# Error bounds for a class of subdivision schemes based on the two-scale refinement equation

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**Abstract**

Subdivision schemes are iterative procedures to construct curves and constitute fundamental tools in Computer Aided Design. Starting with an initial *control polygon*, a subdivision scheme refines the computed values at the previous step according to some basic rules. The scheme is said to be convergent if there exists a limit curve. The computed values define a control polygon in each step. This paper is devoted to estimate error bounds between the "ideal" limit curve and the control polygon defined after $k$ subdivision stages. In particular, a stop criteria of convergence is obtained. The considered refinement rules in the paper are widely used in practice and are associated to the well known two-scale refinement equation including as particular examples Daubechies' schemes. Companies such as Pixar have made subdivision schemes the basic tool for much of their computer graphics-modelling software.

## 1 Introduction

Non-uniform rational B-spline (NURBS) is a mathematical tool widely used in computer graphics for generating curves. The development of NURBS started in the 1950s by engineers who were in need of precise representations of freeform surfaces like those used in car and ship industry.

Other important method for generating smooth curves are the use of subdivision schemes [1]-[3]-[4]. Their flexibility and simplicity are their best

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properties. They have been used in many applications in Computer Graphics and Computer Aided Geometric Design in the last decades. Companies such as Pixar have made subdivision schemes the basic tool for much of their computer graphics-modelling software [10].

Both approaches are included in the control polygon paradigm. In the applications of this type of procedures emerges an important question:

*How well the control polygon approximates the limit curve?.*

Several researchers give several answers in the NURBS framework [5]-[6]-[9]. These methods, for computing the bounds on the approximation of polynomials and splines by their control structures, are based on parameterizations, so that it is very difficult for them to be generalized to the subdivision schemes.

In [8], the authors estimate error bounds for binary subdivision schemes in terms of the maximal differences of the initial control point sequence and constants that depend on the subdivision mask. The technique is independent of parameterizations and therefore it can be easily and efficiently implemented. Nevertheless, there exist widely used masks for which their hypotheses are not satisfied. A very interesting example, commonly used in practice, is the class of subdivision schemes based on the well known two-scale equation (see for instance the Daubechies' filters in [2], p. 195). Our purpose is to obtain error bounds of this general and important class of subdivision schemes. Furthermore, our results will improve the bounds given in [8] (Theorem 1).

### 1.1 The two-scale equation and the associated class of subdivision schemes

The two-scale refinement equation [7], p. 228,

$$\varphi(x) = \sum_{n\in\mathbf{Z}} h_n \, \varphi(2x - n) \tag{1}$$

with

$$\sum_{n\in\mathbf{Z}} h_{2n} = h_{2n+1} = 1, \tag{2}$$

constitutes the starting point in the construction of orthonormal wavelet bases and also it plays a central role in the subdivision schemes for curve generations. The approximation coefficients of the reconstruction algorithm associated to an orthogonal basis are obtained by the use of fast algorithms in terms of convolutions with the filter $h$. More precisely, we use the following notation: The convolution product, $\star$, of two vectors is given by $(u \star h)_i = \sum_{n\in\mathbf{Z}} u_n\, h_{i-n}$. The operation "insertion of zeros" is denoted by:

$$v_i^0 = \begin{cases} v_m & i = 2m \\ 0 & i = 2m+1. \end{cases}$$

Usually, the vector $v_i$ represents the approximation coefficients associated to certain level of resolution $k$, i.e., $v_i = f_i^k$, $k \in \mathbf{N}$, then we will denote $v_i^0$ as $f_i^{k;0}$.

With this notation, the reconstruction algorithm used to define the approximation coefficients at stage $k+1$ in terms of the coefficients at stage $k$ is given by ([7], p. 255):

$$f_i^{k+1} = \sum_{n\in\mathbf{N}} h_{i-2n}\, f_n^k = (f^{k;0} \star h)_i. \tag{3}$$

Equation (3) defines the subdivision rule associated to the two-scale refinement equation. This type of schemes are studied in detail, for instance, in [1]. The sequence of points $\{f_i^k\}$ represents the values associated with the diadic mesh points $x_i^k = \frac{i}{2^k}, i \in \mathbf{Z}$. The initial stage is defined by the set

$$\{f_n^0, \quad n \in \mathbf{Z}\}, \tag{4}$$

with each $f_n^0 \in \mathbf{R}^m$, $m \geq 2$, which is called the initial control polygon and $\mathcal{H} = \{h_n\}_{n\in\mathbf{Z}}$ is usually called the mask associated to the subdivision scheme.

The subdivision scheme with mask $\mathcal{H}$ converges if, for each bounded initial control polygon, there exists a continuous function, $F : \mathbf{R} \mapsto \mathbf{R}^m$, such that

$$\lim_{k\to\infty} \sup_{i\in\mathbf{Z}} \left| F\left(\frac{i}{2^k}\right) - f_i^k \right| = 0.$$

The function $F$ is known as the limit curve associated to the subdivision scheme. This function is usually denoted by $R^\infty$, where $R^k$ represents the interpolating polygonal at the points $\{f_i^k\}_i$.

The previous subdivision procedure can be easily formulated as a binary subdivision scheme (see [2], p. 207). Consider, for each $n \in \mathbf{N}$, $h_n^{[\mathrm{E}]} = h_{2n}$ and $h_n^{[\mathrm{O}]} = h_{2n+1}$. In practice, $h_n$ has finite length. We use the notation $\mathrm{length}\,(h_n) = 2(L+1), L \geq 0$. Then, (3) is equivalent to the following formulation:

$$\begin{cases} f_{2i}^{k+1} = \sum_{m=0}^{L} h_m^{[E]} f_{i-m}^k, \\ f_{2i+1}^{k+1} = \sum_{m=0}^{L} h_m^{[O]} f_{i-m}^k, \end{cases} \tag{5}$$

where, by (2)

$$\sum_{m=0}^{L} h_m^{[E]} = \sum_{m=0}^{L} h_m^{[O]} = 1. \tag{6}$$

Equations (5) define a scheme where $f_{2i}^{k+1}$ replaces the value $f_i^k$ at $x_{2i}^{k+1} = x_i^k$ and $f_{2i+1}^{k+1}$ is inserted at the new point $x_{2i+1}^{k+1} = \frac{x_i^k + x_{i+1}^k}{2}$.

The organization of the paper is as follows. Section 2 is devoted to give the error bounds for $\|R^{k+1} - R^k\|_\infty$. We will use some results which will be proved in a final Appendix. In Section 3 we give the error bounds between subdivision curves and their control polygons, i.e., $\|R^\infty - R^k\|_\infty$. Section 4 is devoted to some numerical experiments and, finally, a complete analysis concerning the main properties of the derived upper bounds will be presented in the Appendix.

## 2 Error bounds for two consecutive subdivision stages

This section is composed of a collection of expressions and inequalities in order to find an upper bound for $\|R^{k+1} - R^k\|_\infty$. We consider the notation described in [8] and that it was also recalled in Section 1.

More precisely, we consider the initial control polygon (4) and the values $f_i^k$, $k \geq 0$, given by (5). As in [8], we introduce

$$\mathcal{E}_k = \max_i \|f_{2i}^{k+1} - f_i^k\|, \tag{7}$$

$$\mathcal{O}_k = \max_i \|f_{2i+1}^{k+1} - \frac{1}{2}(f_i^k + f_{i+1}^k)\|. \tag{8}$$

the maximum difference between $R^{k+1}$ and $R^k$ is attained at a point on the $(k+1)$-th mesh, then

$$\|R^{k+1} - R^k\|_\infty \leq \max\{\mathcal{E}_k, \mathcal{O}_k\}. \tag{9}$$

We denote

$$\tilde{h}_j^{[E]} = \sum_{i=j+1}^{L} h_i^{[E]}$$

and

$$\begin{cases} \tilde{h}_0^{[O]} = \sum_{i=1}^{L} h_j^{[O]} - \frac{1}{2}, \\ \tilde{h}_j^{[O]} = \sum_{i=j+1}^{L} h_i^{[O]}, \qquad j \geq 1. \end{cases}$$

From (5) and (6), by the procedure applied in [8], it follows that

$$f_{2i}^{k+1} - f_i^k = \sum_{j=0}^{L-1} \tilde{h}_j^{[E]} \left( f_{i-j-1}^k - f_{i-j}^k \right), \tag{10}$$

and

$$f_{2i+1}^{k+1} - \frac{1}{2}(f_i^k + f_{i+1}^k) = \sum_{j=0}^{L-1} \tilde{h}_j^{[O]} \left( f_{i-j-1}^k - f_{i-j}^k \right). \tag{11}$$

In order to find upper bounds for $\mathcal{E}_k$ and $\mathcal{O}_k$, it follows from (10) and (11) that we have to control the quantities

$$\max_i \| f_{i-1}^k - f_i^k \|. \tag{12}$$

We analyze the following two cases that correspond to the possibilities to express (12).

By using (6) we obtain the following expressions, analogously to the given ones in [8]:

$$f_{2i}^k - f_{2i+1}^k = \sum_{j=0}^{L} (\tilde{h}_j^{[E]} - \tilde{h}_j^{[O]}) \, (f_{i-j-1}^{k-1} - f_{i-j}^{k-1}), \tag{13}$$

and

$$f_{2i+1}^k - f_{2i+2}^k = \sum_{j=0}^{L} (h_j^{[O]} - (\tilde{h}_j^{[E]} - \tilde{h}_j^{[O]})) \, (f_{i-j-1}^{k-1} - f_{i-j}^{k-1}). \tag{14}$$

From (6) it follows that

$$\tilde{h}_j^{[E]} = 1 - \sum_{n=0}^{j} h_n^{[E]} \quad \text{and} \quad \tilde{h}_j^{[O]} = 1 - \sum_{n=0}^{j} h_n^{[O]}. \tag{15}$$

From (15) it holds that (13) and (14) can be respectively expressed as follows:

$$f^k_{2i} - f^k_{2i+1} = \sum_{j=0}^{L} c_j\,(f^{k-1}_{i-j-1} - f^{k-1}_{i-j}), \qquad (16)$$

$$f^k_{2i+1} - f^k_{2i+2} = \sum_{j=0}^{L} d_j\,(f^{k-1}_{i-j-1} - f^{k-1}_{i-j}), \qquad (17)$$

where

$$c_j = \sum_{n=0}^{j}(h^{[O]}_n - h^{[E]}_n) \quad \text{and} \quad d_j = h^{[O]}_j - c_j. \qquad (18)$$

Expressions (16) and (17) are analogous to the given ones in [8], Eqs. (9) and (10), respectively. The proof of Theorem 1 in [8] strongly uses the conditions

$$sc = \sum_{j=0}^{L} |c_j| < 1 \quad \text{and} \quad sd = \sum_{j=0}^{L} |d_j| < 1. \qquad (19)$$

Our analysis does not require (19). This explains that, in what follows, we will develop another strategy.

We introduce, for each $l \in \mathbf{Z}$,

$$r^k_l = f^k_l - f^k_{l+1}, \qquad (20)$$

and also, for $j = 0, \cdots, L$,

$$g_{2j} = c_j \quad \text{and} \quad g_{2j+1} = d_j. \qquad (21)$$

Then, (16) and (17) take (respectively) the form:

$$r^k_{2i} = \sum_{j=0}^{L} g_{2j}\, r^{k-1}_{i-j},$$

$$r^{k-1}_{2i+1} = \sum_{j=0}^{L} g_{2j+1}\, r^k_{i-j},$$

and also (see (3) and (5)),

$$r^k_{2i} = (r^{k-1;0} \star g)_{2i}, \quad r^k_{2i+1} = (r^{k-1;0} \star g)_{2i+1}.$$

Hence,

$$r^k_i = (r^{k-1;0} \star g)_i. \qquad (22)$$

By the use of the analysis developed in the final Appendix, it follows from Corollary 4 that

$$\max_i \|r_i^k\| \leq G_k \max_i \|r_i^0\|, \tag{23}$$

where $G_k$ is (see Definition 1 and (53)), the associated constant of a $k$-th convolution with filter $g$.

From (7), (8), (9), (10), (11) (20) and (23), we obtain

$$\begin{aligned} \|R^{k+1} - R^k\|_\infty &\leq \max \left\{ \sum_{j=0}^{L-1} |\tilde{h}_j^{[E]}|, \sum_{j=0}^{L-1} |\tilde{h}_j^{[O]}| \right\} \max_i \|r_i^k\| \\ &\leq \max \left\{ \sum_{j=0}^{L-1} |\tilde{h}_j^{[E]}|, \sum_{j=0}^{L-1} |\tilde{h}_j^{[O]}| \right\} G_k \max_i \|r_i^0\|. \end{aligned}$$

By the use of the notation

$$\gamma = \max \left\{ \sum_{j=0}^{L-1} |\tilde{h}_j^{[E]}|, \sum_{j=0}^{L-1} |\tilde{h}_j^{[O]}| \right\}, \quad \beta = \max_i \|r_i^0\|,$$

it holds

$$\|R^{k+1} - R^k\|_\infty \leq \gamma \, \beta \, G_k. \tag{24}$$

Finally, we state the main result of this Section.

**Theorem 1** *Consider the initial control polygon (4) and the values $f_i^k$, $k \geq 1$, recursively defined by (5) with (6). Consider:*

- *$R^k$ the piecewise linear interpolation at the values $f_i^k$.*
- $\tilde{h}_j^{[E]} = \sum_{i=j+1}^{L} h_i^{[E]}$.
- $\tilde{h}_0^{[O]} = \sum_{i=1}^{L} h_j^{[O]} - \frac{1}{2}, \tilde{h}_j^{[O]} = \sum_{i=j+1}^{L} h_i^{[O]}, j \geq 1$.
- $c_j = \sum_{n=0}^{j} (h_n^{[O]} - h_n^{[E]})$, $d_j = h_j^{[O]} - c_j$.
- $g_{2j} = c_j, g_{2j+1} = d_j$.

- $r_l^k = f_l^k - f_{l+1}^k, l \in \mathbf{Z}$.
- $\gamma = \max \left\{ \sum_{j=0}^{L-1} |\tilde{h}_j^{[E]}|, \sum_{j=0}^{L-1} |\tilde{h}_j^{[O]}| \right\}, \quad \beta = \max_i \|r_i^0\|$.
- *$G_k$ is the constant associated to the $k$-th convolution with filter $g$, given by (53).*

*If $g$ is given such that the conditions of Theorem 3 are satisfied, then*

$$\|R^{k+1} - R^k\|_\infty \leq \gamma\, \beta\, G_k.$$

# 3 Error bounds between the limit curve and the $k$-th control polygon

This section is devoted to prove the following result:

**Theorem 2** *Under the same assumptions that Theorem 1, let $R^\infty$ be the limit curve associated to the subdivision process (5) and consider $k_0 \geq 1$, a natural number, satisfying $G_{k_0} < 1$.*

- *If $k_0 = 1$, then*

$$\|R^\infty - R^k\|_\infty \leq \gamma\, \beta\, \frac{G_1^k}{1 - G_1}. \tag{25}$$

- *If $k_0 \geq 2$, then*

$$\|R^\infty - R^k\|_\infty \leq \gamma\, \beta\, G_1 \left( \frac{G_{k_0}^{\alpha}}{1 - G_{k_0}^{1/k_0}} \right),$$

  *where $\alpha = \alpha(k, k_0) = \frac{k - k_0 + 1}{k_0}$.*

**Proof:**
By the use of the triangle inequality, we obtain from (24) that

$$\|R^\infty - R^k\|_\infty \leq \gamma\, \beta \sum_{m=k}^{+\infty} G_m. \tag{26}$$

The series in (26) converges. This fact easily follows by applying, for instance, D'Alembert's criteria. It is possible to find a geometric series that is an upper bound for $\sum_{m=k}^{+\infty} G_k$. From Corollary 5 in the Appendix, it follows that there exists $k_0 \in \mathbf{N}, k_0 \geq 1$, such that

$$G_{k_0} < 1 \tag{27}$$

Let $q = q(m)$ and $r$ be dependent on each $m$ such that

$$m = q\,k_0 + r, \quad r \in \{0, 1, 2, \cdots, k_0 - 1\}. \tag{28}$$

Then

$$G_m \leq \begin{cases} G_{k_0}^q, & \text{if} \quad r = 0, \\ G_{k_0}^q\, G_1, & \text{if} \quad r = 1, \\ G_{k_0}^q\, G_2, & \text{if} \quad r = 2, \\ \cdots & \\ G_{k_0}^q\, G_{k_0-1}, & \text{if} \quad r = k_0 - 1. \end{cases}$$

Hence,

$$G_m \leq \max\{G_{k_0}^q, G_{k_0}^q\, G_1, G_{k_0}^q\, G_2, \cdots, G_{k_0}^q\, G_{k_0-1}\} = \begin{cases} G_1^m & \text{if} \quad k_0 = 1, \\ G_{k_0}^q\, G_1 & \text{if} \quad k_0 \geq 2. \end{cases} \tag{29}$$

By (29) it holds that

$$\sum_{m=k}^{+\infty} G_m \leq \begin{cases} \sum_{m=k}^{+\infty} G_1^m, & \text{if} \quad k_0 = 1, \\ G_1 \sum_{m=k}^{+\infty} G_{k_0}^{q(m)}, & \text{if} \quad k_0 \geq 2. \end{cases} \tag{30}$$

From (28), it follows that

$$\frac{m - k_0 + 1}{k_0} \leq q(m) \leq \frac{m}{k_0}. \tag{31}$$

Denoting $\alpha(m, k_0) = \frac{m-k_0+1}{k_0}$, from (27) and (31), we obtain

$$G_{k_0}^{q(m)} \leq G_{k_0}^{\alpha(m,k_0)}. \tag{32}$$

Hence, by (30) and (32), the proof is complete.

■

It is possible to improve the bound in (25). In order to find a bound smaller than the given one in (25), let us suppose, at first, that $G_2$ is also known. By Theorem 3, we have that $G_2 < G_1 < 1$ and hence:

- If $k = 2j$ with $j \geq 1$, then

$$\begin{aligned}
\sum_{m=2j}^{+\infty} G_m &= G_{2j} + G_{2j+1} + G_{2j+2} + G_{2j+3} + \cdots \\
&\leq G_2^j + G_2^j\, G_1 + G_2^{j+1} + G_2^{j+1}\, G_1 + \cdots \\
&= (1 + G_1) \sum_{m=j}^{+\infty} G_2^m \\
&= (1 + G_1) \left( \frac{G_2^j}{1 - G_2} \right). \qquad (33)
\end{aligned}$$

- If $k = 2j + 1$ with $j \geq 1$, then

$$\begin{aligned}
\sum_{m=2j+1}^{+\infty} G_m &\leq G_2^j\, G_1 + (1 + G_1) \sum_{m=j+1}^{+\infty} G_2^m \\
&= G_2^j\, G_1 + (1 + G_1) \left( \frac{G_2^{j+1}}{1 - G_2} \right). \qquad (34)
\end{aligned}$$

It only remains for analyzing the case $k = 1$. Taking into account that

$$\sum_{m=1}^{+\infty} G_m = G_1 + \sum_{m=2}^{+\infty} G_m, \qquad (35)$$

then, by the use of (33) in the second sum in (35), it holds

$$\begin{aligned}
\sum_{m=1}^{+\infty} G_m &\leq G_1 + (1 + G_1) \sum_{m=1}^{+\infty} G_2^m \\
&= G_1 + (1 + G_1) \left( \frac{G_2}{1 - G_2} \right). \qquad (36)
\end{aligned}$$

**Remark 1** *The bound which improves (25) is given by substituting the term* $\dfrac{G_1^k}{1 - G_1}$ *in (25), by (33) for even* $k$*, or (34) for odd* $k \geq 3$*, or (36) for* $k = 1$*.*

It is possible to give a general result by supposing that $G_1, G_2, \cdots, G_{N_0}$ are known, for some fixed $N_0 > 2$. Then, by Theorem 3, $G_{N_0} < G_{N_0 - 1} <$

$\cdots < G_2 < G_1 < 1$, and

$$G_m \le \begin{cases} G_{k_0}^j, & \text{if} \quad m = N_0\, j, \\ G_{N_0}^j\, G_1, & \text{if} \quad m = N_0\, j + 1, \\ G_{N_0}^j\, G_2, & \text{if} \quad m = N_0\, j + 2, \\ \cdots & \\ G_{N_0}^j\, G_{N_0-1}, & \text{if} \quad m = N_0\, j + (N_0 - 1). \end{cases}$$

In order to simplify the notation, we denote $G_0 = 1$ and consider for $s \ge 0$,

$$\tilde{G}_s = \sum_{n=s}^{N_0-1} G_n.$$

Then, we obtain, for $j \ge 1$,

$$\sum_{m=k}^{+\infty} G_m \le \begin{cases} \tilde{G}_0 \displaystyle\sum_{m=j}^{+\infty} G_{N_0}^m & \text{if} \quad k = N_0\, j, \\ \tilde{G}_1\, G_{N_0}^j + \tilde{G}_0 \displaystyle\sum_{m=j+1}^{+\infty} G_{N_0}^m & \text{if} \quad k = N_0\, j + 1, \\ \tilde{G}_2\, G_{N_0}^j + \tilde{G}_0 \displaystyle\sum_{m=j+1}^{+\infty} G_{N_0}^m & \text{if} \quad k = N_0\, j + 2, \\ \cdots & \\ \tilde{G}_{N_0-1}\, G_{N_0}^j + \tilde{G}_0 \displaystyle\sum_{m=j+1}^{+\infty} G_{N_0}^m & \text{if} \quad k = N_0\, j + (N_0 - 1). \end{cases}$$

It only remains for analyzing the cases $k \in \{1, 2, \cdots, N_0 - 1\}$. Analogously to the particular case $N_0 = 2$, it follows

$$\sum_{m=k}^{+\infty} G_m \le \tilde{G}_k + \tilde{G}_0 \sum_{m=1}^{+\infty} G_{N_0}^m.$$

## 4 Numerical experiments

We have considered the normalized Daubechies's filters (see [2] p. 195), such that (2) is satisfied. According to (18), we have computed $c_j$ and $d_j$. The conditions (19), required in [8], Theorem 1, are not satisfied for the filters corresponding to $db1$, $db6$, $db7$, $db8$, $db9$, and $db10$. These conditions are neither satisfied for $dbN$, $N \ge 11$.

For each Daubechies's filter $db1, \cdots, db10$, we have computed the constants $\{G_i\}_{i=1}^5$, associated to the filter $g$ given by (21). The following table shows $sc$, $sd$, (defined in (19)) and $\{G_i\}_{i=1}^5$ for the normalized Daubechies's filters.

| | $sc$ | $sd$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|---|---|
| $db1$ | 0.0000 | **1.0000** | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| $db2$ | 0.5000 | 0.8660 | 0.8660 | 0.6830 | 0.5123 | 0.3728 | 0.2660 |
| $db3$ | 0.8413 | 0.5406 | 0.8413 | 0.5061 | 0.2977 | 0.1639 | 0.0847 |
| $db4$ | 0.9933 | 0.5657 | 0.9933 | 0.6058 | 0.3430 | 0.1791 | 0.0922 |
| $db5$ | 0.9471 | 0.7832 | 0.9471 | 0.5869 | 0.3247 | 0.1735 | 0.0878 |
| $db6$ | 0.7237 | **1.0087** | 1.0087 | 0.6455 | 0.3263 | 0.1651 | 0.0835 |
| $db7$ | 0.5974 | **1.1499** | 1.1499 | 0.5943 | 0.3241 | 0.1664 | 0.0834 |
| $db8$ | 0.7933 | **1.1532** | 1.1532 | 0.6278 | 0.3167 | 0.1601 | 0.0805 |
| $db9$ | **1.0367** | **1.0007** | 1.0367 | 0.6685 | 0.3405 | 0.1712 | 0.0857 |
| $db10$ | **1.2330** | 0.7079 | 1.2330 | 0.6919 | 0.3569 | 0.1809 | 0.0909 |

Table 1. Constants $sc, sd$ and $G_i$, $i = 1, \cdots, 5$, for Daubechies' filters $db1, \cdots, db10$.

The main purpose of the following example is to improve the bound given in [8], corollary 3, in connection with one of the most used subdivision process. More precisely, in connection with the 4-point interpolatory subdivision rule [3].

The subdivision mask is defined by

$$\begin{aligned} h_n^{[E]} &= (0, 1, 0, 0) \\ h_n^{[O]} &= (-\omega, \frac{1}{2} + \omega, \frac{1}{2} + \omega, -\omega). \end{aligned}$$

We will consider $0 < \omega < (-1+\sqrt{5})/8 \approx 0.1545$ to guarantee a $\mathcal{C}^1$ continuous limit curve.

The subdivision rule is given in [8], eq (1). Hence, the sequences $c_j$ and $d_j$ are given in [8], Eqs. (9) and (10). By the use of $c_j$ and $d_j$ in (21), we obtain the following filter

$$g = (\omega, -\omega, \frac{1}{2}, \frac{1}{2}, -\omega, \omega). \tag{37}$$

From (56), it follows that $G_1 = G_1(\omega) = \frac{1}{2} + 2\,|\omega|$. Note that $G_1 = \delta$ in [8], corollary 3. Hence, the bound in (25),

$$\|R^\infty - R^k\|_\infty \leq \gamma\,\beta\,\frac{G_1^k}{1 - G_1},$$

is the same that the given one in the reference just mentioned. In order to improve the previous bound, we will consider (33) and (34). In this case, we have computed $G_2$ and, taking into account (55) and the range for $\omega$, it follows that $G_2$ is given by

$$G_2 = G_2(\omega) = 2\,|\omega|^2 + \left|\frac{1}{4} - \frac{\omega}{2} - \omega^2\right| + \left|\frac{\omega}{2} + \omega^2\right|. \tag{38}$$

We use the following notation:

$$\begin{aligned} B_1(\omega,k) &= \frac{G_1^k(\omega)}{1-G_1(\omega)}, \\ B_2^{[E]}(\omega,k) &= (1+G_1(\omega))\left(\frac{G_2^{k/2}(\omega)}{1-G_2(\omega)}\right), \quad \text{for even } k, \\ B_2^{[O]}(\omega,k) &= G_2^{(k-1)/2}(\omega)\,G_1(\omega) + (1+G_1(\omega))\left(\frac{G_2^{(k+1)/2}(\omega)}{1-G_2(\omega)}\right), \quad \text{for odd } k \geq 3. \end{aligned}$$

According to Remark 1, the bound in (25) is improved by considering

- $B_2^{[E]}(\omega,k)$ instead of $B_1(\omega,k)$ for even $k$.
- $B_2^{[O]}(\omega,k)$ instead of $B_1(\omega,k)$ for odd $k \geq 3$.

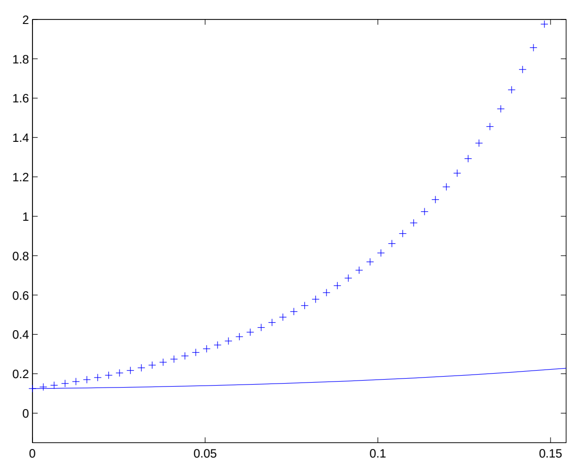


Figure 1: $B_1(\omega,4)$ (with +) and $B_2^{[E]}(\omega,4)$.

Figure 1 shows a comparative between $B_1(\omega,4)$ and $B_2^{[E]}(\omega,4)$, for $0 < \omega < (-1+\sqrt{5})/8 \approx 0.1545$.

We have also computed different values for $B_1(\omega,k)$, $B_2^{[E]}(\omega,k)$ and $B_2^{[O]}(\omega,k)$, in order to complete a comparative between them. We have defined five uniformly distributed values for $\omega$ in (0, 0.1545), i.e.,

$$\omega_1 = 0.0140,\ \omega_2 = 0.0421,\ \omega_3 = 0.0702,\ \omega_4 = 0.0983,\ \omega_5 = 0.1264.$$

and $k = 5, 10, 25, 50$. The results show that the given bounds in [8], corollary 3, are notoriously improved by our analysis. The values are collected in Table 2.

| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|
| $B_1(\cdot,5)$ | 0.0870 | 0.1637 | 0.2997 | 0.5408 | 0.9783 |
| $B_2^{[O]}(\cdot,5)$ | 0.0651 | 0.0721 | 0.0821 | 0.0959 | 0.1145 |
| $B_1(\cdot,10)$ | 0.0035 | 0.0111 | 0.0322 | 0.0887 | 0.2365 |
| $B_2^{[E]}(\cdot,10)$ | 0.0020 | 0.0022 | 0.0026 | 0.0032 | 0.0043 |
| $B_1(\cdot,25)$ | 2.4768e − 007 | 3.5210e − 006 | 4.0419e − 005 | 3.9207e − 004 | 0.0033 |
| $B_2^{[O]}(\cdot,25)$ | 6.3083e − 008 | 7.9240e − 008 | 1.1536e − 007 | 1.9264e − 007 | 3.6391e − 007 |
| $B_1(\cdot,50)$ | 2.8949e − 014 | 5.1540e − 012 | 5.8739e − 010 | 4.6633e − 008 | 2.7663e − 006 |
| $B_2^{[E]}(\cdot,50)$ | 1.8833e − 015 | 2.6821e − 015 | 5.1801e − 015 | 1.3279e − 014 | 4.3889e − 014 |

Table 2. Comparative.

Similar results are obtained for certain values of $\omega$ outside (0.0.1545). More precisely, the functions $B_1(\omega,k), B_2^{[E]}(\omega,k)$ and $B_2^{[O]}(\omega,k)$ are defined on the following domain for $\omega$:

$$\begin{aligned}\{\omega\in\mathbf{R}:\ 1-G_1(\omega)>0\} &\ \bigcap\ \{\omega\in\mathbf{R}:\ 1-G_2(\omega)>0\}\\ =(-0.2500, 0.2500) &\ \bigcap\ (-0.6123, 0.4478).\end{aligned}\tag{39}$$

Note that, the resulting interval in (39), i.e., $(-0.2500, 0.2500)$, is contained in the interval where the application $T$, introduced in Definition 2, is contractive:

$$\{\omega\in\mathbf{R}:\ G_2(\omega)<G_1(\omega)\} = (-0.9478, 0.5756).$$

Figure 2 is an extension of Figure 1. We have represented $B_1(\omega,4)$ and $B_2^{[E]}(\omega,4)$ for $\omega\in(-0.2500, 0.2500)$.

By taking another values for $k$, we obtain analogous graphics and values. The rest of examples given in [8] can be also improved by a similar analysis.

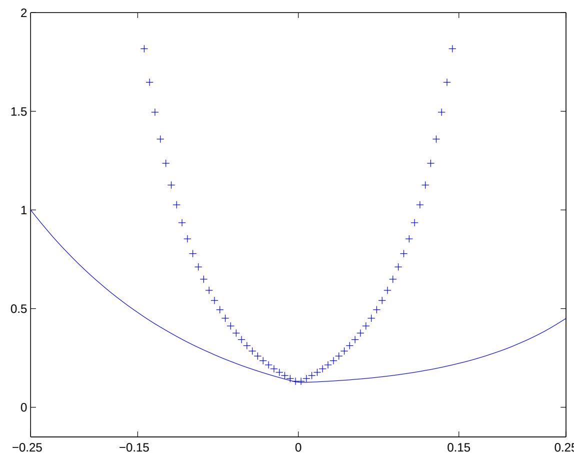


Figure 2: $B_1(\omega,4)$ (with $+$) and $B_2^{[E]}(\omega,4)$.

# 5 Appendix

This appendix is devoted to obtain a sequence, $G_k$, depending on the filter $g$ such that, for a given vector $v$, if $v^{(0)}$ is denoted according to (3), the following $k$ convolutions are bounded as:

$$\|(\cdots(((v^{(0)}\star g)^{(0)})\star g)^{(0)}\star\cdots\star g)^{(0)}\star g\|_\infty \leq \|v\|_\infty\, G_k.$$

In the first part of this section we obtain an useful reformulation of the $k$ convolutions

$$(\cdots(((v^{(0)}\star g)^{(0)})\star g)^{(0)}\star\cdots\star g)^{(0)}\star g. \tag{40}$$

## 5.1 Reformulation of the successive convolutions

The convolution of two vectors $(v_n)_{n\geq 0}, (u_n)_{n\geq 0}$ of finite lengths $L_v$ and $L_u$ defines a new vector $(v\star u)_j$ of length $L_v+L_u-1$ given by

$$(v\star u)_j = \sum_{n=\max\{j-(L_u-1),0\}}^{\min\{j,L_v-1\}} v_n\, u_{j-n}, \quad j=0,1,\cdots,L_v+L_u-2. \tag{41}$$

From now on, we assume that the indexes $n$ and $j$ are as appeared in (41). For simplicity, we start with the case of $k=1$ and $k=2$ convolutions and later on we analyze the general case.

- Case $k = 1$

Let us consider $(v_n)_{n\geq 0}$ as a vector of finite length and $(g_n)_{n\in\mathbf{N}} = (h_n)_{n=0}^{2N-1}$, with $g_n = 0$ if $n \geq 2N$.
From (41), it follows that $(v^{(0)} \star g)_j$ is given by

$$(v^{(0)} \star g)_j = \sum_{n=0}^{[j/2]} v_n g_{j-2n}, \tag{42}$$

where, as usual, $[x]$ denotes the integer part of $x$. Then

$$|(v^{(0)} \star g)_j| \leq \|v\|_\infty \sum_{n=0}^{[j/2]} |g_{j-2n}|$$

and

$$\|(v^{(0)} \star g)\|_\infty \leq \|v\|_\infty \sup_j \{\sum_{n=0}^{[j/2]} |A_{n,j}|\},$$

where, for each $n$ and $j$, the terms $A_{n,j}$ are defined by $g_{j-2n}$. We also introduce $C_{n,j}^{[1]} = A_{n,j}$.

- Case $k = 2$

From (42) we obtain

$$\begin{aligned} ((v^{(0)} \star g)^{(0)} \star g)_j &= \sum_{m=0}^{[j/2]} (v^{(0)} \star g)_m \, g_{j-2m} \\ &= \sum_{m=0}^{[j/2]} \left( \sum_{n=0}^{[m/2]} v_n g_{m-2n} \right) g_{j-2m}, \end{aligned} \tag{43}$$

then (43) is equal to

$$\sum_{m=0}^{[j/2^2]} v_m \left( \sum_{n=2m}^{[j/2]} g_{n-2m} g_{j-2n} \right), \tag{44}$$

therefore

$$
\begin{aligned}
((v^{(0)} \star g)^{(0)} \star g)_j &= \sum_{m=0}^{[j/2^2]} v_m \left( \sum_{n=2m}^{[j/2]} A_{m,n}\, A_{n,j} \right) \\
&= \left( \sum_{n=2m}^{[j/2]} A_{m,n}\, C^{[1]}_{n,j} \right) \\
&= \sum_{m=0}^{[j/2^2]} v_m C^{[2]}_{m,j},
\end{aligned}
$$

where, by definition,

$$
C^{[2]}_{m,j} = \sum_{n=2m}^{[j/2]} A_{m,n}\, C^{[1]}_{n,j}.
$$

Thus,

$$
\begin{aligned}
\|(v^{(0)} \star g)^{(0)} \star g\|_\infty &\leq \|v\|_\infty \sup_j \left\{ \sum_{m=0}^{[j/2^2]} \left| \sum_{n=2m}^{[j/2]} A_{m,n}\, C^{[1]}_{n,j} \right| \right\} \\
&= \|v\|_\infty \sup_j \{ \sum_{m=0}^{[j/2^2]} |C^{[2]}_{m,j}| \}.
\end{aligned}
$$

- General Case

  With a similar process, we obtain the following reformulation for $k$ convolutions:

$$
((\cdots(((v^{(0)} \star g)^{(0)}) \star g)^{(0)} \star \cdots \star g)^{(0)} \star g)_j = \sum_{m=0}^{[j/2^k]} v_m C^{[k]}_{m,j},
$$

  where $C^{[k]}_{m,j}$ is a sequence defined recursively by

$$
\begin{cases}
C^{[1]}_{m,j} = A_{m,j}, \\[2ex]
C^{[k]}_{m,j} = \displaystyle\sum_{p=2m}^{[j/2^{k-1}]} A_{m,p}\, C^{[k-1]}_{p,j}, \quad k \geq 2.
\end{cases}
\tag{45}
$$

Hence,

$$\begin{aligned}
& \|(\cdots(((v^{(0)} \star g)^{(0)}) \star g)^{(0)} \star \cdots \star g)^{(0)} \star g\|_\infty \\
\leq \quad & \|v\|_\infty \sup_j \left\{ \sum_{m=0}^{[j/2^k]} \left| \sum_{n=2m}^{[j/2^{k-1}]} A_{m,n}\, C_{n,j}^{[k-1]} \right| \right\} \\
= \quad & \|v\|_\infty \sup_j \{ \sum_{m=0}^{[j/2^k]} |C_{m,j}^{[k]}| \}, \qquad (46)
\end{aligned}$$

where $j$ depends on $L_v$.

In the following subsection, we present some results concerning the expressions

$$\sup_j \{ \sum_{m=0}^{[j/2^k]} |C_{m,j}^{[k]}| \}.$$

## 5.2 Some previous results

**Lemma 1** *In the above conditions, the constants verify*

$$C_{m,j}^{[k]} = C_{m+1,j+2^k}^{[k]}. \qquad (47)$$

**Proof:**
We consider an induction process over $k$.

- $k = 1$

$$C_{m,j}^{[1]} = A_{m,j} = g_{j-2m} = g_{j+2-2(m+1)} = A_{m+1,j+2} = C_{m+1,j+2}^{[1]}. \qquad (48)$$

In particular, we have also $A_{m+1,j} = A_{m,j-2}$, and so on.

- $k \to k+1$

$$C_{m,j}^{[k+1]} = \sum_{p=2m}^{[j/2^k]} A_{m,p}\, C_{p,j}^{[k]} = \sum_{n=2(m+1)}^{[(j/2^k)+2]} A_{m,n-2}\, C_{n-2,j}^{[k]}.$$

Applying the first case and twice the induction hypothesis we obtain

$$\begin{aligned} C_{m,j}^{[k+1]} &= \sum_{n=2(m+1)}^{[(j/2^k)+2]} A_{m+1,n}\, C_{n-1,j+2^k}^{[k]} \\ &= \sum_{n=2(m+1)}^{[(j/2^k)+2]} A_{m+1,n}\, C_{n,(j+2^k)+2^k}^{[k]} \\ &= \sum_{n=2(m+1)}^{[(j+2^{k+1})/2^k]} A_{m+1,n}\, C_{n,j+2^{k+1}}^{[k]} = C_{m+1,j+2^{k+1}}^{[k+1]}. \end{aligned}$$

■

As consequence of Lemma 1 we have

**Corollary 1**

$$C_{m,j}^{[k]} = C_{m-1,j-2^k}^{[k]}. \tag{49}$$

■

**Lemma 2** *Assume that* $g = (g_0, g_1, \cdots, g_{2N-1})$*, with* $N \in \mathbf{N}$ *and* $\sigma(k,N) = (2^k - 1)(2N-1)$*. Then*

$$C_{0,j}^{[k]} = 0, \quad \textit{for all} \quad j > \sigma(k,N). \tag{50}$$

**Proof:**
We consider an induction process over $k$.

- $k = 1$

$$C_{0,j}^{[1]} = A_{0,j} = g_j = 0, \quad \text{if} \quad j > (2N-1) = \sigma(1,N).$$

- $k \to k+1$

$$
\begin{aligned}
C_{0,j}^{[k+1]} &= \sum_{p=0}^{[j/2^k]} A_{0,p}\, C_{p,j}^{[k]} = \sum_{p=0}^{[j/2^k]} g_p\, C_{p,j}^{[k]} \\
&= \sum_{p=0}^{[j/2^k]} g_p\, C_{p-1,j-2^k}^{[k]} = \sum_{p=0}^{[j/2^k]} g_p\, C_{p-2,j-2\,2^k}^{[k]} \\
&\cdots \\
&= \sum_{p=0}^{[j/2^k]} g_p\, C_{0,j-p\,2^k}^{[k]}.
\end{aligned}
$$

We assume that $0 \le p \le 2N-1$, since otherwise $C_{0,j}^{[k+1]} = 0$ for all $j$. With this we have

$$
j - p\,2^k \ge j - (2N-1)\,2^k.
$$

By the induction hypothesis for $C_{0,j-p\,2^k}^{[k]}$, we obtain $C_{0,j}^{[k+1]} = 0$ if

$$
j - (2N-1)\,2^k > \sigma(k,N).
$$

That is,

$$
j > \sigma(k,N) + (2N-1)\,2^k = \sigma(k+1,N).
$$

■

In particular we have proved (50). Now, applying (47), we obtain the following result.

**Corollary 2**

$$
C_{m,j}^{[k]} = 0, \quad \textit{for all} \quad j > \sigma(k,N) + 2^k\, m. \tag{51}
$$

■

Finally, using (47), (49) and (51) we arrive at the following corollary.

**Corollary 3**

$$
\sup_j \left\{ \sum_{m=0}^{[j/2^k]} |C_{m,j}^{[k]}| \right\} = \sup_{j \in \Sigma(k,N)} \left\{ \sum_{m=0}^{[j/2^k]} |C_{m,j}^{[k]}| \right\},
$$

*where*

$$
\Sigma(k,N) = \{\sigma(k,N) - 2^k + 1, \sigma(k,N) - 2^k + 2, \cdots, \sigma(k,N)\}. \tag{52}
$$

■

We end this section with a definition.

**Definition 1** *We define the associated constant of a $k$-th convolution with filters $g = (g_0, g_1, \cdots, g_{2N-1})$ as*

$$G_k = \sup_{j\in\Sigma(k,N)} \left\{ \sum_{m=0}^{[j/2^k]} |C_{m,j}^{[k]}| \right\}. \tag{53}$$

As consequence of Corollary 3, the quantity $G_k$ is independent of $L_v$. In other words, the supreme of (46) is independent of $L_v$. We remark that the number of summands in $\sum_{m=0}^{[j/2^k]} |C_{m,j}^{[k]}|$ for $j \in \Sigma(k, N)$ is maximum, as in (22) by taking $N = L + 1$.

**Corollary 4** *From (46) , Corollary 3, (53) and taking into account the notation (3), it follows that*

$$\|v^k\|_\infty \leq G_k \, \|v^{(0)}\|_\infty.$$

■

In the next subsection, we analyze the monotonicity of the sequences $\{G_k\}_{k\geq 1}$.

### 5.3 Monotonicity of $\{\mathbf{G_k}\}_{\mathbf{k\geq 1}}$

As we said in the last section, the constants $G_k$ are independent of the finite length $L_v$. In particular, for each $N$, we can compute (independently of $v$) the constants $G_1$ and $G_2$ and check if

$$G_2 < G_1, \tag{54}$$

or

$$G_2 \geq G_1,$$

comparing the sums

$$\sum_{m=0}^{[j/2^2]} \left| \sum_{p=2m}^{[j/2]} g_{p-2m}\, g_{j-2p} \right|, \quad j \in \Sigma(2, N) = \{6N-6, \cdots, 6N-3\} \tag{55}$$

and

$$\sum_{p=0}^{[j/2]} |g_{j-2p}|, \quad j \in \Sigma(1,N) = \{2N-2, 2N-1\}. \tag{56}$$

First we introduce a definition.

**Definition 2** *We say that the transformation*

$$\sum_{m=0}^{[j/2]} |C_{m,j}^{[1]}| \xrightarrow{T} \sum_{m=0}^{[j/2^2]} \left| \sum_{p=2m}^{[j/2]} A_{m,p}\, C_{p,j}^{[1]} \right| = \sum_{m=0}^{[j/2^2]} |C_{m,j}^{[2]}|$$

*is contractive if it verifies (54).*

The following result relates the two definitions introduced before.

**Theorem 3** *Let $N$ be fixed. If $T$ is contractive then the sequence $\{G_k\}_{k\geq 1}$ is decreasing, it converges and*

$$\lim_{k\to+\infty} G_k = 0. \tag{57}$$

**Proof:**
For each $k \geq 1$ we consider the transformation

$$\sum_{m=0}^{[j/2^k]} |C_{m,j}^{[k]}| \xrightarrow{T^{[k]}} \sum_{m=0}^{[j/2^{k+1}]} |C_{m,j}^{[k+1]}|.$$

$T^{[k]}$ is contractive, since

$$T^{[k]} = T \circ T \circ \overset{k}{\cdots} \circ T,$$

and, in particular, the monotonicity holds.

By definition, $0 \leq G_k$ and, since the sequence is decreasing, it is convergent. Therefore, if the constant $G_k$ is reached for $J(k,N) \in \Sigma(k,N)$, then

$$\left\{ \sum_{m=0}^{[J(k,N)/2^k]} |C_{m,J(k,N)}^{[k]}| \right\}_{k\geq 1},$$

is convergent. In particular there exists $\lambda \in \mathbf{R}$ such that

$$\lim_{k\to\infty} C_{m,J(k,N)}^{[k]} = \lambda.$$

Moreover, for each $k \geq 1$,

$$\left[\frac{J(k,N)}{2^{k-1}}\right] \leq \left[\frac{\sigma(k,N)}{2^{k-1}}\right] < 2\,(2N-1).$$

Then, the number of terms in the recursive formula (45) for $C^{[k]}_{m,J(k,N)}$ is finite (it does not increase with $k$) and if we consider the limit in (45) we obtain

$$\lambda = \alpha\,\lambda,$$

with $\alpha \neq 0$. Thus $\lambda = 0$, and (57) holds.

■

**Corollary 5** *There exists $k_0 \in \mathbf{N}$ such that for each $k \geq k_0$ we have*

$$G_k < 1.$$

■

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