The Cournot-Theocharis Problem Reconsidered

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Abstract

In 1959 Theocharis [10] showed that with linear demand and constant marginal costs Cournot equilibrium is destabilized when the competitors become more than three. With three competitors the Cournot equilibrium point becomes neutrally stable, so, even then, any perturbation throws the system into an endless oscillation. Theocharis's argument was in fact proposed already in 1939 by Palander [4]. None of these authors considered the global dynamics of the system, which necessarily becomes nonlinear when consideration is taken of the facts that prices, supply quantities, and profits of active firms cannot be negative. In the present paper we address the global dynamics.

1 Introduction

In a short communication Theocharis (1959) [10] reconsiders the classical Cournot oligopoly problem (see Cournot (1838) [1]), assuming a linear demand function and constant marginal costs. He demonstrates that with three competitors the Cournot equilibrium becomes neutrally stable and

*The authors 1 and 3 are partially supported by the grant FS 00684/PI/04 from Fundación Séneca (Comunidad Autónoma de la Región de Murcia, Spain).

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with four it becomes unstable. This paper has been taken as "seminal" by several authors, so, despite its limited scope, it might be interesting to take a new look at it. See for instance Okuguchi (1976) [7] and Tuinstra (2004) [9](a book review of Puu and Suskho [5]).

As one of the present authors pointed out in another article Puu (2006) [6], the Theocharis argument was stated under more general conditions 20 years earlier by Palander (1939) [4]. Palander's article was published in Swedish, so Theocharis cannot be blamed for rediscovering this, though Palander (1936) [3] gave most of the argument in a presentation at a regular Cowles commission meeting in 1936, so some of Theocharis's supervisors might have had a chance to know the argument.

Our purpose is to consider the global dynamics of this simple model, taking into account that supply quantities cannot become negative, something that makes the model *non*linear. Neither Palander, nor Theocharis, dealt with the global dynamics, so they considered local stability alone. Palander just considered the loss of stability, though for more general conditions, whereas Theocharis proposed a general solution to the linear model, which, however, is not relevant for the global dynamics when the nonnegativity constraints are taken in account.

1.1 The Theocharis 1959 Model

Let us now specify the model. According to the linear demand function, its inverse gives market price:

$$p := a - b \sum_{i=1}^{i=n} q_i \tag{1}$$

as dependent on the total sum of the supplies of the n competitors. Obviously $\sum_{i=1}^{i=n} q_i$ must be less than or at most equal to the ratio a/b. To simplify notation, we can define total supply as

$$Q := \sum_{i=1}^{i=n} q_i \tag{2}$$

and "residual supply" (not under the control of the i:th competitor) as:

$$Q_i := Q - q_i, \qquad i = 1, ...n \tag{3}$$

Total revenue for the i:th competitor hence becomes:

$$R_i = (a - b(Q_i + q_i))\, q_i, \qquad i = 1, ...n \tag{4}$$

whence marginal revenue:

$$R_i' = \frac{\partial R_i}{\partial q_i} = a - bQ_i - 2bq_i, \qquad i = 1, ...n \tag{5}$$

Given constant marginal costs, total costs are

$$C_i = c_i q_i, \qquad i = 1, ...n \tag{6}$$

and marginal costs:

$$C_i' = \frac{\partial C_i}{\partial q_i} = c_i, \qquad i = 1, ...n \tag{7}$$

Equating marginal revenue (5) to marginal cost (7) to obtain profit maximum, we can solve for q_i:

$$q_i = \frac{1}{2}\left(\frac{a - c_i}{b} - Q_i\right), \qquad i = 1, ...n \tag{8}$$

which can either be treated as a simultaneous system of equations, or, else as a system of recurrence equations. Dealing with the first possibility, (8), given definitions (2) and (3), can be solved for the coordinates of the Cournot equilibrium point. To simplify again, define the average marginal cost,

$$c = \frac{1}{n}\sum_{i=1}^{i=n} c_i \tag{9}$$

Then the Cournot point has coordinates:

$$\overline{q}_i = \frac{a + nc - (n+1)c_i}{(n+1)b}, \qquad i = 1, ...n \tag{10}$$

We may also want to know the total output at the Cournot point, which is easily calculated as the sum of (10):

$$\overline{Q} = \frac{n}{n+1}\frac{a - c}{b} \tag{11}$$

It is reassuring that, in the Cournot equilibrium point, price $\overline{p} = a - b\overline{Q} = \frac{1}{n+1}a + \frac{n}{n+1}c > 0$ is always positive.

However, as stated, we may want to treat (8) as a dynamical system. Suppose each firm assumes all the competitors to retain their supplies from the previous period. Then the "best reply" of this firm is:

$$q_i\left(t+1\right) = \frac{1}{2}\left(\frac{a - c_i}{b} - Q_i\left(t\right)\right), \qquad i = 1, ...n \tag{12}$$

This relation (12) is usually called the reaction function. Note that (10) is a particular solution to (12), so, if we want, we can easily restate (10) in terms of deviation variables $q_i\left(t+1\right) - \overline{q}_i$, obtaining a homogenous system.

It is worthwhile to note that the derivatives of (12):

$$\frac{\partial q_i\left(t+1\right)}{\partial Q_i\left(t\right)} = -\frac{1}{2}, \qquad i = 1, ...n \tag{13}$$

are constant, $-\frac{1}{2}$, independent of the coordinates of the Cournot equilibrium point. These derivatives enter the Jacobian determinant of the system in all the off-diagonal elements, whereas the diagonal elements are zero (because the best reply of each firm does not depend on its own supply in the previous period). We have:

$$J = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 \end{bmatrix} \tag{14}$$

Next, the characteristic equation is:

$$|J - \lambda I| = \left(\lambda - \frac{1}{2}\right)^{n-1} \left(\lambda + \frac{n-1}{2}\right) = 0 \tag{15}$$

Hence, there are $n-1$ eigenvalues $\lambda_{1,\ldots n-1} = \frac{1}{2}$ and 1 last eigenvalue $\lambda_n = -\frac{n-1}{2}$. Theocharis concludes that in duopoly there are two eigenvalues $\lambda_1 = \frac{1}{2}$, and $\lambda_2 = -\frac{1}{2}$, both less than unity in absolute value, so the Cournot equilibrium is stable. In triopoly the eigenvalues are $\lambda_{1,2} = \frac{1}{2}$, $\lambda_3 = -1$, so the equilibrium is neutrally stable with a tendency to endless but stationary oscillation. For $n > 3$, $|\lambda_n| = \frac{n-1}{2} > 1$, so the equilibrium is unstable. As usual, the eigenvalues correspond to a diagonalization of the system, the first $n-1$ associated with differences of the phase variables, the last with the sum of all the phase variables.

1.2 Palander's 1939 Argument

In his 1939 article Palander (1939) [4] wrote: "*as a condition for an equilibrium with a certain number of competitors to be stable to exogenous disturbances, one can stipulate that the derivative of the reaction function f' must be such that the condition $|(n-1)\,f'| < 1$ holds. If this criterion is applied to, for instance, the case with a linear demand function and constant marginal costs, the equilibria become unstable as soon as the number of competitors exceeds three. Not even in the case of three competitors will equilibrium be restored, rather there remains an endless oscillation*". As we see, Palander considers not just linear demand (and hence reaction) functions, but stability of the Cournot equilibrium in general.

Further, his main interest is the case dealt with by Robinson (1933) [8] for a monopoly, where the demand function is kinked, due to different demand elasticities of different groups of consumers, and increasing elasticity when a price gets sufficiently low for new groups to afford the commodity.

As realized by Robinson, a monopoly facing such a demand curve will have several local profit maxima, and has to choose the global optimum.

Palander develops this idea to the case of duopoly, and realizes that different supplies by the competitor may cause a firm to choose one or the other of the local optima. The reaction functions hence are composed by disjoint linear segments, which may have several intersections or none at all. Palander discovers multistability and fragmented basins, which is something of an exploit given the computation facilities of the time. We will not enter the details of this more complicated issue, which one of the present authors dealt with extensively in a previous publication (Puu and Sushko (2002)). We only note that Palander considers the simpler case of a linear demand function just *en passant*.

2 Non-linearity and the Global Dynamics

It is now time to consider that the quantities produced by all the firms, $q_i(t+1)$, cannot become negative according to (12). Hence they should be restricted to

$$\frac{a-c_i}{b} \geq Q_i, \qquad i = 1, ...n \tag{16}$$

However, there is another fact we must consider: Profits must be positive, or some of the firms will drop out from the market (temporarily or permanently). So let us consider expected profits as obtained from (4) and (6):

$$\Pi_i = R_i - C_i = (a - c_i - b(Q_i + q_i))\, q_i, \qquad i = 1, ...n$$

Substituting the best reply for q_i from (8), and simplifying, we get:

$$\Pi_i = \frac{b}{4}\left(\frac{a-c_i}{b} - Q_i\right)^2, \qquad i = 1, ...n \tag{17}$$

As the bracketed expression is squared, it might seem that we always have a non-negative profit. However, for cases where (16) is not fulfilled, the non-negativity of profits is due to the fact that, with negative output, positive costs dominate over negative revenues, a mathematically, but not economically, meaningful case. So, we conclude that the condition for non-negative profit is the same as the condition for non-negative output.

Hence we can reformulate the reaction functions as follows

$$q_i(t+1) = f_i(Q_i(t)) := \begin{cases} \frac{1}{2}(\frac{a-c_i}{b} - Q_i(t)) & Q_i(t) \leq \frac{a-c_i}{b}, \\ 0 & Q_i(t) > \frac{a-c_i}{b}, \end{cases} \quad i = 1, ..., n \tag{18}$$

Defining $F = (f_1, f_2, ..., f_n)$, the iteration of this non-linear map is not simple. Preliminary numerical experiments with (18) indicate that eventually the model can go to steady states of a monopoly or a duopoly. With three competitors, a permanent oscillation can appear, and with four or more, oscillations of period two, where all firms drop out every second period, seems to be the rule. However, any number of competitors may drop out, permanently or temporarily and the remaining firm(s) may go to Cournot equilibrium or even a monopolistic state.

2.1 Preliminary Considerations: Existence of Periodic Orbits

It is easy to check that the general solution of the system of difference equations given in (12) is

$$q_i(t) = A_i(\frac{1}{2})^t + B_i(-\frac{n-1}{2})^t + \frac{a + nc - (n+1)c_i}{(n+1)b} \qquad i = 1, 2 \ldots, n \tag{19}$$

where A_i and B_i can be found given the initial values and the fraction on the right hand side is the equilibrium value $\overline{q}_i$.

Then if we look at the general solution for $q_i(t)$ given in (19) and if the number of firms is greater than three, because of the term $B_i(-\frac{n-1}{2})^t$, apart from the Cournot point, for F to have a periodic orbit it is necessary that such a periodic orbit contains a point of the form $(q_1, q_2, ..., q_n)$ such that $q_j = 0$ for some $j \in \{1, 2, \ldots, n\}$. This could make one suspects that the orbit generated by $(0, 0, \ldots, 0)$ would play an important role when studying the dynamics of the Cournot competition.

Next we show that there always exists a periodic orbit of period two. To this end, we are going to consider the orbit generated by $(0, 0, ..., 0)$. For $t > 0$, let

$$(q_1^0(t), q_2^0(t), ..., q_n^0(t)) = F^t(0, 0, ..., 0).$$

Then for any $t \geq 0$ it is easy to check by induction that

$$q_i^0(2t) \leq q_i^0(2t + 2) \leq q_i^0(2t + 3) \leq q_i^0(2t + 1) \tag{20}$$

for $i = 1, 2, ..., n$. Then $(q_i^0(2t))_t$ is an increasing sequence bounded by $\frac{a-c_i}{2b}$ and $(q_i^0(2t + 1))_t$ is a decreasing sequence bounded by 0 and therefore, both of them have a limit point q_i^m and q_i^M, respectively. Note that $q_i^m \leq q_i^M$ for $i = 1, 2, ..., n$. In addition, by the continuity of F we also have that

$$F(q_1^m, q_2^m, ..., q_n^m) = (q_1^M, q_2^M, ..., q_n^M)$$

and

$$F(q_1^M, q_2^M, ..., q_n^M) = (q_1^m, q_2^m, ..., q_n^m).$$

Then either $(q_1^m, q_2^m, ..., q_n^m)$ is a fixed point or a periodic point of period 2.

2.2 The Cournot Equilibrium

In this section we study when the firms reach the Cournot equilibrium. To this end note that the equation of total quantity

$$Q(t+1) = -\frac{n-1}{2}Q(t) + n\frac{a-c}{2b}$$

has an equilibrium point at $\overline{Q} = \frac{a-c}{b}\frac{n}{n+1}$. Then, the hyperplane

$$q_1 + q_2 + ... + q_n = \overline{Q}$$

is invariant for the system of difference equations and hence the set

$$\mathcal{P} = \{(q_1, q_2, ..., q_n) \in \mathbb{R}^n : q_1 + q_2 + ... + q_n = \overline{Q},\ q_i \geq 0\}$$

is invariant for the non–linear system. Moreover, taking into account that $Q_i = \overline{Q} - q_i$, the system restricted to $\mathcal{P}$ can be rewritten as

$$q_i(t+1) = \frac{1}{2}\left(\frac{a-c_i}{b} - Q_i(t)\right) = \frac{1}{2}\left(\frac{a-c_i}{b} - \overline{Q}\right) + \frac{1}{2}q_i(t),\ i = 1, 2, ..., n.$$

The dynamics is then very simple because any individual orbit converges to the equilibrium point

$$\overline{q}_i = \frac{a-c_i}{b} - \overline{Q},\ i = 1, 2, ..., n,$$

and so any orbit which starts from $(q_1, q_2, ..., q_n) \in \mathcal{P}$ converges to the Cournot equilibrium point. We have to point out that the set $\mathcal{P}$ has zero n–dimensional Lebesgue measure, and then from the point of view of experiments the probability of finding an orbit which converges to the Cournot point is zero.

Remark 1 *Note that the existence of a Cournot point in our model is not guaranteed in general. For instance, consider a system such that* $n = 4$, $a = b = 1$, $c_1 = c_2 = c_3 = 0.8$ *and* $c_4 = 0.2$. *Then,* $4c = \sum_{i=1}^{4} c_i = 2.6$ *and the coordinate of the Cournot point for* $i = 1, 2, 3$, *should be*

$$\overline{q}_i = \frac{1}{5}\frac{a-c_i}{b} + \frac{4}{5}\frac{c-c_i}{b} = -0.08,$$

which is impossible because $q_i \geq 0$, $i = 1, 2, 3, 4$.

2.3 The Orbit Generated by $(0, 0, \ldots, 0)$

In this section we discuss whether the point $(0, 0, ..., 0)$ is periodic and which orbits are attracted by the periodic orbit generated by it.

Remember that $F^t(0,0,...,0) = (q_1^0(t), q_2^0(t), ..., q_n^0(t))$. Then for $i = 1, 2, ..., n$,

$$q_i^0(2) = \frac{1}{2}\left(\frac{a-c_i}{b} - \sum_{j\neq i}\frac{a-c_j}{b}\right) = \frac{1}{2}\left(\frac{nc - 3c_i - (n-3)a}{2b}\right),$$

and hence $q_i^0(2) = 0$ if and only if $\frac{a-c_i}{b} - n\frac{a-c}{b} \leq 0$, which gives us the condition

$$nc - 3c_i \leq (n-3)a, \text{ for all } i = 1, 2, \ldots, n. \tag{21}$$

Condition (21) guarantees that $(0,0,...,0)$ is a periodic point of period two always the number of firms is greater than 3, $n > 3$. When $n = 3$, from this condition one immediately deduces that $(0,0,...,0)$ is a periodic point of period two if and only if the three marginal costs are equal, $c_1 = c_2 = c_3$.

Next result shows when an arbitrary orbit is attracted by the 2-period orbit generated by $(0,0,\ldots,0)$.

Theorem 2 *Let $(q_{1_0}, q_{2_0}, ..., q_{n_0}) \in \mathbb{R}^n \setminus \mathcal{P}$ with $q_{i_0} \geq 0$, $i = 1, 2, ..., n$, be arbitrary. Let $Q(t) = q_1(t) + ... + q_n(t)$, $Q_0 = q_{1_0} + \cdots + q_{n_0}$, $c_{max} = \max\{c_1, c_2, \ldots, c_n\}$, and $c_{min} = \min\{c_1, c_2, \ldots, c_n\}$. Let $A_t = \{i|\ q_i(t) = 0\}$ and if B is a subset of $\{1, 2, \ldots, n\}$ we denote $\overline{B} = \{i|\ i \notin B\}$, $|B|$ its cardinality and $c_B = \frac{1}{|B|}\sum_{j\in B} c_j$.*

1. *If $Q(t) < \frac{a-c_{max}}{b}$, then $q_i(t+1) > 0$ for all $i = 1, 2, \ldots, n$.*

2. *There exits t_0 such that $Q(t_0) \geq \frac{a-c_{max}}{b}$.*

3. *If there exists $t_0 \geq 0$ such that $Q(t_0) \geq \frac{3(a-c_{min})}{2b}$, then the 2-period orbit*

$$\{(0,0,\ldots,0), \left(\frac{a-c_1}{2b}, \frac{a-c_2}{2b}, \ldots, \frac{a-c_n}{2b}\right)\}$$

   attracts the orbit generated by $(q_{1_0}, q_{2_0}, \ldots, q_{n_0})$.

4. *Finally assume that $Q(t) \in [\frac{a-c_{max}}{b}, \frac{3(a-c_{min})}{2b}]$.*

   a) If $n \geq 6$ and there exists t_0 such that if

$$a > \max\left\{\begin{array}{l} \frac{1}{n-3}(nc - 3c_i) \text{ for } i = 1, 2, ...n, \\ \frac{1}{n-3}(12(n - |A_{t_0+1}|) + 1)c_{max} - 12(n - |A_{t_0+1}|)c_{\overline{A}_{t_0+1}}, \\ \frac{1}{11n-59}(24nc - 13(n-1)c_{max} - 72c_{min}), \end{array}\right\} \tag{22}$$

   then the 2-period orbit $\{(0,0,\ldots,0), \left(\frac{a-c_1}{2b}, \frac{a-c_2}{2b}, \ldots, \frac{a-c_n}{2b}\right)\}$ attracts the orbit generated by $(q_{1_0}, q_{2_0}, \ldots, q_{n_0})$.

b) If $n = 5$ and there exists t_0 such that if

$$a > \max \left\{ \begin{array}{l} \frac{1}{2}(nc - 3c_i) \text{ for } i = 1, 2, ...n, \\ \frac{1}{2}(12(5 - |A_{t_0+1}|) + 1)c_{max} - 12(5 - |A_{t_0+1}|)c_{\overline{A}_{t_0+1}}, \\ 50c - 19c_{max} - 30c_{min}, \\ \frac{1}{2}(15c - 9c_{min} - 4c_{max}), \end{array} \right\} \tag{23}$$

then the 2-period orbit $\{(0, 0, \ldots, 0), (\frac{a-c_1}{2b}, \frac{a-c_2}{2b}, \ldots, \frac{a-c_n}{2b})\}$ attracts the orbit generated by $(q_{1_0}, q_{2_0}, \ldots, q_{n_0})$.

Proof. See Appendix A. ∎

Note that Theorem 2 states that if the total quantity produced by all firms at a certain time t_0 is greater than $\frac{3(a-c_{min})}{2b}$ or the maximum possible price a is greater than certain amount and $n > 4$ then the 2-period orbit

$$\{(0, 0, \ldots, 0), \left(\frac{a - c_1}{2b}, \frac{a - c_2}{2b}, \ldots, \frac{a - c_n}{2b}\right)\}$$

attracts any orbit generated by a point outside $\mathcal{P}$. This means that all the firms will disappears every two periods appearing in the next period with maximum production $(\frac{a-c_1}{2b}, \frac{a-c_2}{2b}, \ldots, \frac{a-c_n}{2b})$.

In the case in which all the marginal cost are equal, $c_i = c$ for all $i = 1, 2, \ldots, n$, we can prove the following result, that establishes the behavior of any orbit of the system.

Corollary 3 *Assume that $c_i = c$ for all $i = 1, 2, ..., n$ with $n > 4$. For any $(q_{1_0}, q_{2_0}, ..., q_{n_0}) \in \mathbb{R}^n$ with $q_{i_0} \geq 0$ for all $i = 1, 2, \ldots, n$, either $(q_{1_0}, q_{2_0}, ..., q_{n_0}) \in \mathcal{P}$ and its orbit converges to the Cournot fixed point $(\frac{a-c}{b(n+1)}, \frac{a-c}{b(n+1)}, \ldots, \frac{a-c}{b(n+1)})$ or $(q_1, q_2, ..., q_n) \notin \mathcal{P}$ and its orbit converges to the periodic orbit*

$$\left\{(0, 0, ..., 0), \left(\frac{a-c}{2b}, \frac{a-c}{2b}, ..., \frac{a-c}{2b}\right)\right\}.$$

Proof. If $(q_{1_0}, q_{2_0}, ..., q_{n_0}) \in \mathcal{P}$, then its orbit converges to the Cournot fixed point

$$\left(\frac{a-c}{b(n+1)}, \frac{a-c}{b(n+1)}, \ldots, \frac{a-c}{b(n+1)}\right).$$

Now assume $n \geq 6$. Then

$$\frac{1}{n-3}(nc - 3c_i) = c \text{ for } i = 1, 2, ...n,$$

$$\frac{1}{n-3}(12(n - |A_{t_0+1}|) + 1)c_{max} - 12(n - |A_{t_0+1}|)c_{\overline{A}_{t_0+1}} = \frac{1}{n-3}c,$$

and

$$\frac{1}{11n-59}(24nc - 13(n-1)c_{max} - 72c_{min}) = c.$$

Then, by 4.a) of Theorem 2 any orbit with initial condition $(q_{1_0}, q_{2_0}, ..., q_{n_0}) \in \mathbb{R}^n \setminus \mathcal{P}$, with $q_{i_0} \geq 0$ for all $i = 1, 2, \ldots, n$, converges to the periodic orbit generated by $(0, 0, ..., 0)$.

Now, assume $n = 5$. Then

$$\frac{1}{2}(nc - 3c_i) = c \text{ for } i = 1, 2, ...n,$$

$$\frac{1}{2}(12(5 - |A_{t_0+1}|) + 1)c_{max} - 12(5 - |A_{t_0+1}|)c_{\overline{A}_{t_0+1}} = \frac{c}{2},$$

$$50c - 19c_{max} - 30c_{min} = c,$$

and

$$\frac{1}{2}(15c - 9c_{min} - 4c_{max}) = c.$$

Then, by 4.b) of Theorem 2 any orbit with initial condition $(q_{1_0}, q_{2_0}, q_{3_0}, q_{4_0}, q_{5_0}) \in \mathbb{R}^5 \setminus \mathcal{P}$, with $q_{i_0} \geq 0$ for all $i = 1, 2, 3, 4, 5$, converges to the periodic orbit generated by $(0, 0, 0, 0, 0)$, which finishes the proof of the corollary. ∎

Remark 4 *Note that if condition (21) is not satisfied, the orbit generated by* $(0, 0, ..., 0)$ *is not periodic. For instance, consider* $n = 4$*,* $a = b = 1$*,* $c_1 = c_2 = c_3 = 0.8$ *and* $c_4 = 0.2$*. Then*

$$4c - 3c_4 = 2 > 1 = (4-3)a,$$

and condition (21) is not satisfied when $i = 4$*. Hence* $(0, 0, 0, 0)$ *is not a periodic point and moreover, its orbit converges to the fixed point* $(0, 0, 0, 0.4)$*. Again if* $n = 4$*,* $a = b = 1$*,* $c_1 = c_2 = c_3 = 0.5$ *and* $c_4 = 0.2$*. Then*

$$4c - 3c_4 = 1.1 > 1 = (4-3)a,$$

and condition (21) is not satisfied when $i = 4$*. Then the orbit generated by* $(0, 0, 0, 0)$ *converges to the periodic orbit* $\{(0, 0, 0, 0.1), (0.2, 0.2, 0.2, 0.4)\}$*.*

Remark 5 *Even when* $(0, 0, ..., 0)$ *is a periodic point, not all the orbits outside* $\mathcal{P}$ *are attracted by its orbit. For instance, fix* $n = 5$*,* $a = b = 1$*,* $c_1 = c_2 = c_3 = 0.7$*,* $c_4 = 0.5$ *and* $c_5 = 0.3$*. Clearly condition (21) is fulfilled but any point with the form* $(0, 0, 0, 0, q)$*,* $0 \leq q \leq 0.2$*, generates a periodic orbit of period two. Hence, the number of periodic orbits may be infinite, even uncountable.*

Remark 6 *Assume the conditions of Corollary 3 are fulfilled. Then, any orbit is attracted by the Cournot equilibrium point if and only if the initial condition* $(q_{1_0}, q_{2_0}, ..., q_{n_0})$ *fulfils the condition* $q_{1_0} + q_{2_0} + ... + q_{n_0} = \overline{Q}$, *which implies a coordination between all the firms in order to obtain the maximum profit.*

Remark 7 *When* $n = 3$, *then the point* $(0, 0, 0)$ *is periodic if and only if* $c_1 = c_2 = c_3$, *but because of the closed solution of the linear system, there are a lot of orbits, not only those contained in* $\mathcal{P}$, *which do not converge to this periodic point.*

2.4 When does a firm disappear from the market?

One of the most interesting problems in oligopoly dynamics is whether some firms disappear from the market, i.e., their outputs become zero for any time $t \geq t_0 \geq 0$. In connection with this question we have the opposite one, i.e., whether a new firm can entry in the market. This would also give us some information about possible re-enter of firms in an ongoing dynamical process. In this section we develop these problems. We start by studying the simplest case, i.e., the monopoly case.

2.4.1 Monopoly

We analyze the existence of a time t_0 such that if $t \geq t_0$, then $q_i(t) = 0$ for $i = 1, 2, ..., n-1$, and hence only one firm remains, so resulting in monopoly. In this case, note that

$$F^t(0, 0, ..., 0, q_n) = \left(0, 0, ..., 0, \frac{a - c_n}{2b}\right)$$

for all $t \geq 1$. Then the production $q_n(t) = \frac{a-c_n}{2b}$ will be constant. On the other hand, for $i = 1, 2, ..., n-1$

$$q_i(t+1) = \frac{1}{2}\left(\frac{a - c_i}{b} - \frac{a - c_n}{2b}\right) \leq 0,$$

which gives the monopoly condition

$$a + c_n \leq 2c_i \text{ for } i = 1, 2, ..., n-1 \tag{24}$$

which implies that firm n has to have the lower marginal cost among the firms, that is $c_n = c_{min}$. Then, by condition (24), the coordinate of Cournot point of the firm with maximum marginal cost is given by

$$\overline{q}_{max} = \frac{nc + a - (n+1)c_{max}}{b(n+1)} = \frac{a + c_{min} + \sum_{j=1}^{n-1} c_j - (n+1)c_{max}}{b(n+1)} \leq \frac{\sum_{j=1}^{n-1} c_j - (n-1)c_{max}}{b(n-1)} \leq 0.$$

Then $\overline{q}_{max} < 0$ unless $c_1 = c_2 = ... = c_{max} = ... = c_{n-1}$ and $a + c_{min} = 2c_{max}$. Then, there is no Cournot equilibrium point for the nonlinear system except $(0, 0, ..., 0, \frac{a-c_{min}}{2b})$, which is the Cournot point of the limit case of monopoly.

Now, we study the following question. What are the conditions that a new firm has to fulfill to enter in the market?.

A new firm enters in a monopoly. The general Cournot point formula (11) also applies to the case of monopoly, $n = 1$. Then the monopolistic firm produces:

$$\overline{Q} = \frac{1}{2}\frac{a - c_1}{b}$$

as $c = c_1$. Considering the entry of another firm with marginal cost c_2, applying (8), $q_2 = \frac{1}{2}\frac{a-c_2}{b} - \frac{1}{2}Q_2$, would have to be *positive* if the new firm is to enter. As we recall, positivity of the output also guarantees positivity of profit for that firm. In this case, the residual supply equals the supply by the already existent monopoly firm, so $Q_2 = \overline{Q}$. Then

$$\frac{1}{2}\frac{a - c_2}{b} - \frac{1}{2}\left(\frac{1}{2}\frac{a - c_1}{b}\right) > 0$$

must be fulfilled. This obviously simplifies to:

$$c_2 < \frac{1}{2}(a + c_1) \tag{25}$$

Note that the right hand side of (25) is the monopoly price of firm 1. Therefore in order for a new firm to make a positive profit and enter in competition with the monopoly its marginal cost must be lower than the monopoly price. It is not needed that the marginal cost of the entering firm to be lower than the marginal cost for the monopolist. In fact, suppose $a = 1$, $c_1 = 0.5$. Then $c_2 < 0.75$ must hold. If the new firm for instance has $c_2 = 0.6$, then it can enter and make a positive profit. As the Cournot point in duopoly is contractive, with eigenvalues $\pm\frac{1}{2}$, the system will approach the Cournot equilibrium.

Figures 1-2 illustrate these facts. A monopoly is established, and after a while a second firm enters. In Figure 1 we have $c_2 = 0.8 > 0.75$, so the firm does not succeed to remain on the market, in Fig. 2, $c_2 = 0.7 < 0.75$, so the market is transformed into a duopoly.

A natural question that arises is whether the entering firm could even have thrown out the old monopolist, thus establishing a new monopoly. In order for this to happen we would need to have:

$$\frac{1}{2}\frac{a - c_1}{b} - \frac{1}{2}\left(\frac{1}{2}\frac{a - c_2}{b}\right) < 0$$

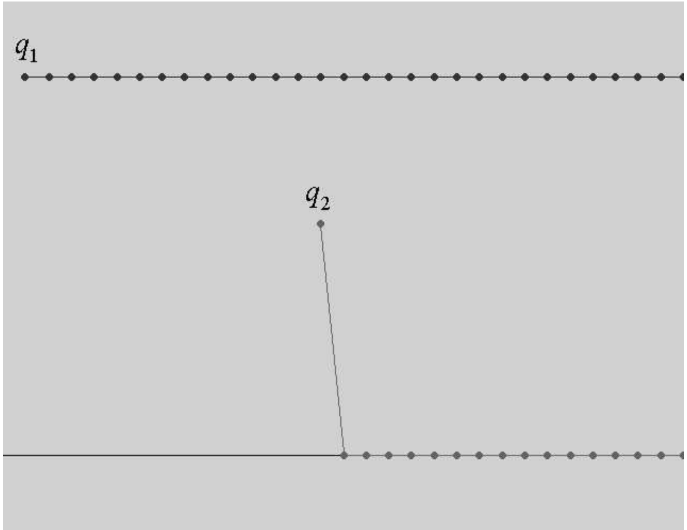

Figure 1: Failing entry attempt of another firm in a monopoly. $c_1 = 0.5$ and $c_2 = 0.8$.

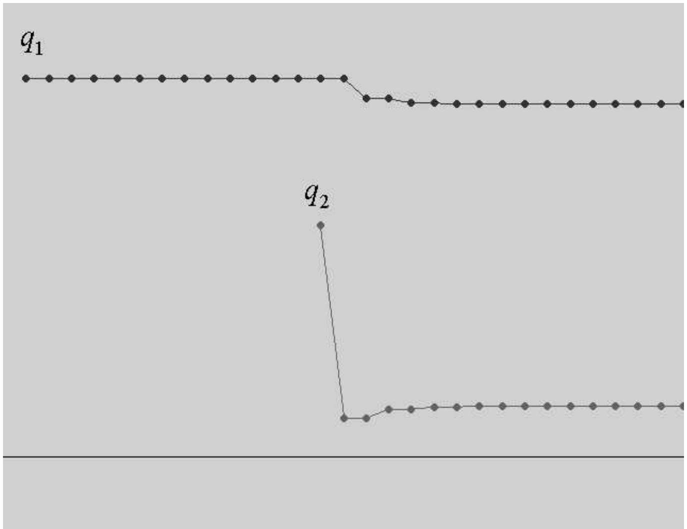

Figure 2: Succeeding entry of another firm in a monopoly. $c_1 = 0.5$ and $c_2 = 0.7$.

that is,

$$a + c_2 < 2c_1$$

which, given $a = 1, c_1 = 0.5$, simplifies to:

$$c_2 < 0$$

which is impossible. This is so because c_1 is so low already. If we had $c_1 = 0.6$, then $c_2 < 0.2$ would do.

Dynamic discussion of the monopolistic system. Recall from Subsection 2.1 that

$$(q_1^0(t), q_2^0(t), ..., q_n^0(t)) = F^t(0, 0, ..., 0).$$

By condition (24), $q_i^0(2t) = 0$ for all $i = 1, 2, ..., n-1$ and $t \geq 0$. Moreover, $q_n^0(2t)$ is an increasing sequence bounded by $\frac{a-c_n}{2b}$, whose limit is q_n^m. Note also that since $q_i^0(2t) = 0$ for $i = 1, 2, ..., n-1$, then $q_n^0(2t+1) = q_n^M = \frac{a-c_n}{2b}$ for all $t \geq 0$. Moreover by the continuity of F we also have that

$$F(q_1^m, q_2^m, ..., q_n^m) = (q_1^M, q_2^M, ..., q_n^M) \text{ and } F(q_1^M, q_2^M, ..., q_n^M) = (q_1^m, q_2^m, ..., q_n^m).$$

Then either $(q_1^m, q_2^m, ..., q_n^m)$ is a fixed point or a periodic point of period 2. So, we have to consider two possible cases.

First, if $q_n^m = q_n^M$, then the orbit of $(0, 0, ..., 0)$ converges to the fixed point $(0, 0, ..., 0, \frac{a-c_n}{2b})$ and all the orbits in the system converge to this point.

Second, if $q_n^m < q_n^M$ additional limit points of orbits are possible. In this case, since $q_i^m = 0$ for $i = 1, 2, \ldots, n-1$ the point $(0, 0, ..., 0, q_n^m)$ will be a periodic point of period two. Taking into account that

$$F^2(0, 0, ..., 0, q_n^m) = \left(0, 0, ..., 0, \frac{n(c-a) + 3(a-c_n)}{4b} + \frac{n-1}{4} q_n^m\right) = (0, 0, ..., 0, q_n^m), \tag{26}$$

we obtain that

$$q_n^m = \frac{n(c-a) + 3(a-c_n)}{(5-n)b}. \tag{27}$$

If $n > 5$, then $q_n^m < 0$ which is impossible. Then, the only possibility is that $q_n^m = 0$ and hence $(0, 0, ..., 0)$ will be periodic of period 2. To this end the by (26) condition

$$\frac{n(c-a) + 3(a-c_n)}{4b} \leq 0$$

must be fulfilled, which gives us

$$3(a-c_n) \leq n(a-c). \tag{28}$$

Note that condition (28) cannot be fulfilled if $n = 2, 3$. If $n = 5$, then q_5^m is not defined because the denominator in (27) vanishes and then $(0, 0, 0, 0, 0)$ is periodic if and only if condition (28) is fulfilled. If $n = 4$, then

$$q_4^m = \frac{4(c-a) + 3(a - c_4)}{b} \geq \frac{a - c_4}{2b},$$

and hence the only possibility to have a periodic point of period two is $(0, 0, 0, 0)$ to be periodic satisfying condition (28). Finally, if $n = 3$

$$q_3^m = \frac{c_1 + c_2 - 2c_3}{2b} \geq \frac{a - c_3}{2b}$$

and if $n = 2$

$$q_2^m = \frac{a + c_1 - 2c_2}{3b} \geq \frac{a - c_2}{2b}.$$

Then, we have proved the following result.

Theorem 8 *Assume that the monopoly condition, (24), is fulfilled and condition (28) is not. Then, for any orbit generated by* $(q_{1_0}, q_{2_0}, ..., q_{n_0}) \in \mathbb{R}^n$, $q_{i_0} \geq 0$, $i = 1, 2, ..., n$, *converges to the fixed point* $\left(0, 0, ..., 0, \frac{a - c_{min}}{2b}\right)$.

Remark 9 *Note that for instance, if* $n = 6$, $a = b = 1$, $c_i = 0.6$ *and* $c_6 = 0.1$, *then* $(0, 0, 0, 0, 0, 0)$ *is periodic and hence, even when the monopoly condition is fulfilled, the system does not go to a monopoly.*

2.4.2 Duopoly

Now we study the existence of a time $t_0 \geq 0$ such that, for $t \geq t_0$ and $i = 1, 2, ..., n-2$, we have $q_i(t) = 0$, i.e., all firms except two will produce nothing in the future. In this case note that

$$\begin{cases} q_{n-1}(t+1) = \begin{cases} \frac{1}{2}\left(\frac{a-c_{n-1}}{b} - q_n(t)\right) & \text{if } q_n(t) \leq \frac{a-c_{n-1}}{b}, \\ 0 & \text{if } q_n(t) > \frac{a-c_{n-1}}{b}, \end{cases} \\ q_n(t+1) = \begin{cases} \frac{1}{2}\left(\frac{a-c_n}{b} - q_{n-1}(t)\right) & \text{if } q_{n-1}(t) \leq \frac{a-c_n}{b}, \\ 0 & \text{if } q_{n-1}(t) > \frac{a-c_n}{b}. \end{cases} \end{cases}$$

The Cournot equilibrium point is

$$(\widetilde{q}_{n-1}, \widetilde{q}_n) = \left(\frac{a - 2c_{n-1} + c_n}{3b}, \frac{a + c_{n-1} - 2c_n}{3b}\right).$$

The eigenvalues of the matrix of the linear system are $\frac{1}{2}$ and $-\frac{1}{2}$ and hence the Cournot point is asymptotically stable. Then, any orbit contained in the set

$$\{(0, 0, ..., 0, q_{n-1}, q_n) \in \mathbb{R}^n : q_{n-1}, q_n \in \mathbb{R}\}$$

converges to the Cournot point $(0, ..., 0, \widetilde{q}_{n-1}, \widetilde{q}_n)$. Now, we consider that for $i = 1, 2, ..., n-2$ the i–th coordinate of $F(0, 0, ..., 0, \widetilde{q}_{n-1}, \widetilde{q}_n)$ is

$$\frac{1}{2}\left(\frac{a-c_i}{b} - \widetilde{q}_{n-1} - \widetilde{q}_n\right),$$

Then we have that for $i = 1, 2, ..., n-2$

$$\frac{a-c_i}{b} - \widetilde{q}_{n-1} - \widetilde{q}_n \leq 0,$$

which can be rewritten as

$$a + c_{n-1} + c_n \leq 3c_i, \text{ for } i = 1, 2, ..., n-2, \tag{29}$$

which is the duopoly condition. Note that for the duopoly condition to be satisfied c_n and c_{n-1} have to be the lower marginal costs, that is, only the firms with lower marginal costs survive in the duopoly.

Recall that the coordinate of the Cournot point of the firm with maximum marginal cost is

$$\begin{aligned}\overline{q}_{max} &= \frac{nc + a - (n+1)c_{max}}{b(n+1)} \\ &= \frac{a + c_{n-1} + c_n + \sum_{j=1}^{n-2} c_j - (n+1)c_{max}}{b(n+1)} \\ &\leq \frac{\sum_{j=1}^{n-2} c_j - (n-2)c_{max}}{b(n+1)} \leq 0.\end{aligned}$$

Then, if $\overline{q}_{max} < 0$, and so there is no Cournot equilibrium point in the nonlinear model with n firms, unless $c_i = c_{max} = \frac{a+c_{n-1}+c_n}{3}$ for $i = 1, 2, ..., n-2$, which gives the Cournot equilibrium point

$$(0, 0, ..., 0, \widetilde{q}_{n-1}, \widetilde{q}_n) = \left(0, 0, ..., 0, \frac{a - 2c_{n-1} + c_n}{3b}, \frac{a + c_{n-1} - 2c_n}{3b}\right).$$

A new firm enters in a duopoly. Suppose the competitor succeeded to intrude the market. Then we, from (11), get a duopoly with total output

$$\overline{Q} = \frac{2}{3}\frac{a-c}{b}$$

where now $c = \frac{1}{2}(c_1 + c_2)$. Considering the entry of a third firm with marginal cost c_3, we must check that output $q_3 = \frac{1}{2}\frac{a-c_3}{b} - \frac{1}{2}Q_3$, and so profits, are positive. Now, $Q_3 = \overline{Q}$, and so

$$\frac{1}{2}\frac{a-c_3}{b} - \frac{1}{2}\left(\frac{2}{3}\frac{a - \frac{1}{2}(c_1+c_2)}{b}\right) > 0$$

must be fulfilled. Again, we easily simplify to:

$$c_3 < \frac{1}{3}(a + c_1 + c_2) \tag{30}$$

Note that the right hand side of (30) is the duopoly price in the duopoly equilibrium. Therefore in order for a new firm to make a positive profit and enter in competition with the duopoly its marginal cost must be lower than the duopoly equilibrium price.

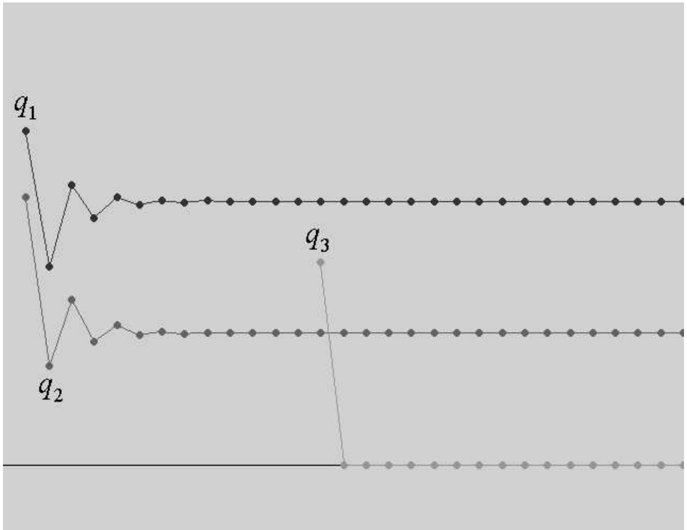

Figure 3: Unsuccessful entry of a third firm into a duopoly. $c_1 = 0.5$, $c_2 = 0.6$ and $c_3 = 0.75$.

Suppose again $a = 1, c_1 = 0.5$, and further $c_2 = 0.6$. Then condition (30) reads $c_3 < 0.7$, and a third firm will enter if, for instance, $c_3 = 0.65$. Provided all firms make positive profits, the system will become oscillating with constant amplitude, because the eigenvalues are now $\frac{1}{2}, \frac{1}{2}, -1$.

Like Figs. 1-2 illustrated the entry of a second firm, Figs. 3-5 illustrate the entry of a third firm when a duopoly has been established. In Fig. 3, $c_3 = 0.75 > 0.7$, so the third firm drops out from the market, whereas in Fig. 4, $c_3 = 0.65 < 0.7$, so the firm remains, and duopoly equilibrium is destroyed.

We can again inquire if it is possible for the newly entering firm to just throw out one of the firms, so remaining a duopolist. This can indeed occur as illustrated in Fig. 5. If the second firm which has the highest marginal cost among the original duopolists be thrown out, it must hold that:

$$\frac{1}{2}\frac{a - c_2}{b} - \frac{1}{2}\left(\frac{2}{3}\frac{a - \frac{1}{2}(c_1 + c_3)}{b}\right) < 0$$

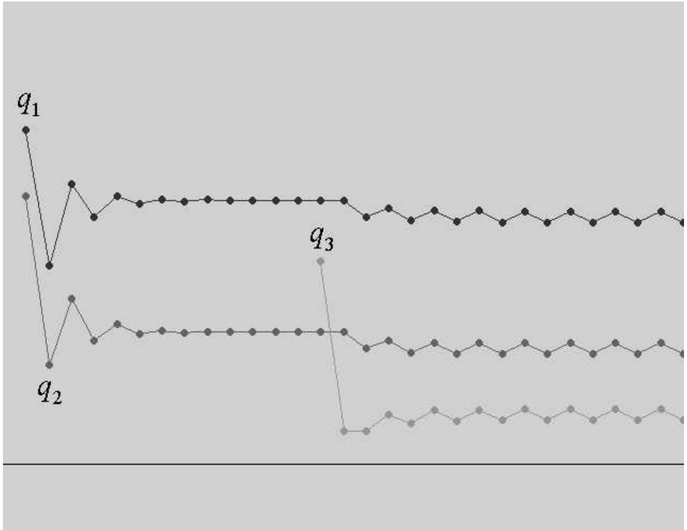

Figure 4: Successful entry of a third firm into a duopoly. $c_1 = 0.5$, $c_2 = 0.6$. and $c_3 = 0.65$.

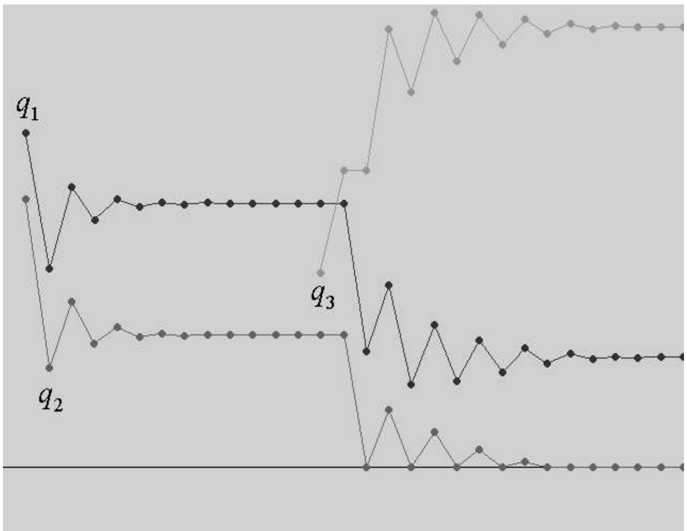

Figure 5: Entry of a third firm into a duopoly, where the duopoly remains but one firm is replaced. $c_1 = 0.5, c_2 = 0.6$. and $c_3 = 0.25$.

which, given $a = 1, c_1 = 0.5$, and further $c_2 = 0.6$ simplifies to:

$$c_3 < 0.3$$

In Fig. 5 we chose $c_3 = 0.25$, so indeed the second firm is thrown out and duopoly reestablished. But it could not establish a monopoly for reasons already given. The marginal cost of the first firm $c_1 = 0.5$ is simply too low, we would need a negative cost for the new firm to become a monopoly.

Dynamic discussion of the system. The description of the duopoly case is more difficult than monopoly case. We only note that not always when the duopoly condition (29) is fulfilled, the system evolve to a duopoly. For instance, assume $n = 4$, $a = b = 1$, $c_1 = c_2 = 0.7$ and $c_3 = c_4 = 0.4$. Then $\{(0, 0, 0, 0), (0.15, 0.15, 0.3, 0.3)\}$ is a periodic orbit which attracts the orbits generated by the points $(0.1, 0.1, 0, 0)$ or $(0.4, 0.1, 0.2, 0.2)$ while the fixed Cournot point $(0, 0, 0.7/3, 0.7/3)$ attracts the orbits generated by $(0.1, 0.1, 0.2, 0.2)$ or $(0.1, 0.2, 0, 0.2)$.

Anyway, we can enunciate the following result whose proof is analogous to that of Theorem 8.

Theorem 10 *Assume that duopoly condition, (29), is fulfilled and that the orbit generated by* $(0, 0, ..., 0)$ *converges to the fixed point* $(0, 0, ..., 0, \widetilde{q}_{n-1}, \widetilde{q}_n)$.

Then, any orbit generated by $(q_{1_0}, q_{2_0}, ..., q_{n_0}) \in \mathbb{R}^n$, $q_{i_0} \geq 0$, $i = 1, 2, ..., n$, *converges to the same fixed point* $(0, 0, ..., 0, \widetilde{q}_{n-1}, \widetilde{q}_n)$..

For instance, when $n = 4$, $a = b = 1$, $c_1 = c_2 = 0.7$ and $c_3 = c_4 = 0.1$, the orbit of $(0, 0, 0, 0)$ converges to $(0, 0, 0.3, 0.3)$, and then any orbit of the system does the same.

2.4.3 Oligopoly

Now, assume that we have an oligopoly with $n > 4$ firms and $n - k$ of them drop out from the market , i.e., the orbit $(q_1(t), q_2(t), ..., q_n(t))$ fulfils $q_i(t) = 0$ for all $t \geq 0$ and any $i = 1, 2, ..., n - k$. Assume also that for $t \geq 0$ and for all $n - k + 1 \leq i \leq n$ the equality $q_i(t) = \widetilde{q}_i$ holds, where $\widetilde{q}_i$ is the i–th Cournot coordinate of the oligopoly for the remaining firms. For $i = 1, 2, ..., n - k$, we have that

$$q_i(t) = \frac{1}{2}\left(\frac{a - c_i}{b} - \widetilde{Q}\right),$$

where $\widetilde{Q} = \sum_{i=n-k+1}^{n} \widetilde{q}_i = \frac{k}{k+1}\frac{a-\widetilde{c}}{b}$. Then, $q_i(1) = 0$ for $i = 1, 2, ..., n - k$ if and only if

$$\frac{a - c_i}{b} \leq \frac{k}{k+1}\frac{a - \widetilde{c}}{b},$$

where $\widetilde{c} = \frac{1}{k}\sum_{i=n-k+1}^{n} c_i$, and then we obtain the oligopoly condition

$$a + k\widetilde{c} \leq (k + 1)c_i \text{ for } i = 1, 2, ..., n - k. \tag{31}$$

In order to discuss whether the above condition implies that $n-k$ firms will abstain from producing in the future, we are going to assume the following: $c_i = c_1$ for $i = 1, 2, ..., n-k$ and $c_j = c_n$ for $j = n-k+1, ..., n$, i.e., only two marginal costs are possible. Then, condition (31) reads as follows

$$a + kc_n \leq (k+1)c_1. \tag{32}$$

Let

$$\mathcal{P}_k = \{(q_{n-k+1}, \ldots, q_n) \in \mathbb{R}^k |\ q_1 + q_2 + \ldots q_k = \widetilde{Q},\ q_i \geq 0\}.$$

Then Corollary 3 states that any orbit lying outside $\mathcal{P}_k$ eventually gives us that $q_i(t_0) = 0$ for $i = n-k+1, ..., n$ and for some $t_0 \in \mathbb{N}$. Hence $q_i(t_0+1) = \frac{a-c_i}{2b} > 0$ for all $i = 1, 2, \ldots, n$. Therefore condition (32) it is not enough for all the firms with the highest marginal costs to disappear from the market. But if an orbit starts from a point in $\mathcal{P}_k$ and the $n-k$ firms cooperate in order to produce $q_{n+k+1}(t) + ... + q_n(t) = \widetilde{Q}$ for any $t \geq 0$ then we have that the remaining firms will disappear.

3 Conclusion

It seems that in terms of asymptotic orbits the global dynamics of the model with linear demand function and constant marginal costs has very simple dynamics, either it goes to a monopoly, or a duopoly, or else to an endless 2-period oscillation where all firms move in phase. It all depends on the marginal costs of the competitors. If the number of remaining competitors is at least three, then the Palander-Theocharis argument that the Cournot equilibrium becomes unstable applies. This would seem to invalidate the idea that, with an increasing number of competitors, the Cournot equilibrium point transubstantiates into a competitive equilibrium.

However, we should bear the following in mind: Considering an increasing number of competitors, we actually want to compare a small number of large competitors to a large number of small competitors. See [6]. Firms with constant marginal cost, i.e. constant returns to scale, are all potentially infinitely large. So in the Theocharis model we are just adding more and more potentially infinitely large firms on the market. That this results in destabilization is not very surprising. Already Edgewotrh (1897) [2] insisted on the importance of adding capacity limits to cases with constant marginal cost. An increasing number of firms should then decrease the capacity limits for the individual firms. A way to do this, through splitting a given total capacity of an industry between more or less numerous firms, was proposed in [6].

In the present model context, introducing decreasing capacity limits with an increasing number of competitors, would mean that each firm either sticks to its output according to (18), or else produces at its capacity limit. In order for the Edgeworth (1897) [2] argument to be relevant with an increasing number of competitors, all the capacity limits would eventually become binding. The analysis of such a case is bound to not be very exciting.

This also hints at a problem with the combination of assumptions. According to elementary text book microeconomics, the demand curve (and hence the marginal revenue curve) become virtually horizontal in the ranges that the small firms can operate. Now, this picture presupposes that we use an increasing marginal cost curve, otherwise the marginal cost curve and the demand curve never intersect. Such a marginal cost curve (which also incorporated an asymptotic capacity limit) was suggested in [6], but there are many alternatives. Anyhow, the destabilization of the Cournot point does not have any great importance in the basic issue of the relation between Cournot oligopoly and perfect competition.

Another issue raised concerning the setup of the model is that a two-period oscillation where all firms drop out every second period is something that will be learned very fast by all the firms, so the naive expectations become untenable, and any firm might try to get out of phase in order to increase long term profits.

4 Appendix A

1. In fact, $q_i(t+1) = \frac{1}{2}\left(\frac{a-c_i}{b} - Q(t) + q_i(t)\right)$ and since by assumption

$$-Q(t) + q_i(t) > -\frac{a - c_{max}}{b} + q_i(t) > -\frac{a - c_{max}}{b}$$

it follows that

$$q_i(t+1) > \frac{1}{2}\left(\frac{a-c_i}{b} - \frac{a-c_{max}}{b}\right) = \frac{1}{2}\left(\frac{c_{max}-c_i}{b}\right) \geq 0$$

for all $i = 1, 2, \ldots, n$.

2. By means of contradiction assume that $Q(t) < \frac{a-c_{max}}{b}$ for all t. Then by the previous claim, $q_i(t) > 0$ for all $t > 0$ and for all $i = 1, 2, \ldots, n$. Thus the total production follows the rule

$$Q(t+1) = \begin{cases} -\frac{n-1}{2}Q(t) + n\frac{a-c}{2b} & \text{if } -\frac{n-1}{2}Q(t) + n\frac{a-c}{2b} \geq 0, \\ 0 & \text{if } -\frac{n-1}{2}Q(t) + n\frac{a-c}{2b} < 0. \end{cases} \tag{33}$$

Hence, the linear difference equation gives us the solution

$$Q(t) = \left(Q_0 + \frac{c-a}{b}\frac{n}{n+1}\right)\left(\frac{1-n}{2}\right)^t + \frac{a-c}{b}\frac{n}{n+1}$$

yielding a contradiction with the assumption that $Q(t) < \frac{a-c_{max}}{b}$ for all $t > 0$.

3. Since $q_i(t_0) \leq \frac{a-c_i}{2b}$ we have that

$$q_i(t_0+1) = \frac{1}{2}\left(\frac{a-c_i}{b} - Q(t_0) + q_i(t_0)\right) \leq \frac{1}{2}\left(\frac{3}{2}(\frac{a-c_i}{b} - \frac{a-c_{min}}{b})\right) \leq 0,$$

for $i = 1, 2, \ldots, n$. Then $q_i(t_0+1) = 0$ for all $i = 1, 2, \ldots, n$ and hence since condition (21) is satisfied, $q_i(t_0+3) = 0$ for all $i = 1, 2, \ldots, n$, as desired.

4.a) Assume that t_0 is the first time in which $Q(t_0) \geq \frac{a-c_{max}}{b}$, that is, $Q(t) < \frac{a-c_{max}}{b}$ for all $t < t_0$. Then

$$Q(t_0+1) = \frac{1}{2}\left((n - |A_{t_0+1}|)\frac{a - c_{\overline{A}_{t_0+1}}}{b} - (n - |A_{t_0+1}| - 1)Q(t_0) - \sum_{j \in A_{t_0+1}} q_j(t_0)\right)$$

Thus, since $-Q(t_0) - \sum_{j \in A_{t_0+1}} q_j(t_0) \leq -\frac{a-c_{max}}{b} - \sum_{j \in A_{t_0+1}} q_j(t_0) \leq -\frac{a-c_{max}}{b}$ it follows that

$$\begin{aligned} Q(t_0+1) &\leq \tfrac{1}{2b}((n - |A_{t_0+1}|)(a - c_{\overline{A}_{t_0+1}}) - (n - |A_{t_0+1}| - 1)(a - c_{max})) \\ &= \tfrac{1}{2b}(a - c_{max} + (n - |A_{t_0+1}|)(c_{max} - c_{\overline{A}_{t_0+1}})) \end{aligned} \tag{34}$$

and hence if

$$a > (12(n - |A_{t_0+1}|) + 1)c_{max} - 12(n - |A_{t_0+1}|)c_{\overline{A}_{t_0+1}} \tag{35}$$

then

$$Q(t_0+1) < \frac{13(a - c_{max})}{24b} \leq \frac{(a - c_{max})}{b}.$$

Then by 1. we have that $q_i(t_0+2) > 0$ for all $i = 1, 2, \ldots, n$ and hence

$$Q(t_0+2) = \frac{1}{2}\left(\frac{n(a-c)}{b} - (n-1)Q(t_0+1)\right) > \frac{1}{2b}\left(n(a-c) - \frac{13}{24}(n-1)(a - c_{max})\right). \tag{36}$$

Therefore $Q(t_0+2) > \frac{3(a-c_{min})}{2b}$ if and only if

$$(11n - 59)a > (24nc - 13(n-1)c_{max} - 72c_{min}). \tag{37}$$

and thus by 3. the 2-period orbit $\{(0, 0, \ldots, 0), (\frac{a-c_1}{2b}), \frac{a-c_2}{2b}, \ldots, \frac{a-c_n}{2b}\}$ attracts the orbit generated by $(q_{1_0}, q_{2_0}, \ldots, q_{n_0})$

Therefore if $n \geq 6$ and there is t_0 such that

$$a > \max \left\{ \begin{array}{l} \frac{1}{n-3}(nc - 3c_i) \text{ for } i = 1, 2, ...n, \\ \frac{1}{n-3}(12(n - |A_{t_0+1}|) + 1)c_{max} - 12(n - |A_{t_0+1}|)c_{\overline{A}_{t_0+1}}, \\ \frac{1}{11n-59}(24nc - 13(n-1)c_{max} - 72c_{min}), \end{array} \right\} \tag{38}$$

we have that the orbit generated by $(q_{1_0}, q_{2_0}, \ldots, q_{n_0})$ is attracted by the periodic orbit generated by $(0, 0, ..., 0)$.

4.b) Now, assume that $n = 5$. Then by (36) we get that $Q(t_0+2) > \frac{1}{12b}(17a+13c_{max}-30c)$. Since $q_i(t_0+2) \leq \frac{a-c_i}{2b}$ for all $i = 1, 2, \ldots, n$ and $q_i(t_0+3) = \frac{1}{2}(\frac{a-c_i}{b} - Q(t_0+2) + q_i(t_0+2))$ we have that

$$q_i(t_0+3) \leq \frac{1}{24b}(a + 30c - 13c_{max} - 18c_i)$$

On the other hand $\frac{1}{24b}(a + 30c - 13c_{max} - 18c_i) < \frac{a-c_{max}}{15b}$ if and only if $a > 50c - 19c_{max} - 30c_i$. Therefore if

$$a > 50c - 19c_{max} - 30c_{min} \tag{39}$$

it follows that $Q(t_0+3) < \frac{a-c_{max}}{3b}$, and hence by the first claim we obtain that $q_i(t_0+4) > 0$ for all $i = 1, 2, 3, 4, 5$. Therefore by (33) we have that $Q(t_0+4) = \frac{5(a-c)}{2b} - 2Q(t_0+3)$. Then

$$Q(t_0+4) > \frac{5(a-c)}{2b} - \frac{2(a-c_{max})}{3b} = \frac{1}{6b}(11a + 4c_{max} - 15c)$$

and $\frac{1}{6b}(11a + 4c_{max} - 15c) > \frac{3(a-c_{min})}{2b}$ if and only if $a > \frac{1}{2}(15c - 9c_{min} - 4c_{max})$.

Therefore we have proved that if $n = 5$ and there exists t_0 such that

$$a > \max \left\{ \begin{array}{l} \frac{1}{2}(nc - 3c_i) \text{ for } i = 1, 2, ...n, \\ \frac{1}{2}(12(5 - |A_{t_0+1}|) + 1)c_{max} - 12(5 - |A_{t_0+1}|)c_{\overline{A}_{t_0+1}}, \\ 50c - 19c_{max} - 30c_{min}, \\ \frac{1}{2}(15c - 9c_{min} - 4c_{max}), \end{array} \right\} \tag{40}$$

then the orbit generated by $(q_1, q_2, ..., q_n)$ is attracted by the periodic orbit generated by $(0, 0, \ldots, 0)$.

References

[1] Cournot A. 1838 *Recherches sur les Principes Mathématiques de la Théorie des Richesses.* Paris: Hachette.

[2] Edgeworth FY 1897. La teoria pura del monopolio. *Giornale degli Economisti*, 15:13-31.

[3] Palander T. 1936. Instability in competition between two sellers. Abstracts of Papers Presented at the Research Conference on Economics and Statistics held by the Cowles Commission at Colorado College, Colorado College Publications, General Series No. 208, Studies Series No. 21.

[4] Palander T. 1939. *Konkurrens och marknadsjämvikt vid duopol och oligopol.* Ekonomisk Tidskrift, 41:124-145, 222-250.

[5] Puu T. and Suskho I. 2002. *Oligopoly Dynamics - Models and Tools.* Springer-Verlag.

[6] Puu T. 2006. On the stability of Cournot equilibrium when the number of competitors increases. (submitted).

[7] Okuguchi K. 1976. *Expectations and Stability in Oligopoly Models*. Springer-Verlag.

[8] Robinson, J. 1933. *The Economics of Imperfect Competition*, Macmillan, London, 1933.

[9] Tuinstra J. 2004. Oligopoly Dynamics: Models and Tools. *Journal of Economic Behavior and Organization* 54:611-614.

[10] Theocharis RD. 1959. On the stability of the Cournot solution on the oligopoly problem. *Review of economic Studies*, 27:133-134.