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ON TOPOLOGICAL SEQUENCE ENTROPY AND CHAOTIC MAPS ON INVERSE LIMIT SPACES

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Abstract. The aim of this paper is to prove the following results: a continuous map $f\colon [0,1] \to [0,1]$ is chaotic iff the shift map $\sigma_f\colon \varprojlim([0,1], f) \to \varprojlim([0,1], f)$ is chaotic. However, this result fails, in general, for arbitrary compact metric spaces. $\sigma_f\colon \varprojlim([0,1], f) \to \varprojlim([0,1], f)$ is chaotic iff there exists an increasing sequence of positive integers A such that the topological sequence entropy $h_A(\sigma_f) > 0$. Finally, for any A there exists a chaotic continuous map $f_A\colon [0,1] \to [0,1]$ such that $h_A(\sigma_{f_A}) = 0$.

1. Introduction

Let (X, d) and $f\colon X \to X$ be a compact metric space and a continuous map respectively. Consider the space of sequences

$$\varprojlim(X, f) = \{\underline{x} = (x_0, x_1, \dots, x_n, \dots) : x_i \in X,\ f(x_i) = x_{i-1}, \text{ for } i = 1, 2, \dots\}.$$

This set is called the **inverse limit space** associated to X and f. Define a new metric $\widetilde{d}$ on $\varprojlim(X, f)$ as

$$\widetilde{d}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

where $\underline{x} = (x_0, x_1, \dots, x_n, \dots)$ and $\underline{y} = (y_0, y_1, \dots, y_n, \dots)$. Then $(\varprojlim(X, f), \widetilde{d})$ is a compact metric space. Consider the natural projection $\pi : \varprojlim(X, f) \to X$ defined by $\pi(x_0, x_1, \dots, x_n, \dots) = x_0$. Note that $\widetilde{d}(\underline{x}, \underline{y}) \geq d(\pi(\underline{x}), \pi(\underline{y}))$ for all $\underline{x}, \underline{y} \in \varprojlim(X, f)$. The **shift map** is a homeomorphism $\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$ defined by

$$\sigma_f(\underline{x}) = \sigma_f(x_0, x_1, \dots, x_n, \dots) = (f(x_0), x_0, x_1, \dots, x_n, \dots).$$

It is clear that $\pi \circ \sigma_f = f \circ \pi$.

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Inverse limit spaces have been studied in the setting of dynamical systems in a large number of papers. In [**6**], Shihai Li proved that some dynamical properties hold at the same time for f and σ_f. In particular, he showed that f is chaotic in Devaney's sense iff σ_f is also like that. He also proved a similar result for a suitable definition of w-chaos. In this paper, a similar result is studied in case of the Li-Yorke's chaos. Recall briefly this definition of chaos.

A point $p \in X$ is **periodic** if there exists a positive integer n such that $f^n(p) = p$. The smallest positive integer satisfying this condition is called the **period of** p. Denote by $\mathrm{Per}(f)$ the set of **periodic points of** f. A point $x \in X$ is said to be **asymptotically periodic** if there exists a $p \in \mathrm{Per}(f)$ such that $\limsup_{n\to\infty} d(f^n(x), f^n(p)) = 0$. A map $f\colon X \to X$ is said to be **chaotic in the sense of Li-Yorke**or simply **chaotic** if there exists an uncountable set $D \subset X \setminus \mathrm{Per}(f)$ such that

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0,$$
$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$

hold for all $x, y \in D$, $x \neq y$. D is called a **scrambled set** of f.

The Li-Yorke's chaos on inverse limit spaces has been studied by Gu Rongbao in [**3**]. In that paper the author attempts to prove that a continuous map f is chaotic iff the shift map σ_f is chaotic. However, in the proof he uses implicitly that f is surjective. As we will see later, this hypothesis on f cannot be removed in the following theorem essentially proved in [**3**].

Theorem 1.1. *Suppose f that is surjective. Then it is chaotic in the sense of Li-Yorke if and only if the map σ_f is chaotic in the sense of Li-Yorke.*

When continuous maps $f\colon [0,1] \to [0,1]$ are concerned, the Li-Yorke's chaos is connected with the notion of topological sequence entropy. Let us recall the definition (see [**2**]). Let $A = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers. Given $\epsilon > 0$, we say that $E \subset X$ is an (A, ϵ, n, f)-separated set if for any $x, y \in E$ with $x \neq y$ there exists $1 \leq k \leq n$ such that $d(f^{a_k}(x), f^{a_k}(y)) > \epsilon$. Denote by $s_n(A, \epsilon, f)$ the cardinality of any maximal (A, ϵ, n, f)-separated set. The **topological sequence entropy** of f is given by

$$h_A(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(A, \epsilon, f).$$

In general,

$$h_A(f) \geq h_A(\sigma_f) \tag{1}$$

for every A. When f is surjective we obtain the equality (see [**2**])

$$h_A(f) = h_A(\sigma_f). \tag{2}$$

The connection between the Li-Yorke's chaos and the topological sequence entropy is established in the following result (see [**1**] and [**5**]).

Theorem 1.2. *Let $c, d \in \mathbb{R}$ and let $f\colon [c,d] \to [c,d]$ be continuous. Then*

(a) *f is chaotic iff there exists an increasing sequence of positive integers A such that $h_A(f) > 0$.*

(b) *For any increasing sequence A there exists a chaotic map $f_A\colon [c,d] \to [c,d]$ such that $h_A(f_A) = 0$.*

Theorem 1.2(a) does not hold in general for continuous maps on arbitrary compact metric spaces as it can be seen in [**8**]. In that paper, on $[0,1] \times [0,1]$ a chaotic map f with $\sup_A h_A(f) = 0$ and a non-chaotic map g with $\sup_A h_A(g) > 0$ are constructed.

The aim of this paper is to prove the following results: $f\colon [0,1] \to [0,1]$ is chaotic iff σ_f is chaotic. Theorem 1.2 holds for maps $\sigma_f\colon \varprojlim([0,1], f) \to \varprojlim([0,1], f)$. Moreover, an example of a chaotic map f for which σ_f is not chaotic is given.

2. Positive Results for One-Dimensional Maps

Let $f\colon [0,1] \to [0,1]$ be continuous. Consider $[a,b] = \bigcap_{n \geq 0} f^n[0,1]$. Then $f|_{[a,b]}\colon [a,b] \to [a,b]$ is obviously surjective.

Proposition 2.1. *Under the above conditions f is chaotic iff $f|_{[a,b]}$ is chaotic.*

Proof. It is clear that if $f|_{[a,b]}$ is chaotic then f is chaotic. Suppose that f is chaotic and let D be a scrambled of f. It is easy to see that $f^n(D)$ is also a scrambled set of f. Let $D_n = f^n(D) \cap [a,b]$. If $\{f^n(x) : n \geq 0\} \cap [a,b] = \emptyset$, then x is asymptotically periodic and then $x \notin D_n$ for all $n \in \mathbb{N}$. So, it must exist a positive integer n_0 such that D_{n_0} is uncountable. Then, $f|_{[a,b]}$ is chaotic. □

Theorem 2.2. *Let $f\colon [0,1] \to [0,1]$ be continuous. Then:*

(a) *f is chaotic if and only if σ_f is chaotic.*

(b) *$\sigma_f\colon \varprojlim([0,1], f) \to \varprojlim([0,1], f)$ is chaotic if and only if there exists an increasing sequence of positive integers A such that $h_A(\sigma_f) > 0$.*

(c) *For any increasing sequence of positive integers A there exists a chaotic map $f_A\colon [0,1] \to [0,1]$ such that $h_A(\sigma_{f_A}) = 0$.*

Proof. It is clear that

$$\varprojlim([0,1], f) = \{(x_0, x_1, \ldots, x_n, \ldots) : x_i \in [a,b],\ f(x_i) = x_{i-1}\} = \varprojlim([a,b], f).$$

First of all we prove (a). Assume that f is chaotic. By Proposition 2.1, $f|_{[a,b]}$ is also chaotic. Applying Theorem 1.1 it follows that σ_f is chaotic. Conversely, suppose that σ_f is chaotic. Applying Theorem 1.1 it follows that $f|_{[a,b]}$ is chaotic. Proposition 2.1 proves that f is chaotic.

Part (b). If σ_f is chaotic, then it follows by (a) that f is chaotic. Hence, by Proposition 2.1, $f|_{[a,b]}$ is chaotic. Applying Theorem 1.2 (a), there exists an increasing sequence of positive integers such that $h_A(f|_{[a,b]}) > 0$. Since $f|_{[a,b]}$ is surjective, by (2), $h_A(\sigma_f) = h_A(f|_{[a,b]}) > 0$. Now suppose that σ_f is non-chaotic. Assertion (a) states that f is non-chaotic. Applying Theorem 1.2 and (1), we conclude that $h_A(\sigma_f) \leq h_A(f) = 0$ for any increasing sequence of positive integers A.

Part (c). Let A be an arbitrary sequence of positive integers. By Theorem 1.2(b), there exists a chaotic map $f_A\colon [0,1] \to [0,1]$ such that $h_A(f_A) = 0$. Since f_A is chaotic, by (a), σ_{f_A} is also chaotic. By (2), $h_A(\sigma_{f_A}) \leq h_A(f_A) = 0$, and the proof ends. □

3. A Counterexample

As usual, $\mathbb{Z}$ will stand for the set of integers, while if $Z \subset \mathbb{Z}$ then Z^n (resp. Z^∞) will denote the set of finite sequences of length n (resp. infinite sequences) of elements from Z. If $\theta \in \mathbb{Z}^n$ or $\alpha \in \mathbb{Z}^\infty$ then we will often describe them through their components as $(\theta_1, \theta_2, \dots, \theta_n)$ or $(\alpha_i)_{i=1}^\infty$, respectively. The **shift map** $\sigma\colon \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ is defined by $\sigma((\alpha_i)_{i=1}^\infty) = (\alpha_{i+1})_{i=1}^\infty$. If $\theta \in \mathbb{Z}^n$ and $\vartheta \in \mathbb{Z}^m$ (with $m \leq \infty$) then $\theta * \vartheta \in \mathbb{Z}^{n+m}$ (where $n+\infty$ means ∞) will denote the sequence λ defined by $\lambda_i = \theta_i$ if $1 \leq i \leq n$ and $\lambda_i = \vartheta_{i-n}$ if $i > n$. In what follows we will denote $\mathbf{0} = (0, 0, \dots, 0, \dots)$ and $\mathbf{1} = (1, 1, \dots, 1, \dots)$, while if $\alpha \in \mathbb{Z}^\infty$, then $\alpha|_n \in \mathbb{Z}^n$ is defined by $\alpha|_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

This section is devoted to construct a chaotic map f on a compact metric space for which σ_f is non-chaotic. In order to do this we will need some information concerning so-called **weakly unimodal maps of type** 2^∞. Recall briefly the definition. We say that a continuous map $f\colon [0,1] \to [0,1]$ is **weakly unimodal** if $f(0) = f(1) = 0$, it is non-constant and there is $c \in (0,1)$ such that $f|_{(0,c)}$ and $f|_{(c,1)}$ are monotone. The map f is said to be of **type** 2^∞ if it has periodic points of period 2^n for any $n \geq 0$ but no other periods.

Weakly unimodal maps of type 2^∞ (briefly, **w-maps**) were studied in [**4**]. In that paper it was proved that for any w-map f it is possible to construct a family $\{K_\alpha(f)\}_{\alpha \in \mathbb{Z}^\infty}$ (or simply $\{K_\alpha\}_{\alpha \in \mathbb{Z}^\infty}$ if there is no ambiguity on f) of pairwise disjoint (possibly degenerate) compact subintervals of $[0,1]$ satisfying the following key properties (P1)–(P4):

- (P1) The interval $K_\mathbf{0}$ contains all absolute maxima of f.
- (P2) Define in $\mathbb{Z}^\infty$ the following total ordering: if $\alpha, \beta \in \mathbb{Z}^\infty$, $\alpha \neq \beta$ and k is the first integer such that $\alpha_k \neq \beta_k$, then $\alpha < \beta$ if either Card $\{1 \leq i < k : \alpha_i \leq 0\}$ is even and $\alpha_k < \beta_k$ or Card $\{1 \leq i < k : \alpha_i \leq 0\}$ is odd and $\beta_k < \alpha_k$. Then $\alpha < \beta$ if and only if $K_\alpha < K_\beta$ (that is, $x < y$ for all $x \in K_\alpha$, $y \in K_\beta$).
- (P3) Let $\alpha \in \mathbb{Z}^\infty$, $\alpha \neq \mathbf{0}$, and let k be the first integer such that $\alpha_k \neq 0$. Define $\beta \in \mathbb{Z}^\infty$ by $\beta_i = 1$ for $1 \leq i \leq k-1$, $\beta_k = 1 - |\alpha_k|$ and $\beta_i = \alpha_i$ for $i > k$

Then $f(K_\alpha) = K_\beta$ and $f(K_{\mathbf{0}}) \subset K_{\mathbf{1}}$.

For any n and $\alpha \in \mathbb{Z}^\infty$, let $K_{\alpha|_n}(f)$ (or just $K_{\alpha|_n}$) be the least interval including all intervals K_β, $\beta \in \mathbb{Z}^\infty$, such that $\alpha|_n = \beta|_n$. Then

(P4) For any $\alpha \in \mathbb{Z}^\infty$, $K_\alpha = \bigcap_{n=1}^\infty K_{\alpha|_n}$.

Additionally, for any fixed n it can be easily checked that the intervals K_θ, $\theta \in \mathbb{Z}^n$, are open and pairwise disjoint and (after replacing ∞ by n, $\mathbf{0}$ by $(0,0,\dots,\overset{n}{0})$ and $\mathbf{1}$ by $(1,1,\dots,\overset{n}{1})$), they also satisfy (P1)–(P3). Observe that if $\theta \in \{-1,0,1\}^n$ and we put $|\theta| := (|\theta_1|, |\theta_2|, \dots, |\theta_n|)$ then $f^{2^n}(K_\theta) \subset K_{|\theta|}$; in particular, $f^{2^n}(K_\theta) \subset K_\theta$ if $\theta \in \{0,1\}^n$.

In the rest of this section $\widetilde{f}$ will denote a fixed w-map with the additional property that $\alpha \in \mathbb{Z}^\infty$ implies $K_\alpha(\widetilde{f})$ is non-degenerate if and only if there is an $n \geq 0$ such that $\sigma^n(\alpha) = \mathbf{0}$. An example of such a map is constructed in [**4**]; it is possible to show that the stunted tent map $\widetilde{f}(x) = \min\{1 - |2x-1|, \mu\}$ ($\mu \approx 0.8249\dots$) from [**7**] is also a w-map with this property.

Bd (Z), Cl (Z) and Int (Z) will respectively denote the boundary, the closure and the interior of Z.

Now, we are ready to construct our counterexample. Consider

$$X = \bigcup_{\alpha \in \{-1,0,1\}^\infty} \text{Bd } (K_\alpha).$$

Let us emphasize that Bd (K_α) consists of both endpoints of K_α if it is non-degenerate and of its only point if it is degenerate. Let $f = \tilde{f}|_X : X \to X$ be the restriction of the above-mentioned w-map $\tilde{f}$ to the set X. The following lemma shows that the above choices make sense.

Lemma 3.1. *X is a compact set and $f\colon X \to X$ is a well-defined continuous map.*

Proof. Since

$$X = \left(\bigcap_{n=1}^\infty \bigcup_{\theta \in \{-1,0,1\}^n} \text{Cl } (K_\theta) \right) \setminus \bigcup_{\alpha \in \{-1,0,1\}^\infty} \text{Int } (K_\alpha),$$

by (P2) and (P4), X is compact.

Recall that if $\mathbf{0} \neq \alpha \in \{-1,0,1\}^\infty$ then $\tilde{f}$ carries the interval K_α onto K_β with β defined as in (P3) (and hence belonging to $\{-1,0,1\}^\infty$). Moreover, $\tilde{f}$ is monotone on K_α because of (P1). So it maps the endpoints of K_α onto the endpoints of K_β. Similarly, since $K_{\mathbf{1}}$ is degenerate both endpoints of $K_{\mathbf{0}}$ are mapped onto its only point. The conclusion is that $f(X) \subset X$ and the map $f : X \to X$ is well–defined (and it is clearly continuous). □

Let $X_1 = \bigcup_{\alpha \in \{0,1\}^\infty} \text{Bd } (K_\alpha)$. Note that, by (P3), $\bigcap_{n \geq 0} f^n(X) = X_1$. Let us see that f is chaotic while $f|_{X_1}$ is non–chaotic.

Theorem 3.2. *$f|_{X_1}$ is non–chaotic and hence σ_f is non–chaotic.*

Proof. Let $x, y \in X_1$ with $x \in K_\alpha$ and $y \in K_\beta$ for some $\alpha \neq \beta$, $\alpha, \beta \in \{0,1\}^\infty$. We will see that there exists a positive real number M satisfying

$$\liminf_{i\to\infty} |f^i(x) - f^i(y)| \geq M,$$

and hence x and y cannot belong to the same scrambled set D. This proves that $\mathrm{Card}(D) \leq 2$ for each scrambled set D of $f|_{X_1}$ and so $f|_{X_1}$ is non–chaotic.

Let j be the first positive integer satisfying $\alpha_j \neq \beta_j$. Suppose, for example, that $\alpha_j = 0$ and $\beta_j = 1$. For any $\theta \in \{0,1\}^{j-1}$ consider the closed interval $A_{\theta*0}$ satisfying $K_{\theta*1} < A_{\theta*0} < K_{\theta*0}$ or $K_{\theta*0} < A_{\theta*0} < K_{\theta*1}$, and let $M = \min\{|A_{\theta*0}| : \theta \in \{0,1\}^{j-1}\} > 0$. By (P3) and (P2), $f^i(x) \in K_{\theta*1}$ and $f^i(y) \in K_{\theta*0}$ or viceversa for all $i \in \mathbb{N}$ and for all $\theta \in \{0,1\}^{j-1}$. This shows that

$$|f^i(x) - f^i(y)| \geq M$$

which concludes the proof. □

For any $\alpha \in \{-1,0,1\}^\infty$ let $\tau(\alpha|_n) = \sum_{i=1}^n |\alpha_i| 2^{i-1}$ for all $n \in \mathbb{N}$.

Theorem 3.3. *$f : X \to X$ is chaotic.*

Proof. Define on $\{-1,1\}^\infty$ the following relation: $\alpha \sim \beta$ if and only if there exists a positive integer k such that $\sigma^k(\alpha) = \sigma^k(\beta)$. Obviously $\sim$ is an equivalence relation. Moreover, for $\alpha \in \{-1,1\}^\infty$ the class of α is given by

$$[\alpha] = \bigcup_{k=0}^{\infty} \{\sigma^{-k}(\sigma^k(\alpha))\} \cap \{-1,1\}^\infty.$$

Since $\{-1,1\}^\infty$ is uncountable and $[\alpha]$ is countable for all $\alpha \in \{-1,1\}^\infty$, the set containing all the equivalence classes $\{-1,1\}^\infty/\sim$ is uncountable.

Let $\mathcal{A}$ be a set containing one and only one representative $\alpha \in [\alpha]$ for all $[\alpha] \in \{-1,1\}^\infty/\sim$, and let D be the set containing exactly one $x \in X \cap K_\alpha$ for all $\alpha \in \mathcal{A}$. We claim that D is a scrambled set for f. In order to see this take $x, y \in D$, $x \in K_\alpha$ and $y \in K_\beta$ with $\alpha, \beta \in \mathcal{A}$, $\alpha \neq \beta$. Then there exists an increasing sequence of positive integers $(k_i)_{i=0}^\infty$ satisfying $\alpha_{k_i} \neq \beta_{k_i}$ and $\alpha_j = \beta_j$ if $j \neq k_i$ for all $i \in \mathbb{N}$. Note that $\tau(\alpha|_n) = \tau(\beta|_n)$ for all $n \in \mathbb{N}$. Suppose, for example, that $\alpha_{k_i} = 1$ and $\beta_{k_i} = -1$ for some i. Then, by (P3),

$$f^{\tau(\alpha|_{k_i})}(x) \in K_{(0,0,\ldots,\overset{k_i-1}{0},1)*\sigma^{k_i}(\alpha)} \text{ and } f^{\tau(\alpha|_{k_i})}(y) \in K_{(0,0,\ldots,\overset{k_i-1}{0},-1)*\sigma^{k_i}(\beta)}.$$

By (P2), $|f^{\tau(\alpha|_{k_i})}(x) - f^{\tau(\alpha|_{k_i})}(y)| \geq |K_{\mathbf{0}}|$ and then

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geq \limsup_{i\to\infty} |f^{\tau(\alpha|_{k_i})}(x) - f^{\tau(\alpha|_{k_i})}(y)| \geq |K_{\mathbf{0}}|.$$

Let now $j \neq k_i$ for all $i \in \mathbb{N}$, and suppose that $\alpha_j = \beta_j = 1$ (the case $\alpha_j = -1$ is analogous). Then $f^{\tau(\alpha|_j)}(y), f^{\tau(\alpha|_j)}(x) \in K_{(0,0,\ldots,0,\overset{j}{1})}$. For any $n \in \mathbb{N}$ let $K^+_{\mathbf{0}|_n}$ and $K^-_{\mathbf{0}|_n}$ be the right and left side components of $K_{\mathbf{0}} \setminus K_{\mathbf{0}|_n}$. Applying (P4), for any $\varepsilon > 0$ there exists a positive integer n_ε such that $\max\{|K^+_{\mathbf{0}|_n}|, |K^-_{\mathbf{0}|_n}|\} < \varepsilon$ for all $n \geq n_\varepsilon$. By (P2) and (P3), $K_{(0,0,\ldots,0,\overset{j}{1})} \subset K^-_{\mathbf{0}|_j}$ or $K_{(0,0,\ldots,0,\overset{j}{1})} \subset K^+_{\mathbf{0}|_j}$, and so for $j \geq n_\varepsilon$ we conclude that $|f^{\tau(\alpha|_j)}(y) - f^{\tau(\alpha|_j)}(x)| < \varepsilon$. This proves that

$$\liminf_{n\to\infty} |f^n(y) - f^n(x)| = 0,$$

and the proof concludes. □

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