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Switched Systems

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Abstract—In this paper, a general method for model order reduction of switched linear dynamical systems is presented. The proposed technique is based on the generalized gramian framework for model reduction. It is shown that different classical reduction methods can be developed into generalized gramian framework. Balanced reduction within specified frequency bound is developed within this framework. In order to avoid numerical instability and also to increase the numerical efficiency, generalized gramian based Petrov-Galerkin projection is constructed instead of the similarity transform approach for reduction. The method preserves the stability of the original switched system under arbitrary switching signal and is applicable to both continuous and discrete time systems. The performance of the proposed method is illustrated by numerical examples.

I. INTRODUCTION

THE ever-increasing need for accurate mathematical modeling of physical as well as artificial processes for simulation and control leads to models of high complexity. This problem demands efficient computational prototyping tools to replace such complex models by an approximate simpler models, which are capable of capturing dynamical behavior and preserving essential properties of the complex one, either the complexity appears as high order describing dynamical system or complex nonlinear structure. Due to this fact model reduction methods have become increasingly popular over the last two decades [1],[2],[3]. Such methods are designed to extract a reduced order state space model that adequately describes the behavior of the system in question.

Most of the studies related to model reduction presented so far have been devoted to linear case and just few methods have been proposed for nonlinear cases which are not strong comparing to linear reduction methods.

On the other hand, most of the methods that are proposed so far for control and analysis of hybrid and switched systems are compelling of suffer from high computational burden when dealing with large-scale dynamical systems. Because of the weakness of nonlinear model reduction techniques and also pressing needs for efficient analysis and control of large-scale dynamical hybrid and switched systems; it is essential to study model reduction of hybrid and switched systems in particular. This fact has motivated the researchers

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in hybrid systems to study model reduction [15]-[23]. Some works have been focused on ordinary model reduction methods that have potential applications in modeling and analysis of hybrid systems [15]-[21] motivated by reachability analysis and safety verification problem. Some researches addresses the problem of model reduction of switched and hybrid systems directly [22], [23]. In [22] it is presented that the state set can be affinely reduced due to non-observability if and only if a subspace of the classical unobservable subspace, characterized using the normal vectors of the exit facets, is nontrivial. This achievement does not provide strong tool for reduction of affine systems because it is an exact reduction and quite restrictive. Exact reduction is very elegant but the class of systems for which this procedure applies is quite small. This method only considers observability for investigating the importance of the states to discard and it has not looked into controllability of the states. The paper [23] is concerned with the problem model reduction for discrete switched system. Two different approaches are proposed to solve this problem. The first approach casts the model reduction into a convex optimization problem, which is the first attempt to solve the model reduction problem by using linearization procedure. The second one, based on the cone complementarity linearization idea, casts the model reduction problem into a sequential minimization problem subject to linear matrix inequality constraints. Both approaches have their own advantages and disadvantages concerning conservatism and computational complexity. These optimization problems will be very hard if not infeasible to solve for a large scale system. This method not only is just applicable to discrete time switched systems but also it does not provide us with any hint about the number of states which is suitable to keep before reduction.

In this paper we propose a method for model reduction of switched system which can be categorized as gramian based model reduction methods. Balanced model reduction is one of the most common gramian based model reduction schemes. It was presented in [4] for the first time.

To apply balanced reduction, first the system is represented in a basis where the states which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations.

Then, the reduced model is obtained by truncating the states which have this property. Balanced model reduction method is modified and developed from different viewpoints [1],[2]. One of the methods that are presented based on balanced model reduction is the method based on the generalized gramians instead of gramians[5]. In this method in order to compute the generalized gramians, one should solve Lyapunov inequalities instead of Lyapunov equations. This method is used to devise a technique for structure preserving model reduction methods in [6].

In this paper we first show that the generalized method in [5] can be extended to various gramian based reduction methods. We also modified the original method in [5] to avoid numerical instability and also to achieve more numerical efficiency by building Petrov-Galerkin projection based on generalized gramians. We propose a method based on the balanced model reduction within frequency bound in this framework. We generalized the framework to model reduction of switched system by solving system of Lyapunov inequalities to find common generalized gramian. The paper is organized as follows: In the next section we review balanced reduction method and balanced reduction technique based on generalized gramian. Section II presents how different gramian based methods can be approximated as generalized gramian based techniques. Balanced reduction within frequency bound based on generalized gramian is also presented in this section. This section ends up with some remarks on numerical implementation of the algorithm and using projection for generalized gramian based reduction methods is suggested intead of balancing and truncation. Section III is devoted to develop generalized gramian based reduction method for model reduction of switched systems, followed by a brief discussion on stability, feasibility and error bound. Section IV presents our numerical results. Section V concludes the paper.

The notation used in this paper is as follows: M^* denotes transpose of matrix if $M \in \mathbb{R}^{n \times m}$ and complex conjugate transpose if $M \in \mathbb{C}^{n \times m}$. The norm $\|.\|_\infty$ denotes the H_∞, norm of a rational transfer function. The standard notation $>$, $\geq (<, \leq)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

II. BALANCED TRUNCATION AND GENERALIZED GRAMIANS

Balanced truncation is a well-known method for model reduction of dynamical systems, see for example [1][2].The basic approach relies on balancing the gramians of the systems. For dynamical systems with minimal realization:

$$G(s) := (A, B, C, D) \qquad (1)$$

where $G(s)$ is transfer matrix with associated state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(t) \in \mathbb{R}^n \\ y(t) = Cx(t) + Du(t) \end{cases} \qquad (2)$$

gramians are given by the solutions of the Lyapunov equations:

$$AP + PA^* + BB^* = 0$$
$$A^*Q + QA + C^*C = 0 \qquad (3)$$

For stable A, they have a unique positive definite solutions P and Q, called the controllability and observability gramians. In balanced reduction, first the system is transformed to the balanced structure in which gramians are equal and diagonal:

$$P = Q = diag(\sigma_1 I_{k_1}, ..., \sigma_q I_{k_q})$$
$$\sum_{j=1}^q k_j = n \qquad (4)$$

where $\sigma_i > \sigma_{i+1}$ and they are called Hankel singular values.

The reduced model can be easily obtained by truncating the states which are associated with the set of the least Hankel singular values. Applying the method to stable, minimal $G(s)$, If we keep all the states associated to $\sigma_m (1 \leq m \leq r)$, by truncating the rest, the reduced model $G_r(s)$ will be minimal and stable and satisfies[1][2]:

$$\|G(s) - G_r(s)\|_\infty \leq 2 \sum_{j=r+1}^q \sigma_j \qquad (5)$$

One of the closely related model reduction methods to the balanced truncation is balanced reduction based on generalized gramian that is presented in [5]. In this method, instead of Lyapunov equations (3), the following Lyaponuv inequalies should be solved:

$$AP_g + P_g A^* + BB^* \leq 0$$
$$A^*Q_g + Q_g A + C^*C \leq 0 \qquad (6)$$

For stable A, they have positive definite solutions P_g and Q_g, called the generalized controllability and observability gramians. Note that these gramians are not unique. The rest of this model reduction method is the same as the aforementioned balanced truncation method, the only difference is that in this algorithm the balancing and truncation are based on generalized gramian instead of ordinary gramian. In this method we have generalized Hankel singular values (γ_i) which are the diagonal elements of balanced gerealized gramians instead of Hankel singular values σ_i which are the diagonal elements of balanced ordinary gramians. For the error bound also the same result holds but in terms of the generalized Hankel singular values instead of Hankel singular values. It is worth to mention that, $\gamma_i \geq \sigma_i$. Therefore the error bound in balanced reduction based on generalized gramian is greater equal than the error bound in ordinary balanced model reduction.

III. GENERALIZED GRAMIAN FRAMEWORK FOR GRAMIAN-BASED MODEL REDUCTION METHODS

In this section we present a general framework to build generalized gramian version of gramian based methods.

Then we present generalized balanced reduction within frequency bound within this framework following by some words about numerical implementation of the algorithm based on projection.

A. Lyapunov Equations, Lyapunov Inequalities and Reduction

Lemma 1: Suppose A is stable, Y is symmetric and

$$A^*Y + AY \leq 0$$
$$A, Y \in \mathbb{R}^{n \times n} \tag{7}$$

is satisfied, then $Y \geq 0$, i.e. Y has to be positive semi definite.

Proof: If $A^*Y + AY \leq 0$, there exists $M \geq 0$ such that:

$$A^*Y + AY + M = 0$$

On the other hand, for any stable A, there exists the following unique solution for the equation above:

$$Y = \int_0^\infty e^{A^*\tau} M e^{A\tau} d\tau$$

In the above structure $M \geq 0$, hence: $Y \geq 0$

$\square$

This lemma leads to the following proposition, that makes the relation between Lyapunov equations and Lyapunov inequalities evident.

Proposition 1[5]: Suppose A is stable and X is the solution of Lyapunov equation:

$$A^*X + XA + Q = 0 \tag{8}$$

where $Q \geq 0$. If a symmetric X_g satisfies:

$$A^*X_g + X_gA + Q \leq 0 \tag{9}$$

Then: $X_g \geq X$.

Proof: It can be proven easily by subtracting (9)-(8) and applying Lemma 1 with $Y = X_g - X$.

$\square$

This proposition is a direct result of Lemma 1.

Proposition 1 shows how the generalized gramian could be an approximation for ordinary gramians. Balanced reduction based on generalized gramian which we reviewed in the last section is based on proposition 1.This method might provide less accurate approximation than its gramian based counterpart but still the approximation error is bounded.

It is possible to propose generalized version of other gramian based reduction methods in this framework. The only step that we need is to derive associated Lyapunov equations and relax them to Lyapunov inequalities. In the following we propose generalized version of balanced reduction within frequency bound.

B. Generalized Balanced Reduction within Frequency Bound

Over the past two decades, a great deal of attention has been devoted to balanced model reduction and it has been developed and improved from different viewpoints. Frequency weighted balanced reduction method is one of the devised gramian based techniques based on ordinary balanced truncation [1],[2],[7]-[9]. In this method by using input and output weights and stressing on certain frequency range more accurate results can be achieved. In many cases the input and output weights are not given and instead the problem is to reduce the model over a given frequency range [1][2]. This is problem can be attacked directly by balanced reduction within frequency bound. This method first proposed in [10] and then modified in [2] in order to preserve the stability of the original system and to provide an error bound for approximation. In this method, for dynamical system (1) the controllability gramian $P(\omega_1, \omega_2)$ and observability gramians $Q(\omega_1, \omega_2)$ within frequency range $[\omega_1, \omega_2]$ are defined as:

$$P(\omega_1, \omega_2) = P(\omega_1) - P(\omega_2)$$
$$Q(\omega_1, \omega_2) = Q(\omega_1) - Q(\omega_2) \tag{10}$$

where:

$$P(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\omega - A)^{-1} BB^* (-Ij\omega - A^*)^{-1} d\omega$$
$$Q(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{\omega} (-Ij\omega - A^*)^{-1} C^*C(Ij\omega - A)^{-1} d\omega \tag{11}$$

In order to show the associated Lyapunov equations, we need some more notations:

$$S(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\omega - A)^{-1} d\omega \tag{12}$$

$$W_c(\omega) =: S(\omega)BB^* + BB^*S^*(-\omega)$$
$$W_o(\omega) =: C^*CS(\omega) + S^*(-\omega)C^*C \tag{13}$$

$$W_c(\omega_1, \omega_2) =: W_c(\omega_2) - W_c(\omega_1)$$
$$W_o(\omega_1, \omega_2) =: W_o(\omega_2) - W_o(\omega_1) \tag{14}$$

The gramians satisfy the following Lyapunov equations[1],[2]:

$$AP(\omega_1, \omega_2) + P(\omega_1, \omega_2)A^* + W_c(\omega_1, \omega_2) = 0$$
$$A^*Q(\omega_1, \omega_2) + Q(\omega_1, \omega_2)A + W_o(\omega_1, \omega_2) = 0 \tag{15}$$

This method is modified in [2] to guarantee stability and to provide a simple error bound. The modified version starts with EVD of $W_c(\omega_1, \omega_2)$ and $W_o(\omega_1, \omega_2)$:

$$W_c(\omega_1, \omega_2) := M \Lambda M^* = M \, diag(\lambda_1, ..., \lambda_n) M^*$$
$$W_o(\omega_1, \omega_2) := N \Delta N^* = N \, diag(\delta_1, ..., \delta_n) N^* \quad (16)$$

where: $MM^* = NN^* = I_n, |\lambda_i| \ge |\lambda_{i+1}| \ge 0, |\delta_i| \ge |\delta_{i+1}| \ge 0$.

Note that since $W_c(\omega_1, \omega_2)$ and $W_o(\omega_1, \omega_2)$ are symmetric decompositions in the form (16) exist. Let:

$$\hat{B} := M \, diag(|\lambda_1|^{1/2}, ..., |\lambda_\xi|^{1/2}, 0, ..., 0)$$
$$\hat{C} := diag(|\delta_1|^{1/2}, ..., |\delta_\rho|^{1/2}, 0, ..., 0) N^* \quad (17)$$

where:

$$\xi = rank(W_c(\omega_1, \omega_2))$$
$$\rho = rank(W_o(\omega_1, \omega_2)) \quad (18)$$

The modified gramians satisfy the following Lyapunov equations instead of (15):

$$A\hat{P}(\omega_1, \omega_2) + \hat{P}(\omega_1, \omega_2)A^* + \hat{B}\hat{B}^* = 0$$
$$A^*\hat{Q}(\omega_1, \omega_2) + \hat{Q}(\omega_1, \omega_2)A + \hat{C}^*\hat{C} = 0 \quad (19)$$

That is all what we need to present the generalized version of this method:

$$A\hat{P}_g(\omega_1, \omega_2) + \hat{P}_g(\omega_1, \omega_2)A^* + \hat{B}\hat{B}^* \le 0$$
$$A^*\hat{Q}_g(\omega_1, \omega_2) + \hat{Q}_g(\omega_1, \omega_2)A + \hat{C}^*\hat{C} \le 0 \quad (20)$$

Then the generalized modified balanced reduction within frequency bound can be obtained by simultaneously diagonalizing $\hat{P}_g(\omega_1, \omega_2)$ and $\hat{Q}_g(\omega_1, \omega_2)$ then by truncating the states associated to the set of the least generalized Hankel singular values.

C. Numerical Issues

Balanced transformation can be ill-conditioned numerically when dealing with the systems with some nearly uncontrollable modes or some nearly unobservable modes. Difficulties associated with computation of the required balanced transformation in [11] draw some attentions to devise alternative numerical methods[12]. Balancing can be a badly conditioned even when some states are much more controllable than observable or vice versa. It is advisable then to reduce the system in the gramian based framework without balancing at all. Schur method and Square root algorithms provides projection matrices to apply balanced reduction without balanced transformation[1],[12]. This method can be easily applied to other Gramian based method. In our generalized method we can use the same algorithm by plugging generalized gramians into the algorithm instead of ordinary gramians. Model Reduction of Switched System

D. Model Reduction of Switched Systems Based on Generalized Gramians

One of the most important subclasses of hybrid systems are Linear switched systems. Linear switched system is a dynamical system specified by the following equations:

$$\Sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \quad (21)$$

where $x(t) \in \mathbb{R}^n$ is the continuous state, $y(t) \in \mathbb{R}^p$ is the continuous output, $u(t) \in \mathbb{R}^m$ is the continuous input, and $\sigma : \mathbb{R}^{\ge 0} \to K \subset \mathbb{N}$ is the switching signal that is a piecewise constant map of the time. K is the set of discrete modes, and it is assumed to that it is finite. For each $i \in K$, A_i, B_i, C_i, D_i are matrices of appropriate dimensions.

In this section we build a framework for model reduction of switched system described by (21). The aim is to find Petrov-Galerkin projectors to project the switched system to lower dimensional subspace.

Petrov-Galerkin projection for a dynamical system[1]:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \quad , x \in \mathbb{R}^n \\ y(t) = g(x(t), u(t)) \end{cases} \quad (22)$$

is defined as a projection $\Pi = VW^*$, where: $W^*V = I_k, \quad V, W \in \mathbb{R}^{n \times k}, k < n$.

The reduced order model using this projection is:

$$\begin{cases} \dot{\hat{x}}(t) = W^* f(V\hat{x}(t), u(t)) \quad , \hat{x} \in \mathbb{R}^k \\ y(t) = g(V\hat{x}(t), u(t)) \end{cases} \quad (23)$$

We can develop generalized gramian framework for model reduction of switched linear system by finding common generalized controllability/observability gramian related to subsystems. To do this we need to solve two systems of lyapunov inequalies, one for finding common generalized controllability gramian and one for common generalized observability gramian. The next step can be simultaneous diagonalization of the common generalized gramians and balancing and reduction all subsystems based on common generalized Hankel singular values. In order to avoid numerical bad conditioning and also to increase the efficiency we use schur or square root algorithm instead of balancing and directly Petrov-Galerkin projection matrices can be computed. In order to clarify the method we extend generalized balanced reduction within frequency bound that is presented in previous section, for model reduction of switched linear system.

First we need to find, common generalized controllability gramian $\hat{P}_{cg}(\omega_1, \omega_2)$ by solving the system of Lyapunov inequalities :

$$\begin{cases} A_\sigma \hat{P}_{cg}(\omega_1, \omega_2) + \hat{P}_{cg}(\omega_1, \omega_2)A_\sigma^* + \hat{B}_\sigma \hat{B}_\sigma^* < 0 \\ \forall \sigma \in K \end{cases} \quad (24)$$

For example in the case of bimodal systems, $K = \{1, 2\}$, so we have to solve:

$$\begin{cases} A_1 \hat{P}_{cg}(\omega_1,\omega_2) + \hat{P}_{cg}(\omega_1,\omega_2)A_1^* + \hat{B}_1 \hat{B}_1^* < 0 \\ A_2 \hat{P}_{cg}(\omega_1,\omega_2) + \hat{P}_{cg}(\omega_1,\omega_2)A_2^* + \hat{B}_2 \hat{B}_2^* < 0 \end{cases} \quad (25)$$

common generalized observability gramian $\hat{Q}_{cg}(\omega_1,\omega_2)$ can be obtained similarly by solving the system of lyapunov inequlalities :

$$\begin{cases} A_\sigma^* \hat{Q}_{cg}(\omega_1,\omega_2) + \hat{Q}_{cg}(\omega_1,\omega_2)A_\sigma + \hat{C}_\sigma^* \hat{C}_\sigma < 0 \\ \forall \sigma \in \mathrm{K} \end{cases} \quad (26)$$

Here also in the case of bimodal systems, $\mathrm{K} = \{1,2\}$, we have:

$$\begin{cases} A_1^* \hat{Q}_{cg}(\omega_1,\omega_2) + \hat{Q}_{cg}(\omega_1,\omega_2)A_1 + \hat{C}_1^* \hat{C}_1 < 0 \\ A_2^* \hat{Q}_{cg}(\omega_1,\omega_2) + \hat{Q}_{cg}(\omega_1,\omega_2)A_2 + \hat{C}_2^* \hat{C}_2 < 0 \end{cases} \quad (27)$$

If we plug in $\hat{P}_{cg}(\omega_1,\omega_2)$ and $\hat{Q}_{cg}(\omega_1,\omega_2)$ to the square root algorithm we can directly obtain projectors for reduction. Note that the results are same as balancing algorithm. A merit of the Square Root method is that it relies on the Cholesky factors of the gramians rather than the gramians themselves, which has advantages in terms of numerical stability.

E. Stability, feasibility and Approximation Error

One of the important issues in model reduction is preservation of the stability. In other words, the question is if the reduction technique method can preserve the stability of the original model in approximation. In the following proposition we show that the proposed framework for model reduction of switched system is stability preserving model reduction method and can preserve the stability of the original system under arbitrary switching.

Proposition 2. If the switched system described in (21) is stable, the generalized gramian based reduced order model is guaranteed to be quadratic stable.

Proof:
In the proposed method, we have:
$$W^*V = I_k, \quad V,W \in \mathbb{R}^{n \times k}, k < n \quad (28)$$
$$\hat{\Sigma}: (\hat{A}_{\sigma(t)} = W^* A_{\sigma(t)} V, \hat{B}_{\sigma(t)} = W^* B_{\sigma(t)}, \hat{C}_{\sigma(t)} = C_{\sigma(t)} V, \hat{D}_{\sigma(t)} = D_{\sigma(t)})$$

which is projected switched system (reduced order model). The outcome of Square root algorithm for projection[1]: $P_g W = V \Sigma_1$ and $Q_g V = W \Sigma_1$, where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is diagonal and positive definite. Since P_g is common generalized gramian : $A_{\sigma(t)} P_g + P_g A_{\sigma(t)}^* < 0$, which implies:

$$W^*(A_{\sigma(t)} P_g + P_g A_{\sigma(t)}^*) W < 0$$

On the other hand,
$$W^*(A_{\sigma(t)} P_g + P_g A_{\sigma(t)}^*) W = W^* A_{\sigma(t)} P_g W + W^* P_g A_{\sigma(t)}^* W$$
$$= W^* A_{\sigma(t)} V \Sigma_1 + \Sigma_1^* V^* A_{\sigma(t)}^* W = \hat{A}_{\sigma(t)} \Sigma_1 + \Sigma_1 \hat{A}_{\sigma(t)}^*$$

Hence:
$$\hat{A}_{\sigma(t)} \Sigma_1 + \Sigma_1 \hat{A}_{\sigma(t)}^* < 0 \quad (29)$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is positive definite.

In stability theory for switched system it is well-known sufficient condition for quadratic stability [13]. Hence, reduced order model is guaranteed to be quadratic stable.

□

The same results hold, if we use balancing transformation instead of projection. The proof is straightforward and it is just based on the fact that for any matix $M \leq 0$, all its leading square partitions are negative semidefinite.

As we can see, the presented framework for model reduction of switched system is stability preserving model reduction method. As we already mentioned the error of approximation for each subsystem is bounded and is in terms of generalized Hankel singular values.

The system of LMIs in our framework is said to be feasible if a common generalized grammian exists. In general existence of a common lyapunov function is not guaranteed for switched systems [13], therefore we can not expect to have common generalized grammian for all linear switched systems. One way to improve the feasibility of the proposed model reduction method is to use recently proposed extended notion of generalized grammian which is called extended grammian [14].

IV. NUMERICAL EXAMPLES

In this section we have applied the proposed method for reduction of two randomly generated bimodal switched linear systems. The first example is of order 5 and the second one is of order 100.

A. Fifth Order Switched linear System:

We consider a randomly generated single-input-single output switched linear of the form (21) for which we have:

$$A_1 = \begin{bmatrix} -4.23 & 0.4654 & 1.305 & 0.313 & -1.461 \\ 0.4654 & -4.418 & 0.8745 & -0.9324 & -0.7062 \\ 1.305 & 0.8745 & -1.839 & -0.0083 & 0.6652 \\ 0.313 & -0.9324 & -0.0083 & -1.801 & -0.4979 \\ -1.461 & -0.7062 & 0.6652 & -0.4979 & -2.355 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -5.055 & 0.4867 & 0.7761 & -3.765 & -2.702 \\ 0.4867 & -3.034 & 0.0537 & 0.6768 & 0.6030 \\ 0.7761 & 0.0537 & -1.392 & -0.0739 & 0.8858 \\ -3.765 & 0.6768 & -0.0739 & -5.26 & -1.886 \\ -2.702 & 0.603 & 0.8858 & -1.886 & -3.909 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.1721 \\ -0.336 \\ 0.5415 \\ 0 \\ -0.5703 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.5081 \\ 0.8564 \\ 0.2685 \\ 0.625 \\ -1.047 \end{bmatrix}$$

$$C_1 = [-1.499 \quad -0.0503 \quad 0.553 \quad 0.0835 \quad 1.578]$$

$$C_2 = [1.536 \quad 0.4344 \quad -1.917 \quad 0 \quad 0], \quad D_1 = D_2 = 0$$

This model is reduced to the following third order switched linear model by applying the presented method over $[\omega_1, \omega_2] = [0.1, 100]$:

$$A_{1r} = \begin{bmatrix} -1.031 & 0.0061 & -0.0811 \\ -0.1413 & -1.606 & 0.7891 \\ -0.1708 & 1.028 & -2.723 \end{bmatrix} \quad A_{2r} = \begin{bmatrix} -0.8714 & 0.0209 & 0.1824 \\ -0.153 & -1.652 & -0.864 \\ 0.0540 & -0.6046 & -2.7 \end{bmatrix}$$

$$B_{1r} = \begin{bmatrix} -0.4154 \\ 0.595 \\ 0.7314 \end{bmatrix}, \quad B_{2r} = \begin{bmatrix} 0.315 \\ 1.136 \\ 2.371 \end{bmatrix} \quad C_{1r} = [-0.2443 \quad -1.076 \quad 0.1176]$$

$$C_{2r} = [0.5949 \quad 0.5316 \quad -0.5847], \quad D_{1r} = D_{2r} = 0$$

Fig. 1 shows the decay rate of the generlized Hankel singular values. The step response of the original and reduced order switched systems associated to randomly generate switching signal of Fig.2 is presented in Fig. 3.
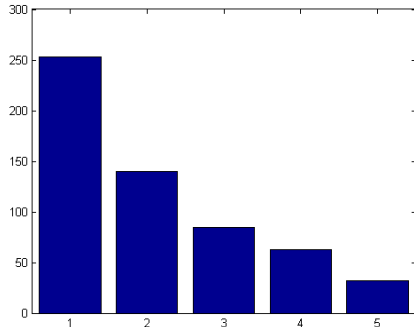

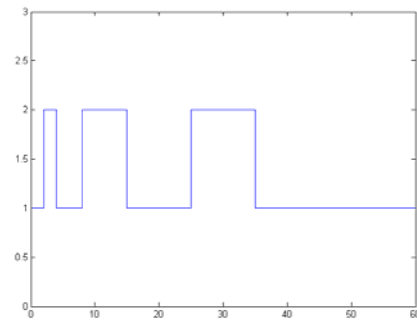Fig. 1. Generalized Hankel Singular Valuse(γ_i)
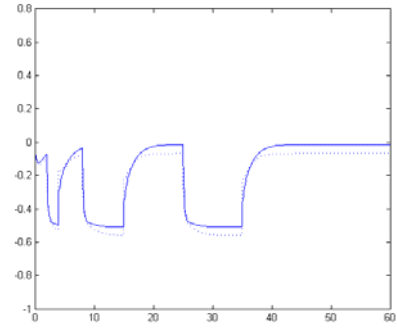

Fig. 2. Randomly generated switching signal

Fig. 3. Step response of original switched linear system(solid line) and the reduced order model (dotted).

Fig. 1. shows that most of the input/output information is in three states of the original systems. The proposed method provides accurate results after reduction of 2 states of the original system (40% of the states) globally (see Fig. 3)

B. Bimodal Switched linear System of order 100:

We consider a randomly generated bimodal switched linear system of order 100. This example shows that the presented method can be applied to fairly large systems. The original system is SISO and it is reduced to 87 using the proposed reduction method over $[\omega_1, \omega_2] = [1, 100]$.

The generlized Hankel singular values are shown in Fig. 4. The step response of the original and reduced order switched systems associated to randomly generate switching signal of Fig.5 is shown in Fig. 6.

The results after reduction of 13 states of the original system (13% of the states) are accurate locally and also globally (see Fig. 6). We already know from "Proposition. 2" that the reduced order switched system is stable. In order to see how the reduction method performs from stability viewpoint, we picked randomly generated subsystems that are stable and their poles are close to imaginary axis. Fig. 6. shows that the stability of the original systems is preserved even in such situations and the step response of the reduced order switched system follows the step response of the original system accurately.
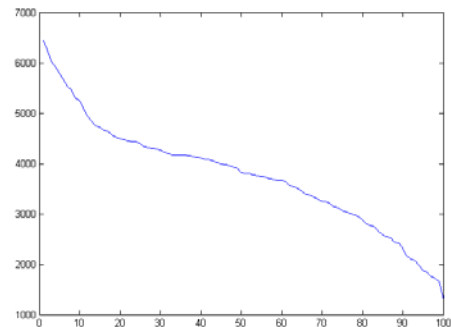

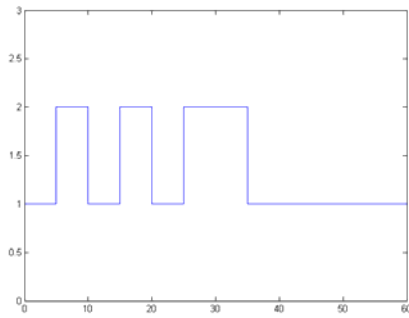Fig. 4. Generalized Hankel Singular Valuse(γ_i)

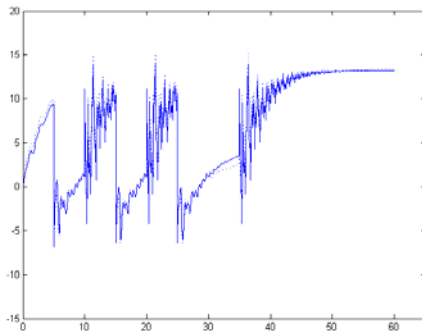Fig. 5. Randomly generated switching signal

Fig. 6.   Step response of original switched linear system (solid line) and the reduced order model (dotted).

V. CONCLUSION

A general framework for model order reduction of switched linear dynamical systems has been presented. In this paper we have reformulated the frequency domain balanced reduction method into this scheme but generally various gramian based reduction methods can be reformulated in the proposed generalized method easily and can be applied for reduction of switched system. The method preserves the stability of the original switched system   under arbitrary switching signal and is applicable to both continuous and discrete time systems. The method has some other merits such as providing more efficient approximation than some nonlinear reduction methods and also offers a reduction procedure which is independent of inputs or snapshots. One of the drawbacks of the method is that it is not always feasible because it is not always possible to find a common Lyapunov function for switched systems. Error is bounded but it is not guaranteed to be always small enough. There are different directions for further extensions such as using optimization, control and also various generalized gramians.

REFERENCES

[1]   A. C. Antoulas. Approximation of Large-Scale Dynamical Systems. Advances in Design and Control, SIAM, Philadelphia, 2005.

[2]   S. Gugercin and A. Antoulas. A survey of model reduction by balanced truncation and some new results. International Journal of Control,77:748–766, 2004.

[3]   Y. Chahlaoui and P. van Dooren. A collection of benchmarks examples for model reduction of linear time invariant dynamical systems.SLICOT Working Note, 2002.

[4]   B. C. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Trans. on Automatic Control*, 26(1):17–32, 1981.

[5]   G. E. Dullerud and E G. Paganini. A course in robust control theory : a convex approach. Springer, New York, 2000.

[6]   Li Li and F. Paganini, "Structured Coprime Factor Model Reduction based on LMIs", *Automatica*, Vol.41, pp:145-151, 2005.

[7]   D.F.Enns, Model Reduction with Balanced Realizations: An Error Bound and a Frequency Weighted Generalization, Proceedings of the *23rd IEEE Conference on Decision and Control,* Las Vegas, 1984, pp.127-132.

[8]   G. Wang, V. Sreeram, and W. Q. Liu, A New Frequency Weighted Balanced Truncation Method and an Error Bound, *IEEE Transactionson Automatic Control*, vol. 44, pp. 1734-1737, 1999.

[9]   V. Sreeram, and A. Ghafoor, "frequency Weighted Model Reduction Technique with Error Bounds", *American Control Conference*, 2005, Portland, OR, USA.

[10] W. Gawronski and J.-N. Juang, Model reduction in limited time and frequency intervals, *International Journal of System Sci.*, Vol. 21(2), pp. 349-376, 1990.

[11] A. J. Laub, M. T. Heath, C. C. Page, and R. C. Ward, "Computation of balancing transformations and other applications of simultaneous diagonalization algorithms, IEEE Trans. Automat. Contr., vol. AC-32, pp. 115-122, 1987.

[12] M.G. Safonov, R.Y. Chiang, A Schur method for balanced-truncation model reduction, IEEE Trans. Automat. Control AC-34 (1989) 729-733

[13] D. Liberzon. Switching in Systems and Control. Birkhauser, Boston, 2003.

[14] H. Sandberg, "Model Reduction of Linear Systems Using Extended Balanced Truncation," *American Control Conference*, 2008, Seattle, Washington, USA.

[15] P. Tabuada, A. Ames, A. A. Julius, and G. J. Pappas, "Approximate reduction of dynamical systems", *Systems and Control Letters*, 57(7):538-545, 2008.

[16] P. Tabuada, A. Ames, A. A. Julius, and G. J. Pappas, "Approximate reduction of dynamical systems", *IEEE Conference on Decision and Control, San Diego, CA,* December 2006.

[17] Z. Han and B. H. Krogh. "Reachability analysis of hybrid systems using reduced-order models." *American Control Conference*, 2004.

[18] H. R. Shaker "Frequency-Domain Balanced Stochastic Truncation for Continuous and Discrete Time Systems" *International Journal of Control, Automation, and Systems*, vol. 6, no. 2, pp.180-185, 2008

[19] H. R. Shaker and R. Wisniewski ."Discussion: "Model reduction of large-scale discrete plants with specified frequency-domain balanced structure", *ASME Int. J. of Dynamic Systems, Measurement and control*, 2008.

[20] H. R. Shaker and M. Samavat, "Accuracy enhancement in HiMAT aircraft controller reduction using a recently developed balanced technique" International Journal of Automatic Control and System Engineering, vol. 5, pp. 31- 36, 2005.

[21] H. R. Shaker and M. Samavat, "Accuracy enhancement in HiMAT aircraft controller reduction," Proc. of 1st Int. Symposium on Systems and Control in Aerospace and Astronautics, Harbin, China, 2006.

[22] Luc C.G.J.M. Habets and Jan H. van Schuppen. Reduction of affine systems on polytopes. *In Proceedings of the International Symposium MTNS* ,2002.

[23] H. Gao, J. Lam, C. Wang "Model simplification for switched hybrid systems" *Systems & Control Letters*, Vol. 55, No. 12. 2006, pp. 1015-1021.