Fluctuations of the correlation dimension at metal-insulator transitions

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We investigate numerically the inverse participation ratio, P_2, of the 3D Anderson model and of the power-law random banded matrix (PRBM) model at criticality. We found that the variance of $\ln P_2$ scales with system size L as $\sigma^2(L) = \sigma^2(\infty) - AL^{-D_2/2d}$, being D_2 the correlation dimension and d the system dimension. Therefore the concept of a correlation dimension is well defined in the two models considered. The 3D Anderson transition and the PRBM transition for $b = 0.3$ (see the text for the definition of b) are fairly similar with respect to all critical magnitudes studied.

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Critical eigenfunctions at metal-insulator transitions show multifractality, whose study has been very intensive in the last two decades [1,2]. Each critical eigenfunction $\phi_\alpha(\mathbf{r})$ can be characterized by a set of inverse participation ratios (IPR)

$$P_q^{(\alpha)} = \int |\phi_\alpha(\mathbf{r})|^{2q} \, d^d r \ , \tag{1}$$

where the index α label the different eigenfunctions and d is the embedding dimension. In a good metal the IPR scale with size L as $P_q^{(\alpha)} \propto L^{-d(q-1)}$, while in an insulator $P_q^{(\alpha)} \propto L^0$. Wegner [1] found, from a renormalization-group treatment of the metal-insulator transition in $2+\epsilon$ dimensions, that the average IPR at criticality show an anomalous scaling of the form

$$P_q \propto L^{-D_q(q-1)} \ , \tag{2}$$

being D_q a set of generalized fractal dimensions.

The IPR fluctuations were studied for 2D systems in the framework of the supersymmetry method [3–5] and it was found that the distribution function of $P_q^{(\alpha)}$ normalized to its typical value $P_2^{(\mathrm{typ})}$ is scale invariant at criticality and inversely proportional to the squared adimensional conductance [4]. Although the 2D case does not present a true Anderson transition, the previous result motivated the conjecture that in general the distribution function of $P_q^{(\alpha)}$ normalized to its typical value, is universal, i.e., size independent for $L \to \infty$. Accordingly, it is assumed that the distribution function of $\ln P_q^{(\alpha)}$ is a universal curve which is just horizontally shifted by changes in L.

Recently it was claimed by Parshin and Schober [6], on the basis of numerical calculations, that the correlation dimension D_2 in the 3D Anderson model at criticality is not well-defined due to strong fluctuations in the IPR. For the sizes considered, it seemed that the standard deviation of the distribution of $\ln P_2^{(\alpha)}$ grows with system size proportionally to $\ln L$. Then, for $L \to \infty$ the correlation dimension, instead of tending to a single value, it would tend to a universal distribution.

If confirmed, the previous result would impose drastic changes in our understanding of critical properties at metal-insulator transitions. Mirlin and Evers [7,8] addressed this problem and study theoretically and numerically the fractal properties of the power-law random banded matrix (PRBM) model at criticality. They found the distribution function of the IPR to be scale independent for all the values of the parameter b characterizing this model (see below). The model describes a whole family of critical theories parameterized by b, in the same way as the dimensionality labels the different Anderson transitions. The standard 3D Anderson transition should be equivalent to a PRBM model with a b of the order of unity, although this belief is not based on any direct knowledge. They finally claimed that the disagreement between their results and those of Ref. [6] is due to the small system sizes used in [6] and not to the different models employed.

Our aim is to perform a careful statistical analysis on the data from numerical calculations of both the 3D Anderson model and the PRBM model at criticality using system sizes larger than in previous calculations [6,7]. We want to study the system size dependence of the fluctuations of the IPR and to elucidate: i) whether the correlation dimension at the Anderson transition is well-defined or alternatively presents an scale invariant distribution, ii) whether this transition is equivalent or not to a PRBM model at criticality.

We first consider the standard Anderson model on a 3D simple cubic lattice, represented by a tight-binding Hamiltonian with matrix elements of the form

$$H_{ij} = \epsilon_i \delta_{ij} + t_{ij} \ , \tag{3}$$

where i, j denote lattice sites. The diagonal (site) energies are randomly distributed with constant probability in the interval $-W/2 < \epsilon_i < W/2$, and the off-diagonal elements t_{ij} are taken equal to unity for nearest neighbors, which sets the energy scale, and to zero otherwise.

The PRBM model, introduced in Ref. [9], describes a 1D sample with random long-range hopping. It can approximately represent a variety of physical systems from an integrable billiard with a Coulomb scattering center [10], to the Luttinger liquid at finite temperatures [11,12]. The model is represented by real symmetric matrices whose entries are randomly drawn from a normal distribution with zero mean and a variance depending on the distance of the matrix element from the diagonal

$$\left\langle (H_{ij})^2 \right\rangle = \frac{1}{1+(|i-j|/b)^{2\alpha}} \ . \tag{4}$$

This model was shown to undergo a sharp transition at $\alpha = 1$ from localized states for $\alpha > 1$ to delocalized states for $\alpha < 1$. This transition is supposed to be similar to an Anderson metal-insulator transition, presenting multifractality of eigenfunctions and non-trivial spectral compressibility at criticality. The parameter b determines the critical dimensionless conductance and so establishes the character of the transition. For $b = 1$ the nearest level spacing distribution differs from the typical one at the 3D Anderson metal-insulator transition [13]. Recent calculations by us have obtained different diverging exponent for the correlation length as the transition is approached from below and from above [14].

We obtain the eigenfunctions and eigenvalues of the Hamiltonian matrix by numerical diagonalization. In the case of the Anderson model, we use techniques for large sparse matrices, in particular a Lanczos tridiagonalization without reorthogonalization method [15], while for the PRBM case we employ standard diagonalization subroutines, since we have to deal with full matrices. For the Anderson model, the system size varies between 5 and 40, and we consider a small energy window $(-1, 1)$ around the center of the band. We take for the critical disorder the value $W_\mathrm{c} = 16.5$. In the PRBM case, the system size ranges between $L = 100$ and 15000 and the energy window considered is $(-0.4, 0.4)$. Reducing the width of the previous windows do not alter the results. The number of random realizations is such that the number of states included for each L is roughly equal to 3×10^5, except for $L = 40$ in the Anderson model where this number is 2×10^4. In order to reduce edge effects, we use periodic boundary conditions in all cases considered.

To elucidate whether the correlation dimension D_2 possesses a well defined single value or alternatively corresponds to a distribution, we have calculated the IPR for the wavefunctions in the energy window considered for many disorder realizations. For each L we obtain the distribution function of $\ln P_2^{(\alpha)}$, since this is a well behaved self-averaging magnitude. At the end of the paper, we will also discuss the distribution of $P_2^{(\alpha)}$. In Fig. 1 we show the evolution of the distribution function $F(\ln P_2^{(\alpha)})$ with L for the 3D Anderson model. The system sizes drawn are $L =$ 5, 6, 8, 10, 12, 15, 17, 20, 25, 30 and 40, from right to left. It is clear that $F(\ln P_2^{(\alpha)})$ changes with size. As L increases this distribution becomes wider and the height of its peak smaller. This is in qualitative, but not quantitative, agreement with the results of Ref. [6].

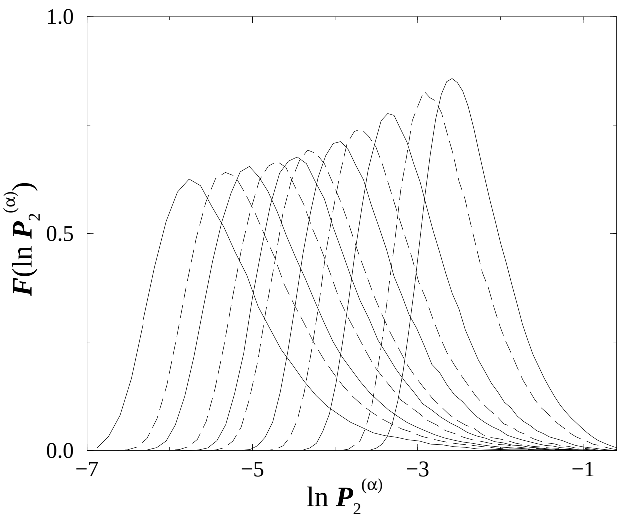

FIG. 1. Distribution function $F(\ln P_2^{(\alpha)})$ for the 3D Anderson model on a logarithmic scale for $L =$ 5, 6, 8, 10, 12, 15, 17, 20, 25, 30 and 40, from right to left.

We characterize the previous distribution by its average value, $\langle \ln P_2^{(\alpha)} \rangle$, and its variance, $\sigma^2(L) = \mathrm{var}(\ln P_2^{(\alpha)})$. This variance increases with L and seems to saturate at a constant value, as one can implicitly appreciate from the peak of the distributions. We try a fit of the form

$$\sigma^2(L) = \sigma^2(\infty) - AL^{-\gamma} \ , \tag{5}$$

with $\sigma^2(\infty)$, A and γ being three adjustable parameters. We found that the exponent γ was always very close to the correlation dimension divided by $2d$. Thus, we fix

$$\gamma = \frac{D_2}{2d} \ , \tag{6}$$

and keep only two free parameters. In the language of scaling theory, this exponent characterizes the behavior of the *irrelevant* length. Our assumption that the irrelevant exponent γ is equal to $D_2/2d$ properly interpolates between two known limiting cases of the PRBM model. For $b \ll 1$, one can get from Eqs. (5) and (6) the $b \ln L$ correction predicted in [8]. Similarly, for $b \gg 1$, the exponent γ in Eq. (6) tends to 1/2, which implicitly coincides with the results of Ref. [8] for this regime.

In Fig. 2 we represent on a log-log scale $\sigma^2(\infty) - \sigma^2(L)$ as a function of L for the 3D Anderson transition. The fitted values of the free parameters are $\sigma^2(\infty) = 1.09$ and $A = 1.24$. The slope of the straight line has not been fitted and corresponds to $D_2/2d$, where $D_2 = 1.4$ has been obtained from the inset of Fig. 2. In this inset we plot $\langle \ln P_2^{(\alpha)} \rangle$ versus $\ln L$. The straight line is a linear fit to the data and its slope is equal to D_2. The value $\sigma^2(\infty) = 1.09$ found is in good agreement with the conjecture $\sigma^2(\infty) \approx 1$ [3].

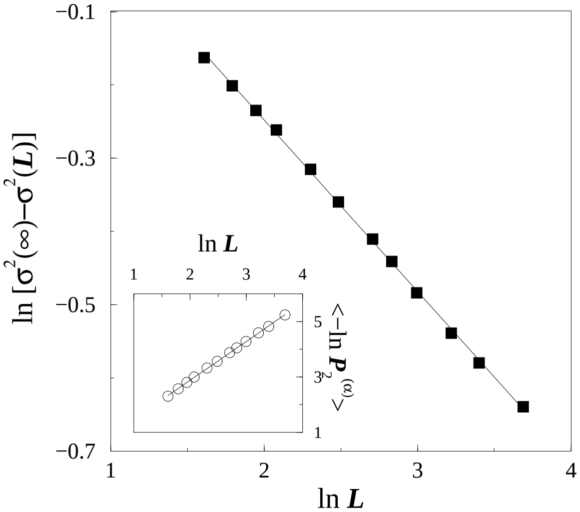

FIG. 2. $\sigma^2(\infty) - \sigma^2(L)$ as a function of L for the Anderson transition on a log-log scale. L ranges between 5 and 40. The straight line is a linear fit to Eq. (5), and the slope is obtained from the inset. Inset: $\langle \ln P_2^{(\alpha)} \rangle$ versus $\ln L$. The straight line is a linear fit to the data and its slope is equal to the (average) correlation dimension D_2.

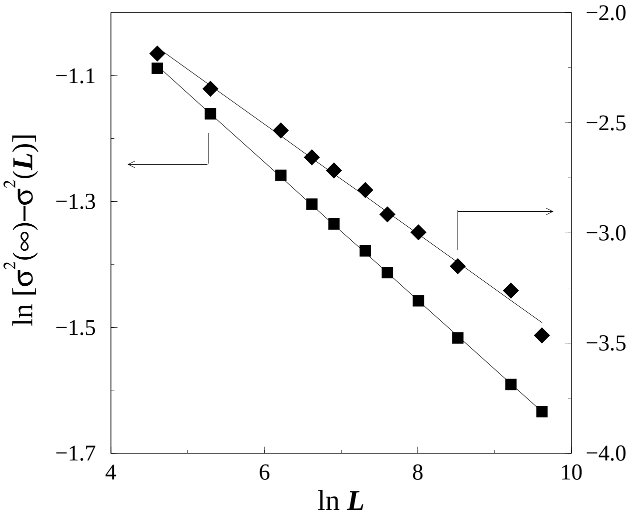

FIG. 3. $\sigma^2(\infty) - \sigma^2(L)$ as a function of L on a log-log scale for the PRBM transition with $b = 0.1$ (squares) and 0.3 (diamonds). Solid lines are fits to Eq. (5).

These results are in contradiction with those of Parshin and Schober [6], who found a linear increase of $\sigma(L)$ with $\ln L$. We have checked that $\sigma(L)$ versus $\ln L$ do not follow a linear behavior. We believe that the disagreement is due to two reasons. Firstly, the relatively small system sizes employed in Ref. [6], and secondly the use of different estimates of an effective IPR, instead of directly studying the width of the $\ln P_2^{(\alpha)}$ distribution. On the other hand, Mirlin and Evers [8] already obtained for the PRBM model similar results to ours.

Eq. (6) may be also valid for other transitions, like the integer quantum Hall transition, where the irrelevant exponent y –which is the same as our γ– is known to be $y = 0.4 \pm 0.1$ and $D_2 = 1.48$. These values are consistent with Eq. (6). Polyakov [16] and Evers and Brenig [17] already obtained a relation between the irrelevant exponent y and the correlation dimension D_2, but our result seems to fit more naturally the reported values of y.

Although D_2 is well defined in the macroscopic limit, it is clear from Fig. 2 that the results of numerical calculations with present computers will drastically depend on the particular definition of correlation dimension adopted. For the sizes available the distributions of $\ln P_2^{(\alpha)}$ are so wide that one can obtain very different numerical values of D_2 depending on the method of calculation, as pointed out in Ref. [6], but this does not imply at all the existence of a distribution of correlation dimensions in the macroscopic limit.

We have now performed similar calculations for the PRBM model, considering several values of the parameter b in the range where the transition could be similar to the 3D Anderson transition. As regards the IPR fluctuations, the results for the PRBM model follow the same behavior, given by Eq. (5), as the Anderson transition. In order to show this, we plot in Fig. 3 the same quantity as in Fig. 2, but for the PBRM model with $b = 0.1$ (squares) and with $b = 0.3$ (diamonds). The data in all cases are well fitted by straight lines, whose slopes are equal to $D_2/2d$. The fitted parameters for the cases shown in Fig. 3 are $\sigma^2(\infty) = 0.55$ and $A = 0.56$ for $b = 0.1$, and $\sigma^2(\infty) = 0.33$ and $A = 0.35$ for $b = 0.3$. The values of D_2 have been obtained by the same procedure as for the Anderson model and are equal to 0.21 and 0.48 for $b = 0.1$ and 0.3, respectively. For $b = 1$, $\sigma^2(L)$ is practically constant, explaining the scale-invariance claim made in Refs. [7,8]. For $b > 1$, Eq. (5) is still valid, but A changes sign. In other words, now the system approaches the asymptotic value $\sigma^2(\infty)$ from above.

Now we focus on the behavior of the asymptotic value $\sigma^2(\infty)$ as a function of b, which is summarized in Fig. 4. We were not able to fit all the data (including large values of b not shown in the figure) with a simple function. Power laws of $1/b$ with different exponents are able to fit extended parts of the curve. For example, the dashed line corresponds to a $1/b$ dependence and fits fairly well the intermediate regime, $b \sim 1$, in which we are most interested. For $4 < b < 12$, the results are in reasonable agreement with the $1/b^2$ prediction of Evers and Mirlin [7], based on the fluctuations of the 2D case and taking into account that the dimensionless conductance is proportional to b.

The distribution of $z \equiv P_2^{(\alpha)}/P_2^{(\mathrm{typ})}$, where the typical IPR $P_2^{(\mathrm{typ})}$ is chosen as the median of the distribution [18], is very different from the distribution of $\ln P_2^{(\alpha)}$ due

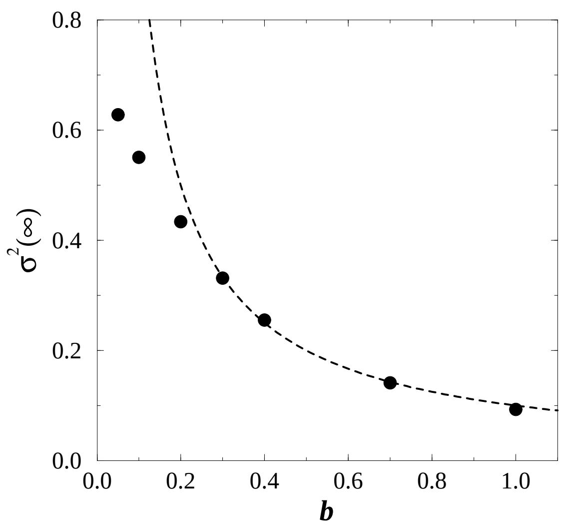

FIG. 4. Asymptotic value of $\sigma^2(L)$ as a function of b. The dashed line is proportional to $1/b$.

to the their strong asymmetries and long tails. We have studied the size dependence of the variance σ_z^2 of z. For $b \ll 1$ we have analytically calculated this dependence using the renormalization group method of Levitov [19]. One combines two blocks of size L to form a block of size $2L$. Some states are assumed to remain practically unchanged in this process, while others combine to form a more extended state. From the evolution equation describing the behavior of the distribution function, one can obtain D_2 in terms of the parameters of the model [8]. We extended the procedure to obtain the following equation for the variance

$$\sigma_z^2(2L) = 2^{-D_2/8}\, \sigma_z^2(L) + \tfrac{b}{2}\, 2^{-2D_2} \; . \tag{7}$$

We have checked that for $b < 0.3$ Eq. (7) fits the variance of z fairly well. On the other hand, the numerical calculations for larger values of b and for the 3D Anderson transition do not follow this behavior. Eq. (7) predicts a power law approach of $\sigma_z^2(L)$ to its asymptotic value with an exponent $-D_2/8$, in contrast with the exponent $-D_2/2d$ observed for the variance of $\ln P_2^{(\alpha)}$.

We have compared the Anderson model with the PRBM model for several values of b through different critical parameters. The case $b = 0.3$ is very similar to the Anderson transition, presenting practically the same values for all the critical magnitudes studied. The correlation dimension divided by the embedding dimension is equal to $1.4/3 = 0.47$ for the Anderson model and to 0.49 for the PRBM model. Correspondingly, the scaling with system size of the fluctuations of $\ln P_2^{(\alpha)}$ is rather similar. The asymptotic value of these fluctuations for the PRBM model is the same as for the Anderson model divided by d. Also, the normalized variance of the nearest level spacing at criticality [20,21], which characterizes the intermediate statistics at the transition [22], is 0.18 for the Anderson model and 0.19 for the PRBM model with $b = 0.3$.

Following the comparison between the two models, we have finally analyzed the large values tail of the distribution of z. Mirlin and Evers [7,8] predicted for the PRBM model a power-law tail $F(z) \propto z^{-1-x_2}$, where x_2 depends on the transition considered. For example, x_2 is equal to 4.2 for $b = 1$ and 2.1 for $b = 0.3$. We have checked that this prediction is also obeyed by the Anderson model, for which we obtained a value of x_2 equal to 1.6, in relative agreement with the PRBM model for $b = 0.3$.

In summary, we found that the variance of the fluctuations of $\ln P_2^{(\alpha)}$ tends to an asymptotic value, implying a well-defined correlation dimension. For the intermediate regime of the PRBM model, $\sigma^2(\infty)$ is proportional to $1/b$. For the Anderson transition and the PRBM model at criticality, the variance tends to its asymptotic value as a power law $L^{-D_2/2d}$. Thus, the irrelevant length scale diverges with an exponent $D_2/2d$, a result that may be extended to other transitions like the integer quantum Hall transition. The behavior of the distributions of $\ln P_2^{(\alpha)}$ and of $P_2^{(\alpha)}/P_2^{(\mathrm{typ})}$ is different. We checked that the 3D Anderson transition and the PRBM transition for $b = 0.3$ are similar for all the critical magnitudes analyzed: the correlation dimension, the scaling of the fluctuations of $\ln P_2^{(\alpha)}$, the exponent characterizing the long z tail of the distribution $F(z)$ and the normalized nearest level variance.

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