□

3.5.2. Proof of Lemma 3.21. First, from the definition of $\tilde{\eta}$ (see (3.21)) we conclude that it would be enough proving a similar statement with η instead of $\tilde{\eta}$. More precisely, we prove two technical results:

Proposition 1. Consider the same conditions as in Theorem 3.11. Then for any $\epsilon>0$ there exists $P_\epsilon>0$ such that for any $\Psi\in F_{N,\omega}$ we have

$$\int_{|\underline{p}|\geq P_\epsilon}|\eta(\Psi)|^2(\underline{p})d\underline{p}\leq\epsilon^2/6.$$

Remark. The above proposition states that the $\mathscr{L}^2$-"tail" of $\eta(\Psi)$ is small, uniformly in Ψ.

The second result states the following:

Proposition 2. Under the same assumptions as above, given $\epsilon>0$ there exists $\delta_4>0$ such that (see (3.25))

$$\int_{|\underline{p}|\leq P_\epsilon}|\eta(\Psi)-\eta(\Phi)|^2(\mathbf{p})<\epsilon^2/3\quad \textit{whenever}\quad \begin{array}{l}\Psi\in F_{N,\omega}\quad \textit{and}\\ |||\Psi-\Phi|||<\delta_4\ .\end{array}$$

Before actually proving these two propositions, let us see why are they implying Lemma 3.21. Indeed, choose a positive ϵ and apply Proposition 1 for getting P_ϵ; then

$$||\eta(\Phi)-\eta(\Psi)||^2_{\mathscr{L}^2(\mathbb{R}^2)}\leq 2\epsilon^2/3+\int_{|\underline{p}|\leq P_\epsilon}|\eta(\Psi)-\eta(\Phi)|^2(\mathbf{p}).$$

Identify δ_3 with δ_4 and we are done. □

3.5.2.1. *Proof of Proposition 1.* The main ingredient we employ is equation (3.27), where Φ should be replaced with Ψ. In other words, we will show that its left hand side has the property stated in the proposition.

Let us start with the first term, i.e. the Fourier transform of κ. Since κ was assumed to belong to $C_0^1(\Omega)$, we employ integration by parts in deriving the next formula:

$$\underline{p}^2 \left| \mathcal{F}^{-1}\kappa \right|^2 (\underline{p}) = \left| \mathcal{F}^{-1}(\partial_1\kappa) \right|^2 (\underline{p}) + \left| \mathcal{F}^{-1}(\partial_2\kappa) \right|^2 (\underline{p}). \tag{3.78}$$

Hence, for any $P > 1$ we get

$$\begin{aligned}\int_{|\underline{p}|\geq P} \left| \mathcal{F}^{-1}\kappa \right|^2 (\underline{p})d\underline{p} &= \frac{1}{P^2}\int_{|\underline{p}|\geq P} [\left| \mathcal{F}^{-1}(\partial_1\kappa) \right|^2 (\underline{p}) + \left| \mathcal{F}^{-1}(\partial_2\kappa) \right|^2 (\underline{p})]d\underline{p} \\ &\leq \frac{1}{P^2}\left[||\partial_1\kappa||^2_{\mathscr{L}^2(\mathbb{R}^2)} + ||\partial_2\kappa||^2_{\mathscr{L}^2(\mathbb{R}^2)} \right] \leq \frac{\mathrm{const}(N,\Lambda)}{P^2}, \end{aligned} \tag{3.79}$$

where in the second line we employed Plancherel's identity and that $k \in W_N$. Clearly, the right hand side of (3.79) can be made arbitrarily small as soon as P is increased, uniformly in κ and therefore in Ψ.

Let us now say a few words about the second term in the left hand side of equation (3.27). One can combine the estimates (3.35) and (3.77) with (3.72) getting

$$\left| \int_\Omega \mathcal{L}_1(\underline{x},\underline{p})\kappa(\underline{x})d\underline{x} \right| \leq \frac{\mathrm{const}}{(1+|\underline{p}|)^{1+\delta}},$$

where the above constant only depends on N and Λ. Now its $\mathscr{L}^2$-tail can be made arbitrarily small, and we are done. □

3.5.2.2. *Proof of Proposition 2.* We will show that a stronger estimate holds, uniformly in $|\underline{p}| \leq P_\epsilon$

$$|[\eta(\Phi)](\underline{p}) - [\eta(\Psi)](\underline{p})| \leq \mathrm{const}_\epsilon \cdot \sup_{\mathbf{y}\in\partial\Lambda'} |\Psi(\mathbf{y};\underline{p}) - \Phi(\mathbf{y};\underline{p})|. \tag{3.80}$$

After we introduce the expression of $u(\Psi)$ from (3.14) in (3.15), we obtain several integrals; one of them will only contain the incident field hence being independent of Ψ, two of them will contain products between (derivatives of) the incident field and (derivatives of) the radiating field $\tilde{u}(\Psi)$. The last integral will only contain the radiating field.

Therefore, the difference $[\eta(\Phi)](\underline{p}) - [\eta(\Psi)](\underline{p})$ will only consists from integrals whose integrands contain at least one term of the form $[\tilde{u}(\Psi)](\mathbf{y};\underline{p}) - [\tilde{u}(\Phi)](\mathbf{y};\underline{p})$ or $[\tilde{u}(\Psi)]^2(\mathbf{y};\underline{p}) - [\tilde{u}(\Phi)]^2(\mathbf{y};\underline{p})$, eventually with some $\mathbf{y}$-derivatives acting on them.

Let us investigate a typical term:

$$\frac{1}{e^{ip_3a}-1}\int_{|\mathbf{y}|=\rho}\{u^{in}(\mathbf{y};\underline{p})[[\partial_\nu\partial_3\tilde{u}(\Phi)](\mathbf{y};\underline{p}) - [\partial_\nu\partial_3\tilde{u}(\Psi)](\mathbf{y};\underline{p})]\}d\sigma(\mathbf{y}). \tag{3.81}$$

First, we want to get rid of the $\mathbf{y}$-derivatives acting on the radiating solutions in the above formula. We intend to apply Theorem 3.9 in [**CK98**]; in order to do that, we introduce a few notations.

Denote by S and D the single- and double-layer operators acting on $C^0(\partial\Lambda')$ and given by (see 3.9):

$$(S\phi)(\mathbf{x}) := 2\int_{\partial\Lambda'} G_0(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}')d\sigma(\mathbf{x}') \tag{3.82}$$

and

$$(T\phi)(\mathbf{x}) := 2\int_{\partial\Lambda'} \frac{\partial G_0}{\partial\nu(\mathbf{x}')}(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}')d\sigma(\mathbf{x}'), \tag{3.83}$$

where $\mathbf{x}$ is restricted to the boundary.

Following [**CK98**], we express the radiating solution as

$$[\tilde{u}(\Psi)](\mathbf{x};\mathbf{p}) = \int_{\partial\Lambda'} \left\{\frac{\partial G_0}{\partial\nu(\mathbf{x}')}(\mathbf{x},\mathbf{x}') - i\alpha G_0(\mathbf{x},\mathbf{x}')\right\}\psi(\mathbf{x}';\underline{p})d\sigma(\mathbf{x}'), \tag{3.84}$$

where $\mathbf{x}\in\mathbb{R}^3\setminus\Lambda'$, $\alpha>0$ is a positive coupling parameter and $\psi(\cdot;\underline{p})\in C^0(\partial\Lambda')$ is an yet unknown continuous function, which is to be found. Indeed, one can prove that the operator $\mathbf{1}+T-i\alpha S$ has a bounded inverse in $B(C^0(\partial\Lambda'))$ and

$$\psi(\cdot;\underline{p}) = 2(\mathbf{1}+T-i\alpha S)^{-1}\Psi(\cdot;\underline{p}). \tag{3.85}$$

Introducing (3.85) in (3.84), we get that for $|\mathbf{y}|=\rho$ i.e. away from the boundary $\partial\Lambda'$ we have

$$\sup_{\mathbf{y}\in\partial\Lambda'}\left|[\partial_\nu\partial_3\tilde{u}(\Phi)](\mathbf{y};\underline{p}) - [\partial_\nu\partial_3\tilde{u}(\Psi)](\mathbf{y};\underline{p})]\right| \le \mathrm{const}||\Psi(\cdot;\underline{p}) - \Phi(\cdot;\underline{p})||_\infty. \tag{3.86}$$

Using the above estimate in (3.81) and remembering that (see (3.7))

$$\sup_{|\mathbf{y}|=\rho}\sup_{|\underline{p}|\le P_\epsilon}\left|u^{in}\right|(\mathbf{y};\underline{p}) = \mathrm{const}(\epsilon,\Lambda),$$

we obtain an estimate as in (3.80).

Besides terms like that one in (3.81), we also have typical “quadratic” terms as the next one:

$$\frac{1}{e^{ip_3a}-1}\int_{|\mathbf{y}|=\rho}\left\{[\tilde{u}(\Phi)]\,[\partial_\nu\partial_3\tilde{u}(\Phi)](\mathbf{y};\underline{p}) - [\tilde{u}(\Psi)]\,[\partial_\nu\partial_3\tilde{u}(\Psi)](\mathbf{y};\underline{p})\right\}d\sigma(\mathbf{y}). \tag{3.87}$$

We split the above term in two, trying to “linearize” it:

$$\frac{1}{e^{ip_3a}-1}\int_{|\mathbf{y}|=\rho}[\tilde{u}(\Phi)](\mathbf{y};\underline{p})\left\{[\partial_\nu\partial_3\tilde{u}(\Phi)](\mathbf{y};\underline{p}) - [\partial_\nu\partial_3\tilde{u}(\Psi)](\mathbf{y};\underline{p})\right\}d\sigma(\mathbf{y}) \tag{3.88}$$

and

$$\frac{1}{e^{ip_3a}-1}\int_{|\mathbf{y}|=\rho}\left\{[\tilde{u}(\Phi)](\mathbf{y};\underline{p}) - [\tilde{u}(\Psi)](\mathbf{y};\underline{p})\right\}[\partial_\nu\partial_3\tilde{u}(\Psi)](\mathbf{y};\underline{p})d\sigma(\mathbf{y}). \tag{3.89}$$

While (3.88) brings nothing new compared to (3.81), equation (3.89) still requires an estimate more; that is, uniformly in Ψ and $|\underline{p}| \leq P_\epsilon$ we have (see also (3.77))

$$\sup_{|\mathbf{y}|=\rho} |\partial_\nu \partial_3 \tilde{u}(\Psi)| \, (\mathbf{y};\underline{p}) \leq \mathrm{const}(\Lambda) ||\Psi(\cdot;\underline{p})||_\infty \leq \mathrm{const}(N,\Lambda).$$

We then conclude that (3.80) holds and so does Proposition 2. □

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Generalized Fourier Transforms Classes

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Abstract. The Fourier class of integral transforms with kernels $B(\omega r)$ has by definition inverse transforms with kernel $B(-\omega r)$. The space of such transforms is explicitly constructed. A slightly more general class of generalized Fourier transforms are introduced. From the general theory follows that integral transform with kernels which are products of a Bessel and a Hankel function or which is of a certain general hypergeometric type have inverse transforms of the same structure. In addition the application of the transformation theory to the acoustic inverse medium problem is presented, with an explicit solution for a linearized problem.

Key Words and Phrases Integral transforms, Fourier transforms

AMS Subjects classification codes 46F12, 42A38

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4.1. Introduction

We denote by a class of integral transforms the set of all mappings $\{\mathcal{B}, \mathcal{B}^{-1}\}$, with kernels $\{B(w,r), \widetilde{B}(r,w)\}$ respectively, that are integral operators mapping $\mathbb{R}_+ \mapsto \mathbb{R}$ and $\mathbb{R} \mapsto \mathbb{R}_+$. We are interested in finding conditions on B such that $\widetilde{B}$ can be explicitly constructed. The classes of integrals transforms that we construct have kernels $\{B(wr), \mathcal{P}B(-wr)\}$ with $\mathcal{P}$ a given operator. Hence a fixed $\mathcal{P}$ defines a class and a fixed B defines a pair $\{\mathcal{B}, \mathcal{B}^{-1}\}$ in the associated class. The class generated by $\mathcal{P} = 1$ we define as the Fourier transform class, and the class generated by $\mathcal{P} = r\partial_r r\omega$ we denote as the $1/x$ Generalized Fourier transform class. The map $\mathcal{B}f : \mathbb{R}_+ \mapsto \mathbb{R}$ is defined by

$$F(\omega) = \mathcal{B}f(\omega) = \int_0^\infty B(\omega r) f(r) dr, \qquad \omega \in \mathbb{R} \tag{4.1}$$

and the inverse map $\mathcal{B}^{-1}F : \mathbb{R} \mapsto \mathbb{R}_+$ is defined by

$$f(r) = \mathcal{B}^{-1}F(r) = \int_{-\infty}^\infty \mathcal{P}B(-\omega r) F(\omega) d\omega, \tag{4.2}$$

Generally it is difficult to find a simple expression for the inverse mapping as an integral operator. A well known example where the inverse map may explicitly be found is the Fourier transform and in this case B is defined by (4.4) with the inverse operator $\mathcal{B}^{-1}$ simply being

$$f(r) = \mathcal{B}^{-1}F(r) = \int_{-\infty}^\infty B(-\omega r) F(\omega) d\omega. \tag{4.3}$$

The map $\mathcal{B}f$ defined by (4.1) is also known to have an inverse map given by $f = \mathcal{B}^{-1}F = \mathcal{M}^{-1}\{[\mathcal{M}B]^{-1}\mathcal{M}F\}$, with $\mathcal{M}$ the Mellin transform. Situations where this inverse mapping will be explicitly expressible will be investigated and characterized when the inverse mapping lead to a simple structure as in (4.2).

The problems to be treated in this paper is therefore to find classes of mappings of the form (4.1) for which the inverse map is given by (4.2). For the Fourier Transform class, generated by $\mathcal{P} = 1$, the canonical element $\{\mathcal{B}, \mathcal{B}^{-1}\}$ has the kernel

$$B_0(x) = \frac{1}{\sqrt{2\pi}} e^{ix}, \tag{4.4}$$

whereas for the $1/x$ Fourier Transform class the canonical element $\{\mathcal{B}, \mathcal{B}^{-1}\}$ has the kernel

$$B_0(x) = \frac{-i}{\sqrt{2\pi x}} e^{ix}. \tag{4.5}$$

For many other appropriately chosen operators $\mathcal{P}$ the construction of Fourier Transform classes may be achieved by the method of this paper. As a third example we have the class generated by $\mathcal{P} = r\omega$ with the canonical element $\{\mathcal{B}, \mathcal{B}^{-1}\}$ having the kernel $B_0(x) = \frac{1}{\sqrt{2\pi x}} e^{i[x - \frac{\pi}{4}]}$.

The achievements of this paper is:

- To construct the class of all $\mathcal{B}$'s satisfying the given generalized Fourier Transform pair of the form (4.1), (4.3) or (4.1), (4.2),$\mathcal{P} = r\partial_r r\omega$, see theorem 4.7 and theorem 4.8 respectively.
- Prove that the structure of the class is $\mathcal{A}B_0$, with $\mathcal{A}$ an Abelian set of bounded operators in a Hilbert space, and B_0 the kernel of the canonical elements for the class. For the Fourier Transform class $\mathcal{A}$ is explicitly constructed, see (4.43), and for the 1/x Generalized Fourier Transform class a large class of kernels is explicitly constructed, see (4.62).
- Define Hilbert spaces such that the mappings $\mathcal{B}$ and $\mathcal{B}^{-1}$ are continuous.
- To apply the theory yielding an explicit solution of a class of integral equations with kernels involving either products of Bessel and Hankel functions, or being a certain general hypergeometric function.
- To apply the solution of integral equations with kernels involving products of Bessel and Hankel functions to solve a linearized inverse medium problem.

We are going to use the Mellin transform of (4.1), (4.3) and (4.2) with $\mathcal{P} = r\partial_r r\omega$ and represent the properties of the kernels by equations for the Mellin transform of the kernels. The key of the analysis is to prove that the transform pair hold if and only if the Mellin transform satisfy a nonlinear equation which for the Fourier transform class is

$$\mathcal{M}^g B(s)\mathcal{M}^g B(1-s)[e^{-i\pi s} - e^{i\pi s}] = 1 \qquad \mathrm{Re}(s) = .5. \tag{4.6}$$

and for the $1/x$ Generalized Fourier Transform class is

$$[1-s]\sin \pi s \mathcal{M}^g B(s)\mathcal{M}^g_a B(2-s) = \frac{1}{2} \quad \mathrm{Re}(s) = .5. \tag{4.7}$$

Equation (4.6) is explicitly solved for the Fourier transform case, see (4.43) and theorem 4.7, and for the $1/x$ Generalized Fourier transform class a large subspace of transform pairs are found, see (4.62) and theorem 4.11.

A similar approach of constructing the inverse maps (4.2) and (4.3) was used in [**SSK98**] for a large class of operators of the form (4.1). As the mapping $\mathcal{B}$ they used the **H**-transform defined by hypergeometric functions and found, for a specific choice of parameters, the transformation pair as

$$\mathcal{B}f = Hf \qquad f(r) = -r\partial_r \int_0^\infty \Phi(\omega r) Hf(\omega)d\omega, \tag{4.8}$$

with the kernel $\Phi(x)$ being given by

$$\Phi(x) = \mathcal{M}^{-1}\{\frac{1}{s\mathcal{M}B(1-s)}\}(x). \tag{4.9}$$

Remark: In (4.8) the **H**-transform is defined for $\omega \in \mathbb{R}_+$ whereas (4.1), (4.3) is defined for $\omega \in \mathbb{R}$.

For the **H**-transform the Mellin transform of the kernel can explicitly be found as

$$\mathcal{M}\mathbf{H} = \mathcal{H}_{p,q}^{m,n}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle| s\right] = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{r=n+1}^{p}\Gamma(a_r+\alpha_r s)\prod_{t=m+1}^{q}\Gamma(1-b_t-\beta_t s)}. \tag{4.10}$$

Thus the functions $\mathcal{M}\mathbf{H}$ and $1/s\mathcal{M}\mathbf{H}(1-s)$ have the same structure and Φ given by (4.9) is again a hypergeometric function. The common idea of [**SSK98**] and the present paper is to represent structure properties of the kernels of $\mathcal{B}$ and $\mathcal{B}^{-1}$ by specifying properties of the Mellin transform of the kernels.

For the Fourier transform class, as well as the 1/x Generalized Fourier Transform class, we will give a number of examples including the hypergeometric functions, showing that the H transform of [**SSK98**] with imaginary argument and properly chosen coefficients, transform by (4.1), (4.3). One of the elements in the 1/x Generalized Fourier Transform Class will be $B(x) = c_{mn}J_m(x)H_n^1(x)$, with H and J Hankel and Bessel functions and c_{mn} a complex constant.

An application with this type of integral equations is the inverse scattering problem in $\mathbb{R}^3$ with a specific spherical incident wave. This Inverse Problem may through the transformation theory be represented as a Second Kind Fredholm problem. The linearized equation is represented as (4.1) with the kernel $B(x) = c_{m0}J_m(x)H_0^1(x)$ or $B(x) = c_{m0}J_m(x)J_0(x)$.

The structure for the solution of $\int_\Gamma f(r)H_0^1 J_n dr = F(\omega) = \mathcal{F}\phi(t)$ is expressed in terms of the Fourier transform, $\mathcal{F}$, as

$$f(r) = 2(-1)^n\{r\partial_r\phi(2r) - n(n+1)\phi(2r) + \int_0^{2r} G_n(t,r)\phi(t)dt\}, \tag{4.11}$$

where $G_n(t,r)$ can be explicitly calculated and is given in (4.77).

In section 4.2 of this paper we start with a review of the generalized Mellin transform theory of distributions, define the spaces to be used and establish the needed Mellin transform of $\mathcal{B}f$. For the Fourier Transform class we find the function $\mathcal{M}\mathcal{B}^{-1}$ in sections 4.2.1, show the key relation (4.38) and construct the space of kernels for the Fourier transform class. Finally in section 4.2.2 we establish similar results for the 1/x Generalized Fourier Transform class. For the two transform classes the continuity of the mappings $\mathcal{B}$ and $\mathcal{B}^{-1}$ are proved.

In section 4.4 an integral equation formulation of the inverse acoustic scattering problem is derived and solved. Basically the problem is from knowledge of trace data of $u(x,\omega)$ to determine the coefficient $k^2(x)$ of an elliptic linear PDE -the Helmholtz equation $(\Delta + k^2(x))u(x,\omega) = 0$. Through the use of the transformation theory an explicit solution for a linearized problem is found, and this can be extended to iteratively solving the associated full problem - that will though not be treated here.

4.2. Generalized Fourier Transform Classes

The classical Mellin transform of a function defined on $\mathbb{R}_+$ is defined through the Fourier Transform $\mathcal{F}$ as

$$\mathcal{M}_\sigma f(s) = \sqrt{2\pi}\mathcal{F}[e^{\sigma u} f(e^u)](y) = \int_0^\infty dr f(r) r^{s-1}, \qquad s = \sigma + iy, \tag{4.12}$$

and the inverse transform is

$$f(r) = \frac{1}{2\pi i} \int ds \mathcal{M}F(s) r^{-s}, \tag{4.13}$$

with the integral along the line $s = \sigma + i\omega$.

In order to define the distributional Mellin transform we will introduce the space of test functions and the topology of that space. Let σ be any real number. Introduce the space $\mathcal{S}_{+\sigma}(\mathbb{R}_+)$ of test functions with support on the positive axis as

$$\mathcal{S}_{+\sigma} = \{\phi \in C^\infty(\mathbb{R}_+) \mid e^{\sigma u}\phi(e^u) \in \mathcal{S}(\mathbb{R})\}. \tag{4.14}$$

For $\mathcal{S}_{+\sigma}$ define the semi norms

$$p_{ab}(\phi) = \sup_{r \in \mathbb{R}_+} \mid ([r\partial_r]^b (\ln r)^a r^\sigma \phi(r) \mid, \tag{4.15}$$

where a, b are non negative integers. Let the space $\mathcal{M}_\sigma \mathcal{S}_{+\sigma}$ be equipped with the induced topology of $\mathcal{S}$, that is the semi norms

$$\pi_{ab}(\mathcal{M}_\sigma \phi) = \sup_{y \in \mathbb{R}} \mid s^a \partial_s^b \mathcal{M}_\sigma \phi \mid, \qquad s = \sigma + iy. \tag{4.16}$$

Then we have the result:

Lemma 4.1. *Let the topology of $\mathcal{M}_\sigma \mathcal{S}_{+\sigma}$ be the induced topology of $\mathcal{S}$. Then the mapping $\mathcal{S}_{+\sigma} \to \mathcal{M}_\sigma \mathcal{S}_{+\sigma} = \mathcal{S}$ is a isomorphism. And, for any $\psi, \phi \in \mathcal{S}_{+\sigma}$ the following equations hold:*

$$\int_0^\infty dr \psi(r)\phi(r) = \frac{1}{2\pi i} \int ds \mathcal{M}_\sigma \psi(1-s) \mathcal{M}_\sigma \phi(s), \tag{4.17}$$

and

$$\int_0^\infty dr \psi(\omega r)\phi(r) = \frac{1}{2\pi i} \int ds \omega^{-s} \mathcal{M}_\sigma \psi(1-s) \mathcal{M}_\sigma \phi(s), \tag{4.18}$$

with the integral along the line $s = \sigma + iy$, where σ is fixed in the complex s plane.

Proof. The mapping $\phi \in \mathcal{S}_{+\sigma} \to e^{\sigma u}\phi(e^u) \in \mathcal{S}$ is an isomorphism of $\mathcal{S}_{+\sigma}$ onto $\mathcal{S}$ with the topologies introduced. And the mapping $\mathcal{S} \to \mathcal{F}\mathcal{S}$ is an isomorphism.

The relations (4.17) and (4.18) are simple computational results. □

The generalized Mellin transform is then defined for any B in the space $\mathcal{S}'_{+\sigma}$ by the continuous linear functional where we affix the superscript "g" to emphasize that it is a generalized function

$$\langle B, \phi \rangle = \frac{1}{2\pi i} \langle \mathcal{M}^g_\sigma B(1-s), \mathcal{M}_\sigma \phi(s) \rangle_y, \quad s = \sigma + iy, \tag{4.19}$$

with the linear functional in the variable y. We have the result:

Lemma 4.2. *The mapping $\mathcal{S}'_{+\sigma} \to \mathcal{M}_\sigma \mathcal{S}'_{+\sigma}$ is an isomorphism of $\mathcal{S}'_{+\sigma}$ onto $\mathcal{M}^g_\sigma \mathcal{S}'_{+\sigma} = \mathcal{S}'$.*

In the remaining part of this subsection the real part of s will be $\frac{1}{2}$, and we will omit the index σ, that is we use $\mathcal{M} = \mathcal{M}_{\frac{1}{2}}$. The next lemma is important for a rigorous definition of the generalized Fourier transform pair (4.1), (4.3). We define the space of tempered C^∞ functions by

$$\mathcal{S}_t = \{\phi \in C^\infty(\mathbb{R}) \mid \exists n(\phi) \|(1+x^2)^{-n}\phi(x)\|_\infty < \infty\}. \tag{4.20}$$

Lemma 4.3. *Let $B(\pm r) \in \mathcal{S}'_{+.5}$ and assume that $\mathcal{M}^g B(\pm r)(s) \in \mathcal{S}_t$. Assume that $f \in \mathcal{S}_{+.5}$. Then $\mathcal{B}f$ defined by (4.21) and (4.24) for ω positive or negative respectively, is a mapping of $\mathcal{S}_{+.5}$ into $\mathcal{S}_{+.5}$, and the generalized Mellin transform of $\mathcal{B}f$ on the positive and negative axis satisfies (4.23) and (4.25) respectively. The functions $\mathcal{M}^g[\mathcal{B}f(\omega)](\frac{1}{2}+iy)$ and $\mathcal{M}^g[\mathcal{B}f(-\omega)](\frac{1}{2}+iy)$ are test functions in the space $\mathcal{S}$.*

Proof. With the assumptions of the lemma we may define for any $\omega > 0$ the linear functional $\mathcal{B}f$ by

$$\mathcal{B}f(\omega) = \langle B(\omega r), f(r) \rangle = \langle B(r), \frac{1}{\omega} f(\frac{r}{\omega}) \rangle. \tag{4.21}$$

The functional (4.21) is a function, which in terms of the Mellin transform may be expressed as the integral along the line $\mathrm{Re}(s) = \frac{1}{2}$

$$\langle B(\omega r), f(r) \rangle = \frac{1}{2\pi i} \langle \omega^{s-1} \mathcal{M}^g B(1-s), \mathcal{M}f(s) \rangle \quad s = \sigma + iy. \tag{4.22}$$

The generalized Mellin transform of $\mathcal{B}f$ is easily proved to be

$$\mathcal{M}^g \mathcal{B}f(s) = \mathcal{M}^g B(s) \mathcal{M}f(1-s) \in \mathcal{S}. \tag{4.23}$$

In a similar way define for $\omega > 0$

$$\mathcal{B}f(-\omega) = \langle B(-\omega r), f(r) \rangle = \langle B(-r), \frac{1}{\omega} f(\frac{r}{\omega}) \rangle. \tag{4.24}$$

The generalized Mellin transform of $\mathcal{B}f$ on the negative axis is:

$$\mathcal{M}^g[\mathcal{B}f(-\omega)](s) = \mathcal{M}^g[B(-r)](s) \mathcal{M}f(1-s) \in \mathcal{S}. \tag{4.25}$$

□

We will now define the function space for the kernels of the Fourier transform class. The canonical kernel $\exp(i\omega z)$ is a holomorphic function of z tending to zero at infinity in the upper half complex plane. The space of kernels is defined such that this property hold for any kernels in the space. This will be the case if the space of kernels is introduced by:

Definition 4.4 (Definition of the space of kernels $\mathcal{A}_0$)**.**

$$\begin{aligned} \mathcal{A}_0 \quad = \quad & \{B \in \mathcal{S}'_{\frac{1}{2}+} \mid \gamma \in [0,\pi] : e^{-i\gamma s}\mathcal{M}^g B(s) \in \mathcal{S}_t^\infty \\ & \wedge \gamma \in]0,\pi[: \|e^{-i\gamma s}\mathcal{M}^g B\|_\infty < \infty\}, \quad s = 0.5 + iy. \end{aligned} \tag{4.26}$$

We will prove:

Lemma 4.5. *Let $B \in \mathcal{A}_0$ then there exists a function $B(z)$ holomorphic in $\mathbb{C}_+$ such that the distribution B on the real positive axis defined by (4.31) is continuous from above, and B may be defined on the negative real axis such that (4.33) hold. For any $f \in \mathcal{S}_{+.5}$ the operators $\mathcal{B}f(\pm\omega) \in \mathcal{S}_{+.5}$, and $\mathcal{M}^g\mathcal{B}f(\pm\omega) \in \mathcal{S}$, and for any $\gamma \in [0,\pi]$ we have:*

$$\mathcal{M}^g B(re^{i\gamma})(s) = e^{-is\gamma}\mathcal{M}^g[B(r)](s) \qquad r > 0. \tag{4.27}$$

Proof. Before defining the analytic continuation of B by (4.30) we will investigate the behavior of $\mathcal{M}^g B$. Let $\gamma \in]0,\pi[$, and assume that $y = \mathrm{Im}(s) > 0$. There exists a positive c such that $\gamma + c < \pi$ and:

$$\mid s^n e^{-i\gamma s}\mathcal{M}^g B(s) \mid \leq \mid s^n e^{-cy} e^{-i[\gamma+c]s}\mathcal{M}^g B(s) \mid . \tag{4.28}$$

From the definition of $\mathcal{A}_0$ the function $e^{-i[\gamma+c]s}\mathcal{M}^g B(s)$ belong to the space $\mathcal{S}_t$, and the right hand side of (4.28) tend to zero as $y \to \infty$. In a similar way for $y < 0$ we have:

$$\mid s^n e^{-i\gamma s}\mathcal{M}^g B(s) \mid \leq \mid s^n e^{\gamma y}\mathcal{M}^g B(s) \mid, \tag{4.29}$$

which tend to zero for $y \to -\infty$. That is the right hand side of (4.27) is in $\mathcal{S}$ for any $\gamma \in]0,\pi[$ and $B(z)$ may be defined in the upper half plane as the inverse Mellin transform

$$B(z) = \frac{1}{2\pi i}\int ds z^{-s}\mathcal{M}^g B(s), \tag{4.30}$$

with the integral along the line $s = \frac{1}{2} + iy$. From this equation it is seen that equation (4.27) hold for $\gamma \in [0,\pi[$. Finally to prove that B on the real axis is the limit from above of $B(z)$, let n be a number such that $[1+y^2]^{-n}\mathcal{M}^g B(\frac{1}{2}+iy)$ is bounded. Then $[1+y^2]^{-n}[1-e^{-i\delta s}]\mathcal{M}^g B(\frac{1}{2}+iy)$ tend uniformly to zero for $\delta \to 0_+$, on any compact interval. The same function tend to zero uniformly at $y = \pm\infty$. Which proves that for any $\phi \in \mathcal{S}_{+.5}$

$$\lim_{\delta\to 0_+}\langle B(e^{i\delta}r) - B(r), \phi\rangle = \frac{1}{2\pi i}\lim_{\delta\to 0_+}\langle \mathcal{M}^g[B(e^{i\delta}r) - B(r)](1-s), \mathcal{M}\phi(s)\rangle = 0. \tag{4.31}$$

This proves that B in the distributional sense is continuous from above on the positive axis. The distribution $B(-r)$ is defined by

$$B(-r) = (\mathcal{M}^g)^{-1}\{e^{-i\pi s}\mathcal{M}^g B\}. \tag{4.32}$$

With that definition equation (4.27) is proved to hold for $\gamma = \pi$. In the same way as used above the limit of the function $B(z)$ on the negative axis is $B(-r)$

$$\lim_{\delta\to 0_+}\langle \mathcal{M}^g B[e^{i(\pi-\delta)}r](1-s) - \mathcal{M}^g B[-r](1-s), \phi\rangle = 0. \tag{4.33}$$

The generalized Mellin Transform of $\mathcal{B}f(\omega)$ for ω positive is given by (4.23), and from the assumptions of the lemma follows that $\mathcal{M}^g\mathcal{B}f \in \mathcal{S}$, and $\mathcal{B}f \in \mathcal{S}_{+.5}$. Similarly (4.25) show that $\mathcal{M}^g\mathcal{B}f(-\omega) \in \mathcal{S}$, and $\mathcal{B}f(-\omega) \in \mathcal{S}_{+.5}$. □

The fundamental properties of the Mellin Transformation that is used for establishing the Fourier Transform class and the 1/x Generalized Fourier Transform class have now been derived.

4.2.1. The Fourier Transform Class. We may now give a rigorous formulation of the Fourier Transform pair (4.1), (4.3). According to lemma 4.3 for any $B \in \mathcal{A}_0$ we may define the Fourier transform on the space $\mathcal{S}_{+\frac{1}{2}}$ by

$$\mathcal{B}f(\omega) = \langle B(\omega r), f(r)\rangle. \tag{4.34}$$

Lemma 4.5 shows that the inverse Fourier Transform (4.3) may be defined on the space $\mathcal{B}\mathcal{S}_{+.5}$ by

$$\mathcal{B}^{-1}F(r) = \langle B(-\omega r), F(\omega)\rangle_{\omega>0} + \langle B(\omega r), F(-\omega)\rangle_{\omega>0}. \tag{4.35}$$

The next result will show that $\mathcal{B}^{-1}$ is indeed the inverse of the mapping $\mathcal{B}$ if and only if the Mellin transform of B satisfies the nonlinear condition (4.38).

Lemma 4.6. *Let $B \in \mathcal{A}_0$ and let $f \in \mathcal{S}_{+.5}$. Assume that $B(r) = \overline{B(-r)}$ for $r \in \mathbb{R}$. Then the Fourier Transform pair (4.34) and (4.35) hold if and only if $\mathcal{M}^g B = \psi\mathcal{M}^g B_0$ with B_0 the kernel of the canonical Fourier transform and $\psi \in \Psi$ with the space Ψ defined by (4.43).*

Proof. In lemma 4.3 it was proved that (4.23) hold. With the definition of the inverse transform $F = \mathcal{B}f \in \mathcal{S}_{+.5}$, and using (4.35), (4.23) and (4.25) it follows that

$$\mathcal{M}^g\mathcal{B}^{-1}F(s) = \mathcal{M}^g F(1-s)\mathcal{M}^g[B(-r)](s) + \mathcal{M}^g[F(-\omega)](1-s)\mathcal{M}^g[B(r)](s). \tag{4.36}$$

For the function $\mathcal{B}f(-\omega)$ equation (4.23) is used to prove that $\mathcal{M}^g F(-\omega)$ will satisfy equation (4.27). Then using (4.27) and (4.23), equation (4.36) reduces to

$$\mathcal{M}^g\mathcal{B}^{-1}F(s) = \mathcal{M}^g B(s)\mathcal{M}^g B(1-s)[e^{-i\pi s} - e^{i\pi s}]\mathcal{M}f(s). \tag{4.37}$$

Hence $f = \mathcal{B}^{-1}\mathcal{B}f$ for any $f \in \mathcal{S}_{+\frac{1}{2}}$ if and only if $\mathcal{M}^g B$ satisfy the equation

$$\mathcal{M}^g B(s)\mathcal{M}^g B(1-s)[e^{-i\pi s} - e^{i\pi s}] = 1. \tag{4.38}$$

The Mellin transform of what is denoted the canonical Fourier Transform kernel may explicitly be found as

$$\mathcal{M}^g B_0(s) = \frac{(i)^s}{\sqrt{2\pi}}\Gamma(s). \tag{4.39}$$

From $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin \pi s}$ follows that $\mathcal{M}B_0$ satisfies the equation (4.38). That is $\mathcal{B}^{-1}\mathcal{B}f = f$ if and only if

$$\mathcal{M}B(s) = \mathcal{M}B_0(s)\psi(s) \tag{4.40}$$

with $\psi \in \mathcal{S}_t$ any solution of

$$\psi(s)\psi(1-s) = 1, \quad \text{Re}(s) = \frac{1}{2}. \tag{4.41}$$

If in addition the distribution B satisfies the condition $B(-x) = \overline{B(x)}$ for $x \in \mathbb{R}$, then the function ψ must also satisfy

$$\overline{\psi(s)} = \psi(\overline{s}), \quad \text{Re}(s) = .5. \tag{4.42}$$

It is easy to see that a function $\psi \in \mathcal{S}_t$ is in the space

$$\Psi = \left\{\psi \in \mathcal{S}_t \middle| \psi(\frac{1}{2} + iy) = \pm \exp\{i\gamma(y)\} \wedge \gamma(y) = \overline{\gamma(y)} = -\gamma(-y), \gamma \in C^\infty\right\} \tag{4.43}$$

if and only if the function satisfies (4.41) and (4.42). □

Thus the set of all Fourier Transform pairs is explicitly given by (4.40) with ψ in the space Ψ.

Next we will investigate the mappings generated by the Fourier Transform class. Introduce the Hilbert spaces:

$$T = \mathscr{L}^2(\mathbb{R}_+), \qquad \Omega = BT, \qquad Y = \mathcal{M}\Omega. \tag{4.44}$$

The inner product of T and Ω are $\mathscr{L}^2$ inner products, and in Y we have an additional factor $\frac{1}{2\pi}$ in the inner product. It is well known that the mapping $\mathcal{F} = \mathcal{B}_0$ is an isomorphism of T onto Ω which is a subspace of $\mathscr{L}^2$, and the Mellin transform is an isomorphism of $\mathscr{L}^2(R_+)$ onto $\mathscr{L}^2(R)$ for $s = \frac{1}{2} + iy$. Using T, Ω and Y as index for the inner products we have for any $f, g \in \mathcal{S}_{+.5}$:

$$(f,g)_T = (\mathcal{B}_0 f, \mathcal{B}_0 g)_\Omega = (\mathcal{M}\mathcal{B}_0 f, \mathcal{M}\mathcal{B}_0 g)_Y + (\mathcal{M}\mathcal{B}_0 \overline{g}, \mathcal{M}\mathcal{B}_0 \overline{f})_Y. \tag{4.45}$$

The main result for the Fourier transform class can now be formulated:

Theorem 4.7. *Assume that the kernel $B \in \mathcal{A}_0$, and $B(-r) = \overline{B(r)}$ for $r \in \mathbb{R}$. Then the transform pair (4.34) and (4.35) hold if and only if $B = (\mathcal{M}^g)^{-1}\psi\mathcal{M}^g B_0$, with ψ any function in the space Ψ defined by (4.43). The mappings $\mathcal{B}$ and $\mathcal{B}^{-1}$ have unique extensions to $\mathscr{L}^2(R_+)$ and $\mathcal{B}\mathscr{L}^2(R_+)$. And the mapping $f \to F = \mathcal{B}f$ is an isomorphism of the Hilbert spaces $\mathscr{L}^2$ into $\mathscr{L}^2$. The class of operators in $\mathcal{A}_0$ which satisfies the transform pair may be constructed by $\{\mathcal{B}\} = \mathcal{A}\mathcal{B}_0$, where $\mathcal{A} = \mathcal{M}^{-1}\Psi\mathcal{M}$ is a set of Abelian bounded operators defined on the space Ω.*

Proof. The first statement of the theorem is proved in lemma 4.6. The isomorphism of the mapping $\mathcal{B}$ on the space $\mathcal{S}_{+\frac{1}{2}}$ follows from (4.45) and $|\ \psi\ |= 1$. The bounded operator has a unique extension to $\mathscr{L}^2(R_+)$. This mapping is an isomorphism, and any transformation has the form indicated in the theorem. □

This conclude the proof of theorem 4.7, and the construction of the Fourier Transform class.

Example 1.

1. Consider the differential equation:

$$\prod_{q=1}^{n}(a_q - xd_x)B = \prod_{q=1}^{n}(a_q + 1 + xd_x)B_0, \tag{4.46}$$

with B_0 the canonical kernel (4.4). A solution of this differential equation is:

$$B = \mathcal{M}^{-1}[\psi(s)\mathcal{M}B_0(s)], \tag{4.47}$$

with

$$\psi(s) = \frac{g(s)}{g(1-s)} \qquad g(s) = \prod_{q=1}^{n}(a_q + 1 - s). \tag{4.48}$$

Clearly B is an example of a kernel of a Fourier Transform pair.

2. Consider a kernel defined by (4.47) and (4.48) but with g defined by:

$$g(s) = \prod_{j=2}^{m}\Gamma(b_j + \beta_j s)\prod_{i=1}^{n}\Gamma(1 - a_i - \alpha_i s). \tag{4.49}$$

A simple calculation show that the function H defined on the imaginary axis by $H(-ix) = B(x)$ and B given by (4.47) and (4.49) is the Hypergeometric function $H(-ix)$ of [**SSK98**] with properly chosen values of the coefficients α_j, β_j, a_j and b_j.

Remark: In the paper [**SSK98**] a condition on $\{a_i, \alpha_i, b_j, \beta_j\}$ is imposed that is not satisfied for the $\{a_i, \alpha_i, b_j, \beta_j\}$ in (4.49), so even though the kernels have similar structures the space of kernels do not overlap. In addition the H function is here defined on the imaginary axis, whereas in [**SSK98**] it is defined on the real axis.

4.2.2. The $1/x$ Generalized Fourier Transform Class. Consider the class of transforms given by the transform pair (4.1), and

$$f(r) = \mathcal{B}^{-1}F = \int_{-\infty}^{\infty} ir\partial_r r\omega B(-\omega r)F(\omega)d\omega. \tag{4.50}$$

which is basically a generalization of the previous theory.

The canonical kernel is (4.5). This function is singular at zero, and the distribution space to be used is $\mathcal{S}_{+(-.5)}$. We are going to construct a second class of kernels which are regular at zero in the sense that the distribution space to be used is $\mathcal{S}_{+.5}$ as in the previous case. We will show that the product of Hankel and Bessel functions $c_{nm}H_n^1J_m$ with $m \geq n$ belong to the class constructed. The main difference of the present section and the previous one is that the nonlinear equation for the Mellin Transform of the kernel will couple the function $\mathcal{M}^gB$ on different lines in the complex s plane, see (4.58).

In order to formulate the theory we introduce an additional condition for the space of kernels assuming that the Mellin Transform of the kernels are holomorphic functions in a strip. Let $\tilde{H}(\Omega)$ denote the set of functions which are holomorphic in the set Ω. The space of kernels will be defined by:

$$\mathcal{A}_1 = \big\{B \in \mathcal{A}_0 \big| \exists\delta \in \mathbb{R}_+ \quad (1-s)\mathcal{M}_a^gB(s) \in \tilde{H}(\{\mathrm{Re}s \in]\frac{1}{2}-\delta, \frac{3}{2}+\delta[\}) \\ \wedge \quad \forall\sigma \in]\frac{1}{2}-\delta, \frac{3}{2}+\delta[\quad \mathcal{M}_a^gB(\sigma+iy) \in \mathcal{S}_t\big\}, \quad (4.51)$$

where $\mathcal{M}_a^gB$ denote the analytic continuation of the function $\mathcal{M}^gB(\frac{1}{2}+iy)$ onto the strip $Re(s) \in]\frac{1}{2}-\delta, \frac{3}{2}+\delta[$. The condition $\mathcal{M}_a^gB \in \mathcal{S}_t$ is by definition uniform in σ that is we assume:

$$\exists n(B), \ \forall\sigma \in]\frac{1}{2}-\delta, \frac{3}{2}+\delta[\ \ \|(1+y^2)^{-n}\mathcal{M}_a^gB(\sigma+iy)\|_\infty < \infty. \quad (4.52)$$

The inverse transform in the distributional sense is introduced as

$$\langle\mathcal{B}^{-1}F, \phi\rangle = \int_{-\infty}^{\infty} d\omega\ \omega F(\omega)\langle rB(-\omega r), \frac{d[ir\phi(r)]}{dr}\rangle. \quad (4.53)$$

In theorem 4.8 it is proved that (4.53) indeed define a linear functional on the space of test functions $\phi \in \mathcal{D}(\mathbb{R}_+)$. The set of kernels in $\mathcal{A}_1$ for which (4.53) is the inverse operator of $\mathcal{B}$ will also be found.

Theorem 4.8. *Let $B \in \mathcal{A}_1$ and let $f \in \mathcal{S}_{+.5}$. Then (4.53) define a linear functional on the space $\phi \in \mathcal{D}(\mathbb{R}_+)$. The $1/x$ Generalized Fourier Transform pair (4.34) and (4.53) hold if and only if $\mathcal{M}^gB$ has the form (4.59) with ψ any function which satisfies the equation (4.60). If in addition $B(r) = \overline{B(-r)}$ for $r \in \mathbb{R}$ then (4.42) hold.*

Proof. First we prove that the integral (4.53) over positive ω exists. With the assumptions of the theorem, the function $F \in \mathcal{S}_{+.5}$. For $\omega > 0$ a similar result to (4.22) is found as

$$2\pi i\langle r\omega B(\pm r\omega), \frac{\partial r\phi(r)}{\partial r}\rangle = \langle\omega^{s-1}\mathcal{M}^g[B(\pm r)](2-s), (1-s)\mathcal{M}\phi(s)\rangle_{s=1.5+iy}. \quad (4.54)$$

The function $\mathcal{M}^g[B(-r)](2-s)$ is expressed in terms of $\mathcal{M}^g[B(r)](2-s)$ by (4.27). The functions $\mathcal{M}\phi \in \mathcal{S}$, and $(1-s)\mathcal{M}^g[B(-r)](2-s) \in \mathcal{S}_t$, are both analytic or has analytic continuations in the strip $\sigma \in]\frac{1}{2}-\delta, \frac{3}{2}+\delta[$. Let $\mathcal{M}_a^g[B(-r)](2-s)$ be the analytic continuation of the function $\mathcal{M}^g[B(-r)](2-s)$. The contour of (4.54) may be deformed onto the line $\mathrm{Re}(s) = \frac{1}{2}$. The part of the integral (4.53) over positive values of ω is then reduced to

$$2\pi i \int_0^\infty d\omega\, \omega F(\omega) \langle rB(-\omega r), \frac{\partial ir\phi(r)}{\partial r}\rangle = \langle \mathcal{M}F[\omega](s)\mathcal{M}_a^g[B(-r)](2-s), (1-s)\mathcal{M}\phi(s)\rangle_{s=.5+iy}. \tag{4.55}$$

In a similar way the corresponding relation for the integral over negative ω is found as

$$2\pi i \int_{-\infty}^0 d\omega\, \omega F(\omega) \langle rB(-\omega r), \frac{\partial ir\phi(r)}{\partial r}\rangle = -\langle \mathcal{M}[F(-\omega)](s)\mathcal{M}_a^g[B(r)](2-s), (1-s)\mathcal{M}\phi(s)\rangle_{s=.5+iy}. \tag{4.56}$$

Using (4.27) in (4.55) and (4.56) it follows that

$$\pi\langle \mathcal{B}^{-1}F, \phi\rangle = \langle \mathcal{M}F(s) \sin \pi s \mathcal{M}_a^g[B(r)](2-s), (1-s)\mathcal{M}\phi(s)\rangle_{s=.5+iy}. \tag{4.57}$$

Collecting the results it is seen that a function in the set $\mathcal{A}_1$ is a kernel of a transform pair if and only if the following relation for the generalized Mellin transform hold

$$[1-s] \sin \pi s \mathcal{M}^g B(s) \mathcal{M}_a^g B(2-s) = \frac{1}{2} \quad \mathrm{Re}(s) = \frac{1}{2}. \tag{4.58}$$

The generalized Mellin transform of the canonical kernel B_0 (4.5) is defined for $\sigma = -\delta < 0$. Let $\mathcal{M}_a^g B_0$ denote the analytic continuation of the generalized Mellin transform of B_0. A simple calculation show that $\mathcal{M}_a^g B_0$ satisfies (4.58). Thus the set of transformations will be given by

$$\mathcal{M}^g B(s) = \psi(s)\mathcal{M}_a^g B_0(s) \qquad \mathcal{M}_a^g B_0 = \frac{-1}{\sqrt{2\pi}} e^{i\frac{\pi}{2}s}\Gamma(s-1), \tag{4.59}$$

with $\psi \in \mathcal{S}_t$ an analytic function in a strip $]\frac{1}{2}-\delta, \frac{3}{2}+\delta[\times\mathbb{R}$ which satisfies the equation

$$\psi(s)\psi(2-s) = 1 \quad \mathrm{Re}(s) \in]\frac{1}{2}-\delta, \frac{3}{2}+\delta[. \tag{4.60}$$

□

Now implicitly all possible functions B in the space $\mathcal{A}_1$, which are kernels of a $1/x$ Generalized Fourier transform pair, are constructed,

The conditions for the functions may seem complex therefore consider the following simple subspace of functions for which the transform pair hold. Let $\tilde{H}^B(]\frac{1}{2}, \frac{3}{2}[)$ denote the space of bounded holomorphic functions in the strip $]\frac{1}{2}, \frac{3}{2}[\times\mathbb{R}$

$$\tilde{H}^B(]\frac{1}{2}, \frac{3}{2}[) = \{\psi \in \tilde{H}(]\frac{1}{2}, \frac{3}{2}[\times\mathbb{R}) \mid \psi \in \mathscr{L}^\infty([\frac{1}{2}, \frac{3}{2}] \times \mathbb{R})\}. \tag{4.61}$$

Introduce the space

$$\Psi_1 = \{\psi \in \tilde{H}^B \mid \psi(s) = \frac{P(s)}{P(2-s)} \wedge \quad \overline{P(\overline{s})} = P(s) \in \tilde{H}\}, \tag{4.62}$$

where $\tilde{H}^B$ and $\tilde{H}$ are bounded holomorphic and holomorphic functions respectively in the strip $]\frac{1}{2}, \frac{3}{2}[$. Let $B = (\mathcal{M}^g)^{-1}\psi\mathcal{M}^g B_0$, with $\psi \in \Psi_1$ and B_0 the canonical $1/x$ Generalized Fourier Transform. Then $B \in \mathcal{A}_1$, and the two equations (4.60) and (4.42) hold; i.e. B is a kernel of a $1/x$ Generalized Fourier Transform pair.

Theorem 4.9. *Assume that $B \in (\mathcal{M}^g)^{-1}\Psi_1\mathcal{M}^g B_0$. Then B is kernel of a $1/x$ Generalized Fourier Transform pair.*

An interesting class of operators for which this apply is the following:

Example 2.

Let the kernel be given by a product of a Bessel and a Hankel function:

$$B(x) = i\sqrt{\frac{\pi}{2}}e^{i\frac{\pi}{2}(\mu-\nu)}H^1_\mu(x)J_\nu(x) \qquad \nu \geq \mu \geq 0 \tag{4.63}$$

This function is in $\mathscr{L}^2$, and B is the limit of B_q, with B_q of the same form, but instead of $H^1_\mu(x)$ we use $H^1_\mu(x + \frac{i}{q})$. The Mellin transform of B_q is found in standard mathematical tables of integrals [**GR79**], and the limit function $\mathcal{M}^g B$ is determined as

$$\mathcal{M}^g_a B(s) = \psi(s)\mathcal{M}^g_a B_0(s) = \frac{P(s)}{P(2-s)}\mathcal{M}^g_a B_0(s),$$

with

$$P(s) = \frac{1}{2^{(1-s)/2}}\Gamma(1-s)\Gamma(\frac{\nu+\mu+s}{2})\Gamma(\frac{\nu+s-\mu}{2}). \tag{4.64}$$

Hence ψ from example 2 is in the space Ψ_1, and the following result has been obtained

Corollary 4.10. *Let $\nu \geq \mu \geq 0$ then the functions B given by (4.63) are kernels of $1/x$ Generalized Fourier Transform, with the inverse transformation given by (4.53).*

Remark: As another example the functions from Example 1.2 (4.49) can be used to construct a kernel that also is a hypergeometric functions for the **H**-transform.

Finally introduce Hilbert spaces with the property that the transform pairs are homeomorphic mappings. In the space $\mathcal{D}(\mathbb{R}_+)$ introduce the inner product by

$$\langle f, g\rangle_T = \int_0^\infty dr\frac{f(r)}{r^2}\overline{g}(r). \tag{4.65}$$

The space T is defined as the smallest Hilbert space, with the inner product (4.65), which contain the space $\mathcal{D}(\mathbb{R}_+)$. In the space $\Omega = \mathcal{B}_0 T$, with B_0 the canonical kernel (4.5) of $\mathcal{B}_0$, the inner product is defined by

$$\langle F, G\rangle_\Omega = \int_{-\infty}^\infty d\omega\ \omega^2 F(\omega)\overline{G(\omega)}. \tag{4.66}$$

The operator $\mathcal{B}_0$ is then an isomorphism of T onto Ω. Finally the inner product for the classical Mellin transform of Ω in the set $s = \frac{3}{2} + iy$ is defined for any $F, G \in \Omega$ by

$$\langle \mathcal{M}F, \mathcal{M}G \rangle_Y = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \mathcal{M}F(3/2 + iy) \overline{\mathcal{M}G(3/2 + iy)}. \tag{4.67}$$

It is easily proved that $\mathcal{M}$ is an isomorphism of Ω onto Y.

Theorem 4.11. *Let $B = (\mathcal{M}^g)^{-1} \psi \mathcal{M}^g B_0$ with $\psi \in \Psi_1$ the kernel of a $1/x$ Generalized Fourier transform $\mathcal{B}$. Then $\mathcal{B}, \mathcal{B}^{-1}$ is a $1/x$ Generalized Fourier Transform pair and $\mathcal{B}$ has a unique extension onto T. The mapping $\mathcal{B} : T \to \mathcal{B}T = \Omega$ is a homeomorphisms of T onto Ω. The transform $\mathcal{B}$ in the space T is found from,*

$$\mathcal{B}f(\omega) = (\mathcal{M}_{\frac{3}{2}})^{-1} \psi(s) \mathcal{M}_a^g B_0(s) \mathcal{M}f(1 - s), \tag{4.68}$$

and the inverse transform defined on Ω is given by (4.69).

Proof. The function $\mathcal{M}_a^g B_0(s) \mathcal{M}f(1-s)$ is in the space Y for all $f \in T$. Since the function ψ is bounded $\psi(s)\mathcal{M}_a^g B_0(s) \mathcal{M}f(1 - s) \in Y$ and $\mathcal{B}$ given by (4.68) has a unique continuous extension to the space T, and $\mathcal{B}f \in \Omega$ will satisfy (4.68) for any $f \in T$. The inverse mapping is given by:

$$\mathcal{B}^{-1} = \mathcal{B}_0^{-1} \mathcal{M}^{-1} \psi^{-1} \mathcal{M}. \tag{4.69}$$

With the assumptions of the theorem the mappings $\psi(\cdot)$ and $\psi^{-1}(\cdot)$, are continuous in the space Y, and it is concluded that B as well as B^{-1} are continuous mappings defined on the spaces T and Ω. This completes the proof. □

We are going to apply the theory for the solution of integral equations with kernels which are products of Bessel and Hankel functions. Thus we will also need the result that corollary 4.10 hold for that case. From (4.64) follows that:

$$\mid \psi(\frac{3}{2} + iy) \mid = c \frac{\mid \nu + \frac{1}{2} + iy \mid}{\mid -1 + 2iy \mid}, \tag{4.70}$$

with c a constant. Hence the assumptions of corollary 4.10 is satisfied for this kernel and we have the result:

Corollary 4.12. *Let the kernel of the generalized Fourier transform be given by (4.63). Then the mapping $\mathcal{B} : T \to \mathcal{B}T = \Omega$ is a homeomorphisms of T onto Ω.*

4.3. Explicit Construction of Solutions

The aim in the remaining part of this chapter is the explicit construction of $f(r)$ from $F(\omega)$, and its applications to an inverse time dependent scattering problem.

4.3.1. Integral Equations - Kernels are Products of Hankel and Bessel Functions. Let B be given by (4.63) and the numbers n and m be nonnegative half-number integers with $n \geq m \geq 0$. This restriction on (n, m) is to allow only the Hankel and Bessel functions that have been treated previously. The fundamental integral equation is

$$\int_0^\infty dr f(r) B(\omega r) = F(\omega), \tag{4.71}$$

From corollary 4.12, follows that this integral equation has a unique solution given by (4.53), with the mapping from F onto f being continuous. We will refer to the solution of this problem as the "frequency space" solution.

The kind of integral equations to be analyzed is (4.71) with the kernels

$$B_n^+(x) = b_n x h_0^1(x) j_n(x) = J_{n+\frac{1}{2}}(x) H_{\frac{1}{2}}^1(x), \qquad b_n = \frac{1}{2\sqrt{2\pi}}(-i)^{n-1}, \tag{4.72}$$

where j_n and h_n denote the spherical Bessel and Hankel functions defined by $j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z)$ and $h_n^1(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^1(z)$. We will show how the problem may be solved in a case relevant for the inverse scattering problem which allow us to introduce proper space restrictions. Let T be the Hilbert space with inner product (4.65).

Theorem 4.13. *Assume that $\phi(t) = \mathcal{F}^{-1}F$ has compact support on $\mathbb{R}_+$, with $\phi, r^{-1}\phi, \phi' \in \mathscr{L}^2(\mathbb{R})$. Then the integral equation (4.71) has a unique solution in the space T. The explicit solution is given by (4.76). Let X and Y be the normed spaces defined by (4.78) and (4.79), then the mapping (4.76) is continuous.*

Proof. Let

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, \phi(t) e^{i\omega t} \tag{4.73}$$

For the Spherical Hankel functions we use the known expansion

$$h_n^{(1)}(r) = (-i)^n \frac{e^{ir}}{ir}\{1 + \sum_{p=1}^{n} \frac{a_{pn}}{r^p}\} \tag{4.74}$$

with a_{pn} known complex coefficients. A corresponding expression as a finite sum exists for the Spherical Bessel function. Using this, the kernel reduces to

$$B_n^+(r) = \sqrt{\frac{2}{\pi}}\{(-1)^{n+1}\frac{e^{i2r}}{ir}[1 + \sum_{p=1}^{n} \frac{a_{pn}}{r^p}] - \frac{1}{ir}[1 + \sum_{p=1}^{n} \frac{\overline{a_{pn}}}{r^p}]\}. \tag{4.75}$$

We now find the formal solution of the integral equation (4.71). That is, we assume that F with the specified assumptions is in the space $\mathcal{B}_n^+ T$. Then F may be calculated by the inverse transformation (4.53). The last term will give zero contribution to the inverse transform, by closing the contour of integration, and the first term (proportional to e^{i2r})

reduces after a few calculations to the expression

$$f(r) = (\mathcal{B}_n^+)^{-1}\mathcal{F}\phi(r) = \\ 2(-1)^n\{r\partial_r\phi(2r) - n(n+1)\phi(2r) + \int_0^{2r} G_n(t,r)\phi(t)dt\}, \quad (4.76)$$

with

$$G_n(t,r) = -r\partial_r \sum_{p=2}^{n} \frac{i^p a_{pn}}{r^p(p-1)!}(t-2r)^{p-1}. \quad (4.77)$$

Using the space assumptions $\phi, r^{-1}\phi, \phi' \in \mathscr{L}^2(\mathbb{R})$ and substituting $t = rt'$ in $G_n(t,r)$ we easily verify that f given by (4.76) is in the space T. That is $F \in \mathcal{B}_n^+T$ and f is the unique solution of the integral equation (4.71).

Assume that the support of ϕ is on $[0, R]$. Define the space

$$X = \{\phi \in H^1([0,R]) \mid \|\phi\| = \max[\|\phi\|_{\mathscr{L}^2([0,R])}, \|\partial_r\phi\|_{\mathscr{L}^2([0,R])}]\}, \quad (4.78)$$

and introduce Y by:

$$Y = \{f \in \mathscr{L}^2[0,a]) \mid \|f\| = \|f\|_{\mathscr{L}^2}\}, \quad (4.79)$$

then the mapping $\phi \in X \to (\mathcal{B}_n^+)^{-1}\mathcal{F}\phi \in Y$ is continuous. □

In the same way the following integral equation may be solved explicitly:

$$\int_0^\infty dr f(r) B_n^-(\omega r) = F(\omega), \quad (4.80)$$

with the kernels:

$$B_n^-(x) = \overline{b_n} x h_0^2(x) j_n(x). \quad (4.81)$$

Assume the right hand side of (4.80) to have support for its Fourier Transform on the negative axis. Let

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 dt\phi(t)e^{i\omega t}. \quad (4.82)$$

Now the complex conjugate of (4.80) has exactly the same form as (4.71), and we find the explicit solution of (4.80) from the previous result as

$$f(r) = (\mathcal{B}_n^-)^{-1}\mathcal{F}\phi(r) = \\ 2(-1)^n\{r\partial_r\phi(-2r) - n(n+1)\phi(-2r) + \int_0^{2r} \overline{G_n(t,r)}\phi(-t)dt\}. \quad (4.83)$$

We have the result:

Theorem 4.14. *Assume that $\phi(-t) = \mathcal{F}^{-1}F$ has compact support on $\mathbb{R}_+$, and $\phi, r^{-1}\phi, \phi' \in \mathscr{L}^2(\mathbb{R})$. Then the integral equation (4.80) has a unique solution in the space T. The explicit solution is given by (4.83). Let X and Y be the normed spaces defined by (4.78) and (4.79), then the mapping (4.83) is continuous.*

This concludes the construction and proof of basic theorems for the integral equations with kernels, that are products of a Hankel and a Bessel function. We note that similar explicit solutions of the form (4.76) or (4.83) may be obtained for integral equations with kernels $xh_n(x)j_m(x)$ and $m \geq n$.

4.3.2. Integral Equations - Kernels are Products of two Bessel Functions. Using the above constructions for solutions of integral equations with kernels that are products of Hankel and Bessel functions, we can find the solution of integral equations with kernels that are products of two Bessel functions. The kind of integral equations to be analyzed are

$$\int_0^\infty drf(r)B_n(\omega r) = F(\omega), \tag{4.84}$$

with the kernels

$$B_n(x) = 2b_n x j_0(x) j_n(x) = \mathcal{B}_n^+(x) + (-1)^n \mathcal{B}_n^-(x), \tag{4.85}$$

where $B_n^\pm$ are the previously defined kernels (4.72) and (4.81).

It is assumed that the functions F and ωF are in $\mathscr{L}^2$. We may decompose F in a sum of two functions with inverse Fourier transforms on a half axis:

$$\int_0^a drf(r)B_n(\omega r) = B^+f + (-1)^n B^- f = F^+(\omega) + F^-(\omega), \tag{4.86}$$

where the projection operator used is $\mathcal{F}\vartheta_{t>0}\mathcal{F}^{-1}$:

$$F^+(\omega) = \mathcal{F}\vartheta_{t>0}\mathcal{F}^{-1}F = \frac{1}{\sqrt{2\pi}}\int_0^\infty dt\phi(t)e^{i\omega t}, \tag{4.87}$$

and

$$F^-(\omega) = [1 - \mathcal{F}\vartheta_{t>0}\mathcal{F}^{-1}]F = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 dt\phi(t)e^{i\omega t}. \tag{4.88}$$

The integral equation is then:

$$\mathcal{B}_n^+ f - F^+ = F^- - (-1)^n \mathcal{B}_n^- f. \tag{4.89}$$

The left hand side of this equation has an inverse Fourier Transform (in the distributional sense) with support on the set $[0, \infty[$, and the right hand side has support on the set $]-\infty, 0]$. Then, the support of (4.89) is on 0. That is $\mathcal{F}^{-1}[\mathcal{B}_n^+ f - F^+]$ is a finite sum of derivatives of delta functions in 0. Now $F^+ \in \mathscr{L}^2(\mathbb{R})$ and we are looking for solutions $F \in T$, i.e. $\omega F \in \mathscr{L}^2(\mathbb{R})$. Then $\mathcal{F}^{-1}[\mathcal{B}_n^+ f - F^+] = 0$, and we have found the following integral equation:

$$\int_0^a drf(r)B_n^+(\omega r) = F^+(\omega), \tag{4.90}$$

with the unique solution $f = (\mathcal{B}_n^+)^{-1}\mathcal{F}\phi$. In the same way we obtain:

$$\int_0^a drf(r)B_n^-(\omega r) = (-1)^n F^-(\omega), \tag{4.91}$$

with the unique solution $f = (-1)^n (B_n^-)^{-1}\phi$.

Since both (4.90) and (4.91) uniquely determines $f(r)$, the solutions of (4.90) and (4.91) must coincide. Using Corollary 4.12 and theorems 4.13 and 4.14 we may prove the result:

Theorem 4.15. *Assume that $\phi = \mathcal{F}^{-1}F$ has compact support on $\mathbb{R}$ and belong to the space $\phi, r^{-1}\phi, \phi' \in \mathscr{L}^2(R)$. Then the integral equation (4.84) has a unique solution in the space T, if and only if, the unique solution of (4.90) coincide with the unique solution of (4.91). The mapping from the space X to Y is continuous and given by*

$$f(r) = -2\{r\partial_r\phi(-2r) - n(n+1)\phi(-2r) + \int_0^{2r} \overline{G_n(t,r)}\phi(-t)dt. \tag{4.92}$$

where the kernel G is defined by (4.77).

4.4. Application to the Inverse Scattering Problem for Low Contrast Mediums

The scattering of an acoustic wave by a penetrable inhomogeneous medium (a scatter) is normally discussed in either the frequency or the time domain. This means that the scatter is either illuminated with a time harmonic/periodic fi eld, or with a time varying fi eld. In the fi rst case, the underlying PDE is the Helmholtz Equation, whereas in the later case, the PDE will be the Wave Equation. For the Inverse Medium Problem properties of the bounded inhomogeneous medium may be found from measuring the fi elds outside the inhomogeneous medium.

When considering medical applications, two different contexts appear naturally. One is where only the location of an object is of interest, and this could be when trying to examine whether there is a tumor in the body and how big it is [**CCM00**]. The other is when internal properties are needed - such as for instance the glucosamine production of a tumor for predicting survival rates and recommending treatment plans for patients. The latter is a much more diffi cult problem, since local variations are needed, in contrast to the fi rst, where only an overall change is of interest.

For the method to be proposed here the forward scattering problem (of illuminating an object) will be discussed in both the frequency and time domain. The incident fi eld will be the composition of an incident spherical fi eld and the scattered fi eld hereof by a homogeneous medium. The mathematical formulation of these two problems are given in table 1

The relationship between the two formulations are that $u(\mathbf{x}, t) = \mathcal{F}_\omega U(\mathbf{x}, \omega)$. If in addition the medium is assumed to be non-absorbing the coeffi cients in the PDE's will

Frequency domain equations	Time domain equations
$(\nabla^2 + k^2(\mathbf{x}))U(\mathbf{x},\omega) = 0$	$(\nabla^2 - c^{-2}(\mathbf{x})\frac{\partial^2}{\partial t^2})u(\mathbf{x},t) = 0$
$U(\mathbf{x},\omega) = U^0(\mathbf{x},\omega) + U^{sc}(\mathbf{x},\omega)$	$u(\mathbf{x},t) = u^0(\mathbf{x},t) + u^{sc}(\mathbf{x},t)$
$U^0(\mathbf{x},\omega) = U^{in}(\mathbf{x},\omega) + U_0^{sc}(\mathbf{x},\omega)$	$u^0(\mathbf{x},t) = u^{in}(\mathbf{x},t) + u_0^{sc}(\mathbf{x},t)$
$(\nabla^2 + k_0^2)U^{in}(\mathbf{x},\omega) = 0$	$(\nabla^2 - c_0^{-2}\frac{\partial^2}{\partial t^2})u^{in}(\mathbf{x},t) = 0$
$\lim_{r\to\infty} r\left(\frac{\partial U_0^{sc}}{\partial r} + ik_0U_0^{sc})\right) = 0$	$\lim_{r\to\infty} r\left(\frac{\partial u_0^{sc}}{\partial r} + ic_0u_0^{sc}\right) = 0$
$\lim_{r\to\infty} r\left(\frac{\partial U^{sc}}{\partial r} + ik_0U^{sc})\right) = 0$	$\lim_{r\to\infty} r\left(\frac{\partial u^{sc}}{\partial r} + ic_0u^{sc}\right) = 0$

Table 1. Table of the mathematical formulation for the acoustic scattering problem in the frequency and the time domain respectively.

satisfy

$$c^2(\mathbf{x}) = \frac{c_0^2}{n(\mathbf{x})} \tag{4.93}$$

$$k^2(\mathbf{x}) = \frac{\omega^2}{c^2(\mathbf{x})} \equiv k_0^2(1 + n(\mathbf{x})\chi_D), \qquad k_0 = \omega/c_0 \tag{4.94}$$

where χ_D denotes the characteristic function for an open set D, c_0 the speed of sound outside the inhomogeneous medium and $n(\mathbf{x})$ the refractive index.

The inverse time dependent problem is from knowledge of the incident and scattered fi elds for all times on a sphere encircling the inhomogeneity to reconstruct the refractive index.

A common way of representing the solutions of the scattering problem in the frequency domain is by the Lippmann-Schwinger integral equation.

$$U(\mathbf{x},\omega) = U^0(\mathbf{x},\omega) - k_0^2 \int_D \Phi(\mathbf{x},\mathbf{y})(1 - n(\mathbf{x}))U(\mathbf{y},\omega)d\mathbf{y} \qquad \mathbf{x} \in \mathbb{R}^3. \tag{4.95}$$

Since $k_0^2 - k^2(\mathbf{x})$ has compact support equation (4.95) is a Second Kind Fredholm Problem for $U(\mathbf{x},\omega)$ with Φ the Green's function for the Helmholtz Equation and U^0 the incident fi eld.

We construct an additional integral equation that can be used for solving the problem. Let $v(\mathbf{x},t), V(\mathbf{x},\omega)$ denote free-space solutions (c^2, k^2 constant respectively). By introducing $w = u - u^0$ and denoting $w * u$ the convolution between w, u it follows from Green's Theorem that

$$\int_{B(0,a)}(1-n)U^0V\,d\mathbf{x}-\int_{B(0,a)}(1-n)(U-U^0)V\,d\mathbf{x} = \frac{1}{k_0^2}\int_{\partial B(0,a)} V\frac{\partial(U-U^0)}{\partial\nu}+(U-U^0)\frac{\partial V}{\partial\nu}d\nu=\eta \quad (4.96)$$

and

$$\int_{B(0,a)}(c_0^{-2}-c^{-2})\frac{\partial^2}{\partial t^2}u^0*v\,d\mathbf{x}-\int_{B(0,a)}(c_0^{-2}-c^{-2})\frac{\partial^2}{\partial t^2}w*v\,d\mathbf{x} = \int_{\partial B(0,a)} v*\frac{\partial w}{\partial\nu}+w*\frac{\partial v}{\partial\nu}d\nu=\hat{\eta} \quad (4.97)$$

Assuming that the contrast of the medium is low and without absorption ($n(\mathbf{x})\approx 1$, $c^2(\mathbf{x})\approx c_0^2$) the second terms in (4.97) and (4.96) respectively are of second order compared to the r.h.s. In physical terms this means that the scattered field is very small, since for a low contrast medium most of the field will be transmitted through the medium. Hence, for mediums with low contrast (4.97), (4.96) reduces to

$$\int_{B(0,a)}(c_0^{-2}-c^{-2})\frac{\partial^2}{\partial t^2}u^0v\,d\mathbf{x}=\hat{\eta}\equiv\phi(t) \quad (4.98)$$

$$\int_{B(0,a)}(1-n(\mathbf{x}))U^0V\,d\mathbf{x}=\frac{1}{k_0^2}\eta \quad (4.99)$$

which are equations that we will be able to construct explicit solutions for.

4.4.1. Reduction to a One-dimensional Integral Equation. After having listed the forward problem in table 1, and defining the necessary integral equations, we introduce the inverse problem of finding $n(\mathbf{x})$ and elaborate on how the transformation theory applies to solving (4.98). Let $n(\mathbf{x})$ be compactly supported in $B(0,a)$, a ball with center at zero and radius a.

Assume that $u(\mathbf{x},t)$ is known for $|\mathbf{x}|=a$ and $t\in\mathbb{R}$. Then $U(\mathbf{x},\omega)$ is known for all $\omega\in\mathbb{R}$. If $u^{in}(\mathbf{x},t)=\delta(t-c_0|\mathbf{x}|)$, then $u^0(\mathbf{x},t)=\delta(t-c_o|\mathbf{x}|)+\delta(t+c_o|\mathbf{x}|)$ and $U^0(\mathbf{x},\omega)=j_0(k_0r)$.

For the 3-dimensional Helmholtz Equation, a general free-space solution $V(\mathbf{x})$ can be written as $V_n(x)=j_n(k_0|x|)Y_n^m(\hat{x})$ where $j_n(x)=\sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)$ are the Spherical Bessel functions and $Y_n^m(\hat{x})$ Spherical Harmonics. Let $(1-n(\mathbf{x}))=\sum_{nm}\tilde{f}_{nm}(|x|)Y_{nm}(\hat{x})$ then using $V(x)=j_n(k_0|\mathbf{x}|)Y_n^m(\hat{x})$ and either $U^0=h_0^1(k_0r)$ or $U^0=j_0(k_0r)$ the r.h.s.

of (4.99) (or (4.96)) reduces to

$$k_0^2 \int_0^a U^0(k_0 r) j_n(k_0 r) \int_{\partial B(0,r)} (n(\mathbf{x}) - 1) Y_n^m(\hat{x}) dS \; r^2 dr$$
$$= \int_0^\infty U^0(k_0 r) j_n(k_0 r) \tilde{f}_{nm}(r) \; r^2 dr \quad (4.100)$$

Our inverse problem of finding $n(\mathbf{x})$ is therefore reduced to that of finding $f_{nm}(r) = r\tilde{f}_{nm}(r)$ compactly supported on $[0, a]$ from the integral equation (4.101)

$$\int_0^\infty k_0 r j_0(k_0 r) j_n(k_0 r) f_{nm}(r) \; dr = \mathcal{F}\phi(t) \quad (4.101)$$
$$\phi(t) = \mathcal{F}^{-1} \frac{1}{k_0} \int_{\partial B(0,a)} V \frac{\partial (U - U^0)}{\partial \nu} + (U - U^0) \frac{\partial v}{\partial \nu} d\nu$$

If $\phi(t)$ satisfies the assumptions of theorem 4.15 then the result of theorem 4.15 applies to $f_{nm}(r)$.

Corollary 4.16. *Assume $\phi(t)$ in (4.101) satisfy the conditions in Theorem 4.15, then equation (4.101) has a unique solution and the permittivity can be found as*

$$\langle (1 - n(\mathbf{x})), Y_n^m \rangle_{\mathscr{L}^2(B(0,a))} = \frac{1}{r} f_{nm}(|\mathbf{x}|) \qquad \forall n, m \quad m \geq n \quad (4.102)$$

4.5. Conclusion

We have constructed classes of integral kernels such that the Generalized Fourier Transform pairs $\{\mathcal{B}, \mathcal{B}^{-1}\}$ are continuous mappings and such that elements in the classes are found through an Abelian map $\mathcal{AB}_0$.

New spaces of **H**-transform with inverse maps have been constructed and a space of kernels that are products of a Bessel and a Hankel function have been constructed.

To the authors knowledge, no theory handling these classes of kernels exist.

The theory's application to the Inverse Medium Problem has been established by solving a first kind problem representation explicitly. This representations of the problem in terms of time dependent data is also new to the authors knowledge.

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The Conductivity Problem

The aim of this chapter is to motivate the Inverse Conductivity Problem, briefly describe the results with many boundary experiments and then focus on the single boundary measurement problem. The chapter concludes with the obtained results of chapters 6 and 7.

The range of possible applications for the Inverse Conductivity problem is large and includes fi elds such as control of sediments, and fi nding structural flaws in metals. The physical aspects of the problem is to induce a current on the boundary of a conducting body, no matter whether it is a human, the earth, or a metal constructions and then from the boundary values of the potential, to determine physical properties of the subject. The inverse conductivity problem was among the mathematical areas that was discussed in the 1996 National Research Council work [**Cou96**] on emerging areas of biomedical imaging. The medical use may be reviewed in the IEEE special issue on "Electrical Impedance Tomography", [**IEE02**].

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded simply connected domain with smooth boundary $\partial\Omega$. The electrical conductivity of Ω is represented by a bounded positive function $\sigma(\mathbf{x}) \geq m > 0$. In the absence of sinks and sources, the equation for the static potential is given by

$$\nabla \cdot (\sigma(\mathbf{x})\nabla u) = 0 \quad \mathbf{x} \in \Omega. \tag{5.1}$$

Let H denote the usual Sobolev space of weak derivatives. Given a voltage potential $f \in H^{1/2}(\partial\Omega)$ on the boundary $\partial\Omega$, the induced potential $u \in H^1(\Omega)$ is a solution of the Dirichlet problem

$$\nabla \cdot (\sigma(\mathbf{x})\nabla u) = 0 \qquad \mathbf{x} \in \Omega, \tag{5.2}$$

$$u = f \qquad \mathbf{x} \in \partial\Omega. \tag{5.3}$$

The Dirichlet to Neumann map, or the voltage to current map, is then given by

$$\Lambda_\sigma(f) = \left(\sigma \frac{\partial u}{\partial \nu^-}\right)\Big|_{\partial\Omega}. \tag{5.4}$$

For every $g \in H^{-1/2}(\partial\Omega)$ with $\int_{\partial\Omega} g = 0$ the Neumann boundary value problem

$$\nabla \cdot (\sigma(\mathbf{x})\nabla u) = 0 \qquad \mathbf{x} \in \Omega, \tag{5.5}$$

$$\sigma \frac{\partial u}{\partial \nu} = g \qquad \mathbf{x} \in \partial\Omega. \tag{5.6}$$

has a unique solution $u \in H^1(\Omega)$. The Neumann to Dirichlet map, or current to voltage map is then defi ned as

$$\Lambda_\sigma^{-1} : H_0^{-1/2}(\partial\Omega) \mapsto H_0^{1/2}(\partial\Omega) \tag{5.7}$$

$$g \mapsto u\big|_{\partial\Omega} \tag{5.8}$$

where the subindex on $H_0^{-1/2}(\partial\Omega)$ means that the integral over $(\partial\Omega)$ is zero.

The present mathematical formulation for the Inverse Conductivity Problem dates back to Calderón [**Cal80**], who asked if the potentials from all possible currents are suffi cient for a unique determination of the conductivity? The question has been answered in the affi rmative for isotropic medias in $\mathbb{R}^3$. In $\mathbb{R}^2$ results exist for a large class of conductivities proving that if the conductivity is in $W^{1,p}(\Omega)$ with $p > 2$, i.e. having one weak derivative in $\mathscr{L}^p$, then uniqueness hold [**BU97**].

Before Calderón proposed this formulation, experiments with and extension of the traditional EKG had been carried out in the late 1970's by Colli Franzone and Colleagues [**Fra79**]. They attached hundreds of electrodes on the surfaces of the torso and measured the potential that the contraction of the heart induces on the surface of the torso. Their results with inferring the electric potential on the surface of the heart where promising in simplistic experiments, but unsuccessful in clinical application. The complexity of the a-priori unknown electrical properties of tissue, essentially made the method unsuccessful.

In 1985, Kohn and Vogelius [**KV85**] proved that if $\partial\Omega \in C^\infty$ and γ is piecewise real analytic, then Λ_γ determines γ uniquely in dimensions $n \geq 2$. In [**SU87**] Sylvester and Uhlmann showed that if $\partial\Omega \in C^\infty$, then Λ_γ determines γ in $C^\infty(\overline{\Omega})$ in dimensions $n \geq 3$. Nachman gave a reconstruction method in [**Nac88**] in dimensions $n \geq 3$ for $\gamma \in C^{1,1}(\Omega)$ and $\partial\Omega \in C^{1,1}$. In [**Nac96**] he then proved that for $n = 2$ uniqueness holds for $\partial\Omega$ Lipschitz and $\gamma \in W^{2,p}(\Omega)$ for $p > 1$. The uniqueness result was extended to $\gamma \in W^{1,p}(\Omega)$ for $p > 2$ in 1997 by Brown and Uhlmann [**BU97**]. The uniqueness result of Nachman [**Nac96**] was proved in a constructive way. Numerical verifi cation of the reconstruction method has been performed by Siltanen for radial symmetric objects on synthetic data [**Sil99**]. In its' current form, the reconstruction method is not a real-time reconstructions method. Suggestion for how to build a reconstruction method from the uniqueness result of Brown and Uhlmann has been undertaken by Knudsen and Tamasan

[**KT01**], but is not yet conclusive. Applications to clinical data of these reconstruction methods are still to be performed.

For clinical trials a vast number of linearized and iterative methods for reconstructing the conductivity has been suggested. References for the mathematical litterature may be found in the survey article [**CIN99**] and for engineering litterature in [**IEE02**]. As an example one experimental setup that has been applied is by a Research group at Rensselaer Polytechnic Institute (RPI). It works in real-time using a one-step Newton method for reconstructing the conductivity, enabling them to achieve an update rate of 60 reconstructions per second.

Another relevant question is how to incorporate a-priori information into the reconstruction, and to develop simple, stable, and fast reconstruction methods. One such question is the study of internal cracks in metal constructions [**FV89**], for which an extensive amount of literature is available. Another is how much can be reconstructed from partial Cauchy data, in the case where only part of the boundary is accessible. A third question is what can be reconstructed from a finite number of experiments. It is found that one experiment can be sufficient for determining internal boundaries for polygons, convex polyhedra, multiple disks, and balls.

Subsequently let the conductivity be a piecewise constant function σ defined as

$$\sigma(\mathbf{x}) = \sigma_0 \chi_{\Omega \setminus D} + \sum_{i=1}^{m} \sigma_i \chi_{D_i}, \quad \partial D_i \cap \partial D_j = \emptyset, i \neq j$$

Assume that $\left\{\Lambda_\sigma^{-1}(g), g\right\}$ is known for one $g \in H_0^{1/2}(\partial\Omega)$. The natural questions to ask are

- Can D_i and σ_i be uniquely determined?
- How to reconstruct D_i and σ?
- Is there stability for reconstructing D_i and σ_i, and how?
- What is the optimal g to use?

Partial answers exist to these questions.

To fix ideas, let $m = 1$, σ_i be fixed (known) and denote u_1 the solution of $\nabla \cdot \sigma \nabla u_1 = 0$ with σ_1 compactly supported on D_1 and denote u_2 the solution when σ_1 is compactly supported on D_2. Let $\left\{\Lambda_{D_i}^{-1}(g), g\right\}$ denote the Cauchy data for u_i.

For discussing uniqueness of different objects the natural tool to use is harmonic continuation. We will use this to describe some of the geometrical constraints for which uniqueness is well-known. The first observation that is made for two different domains D_1 and D_2 is that if their Cauchy data coincide, then $D_1 \cap D_2 \neq \emptyset$. Assume to the contrary that there exist two models with the same Cauchy data, but where the inhomogenities are disjoint when overlayed onto Ω, i.e. $D_1 \cap D_2 = \emptyset$. Then the Cauchy data can be extended from $\partial\Omega$ to $\Omega \setminus (D_1 \cup D_2)$. Since, say, u_1 has a harmonic extension onto

D_2 and u_i is continuous across ∂D_2, $u_1 = u_2$ inside D_2. But then

$$\sigma_1/\sigma_0 \partial_\nu^- u_2 = \partial_\nu^+ u_2 = \partial_\nu^+ u_1 = \partial_\nu^- u_1 = \partial_\nu^- u_2 \quad \mathbf{x} \in \partial D_2 \tag{5.9}$$

and it follows that $\sigma_1 = \sigma_0$ for u_i, contradicting that σ is piecewise constant.

If $D_1 \supset D_2$ and $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$ then Alessandrini found that $D_1 = D_2$. To show this denote σ related to D_i as σ^i, $i = 1, 2$, and assume w.l.o.g. that $\sigma_2 > \sigma_1$. Using Green's formula he wrote the integral identity (with $\sigma_0 = 1$)

$$\int_\Omega \sigma^1 |\nabla u_1|^2 = \int_{\partial\Omega} u_1 \partial_\nu u_1 = \int_{\partial\Omega} u_2 \partial_\nu u_2 = \int_\Omega \sigma^2 |\nabla u_2|^2 \geq \int_\Omega \sigma^1 |\nabla u_2|^2$$

This however, contradicts that u_1 is a solution, since the variational formulation of (5.1) yields that u_1 is the unique minimizer of the energy integral. See Rudnicki et al. [**NRS96**] for a discussion of the Dirichlets principle.

To prove uniqueness of convex polyhedra, we first establish that if two convex polyhedra D_i have a piece of their boundary in common, then $D_1 = D_2$ if their Cauchy data on $\partial\Omega$ coincide. To see this observe that they have harmonic continuation that coincide on $\Omega \backslash (\overline{D}_1 \cup \overline{D}_1)$ and on $D_1 \cap D_2$. Let $D_1 \backslash D_2 \neq \emptyset$, then the boundary consist in part of $\partial(D_1 \cup D_2)$ and $\partial(D_1 \cap D_2)$. Again u_1 and u_2 coincide hereon, and hence inside. Similar to (5.9) u_1 and u_2 coincide globally, hence $D_1 = D_2$. This result can easily be extended to a larger class of domains having a part of the boundary in common. Isakov [**Isa98**] defines this class as i-contact domains.

For the class of convex polygons satisfying $diam(D_i) < dist(D_i, \partial\Omega)$ uniqueness is also established. It is based upon the following lemma for unique continuation across a vertex. We state the lemma since it is important for the indication of which $g \in H^{-1/2}(\partial\Omega)$ to use.

Lemma 5.1 ([**Isa98**, 4.3.6])**.** *Let u solve $\nabla \cdot \sigma \nabla u(\mathbf{x}) = 0$ with $u\big|_{\partial\Omega}$ not a constant, and $\mathbf{x} \in \mathbb{R}^2$. Let the origin be a vertex of a convex polygon D and let u have a harmonic continuation onto a ball $B(0, \epsilon)$. Then there is a rotation of the plane such that u on this ball is invariant with respect to this rotation.*

Since two distinct convex polygons can not have a part of a boundary in common neither be disjoint if their Cauchy data coincide , they must intersect. In this case there exist a vertex for which u_2 has a harmonic continuation, but for which u_1 not readily is harmonic. From lemma 5.1 let the angle of rotation be $2\pi/q$, $q \in \mathbb{Z}$. Since D is convex under the distance condition $diam(D_i) < dist(D_i, \partial\Omega)$, there is a sector S of $B(0, R)$ with $R > diam(D_1)$ and angle $> \pi$ that is contained in $\Omega \backslash D_1$. Rotating S, q times by $2\pi/q$ yield that u_1 has a harmonic continuation from $\Omega \backslash D_1$ onto D_1, hence $u_1 = u_2$, [**Isa98**].

Barceló, Fabes, and Seo [**BFS94**] first relaxed the distance condition by assuming that $\partial_\nu u|_{\partial\Omega} \in \mathscr{L}^\infty$ such that ∇u did not have a harmonic continuation across $\partial\Omega$.

Alessandrini, Powell, and Isakov [**AIP95**] subsequently found that

$$\left\{\sum \nabla u = 0 | \mathbf{x} \in \Omega\right\} = \left\{ \text{ index of } \partial_\nu u \text{ for } \mathbf{x} \in \partial\Omega\right\}$$

A function $\psi \in \mathscr{L}_0^2(\partial\Omega)$ is of index zero if $\{x \in \partial\Omega | \psi(\mathbf{x}) \geq c\}$ is connected for any $c \in \mathbb{R}$. From lemma 5.1 follows that if u has a harmonic continuation across a vertex then $\nabla u = 0$. Hence by assuming $\partial_\nu u$ to be of index zero they established uniqueness for polygons. This result is only valid in $\mathbb{R}^2$ since it relays on techniques from complex analysis.

For multiple disjoint disks, Isakov and Powell [**IP90**] used another result for harmonic continuation. If u is harmonic and bounded on $\Omega \backslash S$ where S is a discrete set of points without any accumulation points, then u has a harmonic extension to Ω. Through a series of complicated geometrical reflection and inversion arguments, they proved that if $\partial_\nu u \in C^1(\partial\Omega)$ has compact support on $\Gamma \subset \partial\Omega$ then uniqueness hold for multiple disk. For one disk however Kang [**KS97**] proved uniqueness through a simpler argument. He found that the weak solution u of $\nabla \cdot \sigma \nabla u = 0$ can be represented as $u = \mathbf{H} + \mathbf{S}_{[D_i,\Gamma]}\xi_{D_i}$ where $\mathbf{H} \in H^2(\Omega)$ and $\mathbf{S}_{[D_i,\Gamma]}\xi_{D_i} \in H^2(\Omega\backslash\partial D)$ a single-layer potential (for the derivation hereof see (6.30) page 79). Relaying on this representation formula he found that for disks

$$\partial_\nu u = \frac{\sigma_0 - \sigma_1}{2(\sigma_0 + \sigma_1)} \partial_\nu \mathbf{H}, \qquad \mathbf{x} \in \partial D.$$

He established that if two disk D_i have the same Cauchy data, i.e. $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$ on $\partial\Omega$, then $D_1 = D_2$.

For balls in $\mathbb{R}^3$ Kang [**KS99**] again used the representation, $u = \mathbf{H} + \mathbf{S}_{[D_i,\Gamma]}\xi_{D_i}$. He proceeded to show that for $D = B(a,d)$ a ball with center a and radius d

$$\mathbf{S}_{[D,\Gamma]}\xi(x) = \frac{d}{|x-a|}\mathbf{S}_{[D,\Gamma]}\xi(x^*(D)), \mathbf{x} \in \mathbb{R}^3,$$

where $\mathbf{x}^*(D)$ is $\mathbf{x}$ reflected over ∂D, has a harmonic continuation onto a neighborhood of ∂D. From somewhat complicated geometrical arguments he concluded that if $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$ on $\partial\Omega$, then $D_1 = D_2$.

A last important result for the single measurement problem, is the local uniqueness that only holds for $\mathbb{R}^2$. If ψ is of index zero, $dist(D_1, D_2)$ is small and $\Lambda_{D_1}(\psi) = \Lambda_{D_2}(\psi)$, then $D_1 = D_2$. In [**Isa98**] arguments for why it may not be readily extended to $\mathbb{R}^3$ are given.

Several search strategies have been proposed for recovering the location and shape of D. In the paper [**KSY02**] numerical experiments for small objetcs with a periodic paramatrization in $\mathbb{R}^2$ are made. They used a Levenberg-Marquadt algorithm for recovering the shape of D. From the result of [**KS01**] for the bounds on D, they made an adequate initial estimate for the location of D. Starting from a disk their algorithm then found the unknown body D. Their numerical experiments have led them to conjecture that uniquenes can be obtained for larger classes than what is currently known.

A final problem to be adressed is the Inverse Problem when both D and σ are unknown. Not many results exist for this problem.

In [**KS99**] it was shown that if D_i are disks and $\Lambda_{\sigma_i,D_i}(\psi) = \Lambda_{\sigma_0,D_0}(\psi)$, then D_1 and D_2 are concentric disks. Also if $\psi \in \mathscr{L}_0^2(\partial\Omega)$ is not continuous at a point $p \in \partial\Omega$, where $\partial\Omega$ is continuously differentiable, then $\Lambda_{\sigma_1,D_1}(\psi) = \Lambda_{\sigma_2,D_2}(\psi)$ implies $\sigma_1 = \sigma_2$ and $D_1 = D_2$.

If D itself consist of two nested domains, i.e. $\Omega \supset D \supset D_1$, uniqueness will not hold for all possible currents. This example is due to Alessendrini. Using a perturbation argument Kang furthermore showed that stability doesn't hold for the class of simply connected domains.

Example 5.2 (non-uniqueness)**.** Assume $\Omega \subset \mathbb{R}^2$. There exist current patterns $\psi \in C(\partial\Omega)$, such that for different domains $D_1 \neq D_2$ with same conductivity σ_1, $\Lambda_{\sigma_1,D_1}(\psi) = \Lambda_{\sigma_1,D_2}(\psi)$. The "classical" example given in [**AIP95**] is depicted in figure 5.1 when $\psi = \cos(n\theta)$. In [**KS97**] it was furthermore proved that if $\lim_{\varepsilon\to 0} D_r^\varepsilon = D_r$ then the example holds for two domains with $\Omega\backslash D_r^\varepsilon$ and $\Omega\backslash D_2$ both connected.

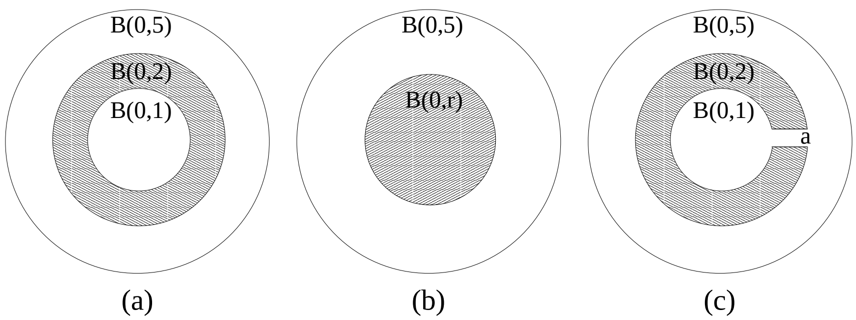

Figure 5.1. Figure (a) shows the domain $D_1 = B(0,2)\backslash B(0,1)$. Figure (b) is the domain $D_2 = B(0,r)$ with $r^2 = \frac{9(4-1)}{9-0.25}$ and Figure (c) is the approximate domain D_1^ε with $\varepsilon = a$.

When the location of D is known, uniqueness results may be established for both two nested inhomogenieties and for multiple disjoint inhomogenieties.

For multiple disjoint inhomogenieties this follows by applying a method of [**Ber02**] for the Inverse Scattering Problem at fixed frequency. The unknown, either field or current, may in Ω be expressed as a combination of a Single-layer Potential and a Harmonic function. This defines a density ξ on ∂D, proportional to the unknown current. The unknown conductivity constants may be uniquely found as the solution of the algebraic equations on ∂D_i,

$$\frac{\sigma_i + \sigma_0}{2(\sigma_i - \sigma_0)}\xi_i - \mathbf{K}^*_{[D_i,G]}\xi_i = \frac{\partial}{\partial\nu_i}\mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi) + \sum_{k\neq i}\frac{\partial}{\partial\nu_i}\mathbf{S}_{[D_k,G]}\xi_k \qquad \mathbf{x} \in \partial D_i.$$

The problem of two nested inhomogenieties, or when σ_0 is unknown, may not be treated as for N disjoint inhomogenieties. The densities ξ_i defined on ∂D_i are not uniquely determined from Cauchy Data. Instead the algebraic expression, for some special known functions v_N and v_D, is

$$\frac{\sigma_0}{\sigma_1}\Big(-\big\langle\sigma_0\partial_\nu u, v_N\big\rangle_{\partial\Omega} + \sigma_0\big\langle u, \partial_\nu v_N\big\rangle_{\partial\Omega}\Big) \;=\; \big\langle\sigma_0\partial_\nu u, v_D\big\rangle_{\partial\Omega} - \sigma_0\big\langle u, \partial_\nu v_D\big\rangle_{\partial\Omega}, \tag{5.10}$$

determines σ_0, σ_1. It has been shown that (5.10) has at most two diffferent pais of σ_0, σ_1 that solves (5.10). The two last results are found in theorem 6.31 and 6.38.

Chapter 6

The Single Measurement Conductivity Problem

When a current is applied to the boundary $\partial\Omega$ of a body Ω the elliptic PDE (Poisson's Equation) modeling the voltage distribution inside Ω is $\nabla \cdot \sigma \nabla u = 0$. Boundary data for the voltage distribution is referring to the current and the associated potential at the boundary $\partial\Omega$.

Let $\sigma(\mathbf{x})$ denote a piecewise constant function describing a conductivity distribution in an object Ω. Assume that the location of the jumps in $\sigma(\mathbf{x})$ is known. The question is whether one set of boundary data is sufficient for the Inverse Conductivity Problem of reconstructing σ. The general boundary value problem, for finding the potential in Ω, will be formulated and uniqueness results for recovering $\sigma(\mathbf{x})$ from one set of boundary data proved when $\sigma(\mathbf{x})$ is either

- N disjoint piecewise constant conductivities in Ω, or as
- two nested piecewise constant conductivities in Ω.

The conditions on both $u(\mathbf{x})$ and $\sigma(\mathbf{x})$ will be made precise in section 6.1, where the forward problem is defined. In section 6.2 the inverse problem is defined and different technical results related to the Cauchy problem is proved. These are used in section 6.3 for establishing an algebraic equation and from this algebraic equation prove uniqueness and continuous dependence of $\sigma(\mathbf{x})$ on the measured data.

6.1. Forward Problem

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, and $D \subset \Omega$ be simply connected domains with Lipschitz boundaries. Let χ_E denote the characteristic function for a set E. For $D_{i,j}$ simply connected domains with Lipschitz boundary $\partial D_{i.j}$ let D be given as

$$D = \bigcup_{i,j=1}^{n,m_n} D_{i,j} \qquad \text{where} \qquad \begin{array}{rclcll} D_{i,p} & \cap & D_{q,j} & = \emptyset & \forall & i \neq q \\ D_{i,p} & \subset & D_{i,j} & & \forall & p \geq j \\ \partial D_{i,p} & \cap & \partial D_{q,j} & = \emptyset & \forall & i \neq q, \text{or } p \neq j. \end{array} \tag{6.1}$$

The first index i in $D_{i,j}$ is henceforth labeling disjoint domains, whereas the second index j is labeling nested sequels of domains. If D does not contain nested sequels of domains, then $j = 0$ and the second index in $D_{i,j}$ is suppressed. Define $\sigma(\mathbf{x})$ as

$$\sigma(\mathbf{x}) = \sigma_{0,0}\chi_{\Omega\setminus D} + \bigcup_{i,j=1}^{n,m_n} \sigma_{i,j}\chi_{D_{i,j-1}/D_{i,j}} \qquad \sigma_{i,j} \in \mathbb{R}_+. \tag{6.2}$$

For a function $f(\mathbf{x})$ defined on the boundary denote the limit value, that if $\mathbf{x} \in \partial D_{i,j}$ and $\nu(\mathbf{x})$ denotes the outward normal derivative of $D_{i,j}$ when it exists, as

$$\lim_{\substack{h\to 0\\ h>0}} f(\mathbf{x} \pm h\nu(\mathbf{x})) = f^{\pm}(\mathbf{x}) \qquad \mathbf{x} \in \partial D_{i,j}.$$

Let $\partial_\nu^\pm$ and $\frac{\partial}{\partial \nu}^\pm$ denote normal derivatives.

For a given $\sigma(\mathbf{x})$ the voltage distribution $u(\mathbf{x})$ in Ω induced by a current flux ψ applied to $\partial\Omega$ satisfies the Neumann Problem

$$P[\sigma, D, \psi] = \begin{cases} \nabla \cdot \sigma \nabla u = 0 & \mathbf{x} \in \Omega \\ \sigma \dfrac{\partial u}{\partial \nu}^- = \psi & \mathbf{x} \in \partial\Omega \quad \psi \in \mathscr{L}_0^2(\partial\Omega) \end{cases} \tag{6.3}$$

where $\mathscr{L}_0^2(\partial\Omega)$ denotes the space of square integrable functions on $\partial\Omega$ satisfying $\int_{\partial\Omega} \psi = 0$. Define the Neumann-Dirichlet map by

$$\Lambda_\sigma(\psi) := u|_{\partial\Omega} \quad \psi \in \mathscr{L}_0^2(\partial\Omega) := \left\{\psi | \psi \in \mathscr{L}^2(\partial\Omega), \int_{\partial\Omega} \psi = 0\right\} \tag{6.4}$$

Definition 6.1 (Forward problem)**.** The forward problem of the Conductivity Problem is from knowledge of σ, D and ψ to find the solution u of $P[\sigma, D, \psi]$.

The problem $P[\sigma, D, \psi]$ is conveniently reformulated as the following transmission value problem, for $u \in H^1(\Omega) \cap H^2(\Omega\setminus\partial D)$, where again H denotes Sobolev spaces.

$$\nabla \cdot \nabla u = 0 \qquad \mathbf{x} \in \Omega \backslash \bigcup_{i,j} \partial D_{i,j} \quad n = 2, 3 \tag{6.5}$$

$$\sigma_{0,0} \frac{\partial u}{\partial \nu} = \psi(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega \quad \psi \in \mathscr{L}_0^2(\partial\Omega) \tag{6.6}$$

$$\int_{\partial\Omega} u = 0 \tag{6.7}$$

$$u^+ = u^- \qquad \mathbf{x} \in \partial D_{i,j} \tag{6.8}$$

$$\sigma_{i,j} \frac{\partial u^+}{\partial \nu} = \sigma_{i,j+1} \frac{\partial u^-}{\partial \nu} \qquad \mathbf{x} \in \partial D_{i,j} \tag{6.9}$$

where (6.9) and (6.9) is to be understood in $\mathscr{L}^2$ sense. The solution $u(\mathbf{x})$ of (6.5)-(6.9) can be expressed as harmonic functions on domains $D_{i,j} \backslash D_{i,j+1}$ etc. We define this type of functions as semi-harmonic.

Definition 6.2 (Semi-harmonic function)**.** Let $\partial D_{i,j}$ and $\partial\Omega$ be of class C^2. A function $u(\mathbf{x}) \in H^1(\Omega) \cap H^2(\Omega \backslash \bigcup \partial D_{i,j})$ is semi-harmonic in Ω if there exist $\lambda_{i,j} \in \mathbb{R}$ such that

$$\begin{array}{rclr} \Delta u & = & 0 & x \in \Omega \backslash \bigcup(\partial D_{i,j}) \\ u^- & = & u^+ & x \in \partial D_{i,j} \\ \lambda_{i,j} \frac{\partial u^-}{\partial \nu} & = & \lambda_{i,j-1} \frac{\partial u^+}{\partial \nu} & x \in \partial D_{i,j} \end{array} \tag{6.10}$$

where $\partial D_{i,j}$ is a finite set of non-intersecting hyper-surfaces. The ratio $\gamma_{i,j} = \lambda_{i,j-1}/\lambda_{i,j}$ denotes the conductivity ratio.

When definition 6.2 is extended to Lipschitz domains, the conditions on $\partial D_{i,j}$ are to be understood as equalities in $\mathscr{L}^2$. The subsequent results will however only be derived for C^2 boundaries.

Define the Neumann Green's function $G(\mathbf{x}, \mathbf{y})$ as the solution of

$$\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = -\delta(\mathbf{x} - \mathbf{y}) \qquad \mathbf{y} \in \Omega \tag{6.11}$$

$$\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \nu(\mathbf{x})} = -\frac{1}{\mu(\partial\Omega)} \qquad \mathbf{y} \in \Omega \quad \mathbf{x} \in \partial\Omega \tag{6.12}$$

$$\int_{\partial\Omega} G(\mathbf{x}, \mathbf{y}) ds(\mathbf{x}) = 0 \qquad \mathbf{y} \in \Omega \tag{6.13}$$

where $\mu(\partial\Omega)$ is the measure of the boundary $\partial\Omega$, [**Hac92**]. The Neumann Green's functions can be expressed as

$$G(x, y) = \Gamma(x - y) + v(x, y) \tag{6.14}$$

where $\Gamma(x - y)$ is the fundamental solution for Laplace's equation and $v(x, y)$ is an analytic function in Ω.

Denote the single- and double-layer potentials for densities $\xi \in \mathscr{L}^2(\partial D)$ and weakly singular kernels G defined on Ω as

$$(\mathbf{S}_{[D,G]}\xi)(\mathbf{x}) = \int_{\partial D} G(\mathbf{x},\mathbf{y})\xi(\mathbf{y})ds(\mathbf{y}), \qquad \xi \in C(\partial D) \tag{6.15}$$

$$(\mathbf{D}_{[D,G]}\xi)(\mathbf{x}) = \int_{\partial D} \frac{\partial G}{\partial \nu(\mathbf{x})}(\mathbf{x},\mathbf{y})\xi(\mathbf{y})ds(\mathbf{y}), \qquad \xi \in C(\partial D) \quad \mathbf{x} \notin \partial D \tag{6.16}$$

which satisfy the well-known jump-conditions

$$\frac{\partial^\pm}{\partial \nu}\mathbf{S}_{[D,G]}\xi = \left(\pm\frac{1}{2} + \mathbf{K}^*_{[D,G]}\right)\xi, \tag{6.17}$$

$$\mathbf{D}^\pm_{[D,G]}\xi = \left(\mp\frac{1}{2} + \mathbf{K}_{[D,G]}\right)\xi, \tag{6.18}$$

where $\mathbf{K}_{[D,G]}$ is the $\mathscr{L}^2$ adjoint of

$$\mathbf{K}^*_{[D,G]}\xi = p.v. \int_{\partial D} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \nu(\mathbf{x})}\xi(\mathbf{y})ds, \qquad \mathbf{x} \in \partial D.$$

$\mathbf{K}^*_{[D,G]}\xi$ is well-defined for C^2 boundaries, hence the principal value only applies to less regular boundaries.

The operator $\mathbf{K}_{[D,G]}$ is a Singular Integral Operator and is bounded on $\mathscr{L}^p$, $(1 < p < \infty)$. If ∂D is of class C^1 then $\mathbf{K}^*_{[D,G]}$ is a compact operator on the space $\mathscr{L}^2(\partial D)$ [**KS97**]. For ∂D Lipschitz and any real number $|\lambda| < 2$, $(I - \lambda \mathbf{K}^*_{[D,\Gamma]})$ is invertible on $\mathscr{L}^2(\partial D)$, [**KS97**]. From [**EFV92**] follows the estimate

$$\|\nabla \mathbf{S}_{[D,\Gamma]} f^\pm\|_{\mathscr{L}^2(\partial D)} + \|\mathbf{S}_{[D,\Gamma]} f\|_{\mathscr{L}^2(\partial D)} \leq C\|f\|_{\mathscr{L}^2(\partial D)} \tag{6.19}$$

where C depends on the Lipschitz character of D.

6.1.1. Representation of the Forward Map for one Inhomogeneity. The weak solution of (6.5)-(6.9) is naturally represented through integral equations. Throughout this section let D be a Lipschitz domain consisting of only one component, i.e. that $n \equiv 1$ and $m \equiv 1$ in (6.1) and $\sigma_{i,j} \equiv \sigma_1$. The general case is discussed in section 6.3.3.

The representation of the forward map based on the Green's function was derived in [**Hof98**]. Hofmann subsequently applied a Newton scheme to find the shape and conductivity from knowledge of the full Neumann-Dirichlet map. To establish if (6.3) is well-posed, existence, uniqueness, and continuous dependence on boundary data is found.

Theorem 6.3 ([**Hof98**])**.** *For $\sigma \in \mathscr{L}^\infty(\Omega)$ with $\inf(ess)_{x\in\Omega}\sigma(x) > 0$ and $\psi \in H_0^{-1/2}(\Omega)$ there exist a unique $u \in H^1(\Omega)$ satisfying $\int_{\partial\Omega} u\, ds = 0$ solving the weak formulation (6.20) of (6.3) given by*

$$\int_\Omega \sigma \nabla u \cdot \nabla v\, ds = \int_{\partial\Omega} \psi v\, ds \quad \forall \quad v \in H^1(\Omega). \tag{6.20}$$

In [**Hof98**] the solution of (6.5)-(6.9) was subsequently sought for $u \in C(\Omega) \cap C^2(\Omega\backslash\partial D)$, $\partial D \in C^2$ and $\Omega \subset \mathbb{R}^2$.

Theorem 6.4 ([**Hof98**] Theorem 1)**.** *The transmission problem (6.5)-(6.9) is uniquely solvable for $u \in C(\Omega) \cap C^2(\Omega\backslash\partial D)$, with $\partial D \in C^2$ provided that $\int_{\partial\Omega} \psi ds = 0$.*

Let G be the Neumann function given by (6.11)-(6.13) on Ω and $u^N = \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)$ the unique harmonic function in Ω with Neumann boundary data $\sigma_0^{-1}\psi$ on $\partial\Omega$. If $\int_{\partial\Omega} u^N ds = 0$, then the solution u can be represented as the sum

$$u = u^N + \mathbf{S}_{[D,G]}\xi_1 \qquad \mathbf{x} \in \Omega, \tag{6.21}$$

where $\xi_1 \in C(\partial\Omega)$ is the unique solution of the boundary integral equation

$$(cI - \mathbf{K}^*_{[D,G]})\xi_1 = \frac{\partial u^N}{\partial \nu}, \quad \mathbf{x} \in \partial D, \qquad c = \frac{\sigma_0 + \sigma_1}{2(\sigma_1 - \sigma_0)}, \tag{6.22}$$

satisfying $\int_{\partial D} \xi_1 ds = 0$.

Proof. Assume $u \in C(\Omega) \cap C^2(\Omega\backslash\partial D)$ solves the transmission problem (6.5)-(6.9) then from Green's theorem follows that for any $v \in H^1(\Omega)$

$$\sigma_0 \int_{\partial\Omega} v\partial_\nu^- u\, ds = \sigma_0 \int_{\Omega\backslash D} \nabla u \cdot \nabla v\, da + \sigma_1 \int_D \nabla u \cdot \nabla v\, da. \tag{6.23}$$

If u is a solution to the homogeneous transmission problem $\psi = 0$ then by setting $v = u$ we obtain $\nabla u = 0$ on $\Omega\backslash\partial D$. Since $u \in C(\Omega)$ and $\int_{\partial\Omega} u = 0$, then $u = 0$ in Ω. Therefore the transmission problem admits at most one solution.

Attempting to represent $u = u^N + v$, with $v = \mathbf{S}_{[D,G]}\xi_1$ the solution u satisfies the transmission-condition (6.9) if and only if v satisfies

$$\sigma_0 \frac{\partial v^+}{\partial \nu} - \sigma_1 \frac{\partial v^-}{\partial \nu} = (\sigma_1 - \sigma_0)\frac{\partial u^N}{\partial \nu} \qquad \mathbf{x} \in \partial D. \tag{6.24}$$

From the jump-condition (6.17) follows that the density ξ_1 satisfies the condition (6.24) if and only if ξ_1 is a solution of (6.22).

From the Green's function properties (6.12) and (6.13) follows that

$$\frac{\partial v}{\partial \nu} = \frac{1}{2\pi} \int_{\partial D} \xi_1\, ds, \quad \mathbf{x} \in \partial\Omega \qquad \text{and} \qquad \int_{\partial\Omega} v\, ds = 0. \tag{6.25}$$

Since u^N and v_- are harmonic in D, and v_+ is harmonic in $\Omega\backslash\overline{D}$ we have

$$\int_{\partial D} \frac{\partial u^N}{\partial \nu_+} ds = \int_{\partial D} \frac{\partial v^-}{\partial \nu} ds = 0 \qquad \int_{\partial\Omega} \frac{\partial v^+}{\partial \nu} ds = \int_{\partial D} \frac{\partial v^+}{\partial \nu} ds = 0 \tag{6.26}$$

From (6.25) follows that the Neumann boundary data of v on $\partial\Omega$ are constant and from (6.26) must be zero.

Therefore if ξ_1 is a solution of the homogeneous equation $(cI - \mathbf{K}^*_{[D,G]})\xi_1 = 0$, $v = \mathbf{S}_{[D,G]}\xi_1$ solves the homogeneous transmission problem (6.5)-(6.9); i.e. that $\frac{\partial v}{\partial \nu} = 0$ for $\mathbf{x} \in \partial\Omega$. Therefore from (6.17), and since the transmission problem has at most one solution, $v = 0$. From the jump conditions (6.17) follows that $\xi_1 = 0$. Hence, the Null space of the integral equation (6.22) is zero and from the Riesz-Fredholm theory [**CK83**] follows that (6.22) is uniquely solvable and that $(cI - \mathbf{K}^*_{[D,G]})^{-1}$ is a bounded operator on $\mathscr{L}^2(\partial D)$. □

We prove that u solving (6.3) is uniform bounded in σ for $\sigma \in [a,b]$ with $a, b \in \mathbb{R}_+$.

Lemma 6.5. *Assume $\psi \in \mathscr{L}^2(\partial\Omega)$ and that $\partial D \in C^2$. Then u solving (6.3) satisfies*

$$\|u\|_{\mathscr{L}^2(\partial D)} \le C\|\partial_\nu^- u\|_{\mathscr{L}^2(\partial\Omega)} \tag{6.27}$$

uniformly in σ for $\infty > N > \sigma_1, \sigma_0 > \delta_0 > 0$.

Proof. Firstly $\mathbf{S}_{[D,G]}\cdot$ is independent of σ_0, σ_1. Let $\lambda = 2(\sigma_0 - \sigma_1)/(\sigma_0 + \sigma_1)$ hence for $\infty > N > \sigma_1, \sigma_0 > \delta_0 > 0$, λ is in a closed interval. For $|\lambda| < 2$, $I - \lambda K$ has a bounded inverse and $\|\xi\|_{\mathscr{L}^2(\partial D)}$ is bounded by $\|\partial_\nu \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)\|_{\mathscr{L}^2(\partial D)}$. Since both $\mathbf{S}_{[D,G]}$ and $\partial_\nu \mathbf{S}_{[D,G]}$ also are bounded on $\mathscr{L}^2(\partial D)$, see (6.19), u satisfies (6.27). □

The extension of ∂D to being Lipschitz follows from Green's Theorem being valid for Lipschitz domains and that the jump-conditions for the Single Layer Potential also hold for Lipschitz domains. However the solution is then no longer a classical solution, but merely a week solution in $H(\Omega) \cap H^2(\Omega \backslash \partial D)$.

6.2. Inverse Problem

This section consists of 2 parts. First some representation formulas will be derived. These are different from the one used in section 6.1 for the forward map, and the equivalence between these different representation formulas is established. Secondly the Cauchy problem for extending Cauchy data from one boundary to another is discussed. It is found that a functional using the trace data on the opposite boundary can depend continuously upon the Cauchy data, even though the problem of extending Cauchy data is severely ill-posed. The latter are technical results that are used in section 6.3 for proving new partial uniqueness results for an Inverse Conductivity Problem.

Let σ and D be defined as $D = D_{1,1}$ and $\sigma = \sigma_0\chi_{\Omega\backslash D} + \sigma_1\chi_D$. Assume $\Omega \in \mathbb{R}^n$, $n = 2, 3$.

Definition 6.6. The problems IP1-IP3 for the Inverse Conductivity Problem with data on $\partial\Omega$ for one experiment $\{\Lambda_\sigma(\psi), \psi\}$, with $\psi \in \mathscr{L}^2_0(\partial\Omega)$ is defined as

IP1 Finding σ_1 and the location of D given σ_0 and D a disk

IP2 Finding σ_1 and σ_0 given D.

Since IP1-IP2 only assumes knowledge of $\{\Lambda_D(\psi), \psi\}$ for one $\psi \in \mathscr{L}^2_0(\partial\Omega)$ more humble results must be expected, than when the full Neumann-Dirichlet map is known. For problems IP1-IP2 the following questions are of interest

- Uniqueness,
- Construction,
- Stability of reconstructing σ, respectively D.

6.2.1. Equivalence of Different Representation Formulas. In this section some fundamental results related to representing the weak solution of (6.3) using integral operators is established. First a common result used for proving uniqueness of balls and disks is repeated in theorem 6.7. An unprecise formulation of this theorem in [**KS97**], [**KSS97**] claims uniqueness of both $\mathbf{\Phi} \in (\mathbb{R}^n \backslash \partial D)$ and $\mathbf{H} \in (\mathbb{R}^n \backslash \partial\Omega)$ to be defined in (6.29). In the recent publication [**AK01**] Kang et al. no longer use that formulation of theorem 6.7. The representation from Theorem 6.7 can not be used for calculating $u \in \Omega$ solving (6.3), it is however of great use when discussing the inverse problem.

In theorem 6.9 a fundamental equality related to seeking the representation of u solving (6.3) using a harmonic function in Ω and a single layer potential is established.

This is the first result relating the equivalence between different single layer potential representations. A different but in spirit similar result is found in [**AK01**]. There, it is found how the Neumann and Dirichlets Green functions at the boundary $\partial\Omega$ are related to the fundamental solution through a Second-kind Fredholm integral operator.

Theorem 6.7 ([**KS97**] Theorem 2.1)**.** *Assume $\partial\Omega$, and ∂D are Lipschitz. Let u be the weak solution of the Neumann problem $P[\sigma, D, \psi]$, with $\phi = u|_{\partial\Omega}$, and $\psi = \partial_\nu u|_{\partial\Omega}$. One representation for u is*

$$u = \mathbf{H} + \mathbf{\Phi}_D \quad \textit{in} \quad \Omega \tag{6.28}$$

with

$$\mathbf{H} = -\mathbf{S}_{[\Omega,\Gamma]}(\sigma_0^{-1}\psi) + \mathbf{D}_{[\Omega,\Gamma]}\phi \qquad \textit{and} \qquad \mathbf{\Phi}_D = \mathbf{S}_{[D,\Gamma]}\xi_2 \tag{6.29}$$

and where $\xi_2 \in \mathscr{L}^2_0(\partial D)$ is the unique solution of

$$(cI - \mathbf{K}^*_{[\Omega,\Gamma]})\xi_2 = \frac{\partial \mathbf{H}}{\partial \nu} \quad \mathbf{x} \in \partial D \qquad c = \frac{\sigma_0 + \sigma_1}{2(\sigma_1 - \sigma_0)}. \tag{6.30}$$

Moreover $\mathbf{H} \in H^1(\mathbb{R}^n \backslash \partial\Omega)$ and $\mathbf{\Phi}_D \in H^1(\mathbb{R}^n \backslash \partial D)$ and

$$0 = \mathbf{H} + \mathbf{\Phi}_D \quad \mathbf{x} \in \mathbb{R}^n \backslash \overline{\Omega}. \tag{6.31}$$

Proof. If u has a representation as $u = \mathbf{H} + \mathbf{\Phi}_D$ then as in theorem 6.4 follows that ξ_2 must be a solution of (6.30). Furthermore $(cI - \mathbf{K}^*_{[\Omega,\Gamma]})$ has a bounded inverse on $\mathscr{L}^2(\partial D)$ for $|c| > 1/2$. Hence ξ_2 is unique. That $u = \mathbf{H} + \mathbf{\Phi}_D$ also solves (6.3), may be found as a computational result. However, consider instead the two functions v_1 and v_2

$$v_1(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \mathbf{x} \in \Omega \\ 0 & \mathbf{x} \in \mathbb{R}^n \backslash \overline{\Omega} \end{cases} \qquad v_2(\mathbf{x}) = -\mathbf{S}_{[\Omega,\Gamma]}(\sigma_0^{-1}\psi) + \mathbf{D}_{[\Omega,\Gamma]}\phi + \mathbf{S}_{[D,\Gamma]}\xi_2 \quad \mathbf{x} \in \mathbb{R}^n.$$

Both v_1 and v_2 may be shown to be weak solutions in $H^1(\mathbb{R}^n\backslash\partial\Omega)$ of the following transmission problem

$$\begin{cases} \nabla\cdot\big((\sigma_0\chi_{\mathbb{R}^n\backslash D}+\sigma_1\chi_D)\nabla v\big)=0 & \mathbf{x}\in\mathbb{R}^n\backslash\partial\Omega & (a)\\ v^- - v^+ = \phi & \mathbf{x}\in\partial\Omega & (b)\\ \dfrac{\partial v^-}{\partial\nu}-\dfrac{\partial v^+}{\partial\nu}=\sigma_0^{-1}\psi & \mathbf{x}\in\partial\Omega & (c)\end{cases} \tag{6.32}$$

The function v_1 satisfies (6.32b) and (6.32c) by construction. Since u is a weak solution of (6.32a) in Ω and zero is a weak solution of (6.32a) in $\mathbb{R}^n\backslash\overline{\Omega}$, the function v_1 satisfies (6.32).

By the jump-conditions (6.17) and (6.18) follows that v_2 satisfies (6.32b) and (6.32c). Since ξ_2 is the unique solution of (6.30) v_2 is a solution of (6.32a) in Ω. Also since $\mathbf{H}$ and $\mathbf{\Phi}_D$ are harmonic in $\mathbb{R}^n\backslash\Omega$, the function v_2 satisfies (6.32).

To prove uniqueness of $v\in H^1(\mathbb{R}^n\backslash\partial\Omega)$ solving (6.32) suppose v is a solution of (6.32) with $\phi=\psi=0$. Then v is a solution to $\nabla\cdot\big((\sigma_0\chi_{\mathbb{R}^n\backslash D}+\sigma_1\chi_D)\nabla u\big)=0$ in $\mathbb{R}^n$. Therefore for large R

$$\begin{aligned}\int_{B_R}|\nabla v|^2 &\le \frac{\sigma_0+\sigma_1}{\sigma_0\sigma_1}\int_{B_R}(\sigma_0+(\sigma_1-\sigma_0)\chi_D)|\nabla v|^2=\frac{\sigma_0+\sigma_1}{\sigma_1\sigma_0}\int_{\partial B_R}v\frac{\partial v}{\partial\nu}\\ &=-\frac{\sigma_0+\sigma_1}{\sigma_1\sigma_0}\int_{\mathbb{R}^n\backslash B_R}|\nabla v|^2\le 0\end{aligned}$$

This holds for all R hence v is constant. Since $v\in H^1(\mathbb{R}^n)$, it follows that $v=0$ establishing uniqueness for the solution of (6.32), and therefore $v_2=v_1$.

Hence $u=\mathbf{H}+\mathbf{\Phi}_D$ indeed is a representation for the weak solution of $P[\sigma,D,\psi]$ and satisfies (6.31). □

Theorem 6.8 ([**KS97**] Theorem 2.1)**.** *Let u be the weak solution of of the Neumann problem $P[\sigma,D,\psi]$. If $\mathbf{\Phi}_D$ is given as $\mathbf{\Phi}_D=\mathbf{S}_{[D,\Gamma]}(\xi_2)$, then when seeking a representation of u as*

$$u=\mathbf{H}+\mathbf{\Phi}_D \quad in \quad \Omega,$$

where $\mathbf{H}\in H(\Omega)\cap H^2(\Omega\backslash\partial D)$ and $\xi_2\in\mathscr{L}_0^2(\partial D)$ are unique in Ω and ∂D, respectively.

Proof. To prove uniqueness of $\mathbf{H}$ and ξ_2, assume that

$$\mathbf{H}_1+\mathbf{S}_{[D,\Gamma]}\tilde{\xi}_2=\mathbf{H}_2+\mathbf{S}_{[D,\Gamma]}\xi_2 \quad \mathbf{x}\in\Omega.$$

Then $\mathbf{S}_{[D,\Gamma]}(\tilde{\xi}_2-\xi_2)$ is harmonic in Ω and the jump-condition for the single-layer potential imply $\tilde{\xi}_2-\xi_2=0$ in Ω, and therefore $\mathbf{H}_1=\mathbf{H}_2$. □

The representation (6.28) may be motivated by the following algebra utilizing the divergence theorem where however the assumptions are not satisfi ed. Firstly formally

$$\frac{1}{\sigma_0}\int_\Omega (\sigma_0\chi_{\Omega\setminus D}+\sigma_1\chi_D)\nabla u\cdot\nabla\Gamma(\mathbf{x}-\mathbf{y})ds(y) = \int_{\partial\Omega}\Gamma(\mathbf{x}-\mathbf{y})\frac{\partial u}{\partial\nu}d\,a(y) = \mathbf{S}_{[\Omega,\Gamma]}(\partial_\nu u)$$

and secondly since formally

$$\frac{1}{\sigma_0}\int_\Omega (\sigma_0\chi_{\Omega\setminus D}+\sigma_1\chi_D)\nabla u\cdot\nabla\Gamma(\mathbf{x}-\mathbf{y})ds(y) = \frac{\sigma_1-\sigma_0}{\sigma_0}\int_D \nabla u\cdot\nabla\Gamma ds(y) + \int_\Omega \nabla u\cdot\nabla\Gamma ds(y) = \mathbf{S}_{[D,\Gamma]}\big(\frac{\sigma_1-\sigma_0}{\sigma_0}\partial_\nu u\big) - u(\mathbf{x}) + \mathbf{D}_{[\Omega,\Gamma]}(u)$$

which motivates (6.28) as a combination of a Single-layer potential and a harmonic function. It must be emphasized that this is not a derivation, since it is not possible to make Γ harmonic in Ω.

Remark In the proof of theorem 6.8 uniqueness of H and ξ is shown given the kernel of the single layer potential $\mathbf{S}_{[D,\Gamma]}$, but the uniqueness of the single layer kernel Γ itself is not and can not be established. Doing this would contradict that both theorem 6.4 and 6.7 are valid. However the uniqueness of $\mathbf{S}_{[D,\Gamma]}$ was claimed by Kang in [**KS97**], [**KSS97**]. However in the recent publication [**AK01**] that claim was removed. If however the additional assumptions

$$\lim_{\mathbb{R}^n\setminus\overline{\Omega}\ni x\to\partial\Omega} u = 0 \qquad \Delta u = 0 \quad \mathbf{x}\in\mathbb{R}^n\setminus\overline{\Omega}$$

are included then the theorems of Kang et al. claiming uniqueness hold.

A central result established in theorem 6.9 for the density ξ of the integral operators $\mathbf{S}_{[D,\Gamma]}\cdot$ and $\mathbf{S}_{[D,G]}\cdot$ is that for both representations the densities are the same.

In section (6.3.2) the inverse problem IP2 is discussed when σ_0 is known. In order to reconstruct σ_1, fi rst a density ξ on ∂D is recovered from anyone of a number of boundary integral representation. Subsequently σ_1 can then be found from an algebraic equation.

The consequence of theorem 6.9 is that the different boundary integral representations all give the same ξ. This in particular means that the different boundary integral representations may be combined to give better reconstruction of ξ.

Theorem 6.9. *Assume $\alpha\in C_0^\infty(\Omega)$ and let Γ be the fundamental solution for Laplace equation. For all functions $v(\mathbf{x},\mathbf{y})$ harmonic in $\Omega\times\Omega$, any solution of (6.3) satisfies in* *the distributional sense that*

$$\Delta u = \Delta \mathbf{S}_{[D,\Gamma+v]}\xi = \xi = \sum_{i,j=1}^{n,m_n} \frac{\sigma_{i,j} - \sigma_{i,j-1}}{\sigma_{i,j}} \chi_{\partial D_{i,j}} \partial_\nu^+ u(\mathbf{x}).$$

Proof. Denote the space of test-functions $\mathcal{D} = \{\alpha | \alpha \in C^\infty, supp(\alpha) \in \Omega\}$, and let $\sigma^{(n)} \in C^\infty$ be an approximation to σ defined by (6.2) such that $\sigma^{(n)} = \sigma$ for all $\mathbf{x}$ with distance larger than δ_n to the boundary ∂D. Firstly, observe that

$$\lim_{n\to\infty} \langle \sigma \nabla u \cdot \nabla ({\sigma^{(n)}}^{-1}) - \Delta u, \alpha \rangle = \lim_{n\to\infty} \langle \nabla ({\sigma^{(n)}}^{-1} \sigma \nabla u - \nabla u), \alpha \rangle = 0,$$

which means that the distributional differential equation of $\nabla \cdot \sigma \nabla u = 0$ is

$$\Delta u = \lim_{n\to\infty} \sigma \nabla u \cdot \nabla ({\sigma^{(n)}}^{-1}) = \lim_{n\to\infty} \nabla ({\sigma^{(n)}}^{-1} \sigma \nabla u).$$

Denoting $\widetilde{\partial D}_{n,m} := supp(\nabla({\sigma^{(n)}}^{-1})) = \{\mathbf{x}\,| dist(\mathbf{x}, \partial D) < \delta_q\}$. It follows from Green's theorem and the continuity of $\sigma \nabla u$ across $\partial D_{n,m}$ that

$$\langle \Delta u, \alpha \rangle = \lim_{n\to\infty} \int_{\widetilde{\partial D}_{n,m}} \nabla ({\sigma^{(n)}}^{-1} \sigma \nabla u) \alpha(\mathbf{x}) ds(\mathbf{x}) = \int_{\partial D_{n,m}} \xi(\mathbf{x}) \alpha(\mathbf{x}) ds, \quad (6.33)$$

where

$$\xi(\mathbf{x}) = \sum_{i,j=1}^{n,m} \left(\frac{\sigma_{i,j} - \sigma_{i,j-1}}{\sigma_{i,j}} \chi_{\partial D_{i,j}} \partial_{\nu_{i,j}}^+ u(\mathbf{x}) \qquad \mathbf{x} \in \partial D_{n,m} \right). \quad (6.34)$$

Since for any function v harmonic in $\Omega \times \Omega$,

$$\langle \Gamma(\mathbf{x} - \mathbf{y}) + v(\mathbf{x}, \mathbf{y}), \Delta_{\mathbf{x}} \alpha(\mathbf{x}) \rangle = \alpha(\mathbf{y}),$$

changing the order of integration proves

$$\langle \Delta(\mathbf{S}_{[D,\Gamma+v]}\xi), \alpha \rangle = \int_{\partial D_{n,m}} \xi(\mathbf{y}) \alpha(\mathbf{y})\ ds(\mathbf{y}). \quad (6.35)$$

Combining (6.33) and (6.35) concludes the proof. □

Theorem 6.9 established the equality between different integral representation formulas, i.e. (6.22) and (6.30).

Corollary 6.10 (The density function)**.** *Assume* $u = \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi) + \mathbf{S}_{[D,G]}\xi_1$ *and* $u = -\mathbf{S}_{[\Omega,\Gamma]}(\sigma_0^{-1}\psi) + \mathbf{D}_{[\Omega,\Gamma]}(\Lambda_\sigma(\psi)) + \mathbf{S}_{[D,\Gamma]}\xi_2$ *are weak solutions of (6.3) for* $\psi \in \mathscr{L}_0^2(\partial\Omega)$. *The densities* ξ_1 *and* ξ_2 *from (6.22) and (6.30), respectively, satisfy*

$$\xi_1 = \frac{(\sigma_1 - \sigma_0)}{\sigma_1} \frac{\partial u^+}{\partial \nu} = \frac{\sigma_1 - \sigma_0}{\sigma_0} \frac{\partial u^-}{\partial \nu} = \xi_2. \quad (6.36)$$

Proof. This follows from theorem 6.9, since ξ is the same for all representations. □

Using the results of theorem 6.4 and theorem 6.7 we have the relation (6.38) between the harmonic functions $\mathbf{H}$ and $v(\mathbf{x}, \mathbf{y})$ at the boundary.

Corollary 6.11. *Let $\{\Lambda_\sigma(\psi), \psi\} = \{\phi, \psi\}$ be Cauchy data for u solving (6.3) with ∂D Lipschitz. For any $v(\mathbf{x}, \mathbf{y})$ harmonic in $(\Omega \times \Omega)$ a weak solution of (6.3) can be represented as $u = \mathbf{H} + \mathbf{S}_{[D,\Gamma+v]}\xi$ where ξ is given by (6.34) and $\mathbf{H}$ is the harmonic function solving*

$$\Delta \mathbf{H} = 0 \qquad \mathbf{x} \in \Omega, \tag{6.37}$$

$$\frac{\partial \mathbf{H}}{\partial \nu} = \sigma_0^{-1}\psi - \frac{\partial}{\partial \nu}\mathbf{S}_{[D,\Gamma+v]}\xi \qquad \mathbf{x} \in \partial\Omega. \tag{6.38}$$

If $v(\mathbf{x}, \mathbf{y})$ is either zero or the harmonic function solving (6.14) then the harmonic function $\mathbf{H}$ is $\mathbf{H} = u^N$ or $\mathbf{H} = -\mathbf{S}_{[\Omega,\Gamma]}\sigma_0^{-1}\psi + \mathbf{D}_{[\Omega,\Gamma]}\phi$ respectively.

A central relation to be established for the inverse problem is (6.39) which holds for one unknown function, with known traces, and all semi-harmonic functions. Besides the unknown σ_i, (6.39) also contains unknown boundary integrals $\langle u, j_{v_{i,j}}\rangle_{\partial D_{i,j}}$. The subsequent section on the Cauchy problem will establish the necessary spaces and functions needed to express these unknown boundary integrals $\langle u, j_{v_{i,j}}\rangle_{\partial D_{i,j}}$ in terms of known boundary integrals. When $\langle u, j_{v_{i,j}}\rangle_{\partial D_{i,j}}$ is replaced by known boundary integrals (6.39) may then be reduced to an algebraic equation for σ_i. This is done for $n = 1$ and $m = 1$, and leads to the new stability estimates for reconstructing σ_i from one boundary experiment.

Theorem 6.12. *Assume $D = \cup D_{i,j} \subset \Omega$, with $\partial D_{i,j}$ Lipschitz. Let $u, v \in H^2(\Omega\backslash\partial D) \cap H^1(\Omega)$ denote two semi-harmonic functions defined on Ω and with conductivity ratios defined with respect to D as $\gamma_{i,j} = \sigma_{i,j-1}/\sigma_{i,j}$ and $\gamma_{i,j} = \lambda_{i,j-1}/\lambda_{i,j}$, respectively. Let $j_{v_{i,j}}$ denote the current of v on the boundaries $\partial D_{i,j}$ then*

$$\sum_{i,j=1}^{n,m_n} \left(\left(\frac{\sigma}{\lambda_i}\right)^+ - \left(\frac{\sigma}{\lambda_i}\right)^- \right) \langle u, j_{v_{i,j}}\rangle_{\partial D_{i,j}} = -\int_{\partial\Omega} u\sigma\partial_\nu v - v\sigma\partial_\nu u \tag{6.39}$$

$$= -\sigma^-\langle u, \partial_\nu v^-\rangle_{\partial\Omega} + \langle v, \sigma^-\partial_\nu u^-\rangle_{\partial\Omega}.$$

Proof. Since $u, v \in H^1(\Omega)$, the gradient of u, v is in $\mathscr{L}^2(\Omega)$. Let D be composed of one component, i.e. let $D = D_{1,1}$, and divide Green's formula as

$$0 = \int_\Omega \nabla \cdot (u\frac{\sigma}{\lambda}\lambda\nabla v - v\sigma\nabla u)\,dx =$$

$$\int_{\Omega\backslash D} \nabla \cdot (u\frac{\sigma}{\lambda}\lambda\nabla v - v\sigma\nabla u)\,dx + \int_D \nabla \cdot (u\frac{\sigma}{\lambda}\lambda\nabla v - v\sigma\nabla u)\,dx. \tag{6.40}$$

Since $\partial\Omega$ and ∂D are Lipschitz, the product $(u\frac{\sigma}{\lambda}\lambda\nabla v - v\sigma\nabla u)$ is differentiable on $\Omega\backslash D$ and D respectively, and Green's formula is valid for Lipschitz domains, [**Gri85**, pp50], an application of the divergence theorem reduces (6.40) to

$$0 = \int_{\partial\Omega} \nu^+ \cdot (u\frac{\sigma}{\lambda}\lambda\nabla v - v\sigma\nabla u) - \int_{\partial D} u\,\nu^+ \cdot (\lambda_+\nabla v)\Big(\Big(\frac{\sigma}{\lambda}\Big)^+_{\Omega\backslash D} - \Big(\frac{\sigma}{\lambda}\Big)_D\Big)^-, \tag{6.41}$$

where ν^+ is the normal derivative into $\mathbb{R}^n\backslash\Omega$ and $\mathbb{R}^n\backslash D$ respectively and $\nu\cdot(\lambda\nabla v)$ is continuous across ∂D. The same result applies to each strip $D_{i,j}\backslash D_{i,j+1}$ and the result (6.39) follows. □

6.2.2. Extending Cauchy data. It is advantageous for extending Cauchy data from one boundary to another to consider the boundary to be of class C^2. Since Green's theorem still hold for Lipschitz domains, [**NRS96**], many of the results may be extended to Lipschitz domains.

Solving the Cauchy Problem means finding a harmonic function u solving

$$\begin{aligned} \Delta u &= 0 && \mathbf{x}\in\Omega\backslash D \\ u &= \tilde{\phi}(\mathbf{x}) && \mathbf{x}\in\partial D \\ \frac{\partial u}{\partial\nu} &= \tilde{\psi}(\mathbf{x}) && \mathbf{x}\in\partial D \end{aligned} \tag{6.42}$$

with Ω and D open sets and, say $\tilde{\psi},\tilde{\phi}\in\mathscr{L}^2(\partial D)$. This is not a well-posed problem, since the solution u (if it exists) does not depend continuously on boundary data, and does in general not even exist [**Had23**]. If, however, a solution exits, then from Holmgreen's Theorem for harmonic continuation, it is also unique. There is a vast number of methods for solving the Cauchy problem. In for instance [**CHWY01b**] analogies to the solution of the moment problem is studied, and in [**CDJP01**] a method for extending $\{\tilde{\phi},\tilde{\psi}\}$ from part of ∂D to ∂D using Tikhonov regularization is discussed.

The notion of solving a Cauchy problem like (6.42), will here be defined by introducing finite dimensional spaces, in which unique optimal solutions for (6.42) exist. These optimal solutions will not solve (6.42), since both boundary conditions will not be satisfied. However, the error hereon may be controlled, by considering a construction of sequences of finite dimensional spaces.

Firstly some denseness results are established.

Theorem 6.13 (Denseness of harmonic functions)**.** *Consider the subset of harmonic functions*

$$X^* = \left\{ u \,\middle|\, \begin{array}{rclll} \Delta u &=& 0, & \mathbf{x}\in\Omega\backslash D, & u\in C^2(\Omega\backslash D), \\ u &=& 0, & \mathbf{x}\in\partial\Omega, & \partial\Omega\in C^2 \end{array} \right\}. \tag{6.43}$$

The normal derivative of all functions in X^* restricted to $\partial\Omega$ is dense in $\mathscr{L}^2(\partial\Omega)$.

Proof. Denote the space of normal derivatives at $\partial\Omega$ for functions in X^* as Y^*,

$$Y^* = \{\psi|\partial_\nu u = \psi, \mathbf{x}\in\partial\Omega, u\in X^*\}. \tag{6.44}$$

If Y^* is not dense in $\mathscr{L}^2(\partial\Omega)$ then there exist a nontrivial $\phi\in\mathscr{L}^2(\partial\Omega)$ such that

$$\int_{\partial\Omega}\phi\,\psi\,ds = 0 \qquad \forall \quad \psi\in Y^*. \tag{6.45}$$

Let v be the solution of

$$\begin{aligned} \Delta v &= 0 && \mathbf{x} \in \Omega\backslash D, \\ v &= \phi && \mathbf{x} \in \partial\Omega, \\ \frac{\partial v}{\partial \nu} &= 0 && \mathbf{x} \in \partial D. \end{aligned} \tag{6.46}$$

From the divergence theorem, and that u and v are harmonic, follows that

$$\int_{\partial D \cup \partial\Omega} u\, \partial_\nu v\, ds = \int_{\Omega\backslash D} \nabla \cdot (v\, \nabla u)\, da = \int_{\partial D \cup \partial\Omega} v\, \partial_\nu u\, ds.$$

Inserting (6.45) and the homogeneous boundary conditions for u and v in this identity implies that

$$0 = \int_{\partial D} v\, \partial_\nu u\, ds, \qquad \forall \quad u \in X.$$

In particular this holds for all $\partial_\nu u \in \mathscr{L}^2(\partial D)$, therefore $v = 0$ for $x \in \partial D$ and since v then has homogeneous Cauchy data on ∂D, $v \equiv 0$ for $\mathbf{x} \in \Omega\backslash D$, contradicting that $\phi \neq 0$. □

A similar theorem is

Theorem 6.14 ([**CHWY01a**]). *Consider the subset of harmonic functions*

$$X = \left\{ u \,\middle|\, \begin{array}{lllll} \Delta u & = & 0, & \mathbf{x} \in \Omega\backslash D, & u \in C^2(\Omega\backslash D) \\ \frac{\partial u}{\partial \nu} & = & 0, & \mathbf{x} \in \partial\Omega & \partial\Omega \in C^2 \end{array} \right\}. \tag{6.47}$$

The trace of all functions in X restricted to $\partial\Omega$ is dense in $\mathscr{L}^2(\partial\Omega)$.

For both theorem 6.13 and 6.14 the denseness results are independent of whether $\partial\Omega$ is the inner or outer boundary.

If $w \in H^2(\Omega\backslash D)$ is a solution of

$$\begin{aligned} \Delta w &= 0 && \mathbf{x} \in \Omega\backslash D, \\ w &= \tilde{\phi} && \mathbf{x} \in \partial D, \\ \frac{\partial w}{\partial \nu} &= 0 && \mathbf{x} \in \partial D, \end{aligned} \tag{6.48}$$

and $v \in H^2(\Omega\backslash D)$ is a solution of

$$\begin{aligned} \Delta v &= 0 && \mathbf{x} \in \Omega\backslash D, \\ v &= 0 && \mathbf{x} \in \partial D, \\ \frac{\partial v}{\partial \nu} &= \tilde{\psi} && \mathbf{x} \in \partial D, \end{aligned} \tag{6.49}$$

then $u = v + w$ is a solution of (6.42). This is the motivation for defining finite dimensional spaces that satisfy a homogeneous boundary condition on ∂D.

Definition 6.15 (Functions spaces X_M Y_M)**.** Let ϕ_n denote an orthonormal basis on $\mathscr{L}^2(\partial\Omega)$ and let v_n solve

$$\Delta v_n = 0 \qquad \mathbf{x} \in \Omega\backslash D, \tag{6.50}$$

$$\partial_\nu v_n = 0 \qquad \mathbf{x} \in \partial D, \tag{6.51}$$

$$v_n = \phi_n \qquad \mathbf{x} \in \partial\Omega. \tag{6.52}$$

Define the space Y_M and X_M for any $M > 0$ as

$$Y_M = \big\{\phi|\phi \in span\{\phi_1, \cdots, \phi_M\}\big\}, \qquad X_M = \big\{u|u \in span\{v_1, \cdots, v_M\}\big\}.$$

The optimal solution of (6.48) obtained by the functions from X_M is defined as:

Definition 6.16. For $\tilde{\phi} \in \mathscr{L}^2(\partial D)$, a function $v \in X_M$ is an optimal solution in X_M of the Cauchy problem

$$\Delta v = 0 \qquad \mathbf{x} \in \Omega\backslash D,$$

$$\partial_\nu v = 0 \qquad \mathbf{x} \in \partial D,$$

$$v = \tilde{\phi} \qquad \mathbf{x} \in \partial D,$$

if v is the unique element in X_M with minimal distance to ϕ in $\mathscr{L}^2(\partial D)$, i.e.

$$\|v - \tilde{\phi}\|^2_{\mathscr{L}^2(\partial D)} = \inf_{\tilde{v} \in X_M} \|\tilde{v} - \tilde{\phi}\|^2_{\mathscr{L}^2(\partial D)}.$$

We now proof a denseness result of the optimal solutions for an arbitrary $\tilde{\phi} \in \mathscr{L}^2(\partial D)$, thereby obtaining control of the error for between the optimal solutions on ∂D and $\tilde{\phi} \in \mathscr{L}^2(\partial D)$.

Theorem 6.17. *Assume $\tilde{\phi} \in \mathscr{L}^2(\partial D)$. For each $M > 0$ there exist a unique harmonic function in $\Omega\backslash D$, $v_M \in X_M$, satisfying $\partial_\nu v_M = 0$ for $\mathbf{x} \in \partial D$, which is the optimal solution in X_M of the Cauchy problem (6.48). The optimal solutions $v_M \in X_M$ satisfies that* $\lim_{M\to\infty} \|v_M - \tilde{\phi}\|_{\mathscr{L}^2(\partial D)} = 0$.

Proof. For any M let v_i and ϕ_i be as in definition 6.15. Define$\hat{\phi}_i := v_i\big|_{\partial D}$. Since $\partial_\nu v_i\big|_{\partial D} = 0$, the functions $\hat{\phi}_i$ must be linear independent. Assume the contrary, then there exist a nontrivial linear combination of elements in X_M, that is harmonic in $\Omega\backslash D$ and has homogeneous Neumann and Dirichlets conditions on ∂D. From Unique Continuation of Cauchy Data, this function is identically zero contradicting that the ϕ_i are orthogonal.

Let $\mathbf{a}_M = (a_1, \cdots, a_M)$ be an M-tuple in $\mathbb{R}^M$ and denote E the functional $\mathbb{R}^M \to \mathbb{R}$.

$$E(\mathbf{a}_M) = \|\tilde{\phi} - \sum_{i=1}^{M}(a_i v_i)\|^2_{\mathscr{L}^2(\partial D)} = \|\tilde{\phi} - \sum_{i=1}^{M}(a_i \hat{\phi}_i)\|^2_{\mathscr{L}^2(\partial D)}. \tag{6.53}$$

Let $\tilde{\mathbf{a}}_M = (\tilde{a}_1, \cdots, \tilde{a}_M)$ be the unique M-tuple minimizing $E(\mathbf{a}_M)$, then $v_M = \sum_{i=1}^M \tilde{a}_i v_i$ defines the optimal solution from definition 6.16.

From theorem 6.14 follows that the space

$$\left\{ u \middle| \begin{array}{rcll} \Delta u & = & 0, & \mathbf{x} \in \Omega \backslash D, \qquad u \in C^2(\Omega \backslash D) \\ \frac{\partial u}{\partial \nu} & = & 0, & \mathbf{x} \in \partial D \qquad \partial \Omega \in C^2 \end{array} \right\},$$

is dense in $\mathscr{L}^2(\partial\Omega)$. Therefore, $\lim_{i\to\infty} span\{\hat{\phi}_i\}$ is dense in $\mathscr{L}^2(\partial\Omega)$.

For increasing M, ε_M is therefore a non-negative monotonic decreasing sequence converging to zero. □

Similar to theorem 6.17 yielding optimal solutions to (6.48) optimal solutions can be found for (6.49). Similar to definition 6.16 define the following finite dimensional space:

Definition 6.18 (Functions spaces X^*_M Y^*_M)**.** Let ϕ_n denote an orthonormal basis on $\mathscr{L}^2(\partial\Omega)$ and let w_n solve

$$\Delta w_n = 0 \qquad \mathbf{x} \in \Omega \backslash D, \tag{6.54}$$

$$w_n = 0 \qquad \mathbf{x} \in \partial D, \tag{6.55}$$

$$\partial_\nu w_n = \psi_n \qquad \mathbf{x} \in \partial\Omega. \tag{6.56}$$

Define the space Y^*_M and X^*_M for any $M > 0$ as

$$Y^*_M = \big\{\psi | \psi \in span\{\psi_1, \cdots, \psi_M\}\big\}, \qquad X^*_M = \big\{u | u \in span\{w_1, \cdots, w_M\}\big\}.$$

Definition 6.19. For $\tilde{\psi} \in \mathscr{L}^2(\partial D)$, a function $w \in X^*_M$ is an optimal solution in X^*_M of the Cauchy problem

$$\Delta w = 0 \qquad \mathbf{x} \in \Omega \backslash D, \tag{6.57}$$

$$w = 0 \qquad \mathbf{x} \in \partial D, \tag{6.58}$$

$$\partial_\nu w = \tilde{\psi} \qquad \mathbf{x} \in \partial D, \tag{6.59}$$

if w is the unique element in X^*_M with minimal distance of $\partial_\nu w$ to ψ in $\mathscr{L}^2(\partial D)$, i.e.

$$\|\partial_\nu w - \psi\|^2_{\mathscr{L}^2(\partial D)} = \inf_{\tilde{w} \in X^*_M} \|\partial_\nu \tilde{w} - \psi\|^2_{\mathscr{L}^2(\partial D)}.$$

Similar to Theorem 6.17 we prove the denseness result using optimal solutions.

Theorem 6.20. *Assume $\tilde{\psi} \in \mathscr{L}^2(\partial D)$. For each $M > 0$ there exist a unique $v_M \in X^*_M$ satisfying $v_M = 0$ for $\mathbf{x} \in \partial D$ which is the optimal solution in X^*_M of the Cauchy problem (6.57)-(6.59). The optimal solutions $v_M \in X^*_M$ satisfies that*

$$\lim_{M\to\infty} \|\partial_\nu v_M - \tilde{\psi}\|_{\mathscr{L}^2(\partial D)} = 0$$

.

Proof. Interchange the roles of $\tilde{\psi}$ and $\tilde{\phi}$ in the proof of theorem 6.17. □

Consider the two Cauchy problems (that generally doesn't have a solution in $H^2(\Omega\backslash D)$)

$$\begin{aligned} \Delta v_{N,M} &= 0 \quad \mathbf{x}\in\Omega\backslash D & \Delta v_{D,M} &= 0 \quad \mathbf{x}\in\Omega\backslash D \\ v_{N,M} &= \tilde\phi \quad \mathbf{x}\in\partial D & v_{D,M} &= 0 \quad \mathbf{x}\in\partial D \\ \tfrac{\partial^+}{\partial\nu} v_{N,M} &= 0 \quad \mathbf{x}\in\partial D & \tfrac{\partial^+}{\partial\nu} v_{D,M} &= \tilde\psi \quad \mathbf{x}\in\partial D. \end{aligned} \tag{6.60}$$

where $v_{N,M}$ and $v_{D,M}$ are optimal solutions as in definition 6.16 and 6.19.

For v solving (6.42), an optimal decomposition into a sum of functions in X_M^* and X_M is $v_M = v_{N,M} + v_{D,M}$. This function, v_M, will not solve (6.42), however it will be a function that in $\mathscr{L}^2(\partial D)$, for increasing M, minimizes the $\mathscr{L}^2$ difference between v_M and $\tilde\phi$ and the $\mathscr{L}^2$ difference between $\partial_\nu v_M$ and $\tilde\psi$.

For M fixed the functions $v_{D,M}$, are harmonic in $\Omega\backslash D$, with one homogenous boundary condition. From Green's formula therefore follows the reduced inequality (6.61).

Lemma 6.21. *If u is a fixed harmonic function in $\Omega\backslash D$ then for $\delta > 0$ there exist an $M_0 > 0$ such that the optimal solution, $v_{D,M}$, as defined in (6.60), minimizing $\|\partial_\nu v_{D,M} - \tilde\psi\|_{\mathscr{L}^2(\partial D)}$ for $\tilde\psi \in \mathscr{L}^2(\partial D)$, satisfies*

$$\left|\langle u,\tilde\psi\rangle_{\partial D} - \langle u,\partial_\nu v_{D,M}\rangle_{\partial\Omega} + \langle\partial_\nu u, v_{D,M}\rangle_{\partial\Omega}\right| < \delta, \qquad \text{for} \quad M > M_0. \tag{6.61}$$

Since $\tilde\psi \in \mathscr{L}^2(\partial D)$ is fixed, a corollary of lemma 6.21 is the convergens of

$$-\langle u,\partial_\nu v_{D,M}\rangle_{\partial\Omega} + \langle\partial_\nu u, v_{D,M}\rangle_{\partial\Omega}. \tag{6.62}$$

Corollary 6.22. *If u is harmonic in $\Omega\backslash D$ and $v_{D,M}$ are optimal solutions, as defined in (6.60), for $\tilde\psi \in \mathscr{L}^2(\partial D)$, then (6.62) converges in $\mathbb{R}$ for $M \to \infty$.*

The boundary integrals in (6.62) are over the opposite boundary of where the splitting in (6.60) is constructed. Besides the convergence of (6.62) also the individual terms herein may converge if further space restrictions are imposed. This is done firstly by considering some special finite dimensional spaces. These spaces will also allow us to prove a continuous dependence in $\mathscr{L}^2(\partial\Omega)$ of the limit of the individual terms in (6.62) upon $\partial_\nu u\big|_{\partial\Omega}$. For the Inverse Conductivity Problem the continuity of the individual terms are attained by assuming that σ_i does not attain an extremal value (in $\mathbb{R}_+$), and that $\partial_\nu u$ lies in a specified finite dimensional space.

6.2.2.1. *Convergence and Continuity of Some Boundary Integrals of Optimal Solutions.* In order to show convergence and continuity of, say $\langle u,\partial_\nu v_{D,M}\rangle_{\partial\Omega}$, it is advantageous for technical reasons, to introduce yet another finite dimensional space.

Definition 6.23 (Function spaces X_M^1 Y_M^1)**.** Let ϕ_n denote an orthonormal basis on $\mathscr{L}^2(\partial D)$, and let $\tilde u_n \in H^2(\Omega\backslash D)$ solve

$$\Delta\tilde u_n = 0 \qquad \mathbf{x}\in\Omega\backslash D, \tag{6.63}$$

$$\tilde u_n = 0 \qquad \mathbf{x}\in\partial\Omega, \tag{6.64}$$

$$\tilde u_n = \phi_n \qquad \mathbf{x}\in\partial D. \tag{6.65}$$

Denote $\tilde{\psi}_n = \partial_\nu \tilde{u}_n(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$ and define

$$Y_n^1 = \left\{ \psi \middle| \psi \in span\{\tilde{\psi}_1, \ldots, \tilde{\psi}_n\} \right\}, \quad X_n^1 = \left\{ u \middle| u \in span\{\tilde{u}_1, \cdots, \tilde{u}_n\} \right\}.$$

Again, the function $\tilde{\psi}_n$ are linear independent, because of unique continuation for harmonic functions. The space X^* defined in (6.43) is the space of harmonic functions for which $u\big|_{\partial\Omega} = 0$. The normal derivative at $\partial\Omega$ of all functions in X^* contains Y_M^1. The space X^* is natural to consider when seeking convergence of the individual terms from lemma 6.21.

Lemma 6.24. *Let $v_{D,M}$ and $v_{N,M}$ be optimal solutions as defined in (6.60). For $\tilde{u} \in X^*$, the sequence of boundary integral $\langle \partial_\nu \tilde{u}, v_{D,M} \rangle_{\partial\Omega}$ converges in $\mathbb{R}$ for $M \to \infty$ to $\langle \tilde{u}, \tilde{\psi} \rangle_{\partial D}$.*

Proof. From Green's formula follows that for $v_{D,M}$, as defined in (6.60), and for $\tilde{u} \in X^*$,

$$\langle \partial_\nu \tilde{u}, v_{D,M} \rangle_{\partial\Omega} = \langle \tilde{u}, \partial_\nu v_{D,M} \rangle_{\partial D}.$$

Since $\|\partial_\nu v_{D,M} - \tilde{\psi}\|_{\mathscr{L}^2(\partial D)} = \varepsilon_{1,M}$, and $\varepsilon_{1,M} \searrow 0$, for fixed $\tilde{u} \in X^*$, the boundary integral $\langle \partial_\nu \tilde{u}, v_{D,M} \rangle_{\partial\Omega}$ converges in $\mathbb{R}$ to $\langle \tilde{u}, \tilde{\psi} \rangle_{\partial D}$. □

Lemma 6.25. *Let $v_{D,M}$ and $v_{N,M}$ be optimal solutions as defined in (6.60). Let $\psi \in Y_n^1 \subset \mathscr{L}^2(\partial\Omega)$ and let u be any harmonic function in $\Omega \backslash D$ with $\partial_\nu u\big|_{\partial\Omega} = \psi$. The limits*

$$B := \lim_{M\to\infty} \langle \partial_\nu u, v_{D,M} \rangle_{\partial\Omega} \quad \text{and} \quad D := \lim_{M\to\infty} \langle \partial_\nu u, v_{N,M} \rangle_{\partial\Omega} \tag{6.66}$$

are continuous from Y_n^1 to $\mathbb{R}$, when Y_n^1 is equipped with the topology of $\mathscr{L}^2$.

Proof. If $\partial_\nu u \in Y_n^1$ then there exist a unique element $\tilde{u} \in X_n^1$ solving

$$\begin{aligned} \Delta \tilde{u} &= 0 && \mathbf{x} \in \Omega \backslash D \\ \tilde{u} &= 0 && \mathbf{x} \in \partial\Omega \\ \partial_\nu \tilde{u} &= \partial_\nu u && \mathbf{x} \in \partial\Omega \end{aligned}$$

Since both B and D does not depend on $u\big|_{\partial\Omega}$ it is sufficient to prove (6.66) for $\tilde{u}$.

Consider first B. Since $\tilde{u} \in X^*$, we have from lemma 6.24 that B converges to $\langle \tilde{u}, \tilde{\psi} \rangle_{\partial D}$.

For $\partial_\nu \tilde{u} \in Y_n^1$, there exist a unique n-tuple such that at $\mathbf{x} \in \partial\Omega$,

$$\partial_\nu \tilde{u} = \sum_{i=1}^{n} a_i \tilde{\psi}_i, \qquad \text{for } \tilde{\psi}_i \in Y_n^1,$$

where the $\tilde{\psi}_i$ are linear independent in $\mathscr{L}^2(\partial\Omega)$. Therefore $a_i \in \mathbb{R}$ depends continuously on $\partial_\nu \tilde{u} \in Y_n^1 \subset \mathscr{L}^2(\partial\Omega)$. Now, Y_n^1 is a finite dimensional space and,

$$\tilde{u}\big|_{\partial D} = \sum_{i=1}^{n} a_i u\big|_{\partial D} \qquad \text{and} \qquad \partial_\nu \tilde{u}\big|_{\partial D} = \sum_{i=1}^{n} a_i \partial_\nu u\big|_{\partial D} \tag{6.67}$$

are both finite series of $\mathscr{L}^2(\partial D)$ functions. Hence $\tilde{u}\big|_{\partial D}$ depends continuously in $\mathscr{L}^2(\partial D)$ on $\partial_\nu \tilde{u}\big|_{\partial\Omega} \in Y_n^1$. Since $\tilde{\psi} \in \mathscr{L}^2(\partial D)$ is fixed, B depends continuously on $\partial_\nu \tilde{u}\big|_{\partial\Omega} \in Y_n^1$ in the $\mathscr{L}^2$ topology.

The proof for D is similar. □

Remark. For each fixed M, the optimal solutions $v_{N,M}$ (respectively $v_{D,M}$) are harmonic functions. Since $v_{D,M}$ does not converge in $\mathscr{L}^2(\partial\Omega)$ the limit of these optimal solutions does in general, not define semi-harmonic functions. This also means that the continuity argument of lemma 6.25 can not be extended from Y_n^1 to $\mathscr{L}^2(\partial\Omega)$. Also, if this was indeed the case, then the Cauchy problem (6.42) would always have a solution.

6.2.3. Denseness of Solutions to the Conductivity Equation. The last lemma is relating the results for the Cauchy problem to the Conductivity problem.

Lemma 6.26. *Let* $\lambda = \lambda_0\chi_{\Omega\backslash D} + \lambda_1\chi_D$ *be fixed with* $\lambda_i \in \mathbb{R}_+$. *For* $\varepsilon > 0$ *and* $\psi \in \mathscr{L}_0^2(\partial D)$ *there exist a* $w \in H^2(\Omega\backslash\partial D) \cap H^1(\Omega)$ *solving*

$$\nabla\cdot\lambda\nabla w = 0 \qquad \mathbf{x} \in \Omega, \tag{6.68}$$

satisfying

$$\|\partial_\nu^+ w - \psi\|_{\mathscr{L}^2(\partial D)} < \varepsilon.$$

Proof. Following the definition of optimal solutions there exist $v_{2,M} \in X_M^*$ solving

$$\Delta v_{2,M} = 0 \qquad \mathbf{x} \in \Omega\backslash D, \tag{6.69}$$

$$v_{2,M} = 0 \qquad \mathbf{x} \in \partial D, \tag{6.70}$$

$$\partial_\nu v_{2,M} = \psi \qquad \mathbf{x} \in \partial D, \tag{6.71}$$

with $\|\partial_\nu v_{2,M} - \psi\|_{\mathscr{L}^2(\partial D)} < \varepsilon_{2,M}$ and $\varepsilon_{2,M} \searrow 0$ for increasing M.

Define for $\mathbf{x} \in D$ the function $v_{1,M}$ as

$$\Delta v_{1,M} = 0 \qquad \mathbf{x} \in D, \tag{6.72}$$

$$\partial_\nu^- v_{1,M} = \frac{\lambda_1}{\lambda_0}\partial_\nu^+ v_{2,M} \qquad \mathbf{x} \in \partial D. \tag{6.73}$$

For $\mathbf{x} \in \Omega\backslash D$ define $v_{1,M} \in H^2(\Omega\backslash D)$ as the solution of

$$\Delta v_{1,M} = 0 \qquad \mathbf{x} \in \Omega\backslash D, \tag{6.74}$$

$$v_{1,M}^+ = v_{1,M}^- \qquad \mathbf{x} \in \partial D, \tag{6.75}$$

$$v_{1,M} = 0 \qquad \mathbf{x} \in \partial\Omega, \tag{6.76}$$

and denote $\tilde{\psi} = \partial_\nu^+ v_{1,M}$ for $\mathbf{x} \in \partial D$.

Following the definition of optimal solutions for $M_1 > 0$, there exist $\tilde{v}_{1,M_1} \in X^*_{M_1}$ solving

$$\Delta\tilde{v}_{1,M_1} = 0 \qquad \mathbf{x} \in \Omega\backslash D, \tag{6.77}$$

$$\partial_\nu\tilde{v}_{1,M_1} = -\tilde{\psi} \qquad \mathbf{x} \in \partial D, \tag{6.78}$$

with $\|\partial_\nu v_{1,M_1} + \tilde{\psi}\|_{\mathscr{L}^2(\partial D)} < \tilde{\varepsilon}_{1,M_1}$. Denote $v_{1,M,M_1} = v_{1,M} + \tilde{v}_{1,M_1}\chi_{\Omega\backslash D}$, then v_{1,M,M_1} satisfies by construction

$$\|\partial_\nu^+ v_{1,M,M_1}\|_{\mathscr{L}^2(\partial D)} < \tilde{\varepsilon}_{1,M} \qquad \text{and} \qquad \|v^+_{1,M,M_1} - v^-_{1,M,M_1}\|_{\mathscr{L}^2(\partial D)} = 0.$$

Assume w solving $\nabla\cdot\lambda\nabla w = 0$ is given as

$$w := v_{1,M,M_1} + v_{2,M}\chi_{\Omega\backslash D} + \mathbf{S}_{[D,\Gamma]}\xi(\mathbf{y}),$$

where Γ is the fundamental solution for Laplace equation, $\xi \in \mathscr{L}^2(\partial D)$, and $\mathbf{S}_{[D,\Gamma]}$ is a single layer potential as defined in (6.15).

Using the same algebra as in the proof of theorem 6.4 for the normal derivative of w, the function w solves $\nabla\cdot\lambda\nabla w = 0$ if and only if ξ solves

$$\big(I - \frac{2(\lambda_0 - \lambda_1)}{\lambda_0 + \lambda_1}\mathbf{K}^*_{[D,\Gamma]}\big)\xi = \frac{2\lambda_0}{\lambda_0 + \lambda_1}\partial_\nu^+ v_{1,M,M_1} = \eta \tag{6.79}$$

Since the spectrum of $\mathbf{K}^*_{[D,\Gamma]}$ is $(-2,2)$ (given on page 76), the density ξ satisfies that $\|\xi\|_{\mathscr{L}^2(\partial D)} \le C\|\eta\|_{\mathscr{L}^2(\partial D)}$. From (6.19) follows that $\|\partial_\nu \mathbf{S}_{[D,\Gamma]}\xi\|_{\mathscr{L}^2(\partial D)} \le C_1\|\partial_\nu^+ v_{1,M,M_1}\|_{\mathscr{L}^2(\partial D)}$. The estimate

$$\|\partial_\nu^+ w - \psi\|^2_{\mathscr{L}^2(\partial D)} \le \|\partial_\nu v_{2,M} - \psi\|^2_{\mathscr{L}^2(\partial D)} + \|\partial_\nu v_{1,M,M_1}\|^2_{\mathscr{L}^2(\partial D)} + C_2\|\xi\|^2_{\mathscr{L}^2(\partial D)} < \varepsilon$$

shows that for any $\varepsilon > 0$, there exist an M_0, such that for an $M > M_0$ and for all $M_1(M) > 0$, the inequality $\|\partial_\nu w - \psi\|^2_{\mathscr{L}^2(\partial D} < \varepsilon$ holds. □

6.3. Uniqueness and Continuous Dependence of σ on Boundary Data

6.3.1. Two Nested Domains with Unknown Conductivity Constants. In theorem 6.12 the fundamental algebraic relation between two semi-harmonic functions was derived as the projection of one onto the other. Consider subsequently two domains $\Omega\backslash D$, and D, with unknown conductivities σ_0, and σ_1, respectively. Assume $v(\mathbf{x})$ is any semi-harmonic function solving (6.3) with conductivities $\lambda = \lambda_0\chi_{\Omega\backslash D} + \lambda_1\chi_D$. We will assume that $\sigma_0, \sigma_1 \in [\delta_0, N]$ for some $\delta_0 > 0$ and $N < \infty$.

For $\psi \in Y^1_n$ let u be a solution of $P[\sigma, D, \psi]$. Assume $u\big|_{\partial\Omega} = \phi$ and denote $Y^{data} = \{(\phi,\psi)\}$. Define the space U of semi-harmonic functions having these traces as

$$U := \left\{ u \;\middle|\; \begin{array}{l} \exists\, \sigma_0, \sigma_1, \sigma = (\sigma_0 \chi_{\Omega \setminus D} + \sigma_1 \chi_D), \\ \nabla \cdot \sigma \nabla u = 0, \mathbf{x} \in \Omega, \\ u = \phi,\ \sigma_0 \partial_\nu^- u = \psi, \quad (\phi, \psi) \in Y^{data} \end{array} \right\}.$$

The formula from theorem 6.12 reduces to

$$\left(\left(\frac{\sigma_1}{\lambda_1} \right) - \left(\frac{\sigma_0}{\lambda_0} \right) \right) \langle u, j_v \rangle_{\partial D} = \sigma_0 \langle u, \partial_\nu v^- \rangle_{\partial\Omega} - \langle v, \sigma_0 \partial_\nu u^- \rangle_{\partial\Omega}, \tag{6.80}$$

for $u \in U$ and v any known semi-harmonic function defined as above. Through the use of results from section 6.2.2 for solving the Cauchy problem

$$\begin{aligned} \Delta v &= 0 && \mathbf{x} \in \Omega \setminus D, \\ v &= \tilde{\phi}(\mathbf{x}) && \mathbf{x} \in \partial D, \\ \frac{\partial v}{\partial \nu} &= \tilde{\psi}(\mathbf{x}) && \mathbf{x} \in \partial D, \end{aligned} \tag{6.81}$$

the unknown boundary integral $\langle u, j_v \rangle_{\partial D}$ in (6.80) can be rewritten in terms of known boundary integrals on $\partial\Omega$.

Let $v_{D,M}$ and $v_{N,M}$ be optimal solution in X_M^* and X_M respectively of

$$\begin{array}{rclll} \Delta v_{N,M} &=& 0 & \mathbf{x} \in \Omega \setminus D & \qquad \Delta v_{D,M} = 0 \quad \mathbf{x} \in \Omega \setminus D, \\ v_{N,M} &=& \tilde{\phi} & \mathbf{x} \in \partial D & \qquad v_{D,M} = 0 \quad \mathbf{x} \in \partial D, \\ \frac{\partial^+}{\partial \nu} v_{N,M} &=& 0 & \mathbf{x} \in \partial D & \qquad \lambda_0 \frac{\partial^+}{\partial \nu} v_{D,M} = \tilde{\psi} \quad \mathbf{x} \in \partial D. \end{array} \tag{6.82}$$

The optimal solutions satisfies that $\|\lambda_0 \partial_\nu v_{D,M} - \tilde{\psi}\|_{\mathscr{L}^2(\partial D)} = \varepsilon_{1,M}$ and $v_{D,M} = 0$, and that $\|v_{N,M} - \tilde{\phi}\|_{\mathscr{L}^2(\partial D)} = \varepsilon_{2,M}$ and $\partial_\nu v_{N,M} = 0$, respectively. Denote

$$\begin{array}{llll} A_M &= \langle \sigma_0 \partial_\nu u, v_{D,M} \rangle_{\partial\Omega}, & B_M &= \langle u, \partial_\nu v_{D,M} \rangle_{\partial\Omega}, \\ C_M &= \langle \sigma_0 \partial_\nu u, v_{N,M} \rangle_{\partial\Omega}, & D_M &= \langle u, \partial_\nu v_{N,M} \rangle_{\partial\Omega}, \\ \delta_{1,M} &= \langle \partial_\nu u, v_{N,M} - \tilde{\phi} \rangle_{\partial D}, & \delta_{2,M} &= \langle u, \lambda_0 \partial_\nu v_{D,M} - \tilde{\psi} \rangle_{\partial D}. \end{array} \tag{6.83}$$

Lemma 6.27. *Assume $\psi \in Y_n^1$, that $\sigma_0, \sigma_1 \in [\delta_0, N]$, and u is a solution of $P[\sigma, D, \psi]$. Then the boundary integrals $A_M, \ldots, D_M$, as defined above, converges for $M \to \infty$ and fixed ψ. The limits of these boundary integrals depends continuously in $\mathscr{L}^2$ upon $\psi \in Y_n^1$.*

Proof. From lemma 6.25 follows that B_M converges and depends continuously on $\sigma_0 \partial_\nu u \in Y_n^1$ in the $\mathscr{L}^2$-topology.

From corollary 6.21 and the boundedness estimate of lemma 6.5 follows that $A_M - \sigma_0 B_M$ converges and depends continuously on $\sigma_0 \partial_\nu u \in \mathscr{L}^2(\partial\Omega)$. In particular for $\sigma_0 \partial_\nu u \in Y_M^1$ then A_M and B_M converges, and their limits depend continuously on $\sigma_0 \partial_\nu u \in Y_M^1$ in the $\mathscr{L}^2$ norm.

A similar argument proves that C_M converges and that the map from ψ is continuous with respect to the induced norm. □

Remark. The boundedness of u in σ_0, σ_1 aids A_M and B_M being convergent sequences, and having continuous limits. For the results of section 6.2.2.1 a similar boundedness argument is not easy and was there substituted through the introduction of the finite dimensional space X_n^1. This has made it necessary here to assume $\sigma_0 \partial_\nu u = \psi \in Y_M^1$ for $\mathbf{x} \in \partial\Omega$.

Theorem 6.28. *For $\psi \in Y_n^1$ and $\sigma = \sigma_0 \chi_{\Omega \setminus D} + \sigma_1 \chi_D$ with $\sigma_0, \sigma_1 \in [\delta_0, N]$, let u be the solution of $P[\sigma, D, \psi]$. For each semi-harmonic function v with $\lambda = \lambda_0 \chi_{\Omega \setminus D} + \lambda_1 \chi_D$ there exist boundary integrals $A_M, \cdots, D_M$ with convergent limits $A, \cdots, D$. The limits for $M \to \infty$ depends continuously on $\psi \in Y_n^1 \subset \mathscr{L}^2(\partial\Omega)$ with the induced topology, and satisfy that*

$$\frac{\sigma_1 \lambda_0}{\sigma_0 \lambda_1}(-C + \sigma_0 D) = A - \sigma_0 B \tag{6.84}$$

Proof. Denote $\psi_1 = \lambda_0 \partial_\nu^+ v$ and let $v_{N,M}$ and $v_{D,M}$ be defined as optimal solutions of (6.82). Repeated applications of Green's formula for fixed M yields that

$$\frac{\sigma_1}{\lambda_1}\langle u, \psi_1 \rangle_{\partial D} = \langle \sigma_0 \partial_\nu u, v_{N,M} \rangle_{\partial\Omega} - \sigma_0 \langle u, \partial_\nu v_{N,M} \rangle_{\partial\Omega} - 1/\lambda_0 \delta_{1,M} \tag{6.85}$$

$$\frac{\sigma_0}{\lambda_0}\langle u, \psi_1 \rangle_{\partial D} = -\langle \sigma_0 \partial_\nu u, v_{D,M} \rangle_{\partial\Omega} + \sigma_0 \langle u, \partial_\nu v_{D,M} \rangle_{\partial\Omega} - 1/\lambda_0 \delta_{2,M} \tag{6.86}$$

with $\delta_{1,M}$ and $\delta_{2,M}$ as defined in (6.83). From lemma 6.27 follows that $A_M, \cdots, D_M$ converges and depends continuously on boundary data ψ. □

Futhermore for any finite M the continuity of $A_M, \cdots, D_M$ on ψ and $u|_{\partial\Omega}$ follows from the continuity of the $\mathscr{L}^2$ scalar product.

Equation (6.84) is the central expression from which to reconstruct σ_0 and σ_1 for one experiment. The fundamental relationship between two semi-harmonic functions (6.80) may also readily be obtained from (6.84). However, from the fundamental relationship it is only through the advantageous definition of optimal solutions that the unknown boundary integral can be analyzed.

Since (6.84) contains both unknowns, two different functions satisfying $\nabla \cdot \lambda \nabla w_i = 0$, $i = 1, 2$ are needed, thereby establishing two nonlinear equations in two unknowns. The solution hereof will be the objective for the remaining of this section.

To fix ideas. Let $v_{N,M,i}$ and $v_{D,M,i}$, $i = 1, 2$ be optimal solutions as defined in (6.82) and such that $v_i = v_{N,M,i} + v_{D,M,i}$ solves $\Delta v_i = 0$ for $\Omega \setminus \partial D$ and $v_i^+ \approx v_i^-$ and $\lambda_0 \partial_\nu^+ v_i \approx \lambda_1 \partial_\nu^- v_i$ for $\mathbf{x} \in \partial D$. The limits of the functions $v_{N,M,i}$ and $v_{D,M,i}$ hence defines

$$\frac{\lambda_0 \sigma_1}{\lambda_1 \sigma_0}(-C_1 + \sigma_0 D_1) = A_1 - \sigma_0 B_1 \tag{6.87}$$

and

$$\frac{\lambda_0 \sigma_0}{\lambda_1 \sigma_1}(-C_2 + \sigma_0 D_2) = A_2 - \sigma_0 B_2. \tag{6.88}$$

To discuss solvability of (6.87) and (6.88) for σ_0 and σ_1, it is necessary to define some mild conditions for the data.

Definition 6.29. For $u \in U$ the triplet $\{D, u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}\}$ is admissible if u_2 solving

$$\Delta u_2 = 0 \qquad \mathbf{x} \in \Omega \backslash D, \tag{6.89}$$

$$u_2 = 0 \qquad \mathbf{x} \in \partial D, \tag{6.90}$$

$$u_2(\mathbf{x}) = u(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega, \tag{6.91}$$

satisfies that $\partial_\nu u_2|_{\partial\Omega} \not\propto \partial_\nu u|_{\partial\Omega}$.

The condition on u_2 in definition 6.29 for admissibility is a condition to guaranty that in (6.87) there exist a function v solving $\nabla \cdot \lambda \nabla v = 0$ such that $A_1, B_1 \neq 0$, but such that (albeit σ_0 is unknown) $A_1 - \sigma_0 B_1 = 0$.

Lemma 6.30. *Assume $u \in U$ and that the triplet $\{D, u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}\}$ is admissible. There exist a v satisfying*

$$\Delta v = 0 \qquad \mathbf{x} \in \Omega \backslash D, \tag{6.92}$$

$$v = 0 \qquad \mathbf{x} \in \partial D, \tag{6.93}$$

and such that

$$\langle u, \partial_\nu v(\mathbf{x}) \rangle_{\partial D} = 0, \tag{6.94}$$

$$\int_{\Omega \backslash D} \nabla u \cdot \nabla v \, d\mathbf{x} \neq 0. \tag{6.95}$$

For $\lambda = \lambda_0 \chi_{\Omega \backslash D} + \lambda_1 \chi_D$ with $\lambda_i \in \mathbb{R}_+$ there exist a semi-harmonic function v_1 in Ω such that for any $\varepsilon > 0$

$$\|\partial_\nu (v_1(\mathbf{x}) - v(\mathbf{x}))\|_{\mathscr{L}^2(\partial D)} < \varepsilon.$$

Proof. Decompose $u \in U$ as $u\chi_{\Omega \backslash D} = u_1 \chi_{\Omega \backslash D} + u_2 \chi_{\Omega \backslash D}$ where the u_i are harmonic in $\Omega \backslash D$ and solves

$$u_1 = u \quad \mathbf{x} \in \partial D, \qquad u_2 = 0 \quad \mathbf{x} \in \partial D, \tag{6.96}$$

$$u_1 = 0 \quad \mathbf{x} \in \partial\Omega, \qquad u_2 = u \quad \mathbf{x} \in \partial\Omega. \tag{6.97}$$

For any v satisfying (6.92) and (6.93), Green's formula readily yields the identities

$$\langle u, \partial_\nu v \rangle_{\partial D} = \langle u_1 \partial_\nu v \rangle_{\partial D} = \langle \partial_\nu u_1, v \rangle_{\partial\Omega}, \tag{6.98}$$

$$\langle u, \partial_\nu v \rangle_{\partial\Omega} = \langle u_2, \partial_\nu v \rangle_{\partial\Omega} = \langle \partial_\nu u_2, v \rangle_{\partial\Omega}. \tag{6.99}$$

For ν denoting the outward normal to $\Omega \backslash D$

$$\int_{\Omega \backslash D} \nabla u \cdot \nabla v d\mathbf{x} = \langle u, \partial_\nu v \rangle_{\partial D} + \langle u, \partial_\nu v \rangle_{\partial\Omega} \tag{6.100}$$

$$= \langle \partial_\nu u_1, v \rangle_{\partial\Omega} + \langle \partial_\nu u_2, v \rangle_{\partial\Omega} \tag{6.101}$$

Since u is admissible $\partial_\nu(u_1+u_2)\big|_{\partial\Omega} \not\propto \partial_\nu u_1\big|_{\partial\Omega}$. Then, $\partial_\nu u_2\big|_{\partial\Omega} \not\propto \partial_\nu u_1\big|_{\partial\Omega}$ and in the orthogonal complement (in $\mathscr{L}^2(\partial\Omega)$) to $\partial_\nu u_1$, there exist an element $\psi^\perp$ such that

$$\langle \partial_\nu u_1, \psi^\perp\rangle_{\partial\Omega} = 0 \qquad \text{and} \qquad \langle \partial_\nu u_2, \psi^\perp\rangle_{\partial\Omega} \neq 0,$$

are satisfied. Define $\tilde{v}$ as solving the boundary value problem consisting of (6.92), (6.93), and $\tilde{v}\big|_{\partial\Omega} = \psi^\perp$. This $\tilde{v}$ satisfies (6.95) and from (6.98) follows that $\tilde{v}$ also satisfies (6.94).

The denseness result follow from lemma 6.26. ☐

We may now prove the main uniqueness theorem for two inhomogeneities.

Theorem 6.31. *Let the triplet $\{D, u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}\}$ be admissible for $\psi \in Y_n^1$, and u the solution of $P[\sigma, D, \psi]$ with $\sigma_0, \sigma_1 \in]\delta_0, N[$. If $A, \ldots, D$ denote limits of decompositions, as in (6.82) of harmonic functions solving $\Delta v = 0$ in Ω, then either:*

(a) *$[A, B] \propto [C, D]$ for all v harmonic in Ω, and then $\{\sigma_0, \sigma_1\}$ is uniquely determined, or*

(b) *There exist a harmonic function v in Ω, such that $[A, B] \not\propto [C, D]$, in which case there exist 2 pairs of $\{\sigma_0, \sigma_1\}$ solving (6.87)-(6.88).*

For both (a) and (b), the coefficients $\{\sigma_0, \sigma_1\}$ depends continuously on $\psi \in Y_n^1$ in the induced topology from $\mathscr{L}^2$.

Proof. Assume the coefficients are proportional for all v solving $\Delta v = 0$, i.e. that $[A, B] \propto [C, D]$. For v_2 a harmonic function in Ω and satisfying

$$\langle u, \partial_\nu v_2\rangle_{\partial D} \neq 0, \tag{6.102}$$

denote $A_2, \ldots, D_2$ the optimal solutions related to the decomposition (6.82) of v_2.

Since $[A_2, B_2] \propto [C_2, D_2]$, with ratio γ_1, then (6.88) defines $\sigma_0/\sigma_1 = \gamma_1^{-1}$. From (6.80) follows that if $v_3 \in U$ with $\lambda_0\backslash\lambda_1 = \gamma_1$ (λ_i known) then

$$0 = \sigma_0\langle u, \partial_\nu v_3^-\rangle_{\partial\Omega} - \langle v_3, \sigma_0\partial_\nu u^-\rangle_{\partial\Omega}.$$

By choosing $\langle u, \partial_\nu v_3\rangle_{\partial\Omega} \neq 0$, σ_0 gets uniquely determined. Hence $\{\sigma_0, \sigma_1\}$ gets uniquely determined. This concludes (a).

Let v_2 be the harmonic function in Ω such that $[A_2, B_2] \not\propto [C_2, D_2]$.

From lemma 6.30 follows that there always exist a harmonic function v in Ω, such that

$$\langle u, \partial_\nu v\rangle_{\partial D} = 0,$$

and for the limit of optimal functions $v_{D,M}$ as in (6.82)

$$A = \lim_{M\to\infty}\langle \partial_\nu u, v_{D,M}\rangle_{\partial\Omega} \neq 0 \qquad \text{and} \qquad B = \lim_{M\to\infty}\langle u, \partial_\nu v_{D,M}\rangle_{\partial\Omega} \neq 0.$$

Then $B_1\sigma_0 = A_1$ and $\sigma_0 C_1 = D_1$ for this v. Therefore the coefficients are linear dependent, say $\gamma[A_1, B_1] = [C_1, D_1]$.

The two equations (6.87)-(6.88) then reduces, by insertion, to the polynomial

$$0 = (-A_1 + \sigma_0 B_1)((C_2 - \sigma_0 D_2) - \gamma(A_2 - \sigma_0 B_2)). \qquad (6.103)$$

By selection $A_1 \neq 0 \neq B_1$, and $[A_2, B_2] \not\propto [C_2, D_2]$. The polynomial therefore is not the zero polynomial, and has at most two distinct solutions σ_0. For each σ_0, with v_2 satisfying (6.102), σ_1 gets uniquely determined. This proves (b).

Since $A, \ldots, D$ depends continuously on $\partial_\nu u \in Y_n^1$, by construction so does $\{\sigma_0, \sigma_1\}$. □

As a corollary we emphasize when the coupled equations (6.87)-(6.88) may define an proper polynomial, i.e. are not just proportional.

Corollary 6.32. *Let the triplet $\{D, u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}\}$ be admissible for $\psi \in Y_M^1$, $\sigma_0, \sigma_1 \in]\delta_0, N]$, and u the solution of $P[\sigma, D, \psi]$. If $A, \ldots, D$ denote limits of decompositions as in (6.82) of harmonic functions solving $\Delta v = 0$, and $[A, B] \not\propto [C, D]$ for one harmonic function, then there exist another harmonic function such that the polynomial $\alpha\sigma_0^2 + \beta\sigma_0 + \gamma = 0$ with*

$$\alpha = D_2 B_1 - B_2 D_1, \qquad \beta = C_2 A_1 - A_2 C_1, \qquad \gamma = (C_2 B_1 - D_2 A_1 - C_1 B_2 - D_1 A_2),$$

is not the zero polynomial.

Theorem 6.33. *Let the triplet $\{D, u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}\}$ be admissible for $\psi \in Y_M^1$, $\sigma_0, \sigma_1 \in]\delta_0, N]$, and u the solution of $P[\sigma, D, \psi]$. If $A, \ldots, D$ denote limits of decompositions as in (6.82) of harmonic functions solving $\Delta v = 0$, that satisfy*

$$\gamma[A_1, B_1] = [C_1, D_1] \text{ and } \gamma_1[A_2, B_2] = [C_2, D_2] \qquad (6.104)$$

where $\gamma \neq \gamma_1$, then $\alpha\sigma_0^2 + \beta\sigma_0 + \gamma = 0$ defines a proper polynomial.

Proof. If the constants of proportionality are different, then at least one of the coefficients α or β will be nonzero. □

6.3.2. IP2 with σ_0 known. Assume furthermore that σ_0 is known for IP2, then there exist an explicit boundary map that uniquely determines σ_1. The well known non-constructive uniqueness result also applies:

Theorem 6.34. *Let $\sigma_i \in \mathbb{R}_+$, $D = \cup_i D_i$ with $D_i \subset \Omega$ disjoint sets with $\partial D_i \in C^2$ and $\psi \in \mathscr{L}^2(\partial\Omega)$. If u_i solves $P[\sigma_i, D, \psi]$, $i = 1, 2$ and $u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega}$ then $\sigma_1 = \sigma_2$.*

Proof. That $u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega}$ implies that there exist a semi-harmonic function in Ω, such that $u\big|_{\partial\Omega} = 0 = \partial_\nu u\big|_{\partial\Omega}$. From unique continuation of harmonic functions follows that $u = 0$ in $\Omega \backslash D$. Therefore since u is semi-harmonic, $\partial_\nu u_1^-\big|_{\partial D} = \sigma_2/\sigma_1 \partial_\nu u_2^-\big|_{\partial D}$, and $u^+\big|_{\partial D} = u^-\big|_{\partial D}$. From the maximum principle follows that since $u_1 = u_2$ on ∂D then $u_1 = u_2$ on D. Hence $\sigma_2/\sigma_1 = 1$. □

In sections 6.1.1 different expressions for $u(\mathbf{x})$ solving (6.5)-(6.9) was described. Using integral equations u was expressed as $u = H + \mathbf{S}\xi$ where H was either (6.29) or $u^N = \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)$ and ξ solved (6.22) or (6.30), respectively. Denote $\phi^N = \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)\big|_{\partial\Omega}$ and $(\mathcal{S}_{[D,\cdot]}\xi)(\mathbf{x}) = \mathbf{S}_{[D,\cdot]}\xi$ for $\mathbf{x} \in \partial\Omega$. The restriction of (6.21) and (6.28) to $\partial\Omega$ are found to be,

$$\mathcal{S}_{[D,G]}\xi = \phi - \phi^n \qquad \mathbf{x} \in \partial\Omega \tag{6.105}$$

$$\mathcal{S}_{[D,\Gamma]}\xi = \frac{1}{2}(\phi - \phi^n) + \mathbf{K}^*_{[\Omega,\Gamma]}(\phi - \phi^n) \qquad \mathbf{x} \in \partial\Omega. \tag{6.106}$$

These boundary maps are ill-posed from $\phi - \phi^N$ to ξ, since $S_{[D,G]}$, respectively $S_{[D,\Gamma]}$, is compact operators on $\mathscr{L}^2(\partial\Omega)$.

Theorem 6.35. *For $\sigma = \sigma_0\chi_{\Omega\setminus D} + \sigma_1\chi_D$ and $\psi \in \mathscr{L}^2_0(\partial D)$, let $\{\phi, \psi\}$ be Cauchy data for u solving $P[\sigma, D, \psi]$. If $u = \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi) + \mathbf{S}_{[D,G]}\xi$ then $\sigma_0\partial_\nu u = \psi$ for $\mathbf{x} \in \partial\Omega$ if and only if $\int_{\partial D}\xi = 0$. There exists a unique $\xi \in \mathscr{L}^2(\partial D)$ with $\int_{\partial D}\xi = 0$ solving $\phi - \mathcal{S}_{[\Omega,G]}(\psi) = \mathcal{S}_{[D,G]}\xi$, and σ_1 is given as the unique solution of*

$$\frac{\sigma_1 + \sigma_0}{2(\sigma_1 - \sigma_0)}\xi = \mathbf{K}^*_{[D,G]}\xi + \frac{\partial}{\partial\nu}\mathbf{S}_{[\Omega,G]}(\psi) \qquad \mathbf{x} \in \partial D \tag{6.107}$$

Proof. Let $h(\mathbf{x}) = \mathcal{S}_{[D,G]}\xi$ for $\mathbf{x} \in \partial\Omega$ and $\xi \in \mathscr{L}^2_0(\partial D)$. From (6.12) is seen that

$$\sigma_0\frac{\partial u}{\partial\nu} = \frac{\partial u^N}{\partial\nu} + \sigma_0\frac{\partial}{\partial\nu}\mathcal{S}_{[D,G]}\xi = \psi + \frac{\sigma_0}{2\pi}\int_{\partial D}\xi \qquad \mathbf{x} \in \partial\Omega.$$

If $h = 0$ for $\mathbf{x} \in \Omega\setminus D$ then from analytic continuation follows that $h(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega\setminus D$. Since $h(\mathbf{x})$ solves the homogeneous interior Dirichlet Problem for the Laplacian, $h = 0$ for $\mathbf{x} \in \Omega$. From the jump condition on ∂D, $\xi = 0$. Equation (6.107) is a simple calculation as derived in (6.22). □

If Ω is considered to be a half-space, i.e. $\mathbb{R}^n_+$ for $n = 2, 3$, then any solution $\xi \in \mathscr{L}^2(\partial D)$ of (6.105) will satisfy that for $u = \mathbf{S}_{[\mathbb{R}_+,G]}(\psi) + \mathbf{S}_{[D,G]}\xi$,

$$u(\mathbf{x}) = \phi \quad \text{for} \quad \mathbf{x} \in \partial\Omega, \qquad \sigma_0\frac{\partial u}{\partial\nu}(\mathbf{x}) = \psi \quad \text{for} \quad \mathbf{x} \in \partial\Omega.$$

This is seen from the definition of the Green's function G, since if Ω is the half-space then the Green's function satisfies

$$\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial\nu(\mathbf{x})} = 0 \qquad \mathbf{x} \in (x_1 = 0) \times \mathbb{R}^{(n-1)} \quad \mathbf{y} \in \mathbb{R}^n_+.$$

Therefore for any $\xi \in \mathscr{L}^2(\partial D)$

$$\frac{\partial}{\partial\nu(x)}\mathbf{S}_{[G,D]}\xi = 0 \qquad \mathbf{x} \in (x_1 = 0) \times \mathbb{R}^{(n-1)}.$$

6.3.3. The Forward and Inverse Map for Multiple Inhomogeneities. The obtained results for IP2 with σ_0 known, generalizes straight forward to n disjoint domains D_i satisfying that $D_i \cap D_j = \emptyset$ if $i \neq j$. For nested domains the result can not be used. The reason therefore is that the boundary maps have a constructive unique way to map Cauchy data from one boundary to the next. For nested domains the method only yields the potential and a proportional current, where the constant of proportionality depends on the unknown conductivity. Hence, techniques as in section 6.2.2 must be applied.

Denote by D_i the set of distinct scatters satisfying (6.1), $D_i \cap D_j = \emptyset$ if $i \neq j$, and define

$$\partial D = \bigcup_{i=1}^{n} \partial D_i \qquad \sigma(\mathbf{x}) = \bigcup_{i=1}^{n} \sigma_i \chi_{D_i} \qquad \xi(\mathbf{x}) = \bigcup_{i=1}^{n} \xi_i \chi_{D_i}. \tag{6.108}$$

The formulation (6.21), (6.22) is for multiple disk replaced by, with $u^N = \mathbf{S}_{[\Omega,G]}\sigma_0^{-1}\sigma_0$,

$$u = u^N + \sum_{j=1}^{n} \mathbf{S}_{[D_j,G]}\xi_j \tag{6.109}$$

where $c_i = (\sigma_{i,1} + \sigma_{i,0})\backslash(2(\sigma_{i,1} - \sigma_{i,0}))$ and ξ_i solves

$$\begin{pmatrix} c_1\xi_1 \\ c_2\xi_2 \\ \vdots \\ c_n\xi_n \end{pmatrix} - \begin{pmatrix} \mathbf{K}^*_{[\Omega,G]} & \frac{\partial}{\partial\nu_1}\mathbf{S}_{[D,G]} & \cdots & \frac{\partial}{\partial\nu_1}\mathbf{S}_{[D_n,G]} \\ \frac{\partial}{\partial\nu_2}\mathbf{S}_{[\Omega,G]} & \mathbf{K}^*_{[D,G]} & \cdots & \frac{\partial}{\partial\nu_2}\mathbf{S}_{[D_n,G]} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial\nu_n}\mathbf{S}_{[\Omega,G]} & \frac{\partial}{\partial\nu_n}\mathbf{S}_{[D,G]} & \cdots & \mathbf{K}^*_{[D_n,G]} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \partial u^N/\partial\nu_1 \\ \partial u^N/\partial\nu_2 \\ \vdots \\ \partial u^N/\partial\nu_n \end{pmatrix} \tag{6.110}$$

Theorem 6.36 (Multiple domains)**.** *Assume u^N is as in theorem 6.4 and $\sigma = \sigma_{i,j}$, $j = 0, 1$, then the transmission problem (6.5)-(6.9) has a unique solution given as*

$$u(\mathbf{x}) = u^N + \sum_{n=1}^{N} \mathbf{S}_{[D_j,G]}\xi_j. \tag{6.111}$$

if and only if the densities ξ_j solves

$$\frac{\sigma_{i,1} + \sigma_{i,0}}{2(\sigma_{i,1} - \sigma_{i,0})}\xi_i - \mathbf{K}^*_{[D_i,G]}\xi_i = \frac{\partial u^n}{\partial\nu_i} + \sum_{k\neq i} \frac{\partial}{\partial\nu_i}\mathbf{S}_{[D_k,G]}\xi_k \quad \mathbf{x} \in \partial D_i \quad \forall \quad i = 1, \ldots, n \tag{6.112}$$

The solution $u(\mathbf{x})$ of (6.5)-(6.9) may therefore always be represented as the semi-harmonic function (6.109).

Definition 6.37 (IP2a)**.** The Inverse Conductivity Problem with one experiment, IP2a, is defined as recovering σ_i for $i = 1, \ldots, n$ from knowledge of $\{\Lambda_{\mathbf{D}}(g), g\}$ for **one** g, given that D_i and σ_0 are known.

The inverse problem IP2a of finding the conductivities are found from the boundary maps (6.105) or (6.106) with the appropriate extensions, as the solution of (6.112), compare with theorem 6.35.

Theorem 6.38 (Uniqueness for IP2a)**.** *For $\sigma = \sigma_0\chi_{\Omega\setminus D} + \sum_i \sigma_i\chi_{D_i}$ and $\psi \in \mathscr{L}^2(\partial D)$, let $\{\phi, \psi\}$ be Cauchy data for u solving $P[\sigma, D, \psi]$. Then there exist unique $\xi_i \in \mathscr{L}^2(\partial D)$ with $\int_{\partial D_i} \xi_i = 0$ solving*

$$\phi - \mathcal{S}_{[\Omega,G]}(\sigma_0^{-1}\psi) = \sum_{i=1}^{n} \left(\mathbf{S}_{[D_i,G]}\xi_i\right)\big|_{\partial\Omega} = \sum_{i=1}^{n} \mathcal{S}_{[D_i,G]}\xi_i \tag{6.113}$$

The potential $u = \mathbf{S}_{[G,\Omega]}(\sigma_0^{-1}\psi) + \mathbf{S}_{[G,D]}\xi$ solves (6.5)-(6.9) where σ_i is given as the unique solution of (6.112).

The method for determining σ_i from an equation on ∂D_i involving the density ξ from boundary data, has been applied by [**Ber02**] to the Inverse Medium Problem with N disjoint homogeneous scatterer. He used it to prove uniqueness of the permittivity in such a finite set of homogeneous disjoint scatters with fixed location embedded in a half-space, from one boundary measurement.

Numerical Calculations for the Inverse Conductivity Problem

In this chapter the reconstruction that was established for the Inverse Conductivity Problem with one measurement, using First-kind Integral Equations will be numerically tested in $\mathbb{R}^2$. Two phantoms to be used, will consist of both one and two disjoint obstacles D_i, embedded in a body $\Omega \in \mathbb{R}^2$, both shown in figure 7.1.

To solve the Inverse Conductivity Problem for N disjoint piecewise constant inhomogeneities the measured Cauchy data on $\partial\Omega$ should be mapped to densities defined on the boundary of the inhomogeneities. From these densities the unknown conductivity constant can then be found from the algebraic equation (7.3). For simplicity the obstacles D_1, D_2 and Ω are chosen as disks.

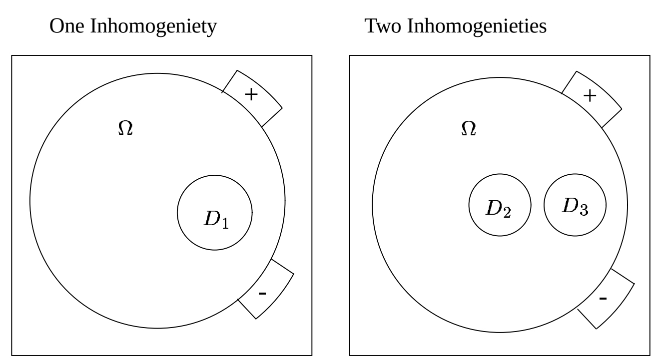

Figure 7.1. The two types of inhomogeneities to be considered. The plus and the minus indicates where the current ψ will have support, and the sign of ψ thereon.

7.1. Different Boundary Maps.

The conductivity problem is for given ψ and σ_i to solve

$$\nabla \cdot (\sigma_0 \chi_{\Omega \setminus D} + \sigma_1 \chi_D) \nabla u = 0 \qquad \mathbf{x} \in \Omega, \tag{7.1}$$

$$\sigma_0 \partial_\nu u = \psi \in \mathscr{L}^2(\partial\Omega) \qquad \mathbf{x} \in \partial\Omega. \tag{7.2}$$

In theorem 6.9 it was proved that the solution of the conductivity problem, (7.1)-(7.2), may be represented as the sum of a harmonic function $\mathbf{H}$ on Ω and a Single-layer potential $\mathbf{S}_{[D,\Gamma+v]}\xi$ on ∂D. If Γ is the fundamental solution for the Laplacian and v any harmonic function on Ω then the density, ξ, for the Single-layer potentials is the same for all kernels.

For the Inverse Conductivity Problem, first ξ was determined from

$$u\big|_{\partial\Omega} - \mathbf{S}_{[\Omega,G]}\psi\big|_{\partial\Omega} = \mathbf{S}_{[D,G]}\xi\big|_{\partial\Omega}$$

where G is the Neumann Green's function, which for a disk is

$$G(\mathbf{x},\mathbf{y}) = \log(|\mathbf{x}-\mathbf{y}|) - \log\left(\left||\mathbf{y}|\mathbf{x} - \frac{\mathbf{y}}{|\mathbf{y}|}\right|\right).$$

From ξ, the conductivity σ_i was then found from the algebraic equation

$$q(\sigma)\xi_i = \frac{\sigma_i + \sigma_0}{2(\sigma_i - \sigma_0)}\xi_i = \partial_{\nu(\mathbf{x})}u^N + \mathbf{K}^*_{[G]}\xi_i + \partial_{\nu(\mathbf{x})}\mathbf{S}_{[D_j,G]}, \quad \mathbf{x} \in \partial D_i, \quad i \neq j. \tag{7.3}$$

Since there are many different integral representations for u, and hence for ξ, from just (6.22) and (6.30) derived in Chapter 6 the following four explicit boundary maps are given

$$A_1: \qquad \mathcal{S}_{[D,G]}\xi\Big|_{\partial\Omega} = \phi - \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi), \tag{7.4}$$

$$A_2: \qquad \mathcal{S}_{[D,\Gamma]}\xi\Big|_{\partial\Omega} = \frac{1}{2}(\phi - \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)) + \mathbf{K}^*_{[\Omega,\Gamma]}(\phi - \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)), \tag{7.5}$$

$$A_3: \qquad \mathcal{S}_{[D,G-\Gamma]}\xi\Big|_{\partial\Omega} = \frac{1}{2}(\phi - \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)) - \mathbf{K}^*_{[\Omega,\Gamma]}(\phi - \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)), \tag{7.6}$$

$$A_4: \quad \frac{\partial}{\partial\nu}\mathcal{S}_{[D,G-\Gamma]}\xi\Big|_{\partial D} = \frac{\partial}{\partial\nu}(\mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi) - (-\mathbf{S}_{[\Omega,\Gamma]}(\sigma_0^{-1}\psi) + \mathbf{D}_{[\Omega,\Gamma]}\phi))\Big|_{\partial D}. \tag{7.7}$$

For the remaining of this chapter we will refer to these maps as A_1 through A_4, and consider these compact integral operators as being replaced by discretized operators, i.e. finite dimensional matrices. Since both G and Γ are smooth on ∂D for $\mathbf{x} \in \partial\Omega$, the discretization of the kernel is made using Simpson's quadrature rule, [**Ste96**].

Since our inverse method is closely related to the integral formulation of theorem 6.4, the solution of (7.1)-(7.2) will be found using a Finite Element Method formulation.

The current, ψ, will be a piecewise constant function, with compact support on roughly 10 percent of the boundary $\partial\Omega$. The current, ψ, and the solution u of (7.1)-(7.2) with $\sigma_1 = 6$, $\sigma_0 = 1$, $D_1 = B(1/2, 0.2)$ and $\Omega = B(0,1)$ is plotted in figure 7.2.

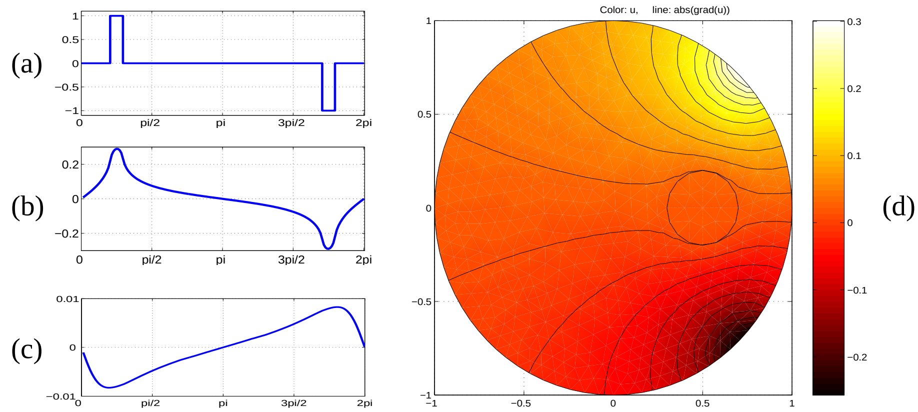

Figure 7.2. (a), the current ψ. (b) the potential on $\partial\Omega$ given ψ. (c) the difference $\phi - \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)$ at $\mathbf{x} \in \partial\Omega$. (d) the solution of forward problem, with $\sigma_1 = 6$ and $\sigma_0 = 1$.

7.2. Regularized Solutions

Since the A_i are compact operators on $\mathscr{L}^2(\partial D)$, the inverse operators A_i^{-1} are unbounded and the problem is ill-posed. For each A_i there exists a Single Value Decomposition (SVD) $(\mu_{ji}, x_{ji}, y_{ji})$, such that $\|x_{ji}\|_{\mathscr{L}^2} = 1 = \|y_{ji}\|_{\mathscr{L}^2}$ and $\mu_{ji} \in \mathbb{R}$. The functions x_{ji} and y_{ji} are the eigenfunctions for A_i and the adjoint A_i^*, respectively. The $\mu_j = \sqrt{\lambda_j}$ are the singular values, where $\sqrt{\lambda_j}$ are the eigenvalues for $A_i^* A_i$. The singular values decays rapidly to 0. This decomposition admits the representation of A_i as

$$A_i(\cdot) = \sum_{j=1}^{N} \mu_{ji}(x_{ji}, \cdot)y_{ji}$$

The eigenfunctions will for increasing j become more oscillatory. Let $Af = g$. If $f \in \mathscr{L}^2$ then for a compact operator A the estimate $\|\mu_j(x_j, f)y_j\|_{\mathscr{L}^2} \leq \mu_j\|f\|_{\mathscr{L}^2}$ shows that Af will reduce the high frequency components of f.

From the decomposition of A_i follows that $f_N^r := A_i^{-1}g$ is given as

$$f_N^r = \sum_{j=1}^{N} \frac{1}{\mu_{ji}}(g, y_{ji})x_{ji}. \tag{7.8}$$

Since the μ_{ji} decay rapidly to zero the inner products (g, y_{ji}) should also decay rapidly in order for $\lim_{N\to\infty} f_N x^r = f$ to hold in $\mathscr{L}^2$. The finite series will always exist, however the norm of f_N^r increases exponentially for these problems, and there is no stability for f_N^r on g. To ensure both convergence and stability of such a series, a filter function depending on a parameter δ, say $q(\delta, \mu_{ji})$, is introduced as

$$f_\delta = \sum_{j=1}^{N} \frac{q(\delta, \mu_{ji})}{\mu_{ji}} (g, y_{ji}) x_{ji},$$

such that for large N, $q(\delta, \mu_{ji}) \searrow 0$. When the filter function is $\mu_{ji}^2/(\delta + \mu_{ji}^2)$, this problem is equivalent to the formulation considered by Tikhonov [**Kir96**],

$$\min_{f \in \mathscr{L}^2} \|A_i f_i - g\|^2_{\mathscr{L}^2} + \delta \|f_i\|^2_{\mathscr{L}^2} \tag{7.9}$$

where δ penalizes the norm of f.

Instead of regularizing with just $\delta\|f\|$, any reasonable operator $\delta\|Bf\|$ that applies to the problem (7.9), which might be the first or second derivative, may be used. The choice of δ determines how many of the high-frequency components in f should be ignored, and the filter-function tries to tweak out some more information from g than just truncating the series.

One way to determine the value of δ is by inspection of the Picard plot. This is a plot of $(\langle g, y_{ji}\rangle)/\mu_{ji}$ for increasing values of j. If the series (7.8) is to converge, then on average the coefficients $(\langle g, y_{ji}\rangle)$ should decay faster than μ_{ji}. For (7.4) with $\phi - \mathbf{S}_{[\Omega,G]}(\sigma_0^{-1}\psi)$ as in figure 7.2, the Picard plot is illustrated in figure 7.3. It is observed that for $j > 23$ the products $(\langle g, y_j\rangle)/\mu_j$ increases, hence the value of δ should be such that it damps all the coefficients for $j > 23$. For noisy data $j \ll 23$ should be expected.

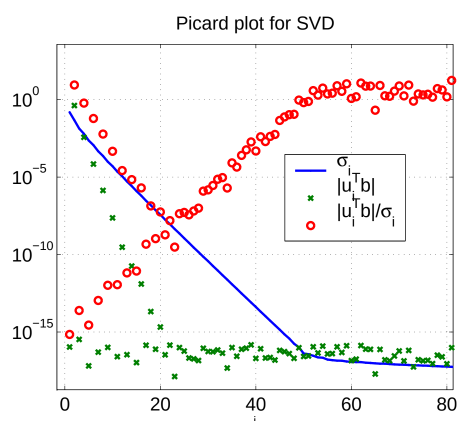

Figure 7.3. The Picard plot for the SVD decomposition of $A_1 f = g$ using g as in figure 7.2c

7.3. The Regularized Solution

The solution of anyone of (7.4)-(7.7) for determining ξ is considered as solving for any i,

$$\min_{\xi} \|A_i\xi - \eta_i\|_{\mathscr{L}^2(\partial\Omega)} + \delta_i\|\xi\|_{\mathscr{L}^2(\partial D)}. \tag{7.10}$$

Subsequently comparison between Conjugated gradient, Tikhonov regularization and Truncated SVD to solve this minimization problem will be made.

For all methods either a penalizer as in (7.10), or equivalently a stopping-rule for an iterative scheme is defined. Different discrepancy principles can be used for defining the penalizing parameter. If the level of noise is a-priori known, this can be implemented a-priori in Mozorow's discrepancy principle. Two other methods are the L-curve method and the Regińska principle. The latter two are both based on a log-log plot of $\|Ax - b\|$ versus $\|x\|$ for different values of δ. This yields a plot like figure 7.4, where the L-curve

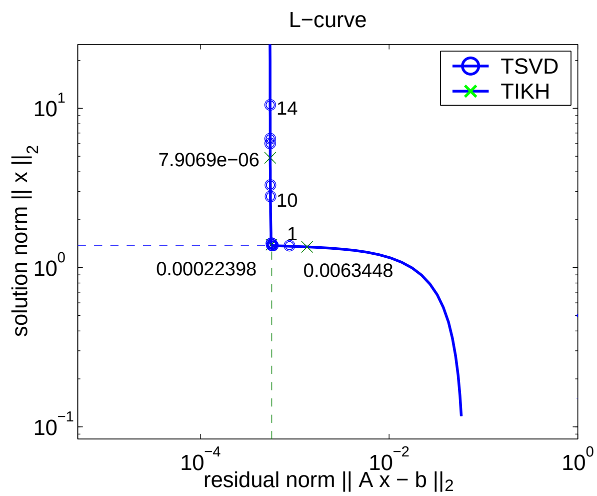

Figure 7.4. The L-curve plot for TSVD and Tikhonov regularization.

for Truncated SVD is discrete and the L-curve for Tikhonov regularization is continuous. The same plot may be readily constructed for any iterative solution strategy of (7.10).

The problem is how to define what the optimal solution is. From geometrical observations it is at the "corner" that the transition from over to under regularization occur. But where? The choice of δ should be chosen in such a way that as much information as possible from g gets extracted, and such that noise in g gets filtered out. The L-curve method defines the optimal point as where the curvature is greatest. The method of Regińska is on the contrary to find when a line with ratio -1 intersects the L-curve. We will consider the optimal regularization parameter as where the curvature of the L-curve in the log-log plot is greatest.

For the integral operators $A_1, \cdots A_4$, the decay of the singular values depend on where D is located. The further away D is from the boundary $\partial\Omega$, the more ill-posed the problem becomes. The decay rate for the singular values of the operators A_1, A_4 as-well as for different depths and only partial Cauchy data are plotted in figure 7.5. It is

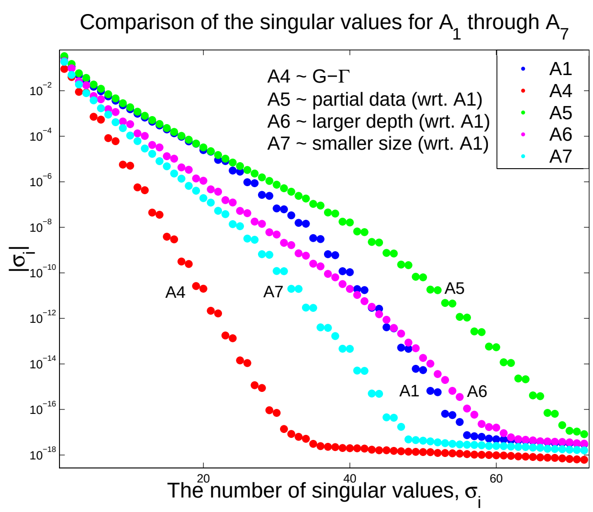

Figure 7.5. The decay of the singular values for A_1 and $A_4, \ldots A_7$. The singular values for A_2 and A_3 are the same as those for A_1.

observed that the singular values decay the slowest when partial data is used. The decay rate for the singular values of A_2 and A_3 are the same as those those of A_1, since Ω is a disk. This may be observed by considering G to be the half-space Neumann Greens function.

7.4. Reconstructing ξ

For $u|_{\partial\Omega}$ as in figure 7.2, the reconstruction of ξ using Truncated SVD, Tikhonov Regularization, and Conjugated Gradient Least Squares iteration is displayed in figure 7.6.

For most applications the data has some inherent noise. For Cauchy data with 10% noise the reconstruction of ξ is plotted in figure 7.6 using Truncated SVD, Tikhonov regularization, and Conjugated Gradient Least Squares iteration

7.5. Reconstructing σ

Using the L-curve stopping criteria for reconstruction ξ from forward data, calculated using a FEM method, and subsequent solving the algebraic equation (7.3) is done for different values of σ_1. The result hereof is plotted in figure 7.7. It is apparent that for

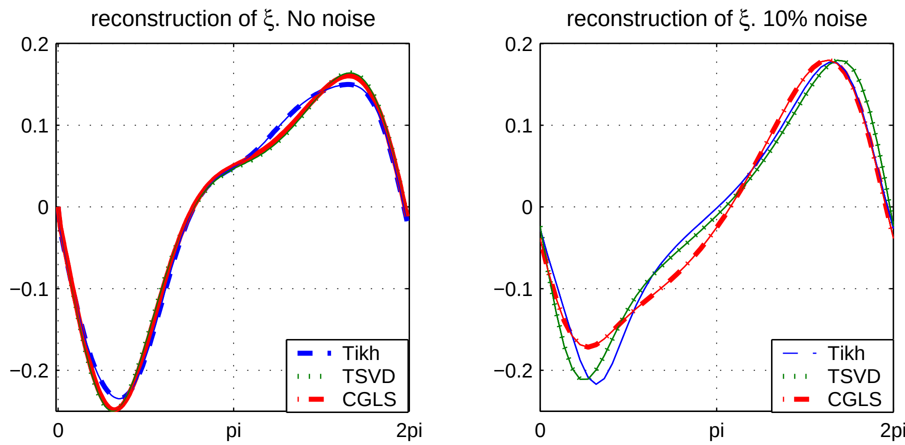

Figure 7.6. The reconstruction of ξ using Truncated SVD, Tikhonov regularization, Conjugated Gradient Least squares, when D is a the disk $B([0.5, 0.3], 0.2)$.

larger values of σ the reconstructions becomes worse (roughly 3 % error). The reason for this should be seen in the least squares estimate of q in

$$\min_{q\in\mathbb{R}} \|(\partial_\nu u^N + \mathbf{K}^*_{[D,G-\Gamma]}\xi) - q\xi\|_{\mathscr{L}^2(\partial D)}.$$

Since $\sigma = (2q + 1)/(2q - 1)$ is nonlinear, errors in σ will be amplifi ed the smaller the value of q is.

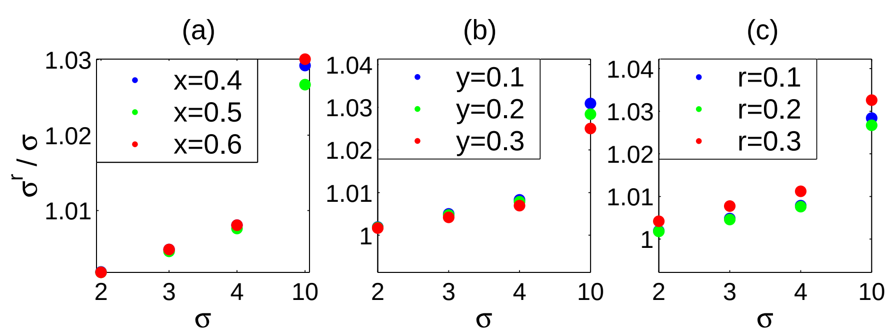

Figure 7.7. Reconstruction of σ for different values of σ, sizes and locations. (a) is for a disk located at $y = 0$ and $r = .2$. (b) for a disk located at $x = 0.5$ and $r = 0.2$. (c) for a disk located at $x = 0.5$ and $y = 0$.

Assume that the noise is additive white noise. Reconstruction based on noisy data is made when the Cauchy data is known only on part of $\partial\Omega$ and when there are two inhomogeneities. The reconstructions are made for noise-levels 1%, 5% and 10% and displayed in fi gure 7.8. The reconstruction of σ is for both 1%, 5% and 10% at the same levels as for unperturbed data. It is observed, that generally Tikhonov regularization

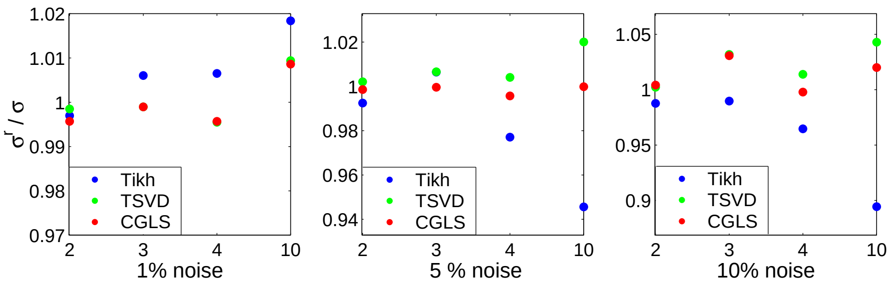

Figure 7.8. The reconstruction from noisy data for a disk $B([0.5, 0.1], 0.2)$, with $\sigma = [2, 3, 4, 10]$, respectively.

yields a reconstructed value of σ that is more erroneous than that of both Truncated SVD and Conjugated Gradient Least Squares. The latter two methods generally agree and achieve good reconstructions of σ.

7.6. Partial Data

Using the integral formulation (7.5) it is apparent that to calculate $\mathbf{H} = -\mathbf{S}_{[\Omega,\Gamma]}(\sigma_0^{-1}\psi)+\mathbf{D}_{[\Omega,\Gamma]}(u\big|_{\partial\Omega})$ the Cauchy data are needed on all of $\partial\Omega$. For the integral formulation (7.4) however it is possible to use only partial Cauchy data. Considering the decay rate of the singular values in fi gure 7.5 is appears that A_5 has a slower decay rate than say A_1, and therefore is expected to reconstruct ξ better. However from the Picard plot and the experiment with full Cauchy data it is seen that only a small number of eigenfunctions are needed, less than 23.

From the forward data plotted in fi gure 7.2 it is seen that the extrema of $u|_{\partial\Omega}$ is located near the support of $\partial_\nu u|_{\partial\Omega}$ because of the particular choice of $\partial_\nu u|_{\partial\Omega}$. The three different sizes of partial Cauchy data that are tested are

$\theta_1 = [-\tan^{-1}(3/4); \tan^{-1}(3/4)]$**:** The angle between the support of $\partial_\nu u|_{\partial\Omega}$.

$\theta_2 = [-\tan^{-1}(4/3); \tan^{-1}(4/3)]$**:** θ_1 and the support of $\partial_\nu u|_{\partial\Omega}$.

$\theta_3 = [-\pi/2; \pi/2]$**:** A subset of $\partial\Omega$ containing θ_2.

The reconstruction for one conductivity and the dependence on location and size using partial Cauchy data is plotted in fi gure 7.9. Again, it is apparent that for noise free data good reconstructions can be achieved for smaller values of σ.

7.6.1. Two inhomogeneities. For two inhomogeneities $\mathbf{S}_{[D_1\cup D_2,G]}\xi = \eta$, with $\xi = \xi_1\chi_{\partial D_1} + \xi_2\chi_{\partial D_2}$ is the integral operator A_1. If the number of evaluation points on $\partial\Omega$ is held fi xed ξ_1 and ξ_2 must together have less evaluation points. This is just an artifi cial problem since $u|_{\partial\Omega}$ can be interpolated onto more points on $\partial\Omega$. The same does however not hold for $\partial_\nu u\big|_{\partial\Omega}$, since $\partial_\nu u\big|_{\partial\Omega}$ may not be continuous, thereby introducing large interpolation errors.

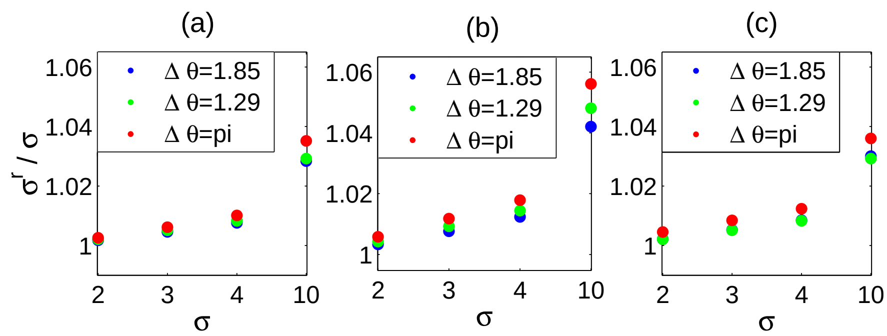

Figure 7.9. The reconstruction using Truncated SVD for partial Cauchy data with noise free data. (a) is a disk at $(0.5, 0.3)$ and radius 0.2. (b) is a disk at $(0.5, 0)$ and radius 0.1. (c) is a disk at $(0.4, 0.0)$ and radius 0.2.

The equation for σ_i is now

$$q(\sigma)\xi = \frac{\sigma_i + \sigma_0}{2(\sigma_i - \sigma_0)}\xi_i = \partial_\nu u^N + \mathbf{K}^*_{[D_j,G]}\xi_i + \partial_{\nu(\mathbf{x})}\mathbf{S}_{[D_j,G]} \quad \mathbf{x} \in \partial D_i, \quad i \neq j \quad (7.11)$$

Consider the geometry of two inhomogeneities in figure 7.1. D_1 can be interpreted as shadowing for D_2 in this model. The reconstruction of $\sigma|_{D_1}$ and $\sigma|_{D_2}$ is tabulated in table 1 for different values of $\sigma|_{D_2}$ and different locations of D_2, where D_2 is translated in the up-down direction, and where the D_i are of same size.

Again good reconstructions of σ_1 and σ_2 is achieved for smaller values of σ_i. If either σ_1 or σ_2 is erroneous, then also the other is determined poorly.

y	σ_1^{true}	$\sigma_1^{recon}/\sigma_1^{true}$	σ_2^{true}	$\sigma_2^{recon}/\sigma_2^{true}$
0	3	1	2	1.03
0	3	1.02	3	1
0	3	1.03	4	0.98
0	3	1.09	10	0.91
0.3	3	0.96	2	1.07
0.3	3	0.98	3	1.03
0.3	3	1	4	0.99
0.3	3	1.07	10	0.87

Table 1. Reconstructed values of σ for two inhomogeneities

The functions from (7.11) involved in finding an estimate for $f(\sigma_i)$ are plotted in figure 7.10. The $\xi|_{\partial D}$, for $i = 1, 2$, appear to have more oscillatory behavior then the ξ in figure 7.6

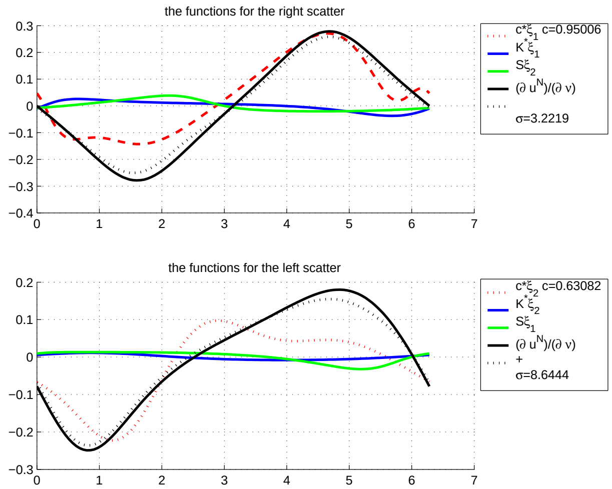

Figure 7.10. The functions involved in estimating σ_i for two inhomogeneities. The true values are $\sigma_1 = 3$ and $\sigma_2 = 10$ and the reconstructed values are $\sigma_1^r/\sigma_1 = 1.07$ and $\sigma_2^r/\sigma_2 = 0.87$

Conclusion

The numerical investigation of the Inverse Conductivity Problem with piecewise constant conductivity show that σ may be reconstructed well for small values of σ. Both the Conjugated Gradient Least Squares and Truncated Singular Value Decomposition appear to be superior to Tikhonov regularization when the L-curve method is used for discrepancy principle. Extending the method of section 6.3.3 to nested domains, the non-uniqueness of the ξ_i are observed. Hence, the proposed method for disjoint inhomogeneities can not be extended to nested objects.

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