# QUANTIZATION ON THE TORUS AND MODULAR INVARIANCE

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## Abstract

The implementation of modular invariance on the torus as a phase space at the quantum level is discussed in a group-theoretical framework. Unlike the classical case, at the quantum level some restrictions on the parameters of the theory should be imposed to ensure modular invariance. Two cases must be considered, depending on the cohomology class of the symplectic form on the torus. If it is of integer cohomology class $n$, then full modular invariance is achieved at the quantum level only for those wave functions on the torus which are periodic if $n$ is even, or antiperiodic if $n$ is odd. If the symplectic form is of rational cohomology class $\frac{n}{r}$, a similar result holds –the wave functions must be either periodic or antiperiodic on a torus $r$ times larger in both direccions, depending on the parity of $nr$. Application of these results to the Abelian Chern-Simons is discussed.

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# 1 Introduction

Since the pioneer work by Dirac [1] on the quantization of constrained systems, a lot of work has been done on this subject, and plenty of methods have been developed to face this interesting and, many times, difficult problem. Roughly speaking, the different methods can be classified into two types, depending on whether the quantization of the corresponding unconstrained system is first performed and then the constraints imposed at the quantum level ("quantize-fist" method) or the constraints are first imposed and then the quantization of the resulting "reduced" system is performed ("constrain-first" method). An example of the former is given by the abovementioned paper by Dirac [1], while the latter was originated by the work of Faddeev [2]. Many other procedures derive from these two, adapted to the properties of the particular system under consideration. Thus, for instance, the BRST quantization is a "quantize-first" technique adapted to the covariant quantization of gauge invariant systems [3]. Also, the method proposed by Ashtekar [4] was designed to simplify the form of the quantum constraints in quantizing Gravity. Alternatively, symplectic or Marsden-Weinstein reduction [5] is a specific technique developed to obtain a reduced classical phase space, which must be the starting point for (some sort of) Geometric Quantization [6].

The main drawback of the "constrain-first" method lies in that the classical phase space could be not properly defined as a differential manifold or, even more, the classical equation of motion might have no general solution. In addition, all the problems that Geometric Quantization encounters in dealing with non-trivial phase spaces must be considered (anomalies, i.e the lack of invariant polarizations, the search for operators compatible with the polarization, etc.).

The troubles with the "quantize-first" methods appears in the implementation of the quantum constraints; only quadratic constraints can be directly imposed due to normal-order ambiguities. Besides, finding the operators which preserve the quantum constraints is a non-trivial problem.

In [7] a method for studying quantum systems with constraints on a group-theoretical framework, Algebraic Quantization on a Group (AQG), was introduced. AQG is a "quantize first" method in which both the unconstrained systems and the constraints are supposed to be dealt in a group setting. This could seem at first instance a severe restriction but, in practice, most of the interesting cases can be treated with this formalism, and the advantages it provides are numerous. In particular, there are no ambiguities in the imposition of quantum constraints (even for non-polynomical ones), and there is a characterization for the operators that preserve the quantum constraints.

Another advantage of AQG is the possibility of implementing the non-trivial topology of the configuration or phase space as contraints, taking into account that these are discrete transformation, which can be easily addressed in this formalism.

In [8] the quantization of the Heisenberg-Weyl (H-W) group with constraints was considered and the particular case of the H-W group on the torus was studied. Now, we

wish to implement modular invariance on the torus at the quantum level. In general, the modular invariance of a conformal theory formulated on a Riemannian surface of genus $g$, $\Sigma_g$, refers to the quotient group $Diff(\Sigma_g)/Diff_0(\Sigma_g)$, where $Diff(\Sigma_g)$ is the group of diffeomorphisms of $\Sigma_g$ and the subscript 0 designs the normal subgroup of diffeomorphisms connected to the identity (see, for instance, [9, 10]).

Clearly, modular transformations on the torus are the $SL(2,Z)$ subgroup of the group $SL(2,R) \approx Sp(2,R)$ of linear symplectic transformations of the plane that preserves the torus. Therefore we can implement them in the formalism of Algebraic Quantization on a Group by considering the Schrödinger group (or Weyl-Symplectic group, see [11]) $WSp(2,R)$ as the symmetry group of the unconstrained system and imposing the appropriate constraints to obtain a torus as the (reduced) symplectic manifold, pretty much in the same manner as in [8]. Then we expect to obtain modular transformations as good operators, i.e. those preserving the Hilbert space of wave functions satisfying the constraints. However, to obtain full modular invariance, we must impose some restrictions on the parameters of the theory. As in [8], three different cases should be considered, depending on the cohomology class of the symplectic form on the torus, which can be either integer, fractional or irrational. Only the integer and fractional cases will be considered here, since the irrational one requires techniques from Non-Commutative Geometry [12] and lies beyond the scope of this paper.

These results are applied to $2+1D$ Abelian Chern-Simons theory and compared with the ones obtained in the literature.

The present paper is organized as follows: In section 3 we study the Schrödinger group without constraints and compute the metaplectic (or spinor) representation with the help of a higher-order polarization. Section 4 is devoted to the determination of the constrained Hilbert space and good operators when the phase-space is constrained to be a torus. Two cases are considered, the one for which the symplectic form on the torus is of integer cohomology class $n$ (section 4.1), where full modular invariance is obtained only when the wave functions are periodic for $n$ even or antiperiodic for $n$ odd, and the case of symplectic form of rational cohomology class $\frac{n}{r}$ (section 4.2), where full modular invariance is obtained only when the wave functions are periodic for $nr$ even or antiperiodic for $nr$ odd. Here periodicity and antiperiodicity are understood in a torus which is r times larger in both directions. Finally, section 5 is devoted to the application of our study to $2+1D$ Abelian Chern-Simons theory.

In a separate Appendix, we study the representations of the subgroup $T$ both for the integral and fractional case.

# 2 Algebraic Quantization on a Group

Algebraic Quantization on a Group (AQG) (see [7, 8]) is a group-theoretical procedure developed for quantizing systems with constraints (both first- and second-class) in a first-

quantize-then-constrain basis. The starting point is the group $\widetilde{G}$ of quantum symmetries of the unconstrained system, which is a central extension by $U(1)$ of the group $G$ of classical symmetries of the unconstrained system. From $\widetilde{G}$, a subgroup $T$, called the structure group, is selected for defining the constraints. For convenience, $T$ is chosen to include the $U(1)$ subgroup of the central extension, which accounts for the phase-invariance of Quantum Mechanics ($U(1)$-equivariance), in such a way that $\widetilde{G}/T$ is the classical reduced phase-space of the constrained system[1].

The quantum Hilbert space $\mathcal{H}_T$ for the constrained system is defined by selecting, from the Hilbert space $\mathcal{H}$ associated with a unitary irreducible representation $U(\widetilde{G})$ of $\widetilde{G}$, those wave functions that transform irreducibly under a given unitary irreducible representation $D(T)$ of $T$. We shall say that these wave functions satisfy the $T$-function condition (or $T$-equivariance condition), which has the general form:

$$\Psi^\alpha(g_T * g) = D^\alpha(g_T)\Psi^\alpha(g)\,, \quad \forall g_T \in T\,, \tag{1}$$

where the index $\alpha$ in $D$ ranges over the set $\hat{T}$, the Pontryagin dual of $T$ –that is, the set of all unitary irreducible representations of $T$. Precisely stated, $\alpha$ will be allowed to vary along the subset $\hat{T}_U \subset \hat{T}$ of those representations which are contained in the restriction of $U(\widetilde{G})$ to $T$; otherwise the constraints would be inconsistent and the constrained Hilbert space $\mathcal{H}_T$ would be trivial. In particular, the representation $D^\alpha$, when restricted to the subgroup $U(1) \subset T$, should be the natural (faithful) representation of $U(1)$, $D^\alpha(\zeta) = \zeta$, $\forall \zeta \in U(1)$. That is, the $T$-equivariance condition must contain the $U(1)$-equivariance condition. Complex functions on the group satisfaying the $T$-equivarance condition can be identified with sections of the vector bundle associated with the principal bundle $T \to \widetilde{G} \to \widetilde{G}/T$ through the representation $D^\alpha$ of $T$ [13].

Both the unitary irreducible representations $U(\widetilde{G})$ and $D(T)$ can be obtained, for instance, by using the Group Approach to Quantization (GAQ) technique (see [7] and references therein), which uses the method of polarizations (see below) to reduce the left-regular representation of the group acting on $U(1)$-equivariant complex functions on the group $\widetilde{G}$.

An important concept that we are forced to introduce is the notion of *good operators*, defined as those preserving the constrained Hilbert space $\mathcal{H}_T$. It is clear that, since $\mathcal{H}_T$ is in general smaller than $\mathcal{H}$, not all operators in $\widetilde{G}$ will preserve it; otherwise the representation $U(\widetilde{G})$ would be reducible. It is difficult to give a general characterization of these operators (for instance, there can be operators preserving $\mathcal{H}_T$ which belong neither to $\widetilde{G}$ nor to its enveloping algebra, escaping to any algebraic or differential characterization), but we can find all good operators in $\widetilde{G}$ simply by considering the *little group* of the representation $D^\alpha(T)$ of $T$ –that is, the subgroup $G_{\text{good}}$ of elements $g_g$ that send the representation

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[1]To be precise, $\widetilde{G}$ contains in general symmetries without symplectic content, like time translations or rotations, so that $\widetilde{G}/T$ is the reduced pre-symplectic manifold of the constrained system.

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$D^\alpha(T)$ to an equivalent one under the adjoint action:

$$D^\alpha_{g_g}(g_T) \equiv D^\alpha(g_g * g_T * g_g^{-1}) \approx D^\alpha(g_T) \quad \forall g_T \in T \,, \ \forall g_g \in G_{\text{good}} \,. \tag{2}$$

Note that this definition generalizes the (sufficient) ones given in [7] and [8]. For instance, in the case in which the representation $D^\alpha(T)$ is one-dimensional (in particular, if $T$ is Abelian), the definition above gives $D^\alpha_{g_g}(T) = D^\alpha(T)$, and the sufficient condition given in [8],

$$[G_{\text{good}}, T] \subset \ker D^\alpha(T) \,, \tag{3}$$

also proves to be necessary. This characterization reproduces the standard one for the case of first-class constraints, for which $T = C \times U(1)$, where $C$ is the subgroup of constraints ($U(1)$ only accounts for the phase-invariance of Quantum Mechanics). If we choose for $C$ the trivial representation (for $U(1)$ the natural representation must always be chosen), then

$$[G_{\text{good}}, C] \subset C \,. \tag{4}$$

This condition gives $G_{\text{good}}$ as the normalizer of the constraints, as it is usually the case (see, for instance, [4]). However, if a non-trivial representation of $C$ is chosen, the subgroup of good operators can be smaller that the normalizer of the constraints, revealing a strong dependence of $G_{\text{good}}$ on the representation $D^\alpha(T)$ of $T$ and, therefore, we should use the more precise notation $G^\alpha_{\text{good}}$ for the subgroup of operators preserving the reduced Hilbert space $\mathcal{H}^\alpha_T$. Note that, from the very definition of little group, $G^\alpha_{\text{good}} \subset N_T$, $\forall \alpha \in \hat{T}_U$, where $N_T$ is the normalizer of T in $\widetilde{G}$ , so that the appropriate place to look for good operators will be in $N_T$.

It is useful to examine the case in which $C$ is an invariant subgroup of $\widetilde{G}$ and we choose $D(T)$ to be the restriction of $U(\widetilde{G})$ to $T$ (or $U(\widetilde{G})$ to be the induced representation by $D(T)$). Then the constraints are trivial; i.e., they do not imply additional restrictions on the wave functions, and the constrained and unconstrained Hilbert spaces coincide. Moreover, the subgroup of good operators turns out to be the whole $\widetilde{G}$ . In this case, $C$ is called a gauge group (see [14]).

A separate study is warranted by the case when $T$ cannot be written as $C \times U(1)$, for instance when $T$ is a non-trivial central extension of $C$ by $U(1)$. In this case, $C$ contains canonically conjugated variables, and the constraints are of *second class*. This case, also contemplated in [7, 8], will be studied in section 4.2.

It should be noted that the same programme can be carried out considering the Lie algebras $\tilde{\mathcal{G}}$ of $\widetilde{G}$ and $\mathcal{T}$ of $T$, when these are simply connected groups. In this case, the treatment becomes simpler, since the representations $dU(\tilde{\mathcal{G}})$ and $dD(\mathcal{T})$ are easier to obtain. In general, however, the treatment is more involved, not only because the good operators can lie in the enveloping algebra, but also because the constraints themselves can be defined through higher-order differential equations [15]. But all these cases can be handled with a direct generalization of AQG.

Thus, AQG can be applied to constrained systems, irrespective of the type (first- or second-class) of constraints. Some examples of application of AQG can be found in [7], where parity in a two-particle system was introduced to obtain both bosonic and fermionic quantizations, and diffeomorphisms constraints to obtain the bosonic string.

Other interesting examples for applying AQG are those systems in which the configuration or phase-spaces are multiply connected and the group $\widetilde{G}$ of quantum symmetries of the simply connected counterpart (universal covering) is known. If $P$ is a multiply connected phase space which is homogeneous under a group $G$ of symmetries, then $P$ is locally diffeomorphic to a coadjoint orbit of $G$, or to a coadjoint orbit of a central extension of $G$ by $U(1)$ or $R$, $\widetilde{G}$ [16]. For the first case, if $H$ is the isotropy group of $P$, $G/H$ is locally diffeomorphic to $P$. If we choose $G$ appropriately (taking coverings, if necessary) in such a way that $G/H$ is simply connected, then $P$ is the quotient of $G/H$ by $\pi_1(P)$, the first homotopy group of $P$. For the cases in which $P$ is locally diffeomorphic to a coadjoint orbit of a central extension $\widetilde{G}$ of G, and if $\widetilde{G}$ is choosen (taking coverings) in such a way that this orbit is simply connected, then $P$ is the quotient of $\widetilde{G}/H$ by $\pi_1(P) \times U(1)$ (or $R$). Then $C = \pi_1(P)$ and $T = C \times U(1)$.

However, if $P$ is not the cotangent bundle of any configuration space (as, for instance, the sphere or the torus as symplectic manifolds), then it could well happen that $\pi_1(P)$, as a subset of $\widetilde{G}$ (we should not forget that all operations of taking quotients are done in $\widetilde{G}$ , and therefore we must consider the embeding of $\pi_1(P)$ in $\widetilde{G}$ , and this could be not a group), contains canonically conjugated pairs. In this case, $T$ is a central extension of $C$ by $U(1)$ and the constraints are of second class. However, if the representation $D$ of $T$ is finite-dimensional (see Appendix A), even though $T$ defines second-class constraints, the treatment follows as though they were first-class, yet non-Abelian.

# 3   The Schrödinger group

As mentioned in the Introduction, we shall replace the H-W group used in [8] with the Schrödinger group, which coincides with the Weyl-Symplectic group $WSp(2,R)$ in 1 dimension. Is was first studied by Niederer [17] as the maximal kinematical invariance group of the Schrödinger equation with general quadratic potential. The complete classification of its unitary irreducible representations was given in [18]. Mathematically it can be obtained from the Galilei (or from the Newton) group by replacing the time parameter with the three-parameter group $SL(2,R)$. The interest of the $SL(2,R)$ group in the present work lies in that it constitutes the maximal finite subgroup of the diffeomorphisms (in fact symplectomorphisms) group of the phase space $R^2$ (see [19], where some physical meaning is given to the representations considered "unphysical" in [18]).

To perform a global-coordinate treatment of the problem, we shall start by considering matrices $S \in GL(2,R)$ instead of $SL(2,R)$, and the condition for these matrices to belong to $SL(2,R)$ will appear naturally. A group law for the Schrödinger group can be written

as:

$$
\begin{aligned}
\vec{x}\,'' &= \vec{x}\,' + \frac{S'}{|S'|^{1/2}}\vec{x} \\
S'' &= S'S \\
\zeta'' &= \zeta'\zeta \exp\frac{im\omega}{2\hbar}\left[\frac{-A'x_2'x_1 - B'x_2'x_2 + C'x_1'x_1 + D'x_1'x_2}{|S'|^{1/2}}\right],
\end{aligned}
\tag{5}
$$

where $\vec{x} = (x_1, x_2) \in R^2$, $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2, R)$, $|S| \equiv AD - BC$ and $\frac{m\omega}{\hbar}$ is an adimensional constant parametrizing the central extensions of the H-W group (we write it in this form for later convenience). The factor $|S'|^{-1/2}$ in the semidirect action of $GL(2, R)$ is needed in order to have a proper central extension.

Let us quantize this system using GAQ, whose principal ingredients will be introduced as needed (see [7] for details). From the group law, the left-invariant vector fields associated with the coordinates $x_1, x_2, A, B, C, D, \zeta$,

$$
\begin{aligned}
\tilde{X}_{x_1}^L &= |S|^{-1/2}\left[A\frac{\partial}{\partial x_1} + C\frac{\partial}{\partial x_2} + \frac{m\omega}{2\hbar}(-Ax_2 + Cx_1)\Xi\right] \\
\tilde{X}_{x_2}^L &= |S|^{-1/2}\left[D\frac{\partial}{\partial x_2} + B\frac{\partial}{\partial x_1} + \frac{m\omega}{2\hbar}(-Bx_2 + Dx_1)\Xi\right] \\
\tilde{X}_A^L &= A\frac{\partial}{\partial A} + C\frac{\partial}{\partial C} \\
\tilde{X}_B^L &= A\frac{\partial}{\partial B} + C\frac{\partial}{\partial D} \\
\tilde{X}_C^L &= B\frac{\partial}{\partial A} + D\frac{\partial}{\partial C} \\
\tilde{X}_D^L &= B\frac{\partial}{\partial B} + D\frac{\partial}{\partial D} \\
\tilde{X}_\zeta^L &= i\zeta\frac{\partial}{\partial \zeta} \equiv \Xi,
\end{aligned}
\tag{6}
$$

as well as the right-invariants ones,

$$
\begin{aligned}
\tilde{X}_{x_1}^R &= \frac{\partial}{\partial x_1} + \frac{m\omega}{2\hbar}x_2\Xi \\
\tilde{X}_{x_2}^R &= \frac{\partial}{\partial x_2} - \frac{m\omega}{2\hbar}x_1\Xi \\
\tilde{X}_A^R &= A\frac{\partial}{\partial A} + B\frac{\partial}{\partial B} + \frac{1}{2}x_1\frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_2} \\
\tilde{X}_B^R &= C\frac{\partial}{\partial A} + D\frac{\partial}{\partial B} + x_2\frac{\partial}{\partial x_1}
\end{aligned}
\tag{7}
$$

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$$\tilde{X}_C^R = A\frac{\partial}{\partial C} + B\frac{\partial}{\partial D} + x_1\frac{\partial}{\partial x_2}$$

$$\tilde{X}_D^R = C\frac{\partial}{\partial C} + D\frac{\partial}{\partial D} - \frac{1}{2}x_1\frac{\partial}{\partial x_1} + \frac{1}{2}x_2\frac{\partial}{\partial x_2}$$

$$\tilde{X}_\zeta^R = \Xi, \tag{8}$$

can be obtained. The commutation relations for the (left) Lie algebra are:

$$
\begin{aligned}
\left[\tilde{X}_A^L, \tilde{X}_B^L\right] &= \tilde{X}_B^L & \left[\tilde{X}_A^L, \tilde{X}_{x_2}^L\right] &= -\tfrac{1}{2}\tilde{X}_{x_2}^L \\
\left[\tilde{X}_A^L, \tilde{X}_C^L\right] &= -\tilde{X}_C^L & \left[\tilde{X}_B^L, \tilde{X}_{x_1}^L\right] &= 0 \\
\left[\tilde{X}_A^L, \tilde{X}_D^L\right] &= 0 & \left[\tilde{X}_B^L, \tilde{X}_{x_2}^L\right] &= \tilde{X}_{x_1}^L \\
\left[\tilde{X}_B^L, \tilde{X}_C^L\right] &= \tilde{X}_A^L - \tilde{X}_D^L & \left[\tilde{X}_C^L, \tilde{X}_{x_1}^L\right] &= \tilde{X}_{x_1}^L \\
\left[\tilde{X}_B^L, \tilde{X}_D^L\right] &= \tilde{X}_B^L & \left[\tilde{X}_C^L, \tilde{X}_{x_2}^L\right] &= 0 \\
\left[\tilde{X}_C^L, \tilde{X}_D^L\right] &= -\tilde{X}_C^L & \left[\tilde{X}_D^L, \tilde{X}_{x_1}^L\right] &= -\tfrac{1}{2}\tilde{X}_{x_1}^L \\
\left[\tilde{X}_{x_1}^L, \tilde{X}_{x_2}^L\right] &= \frac{m\omega}{\hbar}\Xi & \left[\tilde{X}_D^L, \tilde{X}_{x_2}^L\right] &= \tfrac{1}{2}\tilde{X}_{x_2}^L. \\
\left[\tilde{X}_A^L, \tilde{X}_{x_1}^L\right] &= \tfrac{1}{2}\tilde{X}_{x_1}^L
\end{aligned}
\tag{9}
$$

From these commutation relations we see that two linear combinations of vector fields can be introduced, $\tilde{X}_A^L - \tilde{X}_D^L$ and $\tilde{X}_A^L + \tilde{X}_D^L$ (the same for the right-invariant vector fields), in such a way that $\tilde{X}_A^L + \tilde{X}_D^L$ is a central generator, which is also horizontal (see below), and therefore is a gauge generator (see [14]). In fact, it coincides with its right version, as is always the case for a central generator.

We define the Quantization 1-form $\Theta$ as the vertical component (dual to the vertical generator $\Xi$, in this basis) of the canonical 1-form of the Lie algebra:

$$\Theta = \frac{m\omega}{2\hbar}(x_2 dx_1 - x_1 dx_2) + \frac{d\zeta}{i\zeta}. \tag{10}$$

The 2-form $d\Theta$ defines a presymplectic form on $\widetilde{G}$, and its value at the identity, $\Sigma = d\Theta|_e$, is a 2-co-cycle of the Lie-algebra, and it can be used to characterize the central extension (when the group $\widetilde{G}$ is simply connected). A subalgebra is said to be *horizontal* if it lies in the kernel of $\Theta$. The *characteristic subalgebra* is defined as $\mathcal{G}_\Theta = \mathrm{Ker}\Theta \cap \mathrm{Ker}d\Theta$, and in this case it has the form:

$$\mathcal{G}_\Theta = <\tilde{X}_A^L + \tilde{X}_D^L, \tilde{X}_A^L - \tilde{X}_D^L, \tilde{X}_B^L, \tilde{X}_C^L> . \tag{11}$$

Note that $d\Theta/(\mathrm{Ker}d\Theta)$ defines a true symplectic form in $R^2$.

We define the representation $U(\widetilde{G})$ of $\widetilde{G}$ to be given by the left regular representation on complex wave functions over $\widetilde{G}$, satisfying the $U(1)$-function condition $\Xi\Psi = i\Psi$ (phase invariance of Quantum Mechanics). This representation is obviously reducible, and additional restrictions should be imposed on the wave functions in order to obtain

an irreducible representation. These are accomplished by the polarization $\mathcal{P}$, defined as a maximal horizontal left subalgebra of $\tilde{\mathcal{G}}$. The condition $X^L\Psi = 0, \forall X^L \in \mathcal{P}$ leads, in most of the cases, to an irreducible representation $U(\widetilde{G})$ acting on the Hilbert space $\mathcal{H}$ of complex polarized functions on the group satisfying the $U(1)$-function condition.

However, there are groups, called anomalous (see [11]), for which this representation $U(\widetilde{G})$ so obtained is not irreducible, and a generalization of the concept of polarization is required for them. This task is acomplished by means of higher-order polarizations (see [11, 20, 21]), which admit elements of the left enveloping algebra to enter into them.

The system we are studying is an example of anomalous system (see [11, 21]), and a higher-order polarization is required to obtain an irreducible representation. There are essentially two of them[2], given by:

$$
\begin{aligned}
\mathcal{P}^{HO} \;=\; &< \tilde{X}_A^L + \tilde{X}_D^L, \tilde{X}_A^L - \tilde{X}_D^L - \frac{i\hbar}{2m\omega}\left(\tilde{X}_{x_1}^L \tilde{X}_{x_2}^L + \tilde{X}_{x_2}^L \tilde{X}_{x_1}^L\right), \tilde{X}_B^L + \frac{i\hbar}{2m\omega}\left(\tilde{X}_{x_1}^L\right)^2, \\
&\tilde{X}_C^L - \frac{i\hbar}{2m\omega}\left(\tilde{X}_{x_2}^L\right)^2, \tilde{X}_{x_1}^L \text{ or } \tilde{X}_{x_2}^L > .
\end{aligned}
\tag{12}
$$

If we choose, for instance, $\tilde{X}_{x_1}^L$ to be in the polarization, the polarization equations are:

$$
\begin{aligned}
\left(\tilde{X}_A^L + \tilde{X}_D^L\right)\Psi &= 0 \\
\tilde{X}_B^L\Psi &= 0 \\
\left(\tilde{X}_A^L - \tilde{X}_D^L\right)\Psi &= -\frac{1}{2}\Psi \\
\tilde{X}_{x_1}^L\Psi &= 0 \\
\tilde{X}_C^L\Psi &= \frac{i\hbar}{2m\omega}\left(\tilde{X}_{x_2}^L\right)^2\Psi,
\end{aligned}
\tag{13}
$$

The first of these equations has as solutions those complex wave functions on the group $GL(2,R)$ which are defined on $SL(2,R)$, as expected. Therefore, the solutions of this equation have the form:

$$
\Psi = \Psi(a,b,c,d,x_1,x_2),
\tag{14}
$$

where $a \equiv \frac{A}{\sqrt{AD-BC}}, b \equiv \frac{B}{\sqrt{AD-BC}}, c \equiv \frac{C}{\sqrt{AD-BC}}$ and $d \equiv \frac{D}{\sqrt{AD-BC}}$, with $ad - bc = 1$.

To proceed further in solving the polarization equations, it is convenient to introduce local charts on $SL(2,R)$. We choose them to be the ones defined by $a \neq 0$ and $c \neq 0$, respectively[3]. The first chart contains the identity element $I_2$ of $SL(2,R)$, and the second contains $J \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

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[2]There are another two, if we allow for complex coordinates, but all of them lead to equivalent representations

[3]Certainly they really correspond to four contractible charts: $a > 0, a < 0$ and $c < 0, c > 0$, but the transition functions between each pair of these charts are trivial, so we shall consider them as only one chart.

The solutions to the the polarization equations are given by:

- For $a \neq 0$:
$$\Psi = \zeta a^{-1/2} e^{\frac{im\omega}{2\hbar} xy} \chi(\tau, y),$$
(15)

where $x \equiv x_1$, $y \equiv x_2 - \tau x_1$ and $\tau \equiv \frac{c}{a}$, with $\chi$ satisfying the Schrödinger-like equation
$$\frac{\partial \chi}{\partial \tau} = \frac{i\hbar}{2m\omega} \frac{\partial^2 \chi}{\partial y^2}.$$
(16)

- For $c \neq 0$:
$$\widetilde{\Psi} = \zeta c^{-1/2} e^{-\frac{im\omega}{2\hbar} \tilde{x}\tilde{y}} \widetilde{\chi}(\tilde{\tau}, \tilde{y}),$$
(17)

where $\tilde{x} \equiv x_2$, $\tilde{y} \equiv x_1 - \tilde{\tau} x_2$ and $\tilde{\tau} \equiv \frac{a}{c}$, with $\widetilde{\chi}$ satisfying the Schrödinger-like equation
$$\frac{\partial \widetilde{\chi}}{\partial \tilde{\tau}} = -\frac{i\hbar}{2m\omega} \frac{\partial^2 \widetilde{\chi}}{\partial \tilde{y}^2}.$$
(18)

The element $J$ represents a rotation of $\frac{\pi}{2}$ in the plane $(x_1, x_2)$, and takes the wavefunction from one local chart to the other[4]. Obviously, $J^4 = I_2$, but acting with $J$ on the wavefunctions we obtain:
$$\Psi(J * g) = (-1)^{1/4} \widetilde{\Psi}(g),$$
(19)

from which the result $\Psi(J^4 * g) = -\Psi(g)$ follows, that is, the representation obtained for the subgroup $SL(2, R)$ is two-valued. This representation is the well-known *metaplectic* or *spinor representation*. The metaplectic representation is for $SL(2, R)$ as the $\frac{1}{2}$-spin representation is for $SO(3)$ (see [22] and references therein, and also [16]). We refer the reader to [19] for a detailed study of the Schrödinger group, including the non-anomalous representations and a physical interpretation for them.

# 4  The Schrödinger group on the torus

Once we have obtained the polarized wave functions and therefore fixed the unitary and irreducible representation $U(\widetilde{G})$ of $\widetilde{G}$ and the unconstrained Hilbert space $\mathcal{H}$, we have to impose the appropriate constraints to reduce the phase space to a torus. This task is achieved by the structure group $T$, which is a fibre bundle with base $\Gamma_{\vec{L}} \equiv \{e_{\vec{k}}, \vec{k} \in Z \times Z\}$ and fibre $U(1)$, where $e_{\vec{k}}$ are translations of $\vec{x}$ by an amount of $\vec{L}_{\vec{k}} \equiv (k_1 L_1, k_2 L_2)$, in such a way that $\widetilde{G}/T$ is essentially the torus.[5] The fibration of $T$ by $U(1)$ depends on the values of $m, \omega, L_1$ and $L_2$, and is, in general, non-trivial.

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[4]In fact, up to a factor, $J$ represents the Fourier transform passing from the $x_1$ representation to the $x_2$ representation.

[5]As was commented before, $\widetilde{G}/T$ in this case is a presymplectic manifold, which, once the kernel of the (pre)symplectic form $d\Theta$ (containing the $SL(2, R)$ subalgebra) is removed, turns out to be a torus.

The following task is to obtain the irreducible representations of $T$. These are studied in detail in Appendix A, and here we shall report only the main results. The form of the representations of $T$ depends strongly on its structure as $U(1)$ bundle with base $\Gamma_{\vec{L}}$ (which plays the role of constraints $C$), and this is determined by the character of the adimensional parameter $\frac{m\omega L_1 L_2}{2\pi\hbar}$, in such a way that:

i) INTEGER CASE: $\frac{m\omega L_1 L_2}{2\pi\hbar} = n \in Z$. In this case $T$ is Abelian, $T = \Gamma_{\vec{L}} \times U(1)$, and therefore all its representations are 1-dimensional.

ii) FRACTIONAL CASE: $\frac{m\omega L_1 L_2}{2\pi\hbar} = \frac{n}{r}$, where $n$ and $r$ are relative prime integers (with $r > 1$). In this case $T$ is not Abelian, but its representations are of finite dimension.

iii) IRRATIONAL CASE: $\frac{m\omega L_1 L_2}{2\pi\hbar} = \rho$, where $\rho$ is an irrational number. In this case, $T$ is not Abelian and possesses representations (the ones which are compatible with the $U(1)$-function condition) of infinite dimension.

The irrational case will not be considered here, since its study requires techniques from Non-Commutative Geometry [12], and therefore lies beyond the scope of the present work. The most interesting properties of this case, in particular, of the group $C^*$-algebra generated by the elements of $T$, denoted *irrational rotation algebra*, is that it is not a type I algebra [23] (in fact it is a type $\mathrm{II}_\infty$ algebra).

Although normally the integer and fractional cases are used in physical applications, like Abelian Chern-Simons theory (see section 5) or the Quantum Hall Effect (Integer and Fractional), there exists works [24] where the irrational case has been use to study the Quantum Hall Effect, using techniques of $C^*$-algebras and cyclic cohomology to explain the integrality of the conductance on the Quantum Hall Effect (see also [12]).

## 4.1 The Integer case

We shall consider first the integer case, for which $\frac{m\omega L_1 L_2}{2\pi\hbar} = n \in Z$ and the structure group is $T = \Gamma_{\vec{L}} \times U(1)$, $\Gamma_{\vec{L}}$ being a subgroup isomorphic to $Z \times Z$. This case leads to a symplectic form on the torus of integer cohomology class $n$ (and therefore the torus is quantizable according to Geometric Quantization), and $n$ can be interpreted as the Chern number of a $U(1)$-bundle over the torus (see [8]).

The representations of $T$ (compatible with the $U(1)$-function condition) for the integer case are easily computed (see Appendix A), and have the form:

$$D^{\vec{\varphi}}(I_2, k_1 L_1, k_2 L_2, \zeta) = \zeta e^{i(\varphi_1 k_1 + \varphi_2 k_2)} e^{-i\pi n k_1 k_2} , \tag{20}$$

where $\varphi_1, \varphi_2 \in [0, 2\pi)$ parameterize the inequivalent representations of the subgroup $\Gamma_{\vec{L}} \approx Z \times Z$. They are the analogue of vacuum angles in Quantum Chromodynamics (see, for instance, [25]).

The $T$-function conditions are written as $\Psi^{\vec{\varphi}}(g_T * g) = D^{\vec{\varphi}}(g_T)\Psi^{\vec{\varphi}}(g)$, $\forall g_T \in T$. They can be interpreted as periodic boundary conditions, selecting those wave functions in $\mathcal{H}$ which are quasi-periodic; i.e., picking up a phase $e^{i\varphi_1}$ when translated by $L_1$ and $e^{i\varphi_2}$ when translated by $L_2$. This condition reduces to $\Psi^0(g_T * g) = \zeta\Psi^0(g)$ if the trivial representation for $\Gamma_{\vec{L}}$ is chosen (strictly periodic boundary conditions). As in [8], the rest of non-equivalent representations can be obtained by acting with those finite translations which are not good operators. We are not interested in their explicit form, so we refer the interested reader to [8] for the details of the computations.

The solutions to the $T$-function condition for the trivial representation are those functions $\Psi$ of the form (15) for which $\chi(\tau, y)$ is of the form:

$$\chi^0(\tau, y) = \sum_{k=0}^{n-1} a_k \Delta_k^0(\tau, y), \tag{21}$$

with

$$\Delta_k^0(\tau, y) = e^{i2\pi k(\frac{y}{L_2} - \frac{\delta}{2n}\tau k)} \sum_{q \in Z} e^{i2\pi nq[\frac{y}{L_2} - \frac{\delta}{2n}\tau(k^2+nq)]}, \tag{22}$$

$\delta \equiv \frac{L_1}{L_2}$ and $a_k$, $k = 0, 1, \ldots, n-1$ being arbitrary coefficients. We can write these in a form which resembles the one obtained in [8], where we considered only the Heisenberg-Weyl subgroup:

$$\Delta_k^0(\tau, y) = e^{i\frac{\delta L_2^2 \tau}{4\pi n}\frac{d^2}{dy^2}} \Delta_k^0(y), \tag{23}$$

with $\Delta_k^0(y) = e^{i2\pi k\frac{y}{L_2}} \sum_{q \in Z} \delta(y - \frac{q}{n}L_2)$.

For the local chart at $J$ ($c \neq 0$), we could follow the same procedure or simply transform the wave function acting with $J$. The result obtained is completely analogous to the one obtained in the local chart at the identity. Therefore, the constrained Hilbert space $\mathcal{H}_T$ is finite dimensional, with a basis of $n$ independent functions, $\{\Delta_k^0\}_{k=0}^{n-1}$.

Now we have to compute the *good operators*, those preserving the Hilbert space $\mathcal{H}_T$ of polarized wave functions verifying the $T$-function condition. We should look for good operators in the normalizer of $T$ in $\widetilde{G}$. In this case (this result is also valid for the fractional case), we have:

$$N_T = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array}\right), x_1, x_2, \zeta \right) \in \widetilde{G} \ / \ a, b\delta^{-1}, c\delta, d \in Z, \ x_1, x_2 \in R, \ \zeta \in U(1) \right\}, \tag{24}$$

which implies that $N_T$ is the semidirect product of $SL(2, Z)$ by the H-W group.

Since $T$ is Abelian, the caracterization (2) reduces to (3), and this leads to the condition:

$$(a-1)\frac{\varphi_1}{2\pi} + c\delta\frac{\varphi_2}{2\pi} + n(-a\frac{x_2}{L_2} + c\delta\frac{x_1}{L_1} - \frac{1}{2}ac\delta) = k \in Z$$

$$(d-1)\frac{\varphi_2}{2\pi} + b\delta^{-1}\frac{\varphi_1}{2\pi} + n(-b\delta^{-1}\frac{x_2}{L_2} + d\frac{x_1}{L_1} - \frac{1}{2}db\delta^{-1}) = k' \in Z. \tag{25}$$

With regard to the H-W subgroup (i.e. with $a = d = 1$ and $b = c = 0$), we get the same result as in [8]: $x_1 = k_1 \frac{L_1}{n}$ and $x_2 = k_2 \frac{L_2}{n}$, with $k_1, k_2 \in Z$. This implies that

$$W \equiv \left\{ \zeta (\hat{\eta}_1)^{\frac{k_1}{n}} (\hat{\eta}_2)^{\frac{k_2}{n}}, \; k_1, k_2 \in Z, \; \zeta \in U(1) \right\} \subset G_{\text{good}}^{\vec{\varphi}}, \tag{26}$$

with $\hat{\eta}_1 \equiv e_{(1,0)}$ and $\hat{\eta}_2 \equiv e_{(0,1)}$, for any values of the vacuum angles $\varphi_1$ and $\varphi_2$. These operators can be interpreted as the Wilson loops in a Chern-Simons theory on the torus (see section 5 and [25, 26]).

When studying the $SL(2, R)$ subgroup (i.e. with $x_1 = x_2 = 0$), we can proceed in two ways. Either we can determine for which values of $\varphi_1$ and $\varphi_2$ we obtain the full modular group $SL(2, Z)$ as good operators, or we can compute $G_{\text{good}}^{\vec{\varphi}}$ for given values of $\varphi_1$ and $\varphi_2$.

In the first case, from (25) we easily deduce that modular invariance is achieved for $\varphi_1 = 2\pi m_1$, $\varphi_2 = 2\pi m_2$ if $n$ is even and for $\varphi_1 = \pi(2m_1 + 1)$, $\varphi_2 = \pi(2m_2 + 1)$ if $n$ is odd, with $m_1, m_2 \in Z$. Clearly, since the vacuum angles are defined modulo $2\pi$ these correspond to $\varphi_1 = \varphi_2 = 0$, or periodic boundary conditions for $n$ even and to $\varphi_1 = \varphi_2 = \pi$, or antiperiodic boundary conditions for $n$ odd. This is an interesting result, since it reflects the fact that good operators really depend on the particular representation $D^{\vec{\varphi}}$ of $T$ we are considering.

The group $G_{\text{good}}$ of good operators for these cases would be obtained by taking the product of elements of $SL(2, Z)$ with those of $W$ given by (26). But from (25) we see that there are a few more good operators which cannot be obtained in this way. Altogether, we obtain the following group of good operators for $\varphi_1 = \varphi_2 = 0$ with $n$ even and for $\varphi_1 = \varphi_2 = \pi$ with $n$ odd. :

$$G_{\text{good}} = \left\{ (S, \frac{1}{n} S J^m \vec{L}_{\vec{k}}, \zeta) \; / \; S \in SL(2, Z), \; m = 0, 1, 2, 3, \; \vec{k} \in Z \times Z, \; \zeta \in U(1) \right\} \tag{27}$$

The computation of $G_{\text{good}}^{\vec{\varphi}}$ for arbitrary values of $\varphi_1, \varphi_2$ is a bit more involved. We have seen that the subgroup $W$ given in (26) is always included in $G_{\text{good}}^{\vec{\varphi}}$, so we have only to consider the $SL(2, Z)$ subgroup. It is easy to see that if both $\frac{\varphi_1}{2\pi}$ and $\frac{\varphi_2}{2\pi}$ are irrational, then only the identity matrix in $SL(2, Z)$ is a good operator, so there is no hint of modular invariance for this case. If $\frac{\varphi_1}{2\pi}$ is irrational and $\frac{\varphi_2}{2\pi} = \frac{p}{q}$ is rational (the case obtained by interchanging 1 and 2 is analogous), then only the subgroup of modular transformations of the form $\begin{pmatrix} 1 & \epsilon q \delta^{-1} k \\ 0 & 1 \end{pmatrix}$ are good operators, with $k \in Z$ and with $\epsilon = 1$ for $n$ even and $\epsilon = 2$ for $n$ odd. If $\frac{\varphi_2}{2\pi} = \frac{p_1}{q_1}$ and $\frac{\varphi_2}{2\pi} = \frac{p_2}{q_2}$ are rational, then the good operators are given by the subgroup of modular transformations satisfying the following diophantine equations:

$$(a - 1)\frac{p_1}{q_1} + c\delta \frac{p_2}{q_2} - n\frac{ac\delta}{2} = k \in Z$$

$$b\delta^{-1}\frac{p_1}{q_1} + (d - 1)\frac{p_2}{q_2} - n\frac{db\delta^{-1}}{2} = k' \in Z. \tag{28}$$

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## 4.2 The Fractional case

For the fractional case, we shall restrict ourselves to the determination of the subgroup of good operators. The computation of the explicit form of the constrained wave functions can be performed along the guidelines of the previous section (they are essentially the ones given in [8] for the H-W group), using the representations of $T$ given in Appendix A. The dimension of the Hilbert space turns out to be $nr$, and it can be considered to be a $n$ dimensional Hilbert space made of vector-valued wave functions, $r$ being the dimension of the vector space.

To determine the subgroup $G_{\text{good}}$ of good operators, we make use of the caracterization (2) for the litle group, where now, since the representations are of dimension $r$, the equivalence can be established through a non-trivial unitary matrix $V(g_g)$.

First, we compute, for $g_g = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x_1, x_2, \zeta') \in N_T$,

$$g_g * (I_2, k_1 L_1, k_2 L_2, \zeta) * g_g^{-1} = (I_2, (ak_1 + b\delta^{-1}k_2)L_1, (c\delta k_1 + dk_2)L_2,$$
$$\zeta e^{i2\pi \frac{n}{r}[(-a\frac{x_2}{L_2} + c\delta \frac{x_1}{L_1})k_1 + (-b\delta^{-1}\frac{x_2}{L_2} + d\frac{x_1}{L_1})k_2]}), \qquad (29)$$

and then we must find for which $g_g \in N_T$ we have

$$D^{\vec{\varphi}}(g_g * g_T * g_g^{-1}) = V(g_g)D^{\varphi}(g_T)V(g_g)^{\dagger} \ , \ \forall g_T \in T \,, \qquad (30)$$

where the representations $D^{\vec{\varphi}}$ for the fractional case (obtained in Appendix A), are given by:

$$D^{\vec{\varphi}}(I_2, k_1 L_1, k_2 L_2, \zeta) = \zeta e^{i(\varphi_1 k_1 + \varphi_2 k_2)} e^{-i\pi \frac{n}{r} k_1 k_2} A_r^{k_1} B_r^{k_2} \,. \qquad (31)$$

We proceed as in the integer case, computing firstly the good operators in the H-W subgroup. Then the previous equation is written:

$$e^{i2\pi \frac{n}{r}(-\frac{x_2}{L_2} k_1 + \frac{x_1}{L_1} k_2)} A_r^{k_1} B_r^{k_2} = V(g_g) A_r^{k_1} B_r^{k_2} V(g_g)^{\dagger} \,. \qquad (32)$$

This equation is the same one which states the equivalence of the representations $D^{(\mu_1, \mu_2)}$ and $D^{(0,0)}$ and, therefore, making use of the results given in Appendix A, we find that $x_1 = k\frac{L_1}{n}$ and $x_2 = k'\frac{L_2}{n}$, with $k, k' \in Z$. This implies that the subgroup $W$ given in (26) is included in $G_{\text{good}}^{\vec{\varphi}}$ for all values of $\varphi_1, \varphi_2 \in [0, \frac{2\pi}{r})$.

As far as the $SL(2, Z)$ subgroup is concerned, we shall determine only the conditions under which full modular invariance is obtained as good operators, and for this purpose we shall make use of the fact that $SL(2, Z)$ is generated by two modular transformations:

$$g_1 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \qquad (33)$$

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Determining under which conditions these two transformations are good operators will tell us when the theory is fully modular-invariant. For $g_1$ we obtain the condition:

$$e^{i2\pi k_2(\frac{\varphi_1}{2\pi} - \frac{nk_2}{2r})} A_r^{k_1+k_2} B_r^{k_2} = V(g_1) A_r^{k_1} B_r^{k_2} V(g_1)^\dagger, \quad \forall k_1, k_2 \in Z. \tag{34}$$

For this condition to hold, it is necessary that $\varphi_1 = 0$ if $nr$ is even, or $\varphi_1 = \frac{\pi}{r}$ if $nr$ is odd. For the first case, the unitary matrix $V(g_1)$ has the form $V(g_1)_{ij} = \omega_r^{\frac{(i-1)^2}{2}} \delta_{ij}$, and, for the second, we have $V(g_1)_{ij} = \omega_r^{\frac{i-1}{2n}} \omega_r^{\frac{(i-1)^2}{2}} \delta_{ij}$.

For $g_2$ to be a good operator, we obtain the condition:

$$e^{i2\pi k_1(\frac{\varphi_2}{2\pi} - \frac{nk_1}{2r})} A_r^{k_1} B_r^{k_1+k_2} = V(g_2) A_r^{k_1} B_r^{k_2} V(g_2)^\dagger, \quad \forall k_1, k_2 \in Z. \tag{35}$$

Again, for this condition to hold it has to be $\varphi_2 = 0$ if $nr$ is even or $\varphi_2 = \frac{\pi}{r}$ if $nr$ is odd. The unitary matrix $V(g_2)$ has the form:

$$V(g_2) = V = \frac{1}{\sqrt{r}} \begin{pmatrix} 1 & \omega_r^{\frac{(r-1)(r-2)}{2}} & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & \\ \omega_r & 1 & \dots & \dots & \\ \omega_r^3 & \omega_r & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \omega_r^{\frac{(r-1)(r-2)}{2}} \\ \omega_r^{\frac{(r-1)(r-2)}{2}} & \dots & \dots & 1 & 1 \end{pmatrix} \quad \text{if } nr \text{ is even}$$

$$V(g_2) = A_r^{\frac{1}{2n}} V \quad \text{if } nr \text{ is odd},$$

where $(A_r^{\frac{1}{2n}})_{ij} = e^{i\pi\frac{i-1}{r}} \delta_{ij}$.

It should be stressed that the values of $\vec{\varphi}$ for which full modular invariance is obtained correspond to wave functions which are periodic if $nr$ is even, or antiperiodic if $nr$ is odd, where these boundary conditions should be understood with respect to translations by $rL_1$ and $rL_2$.

Note also that the matrix representation $V(g_1)$ and $V(g_2)$ obtained for $g_1$ and $g_2$ (and therefore for the whole $SL(2, Z)$ group) corresponds to their action on the $r$-dimensinal vector space. The complete action of any modular transformation on the wave functions (through $nr \times nr$ matrices) decomposes, thus, in a tensor product of a $n \times n$ matrix and a $r \times r$ matrix, each one acting on different indices of the wave functions [26].

This structure of tensor product of the Hilbert space suggests a duality under the interchange of $n$ and $r$. Indeed, the set of Wilson loops (26) for the theories characterized by $n/r$ and $r/n$ are isomorphic. Since all the information of the theory is contained in the Wilson loops, we could say that the two theories are equivalent. The case $n/r = 1$ would, of course, be self-dual. Moreover, as pointed out in [27], if we denote by $\mathcal{A}_{\frac{n}{r}}$ the (group) algebra generated by $A$ and $B$ satisfying

$$AB = e^{i2\pi\frac{n}{r}} BA, \tag{36}$$

then we have $\mathcal{A}_{1/(nr)} = \mathcal{A}_{\frac{n}{r}} \times \mathcal{A}_{\frac{r}{n}}$. Therefore, the algebra of Wilson loops, besides being the same for a theory with $T = \mathcal{A}_{\frac{n}{r}}$ and $T = \mathcal{A}_{\frac{r}{n}}$, is given by the direct product of both (commuting) algebras. From the point of view of non-commutative $C^*$-algebras, the algebras $\mathcal{A}_{\frac{n}{r}}$ and $\mathcal{A}_{\frac{r}{n}}$ are strongly Morita equivalent, which means, in particular, that they possess the same representation theory [23] (see also [12]).

# 5   $2+1$D Abelian Chern-Simons Theory

As a first application of our results, let us consider a pure topological field theory on the torus.

Let $M$ be a globally hyperbolic three-dimensional manifold, $M = \Sigma \times R$, where $\Sigma$ is an orientable two-dimentional manifold.

The action for an Abelian Chern-Simons (ACS) theory is given by [28, 25, 26]:

$$S_{ACS} = \frac{k}{4\pi} \int_M (A \wedge dA) , \tag{37}$$

where $A$ is a one-form in $M$ which takes values on the Lie algebra $\mathcal{K}$ of an Abelian Lie group $K$. It is straightforward to check that the action $S_{ACS}$ is invariant under gauge transformations $A \to A + ig^{-1}dg$ for any (single-valued) $g : M \longrightarrow K$.

The equations of motion are:

$$dA \equiv F = 0 , \tag{38}$$

the solution of which is the vector space $\mathcal{V}_{ACS}$ of all flat connections on $M$. A generic element $A \in \mathcal{V}_{ACS}$ can be written in the form $(A_0, ig^{-1}\nabla g + a(t))$, where $a$ is a map from $R$ to the fibre of $T^*(\Sigma) \otimes \mathcal{K}$.

This vector space of solutions can be endowed with a (pre-)symplectic structure by means of a (pre-)symplectic form

$$\Omega_{ACS}(A', A) = \int_\Sigma J = \frac{k}{4\pi} \int_\Sigma A' \wedge A , \tag{39}$$

where $J^\mu \equiv \frac{k}{4\pi}\epsilon^{\mu\nu\sigma} A'_\nu A_\sigma$ is a divergenceless current which ensures the independence of $\Omega_{ACS}(A', A)$ on the chosen Cauchy factorization of $M$, $M = \Sigma \times R$.

Since the exterior derivative $d$ commutes with the pullback operator $*$, if $f$ is a diffeomorphism of $M$, and $A', A \in \mathcal{V}_{ACS}$, then $A' + f^*A$ is also a solution of (38).

With this information, we can propose a quantizing group $\widetilde{G}_{ACS}$ for this theory, the composition law of which is:

$$\begin{aligned} f'' &= f' \circ f , \quad f, f', f'' \in \mathrm{Diff}(M) \\ A'' &= f^{-1*}A' + A \\ \zeta'' &= \zeta\zeta' \exp \Omega_{ACS}\left(f^{-1*}A', A\right) , \end{aligned} \tag{40}$$

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i.e., the extension by $U(1)$ of the semidirect product $\mathcal{V}_{ACS} \otimes_s \text{Diff}(M)$. The characteristic subgroup (generated by the kernel of $\Omega_{ACS}$, see Sec. 3) of this group proves to be $G_\Omega = \{(f, A, 1)/A = (A_0, ig^{-1}\nabla g) \quad \text{for some} \quad g : M \to K\} \subset \widetilde{G}_{ACS}$, which contains the *gauge group* $G_{\text{gauge}}$ of the theory, constituted by all (single-valued) $g : M \to K$ [To be precise, $G_{\text{gauge}}$ has not been included in $\widetilde{G}_{ACS}$, but rather an orbit of it under the action $A \to A + ig^{-1}dg$. Including the group $G_{\text{gauge}}$ explicitly in $\widetilde{G}_{ACS}$ requires a slight modification of the notion of gauge transformation [29]. For the present case, we shall assume the existence of a group $G_{\text{gauge}}$ and a subgroup of $\widetilde{G}_{ACS}$ directly related to it]. Thus, the polarization conditions (which contains the characteristic subgroup) imply that wave functions depend only on topological and gauge invariant quantities. For this kind of theory, standard approaches claim that all gauge-invariant information of a connection can be extracted from the *Wilson loops* defined by:

$$W(A, \gamma) = \exp \int_\gamma A \,, \tag{41}$$

for any loop $\gamma$ in $\Sigma$. Since connections $A$ are flat, the Wilson loops will depend only on the homotopy class $[\gamma] \in \pi_1(\Sigma)$ of the corresponding loop $\gamma$. For this reason the normal subgroup $Diff_0(M) \subset Diff(M)$ of diffeomorphism of $M$ connected to the identity acts trivially on the Wilson loops. Therefore the diffeomorphisms that really matter in (40) are the quotient $Diff(M)/Diff_0(M)$ called the *modular group* (see [9, 10]) of the Riemann surface $\Sigma$ (note that all diffeomorphisms of the $R$ part of $M$ are connected to the identity).

It should be stressed that if $\pi_1(\Sigma) = 0$ (what implies that $H^1(\Sigma) = 0$) then the ACS theory is trivial since all connections are of the form $A = ig^{-1}dg$ for some (always single-valued) $g : M \to K$. This implies that $G_\Omega = G_{ACS} \equiv \mathcal{V}_{ACS} \otimes_s \text{Diff}(M)$ and $\widetilde{G}_{ACS} = G_{ACS} \times U(1)$, that is, the central extension is trivial (another way of seeing this is that since the homotopy group of $M$ is trivial, all Wilson loops are trivial). Therefore we assume that $\Sigma$ is a multiply-connected oriented two-dimensional manifold. What makes the theory non-trivial in this case is the fact that the gauge group, $G_{\text{gauge}}$, is smaller than its simply-connected counterpart in the universal covering space of $M$, $\bar{G}_{\text{gauge}}$, constituted by all $\bar{g} : \bar{M} \to K$, with $\bar{M} = \bar{\Sigma} \times R$, and $\bar{\Sigma}$ the universal covering space of $\Sigma$. In fact, the group $G_{\text{gauge}} \subset \bar{G}_{\text{gauge}}$ is made of those elements $\bar{g} \in \bar{G}_{\text{gauge}}$ verifying $\bar{g} \circ [\gamma] = \bar{g}$, where here $[\gamma]$ represents the natural action (as diffeomorphism) of the homotopy class $[\gamma] \in \pi_1(M)$ on $\bar{M}$.

For the present case, $\Sigma = S^1 \times S^1$ and $K = U(1)$. The space $\mathcal{V}_{ACS}$ is made of connections of the form $(A_0, ig^{-1}\nabla g + a(t))$, where $g$ is single-valued on the torus. The solution manifold, that which remains once the quotient by the characteristic subgroup $G_\Omega$ is taken, is parameterized by the variables $a_1(t), a_2(t)$ modulo an integer, defining a torus. The reason is that $G_\Omega$ also contains the *global (large) gauge transformations* (see for instance [26] and references therein),

$$a_j \to a_j + k_j \,, \ k_j \in Z \,, \tag{42}$$

This large gauge transformations are clearly seen to come from transformations of the form $g = exp(ik_j x^j)$, with $k_j \in Z$ and $\{x^j\}$ a set of local coordinates on the torus, in such a way that 0 and $2\pi$ are identified. The reason of the restriction of $k_j$ to integers is the condition $\bar{g} \circ [\gamma] = \bar{g}$. This indicates that the gauge group $G_{\text{gauge}}$ is a disconnected group, with $G_{\text{gauge}}/G^0_{\text{gauge}} = Z \times Z$, $G^0_{\text{gauge}}$ being the connected component of the identity.

For this reason, in the quantum theory the operator associated with the variables $A$ (more precisely, $a$) are not properly defined (they are *bad operators*, see Sec. 2). We must resort to single-valued (*good*) operators of the form:

$$W(A) = e^{2\pi i \sum_{j=1}^{2} n_j a_j} , \tag{43}$$

where $n_j \in Z$ should be interpreted as the winding number of a path $\gamma$ around the cycle $j$. Remember that for the torus, the homotopy classes $[\gamma]$ are generated by two elements, $[\gamma_j]$ $j = 1, 2$, representing loops (with winding number one) around each one of the two cycles of the torus. The modular group proves to be $Diff(T^2)/Diff_0(T^2) = SL(2, Z)$.

At this point it should be stressed that the resulting theory corresponds to a quantum mechanical system with phase space a torus parameterized by $(q, p) \equiv (a_1 \bmod 1, a_2 \bmod 1)$.

According to this equivalence, we could have studied this system in the framework of AQG by starting with the group H-W$\otimes_s Diff(R^2 \times R)$ with structure group $T$ a fibre bundle with base $Z \times Z$ and fibre $U(1)$, where $Z \times Z$ is the subgroup of $Diff(R^2 \times R)$ of translations by $(k_1 L_1, k_2 L_2)$, with $k_1, k_2 \in Z$. Since the only relevant diffeomorphisms at the final theory on the torus will be the modular transformations $SL(2, Z) \subset SL(2, R)$, it is enough to start with H-W$\otimes_s SL(2, R)$, which is the Schrödinger group. Thus, all the results of Sec. 4 apply here. Of course, we could have started directly with H-W$\otimes_s SL(2, Z)$, but this group, being disconneted, is more difficult to quantize than the Schrödinger group (in particular, finding a polarization for this group is a difficult task). In addition, we think that showing how $SL(2, Z)$ emerges as good operators is a very illustrative way of studying the problem.

*In summary*:

- The coupling constant $k$ plays the same role as the quantity $\phi \equiv \frac{m\omega L_1 L_2}{2\pi\hbar}$ in the Schrödinger group on the torus, determining the character of the resulting (finite-dimensional) Hilbert space.

- The set of Wilson loops (43) takes part of the set of good operators in our language. More precisely, they are the analogue of the set $W$ given in (26).

- The group of large gauge transformations is the analogue of the structure group $T$. When the coupling constant $k$ is fractional, this gauge group is called *anomalous* [26] because of its non-Abelian character due to the non-trivial fibration for this case, as oposed to the original Abelian gauge group $K$.

- The non-equivalent representations of $T$, parametrized by the indices $\varphi_1$, $\varphi_2$ (vacuum angles), characterize the non-equivalent quantizations of the theory.

The Chern-Simons theory constitutes a particular example of a drastic reduction of the number of original infinite (field) degrees of freedom to a finite number (which, in addition, contain a finite number of states, due to the compactness of the phase space when restricted to the torus), as a consequence of a huge gauge invariance which kills all of them except for the topologic ones.

## 5.1 Further comments

Comparing our results with those in the literature, we find full agreement with Ref. [25], in the context of $U(1)$ Chern-Simon theory on the torus, as far as the integer case is concerned. For the fractional case, an apparent discrepance with the results in [25] appears: in our notation, modular invariance is obtained only for $n$ even (and any value of $r$) and vacuum angles $\varphi_1 = \varphi_2 = 0$. However, the agreement is achieved if we realize that the proper range of inequivalent vacuum angles in [25] should be $[0, \frac{2\pi}{r})$.

This problem was also studied in Ref. [26] (also in the context of $U(1)$ Chern-Simons theory and anyons on the torus), where full modular invariance was obtained for both the integer and fractional case, but they claimed that the vacuum angles always have to be $\frac{1}{2}$ disregarding the parity of the coupling constant (the equivalent of our $\frac{n}{r}$). A more detailed analysis of their results reveals that the vacuum angles they introduce are defined modulo $\frac{1}{nr}$, and $(\frac{1}{2} \bmod \frac{1}{nr})$ is 0 for even $nr$ and $\frac{1}{2nr}$ for odd $nr$, corresponding to periodic and antiperiodic boundary conditions, respectively. Therefore, their results completely agree with ours.

In [27], a non-Abelian Chern-Simons theory is considered, with gauge group $SL(2, R)$. When restricted to the torus, they obtained essentially the same results as ours and those of [25, 26] (with the abovementioned remarks) with respect to the Hilbert space and the set of observables (good operators), because the reduced phase space of the theory is almost the space of flat connections of an Abelian gauge group.

# Aknowlodgements

# A    Appendix:  Unitary  and  Irreducible  representations of $T$

The structure subgroup $T$, as defined in section 4, is a $U(1)$ bundle with base $\Gamma_{\tilde{L}}$, and can be written as:

$$T = \{(I_2, k_1 L_1, k_2 L_2, \zeta) \in \tilde{G} \mid k_1, k_2 \in Z \,,\, \zeta \in U(1)\} \,, \tag{44}$$

with group law derived from the group law of the Schrödinger group:

$$(I_2, k_1' L_1, k_2' L_2, \zeta') * (I_2, k_1 L_1, k_2 L_2, \zeta) = (I_2, (k_1'+k_1)L_1, (k_2'+k_2)L_2, \zeta' \zeta e^{i \frac{m\omega L_1 L_2}{2\hbar}(k_1' k_2 - k_2' k_1)}) \,. \tag{45}$$

To determine the structure of $T$, we compute the group commutator of two elements:

$$[(I_2, k_1' L_1, k_2' L_2, \zeta'), (I_2, k_1 L_1, k_2 L_2, \zeta)] = (I_2, 0, 0, e^{i \frac{m\omega L_1 L_2}{\hbar}(k_1' k_2 - k_2' k_1)}) \,, \tag{46}$$

from which we see that its structure depends on the value of $\frac{m\omega L_1 L_2}{2\pi\hbar}$, in such a way that there are three possibilities:

*i)* INTEGER CASE:  $\frac{m\omega L_1 L_2}{2\pi\hbar} = n \in Z$.

*ii)* FRACTIONAL CASE:  $\frac{m\omega L_1 L_2}{2\pi\hbar} = \frac{n}{r}$ ,  $n, r \in Z$ and relative prime (with $r > 1$).

*iii)* IRRATIONAL CASE:  $\frac{m\omega L_1 L_2}{2\pi\hbar} = \rho$, with $\rho$ an irrational number.

Let us study the integer and fractional case separately. The irrational case will not be considered here (see [12] for a detailed study of this case).

*INTEGER CASE:*

In this case, $T$ is an Abelian group, and therefore $T = \Gamma_{\tilde{L}} \times U(1)$ and all its representations are of dimension 1. As stated above, we shall consider only those representations, which restricted to $U(1)$, are the natural representations, and these have the form:

$$D^{\vec{\varphi}}(I_2, k_1 L_1, k_2 L_2, \zeta) = \zeta e^{i(\varphi_1 k_1 + \varphi_2 k_2)} e^{-i\pi n k_1 k_2} \,, \quad \forall k_1, k_2 \in Z, \ \forall \zeta \in U(1), \tag{47}$$

where the range of inequivalent representations, since they are 1-dimensional, is given simply by $\varphi_1, \varphi_2 \in [0, 2\pi)$. Note that, except for the term $e^{-i\pi n k_1 k_2}$, this is the product of the natural representation of $U(1)$ times a representation of $\Gamma_{\tilde{L}} \approx Z \times Z$. This extra term is only a coboundary coming from the fact that we have used Bargmann's cocycle in the group law of the Schrödinger group, and Bargmann's cocycle does not satisfy the conditions given in [8] for the possible cocycles for the H-W group on the torus. Note, thus, that this restriction can be relaxed by introducing this coboundary term in the representations of $T$.

*FRACTIONAL CASE:*

In this case, $T$ is not Abelian, and the commutator of two elements has the form:

$$[(I_2, k_1'L_1, k_2'L_2, \zeta'), (I_2, k_1L_1, k_2L_2, \zeta)] = (I_2, 0, 0, \omega_r^{k_1'k_2 - k_2'k_1}), \tag{48}$$

where $\omega_r \equiv e^{i2\pi\frac{n}{r}}$ is a $r^{th}$ root of unity. Note that if $|n| > r$, then $w_r = e^{i2\pi\frac{n}{r}} = e^{i2\pi\frac{q}{r}}$, where $q = n \bmod r$. Since $n$ and $r$ are relative prime, $q$ and $r$ turn out also to be relative prime and, therefore, we can use either of the two pairs to characterize $T$.

The group $T$ admits a non-trivial characteristic subgroup (see [8]), of the form:

$$G_C = \{(I_2, rk_1L_1, rk_2L_2, e^{i\pi nrk_1k_2}) \,/\, k_1, k_2 \in Z\}. \tag{49}$$

The characteristic subgroup can be identified in this case with the Casimir elements of $T$ which are not in $U(1)$, i.e. those elements of $T$ (not belonging to $U(1)$) which commute with all other elements in $T$. In fact, the centre of $T$ is given by $G_C \times U(1)$.

If we quotient $T$ by $G_C$, we obtain a group which is a *generalized Clifford group* $G_2^r$ (see [30] for the definition and the study of representations of generalized Clifford groups) times $U(1)$. Therefore, the representations of $T$ can be obtained from those of $G_C$ and $G_2^r$ (and the natural representation of $U(1)$).

The representations of $G_C$, being isomorphic to $Z \times Z$, are characterized by two "vacuum angles" $\varphi_1, \varphi_2$, whose range of non-equivalence should be determined. The representations of $G_2^r$ are studied in detail in [30], so that here we shall give only the results. It should be remarked, however, that in [30] $\omega_r$ is an arbitrary $r^{th}$ root of unity, and different choices for it give inequivalent representations of $G_2^r$, whereas here the value of $\omega_r$ is given *a priori* (it is determined by the fact that $T$ is a subgroup of $\widetilde{G}$ ), so that the representation of $G_2^r$ is uniquely determined. In addition, since $n$ and $r$ are relative prime, $\omega_r$ is a primitive $r^{th}$ root of unity, implying that the representation of $G_2^r$ associated with it is of dimension $r$, either for prime or non-prime $r$ .

The $r$-dimensional unitary irreducible representation of $G_2^r$ can be constructed with the aid of two $r \times r$ matrices, $A_r$ and $B_r$:

$$(A_r)_{ij} = \omega_r^{i-1}\delta_{ij} \ , \ \ (B_r)_{ij} = \delta_{i,(j \bmod r)+1}, \ \ i,j = 1, 2, ..., r \,, \tag{50}$$

verifying $A_rB_r = \omega_r B_r A_r$, and $A_r^r = B_r^r = I_r$. Putting together this representation of $G_2^r$ and that of $G_C \approx Z \times Z$, we can build a representation for the entire $T$, of the form:

$$D^{\vec{\varphi}}(I_2, k_1L_1, k_2L_2, \zeta) = \zeta e^{i(\varphi_1 k_1 + \varphi_2 k_2)} e^{-i\pi\frac{n}{r}k_1k_2} A_r^{k_1} B_r^{k_2} \,, \ \ \forall k_1, k_2 \in Z, \ \forall\zeta \in U(1)\,. \tag{51}$$

One would naïvely expect that the range of non-equivalent representations would be $\varphi_1, \varphi_2 \in [0, 2\pi)$, as in the integer case. However, since the representations are not 1-dimensional, there could be non-trivial unitary transformations relating representations in this interval and, therefore, reducing the range on non-equivalence.

Thus, we have to determine the minimum values of $\mu_1, \mu_2$ for which the representation $D^{(\mu_1,\mu_2)}$ is equivalent to the trivial representation $D^{(0,0)}$; i.e. there exists a unitary matrix $V$ such that $D^{(\mu_1,\mu_2)} = V D^{(0,0)} V^\dagger$. Studying separately the cases $(\mu_1, 0)$ and $(0, \mu_2)$, and after a few computations, we obtain:

(i) $D^{(\mu_1,0)}$ is equivalent to $D^{(0,0)}$ for $\mu_1 = \frac{2\pi}{r}$, with $V = B_r^{m_0}$.

(ii) $D^{(0,\mu_2)}$ is equivalent to $D^{(0,0)}$ for $\mu_2 = \frac{2\pi}{r}$, with $V = A_r^{\frac{1}{n}}$.

Here $(A_r^{\frac{1}{n}})_{ij} \equiv \omega_r^{\frac{i-1}{n}} \delta_{ij} = e^{i2\pi \frac{i-1}{r}} \delta_{ij}$, and $0 < m_0 < r$ is an integer solution of the diophantine equation:

$$1 + n m_0 = rk \ , k \in Z \,, \tag{52}$$

which has always two solutions in the range $\{-(r-1), \ldots, 0, \ldots, r-1\}$, provided $n$ and $r$ are relative prime (this is a particular case of the Bezout lemma, for $gcd(n,r) = 1$, which in turn can be proven using Euclidean division of integers, see, for instance, [31]). Note that $(A_r^{\frac{1}{n}})^n = A_r$ and $(B^{m_0})^n = B_r^{-1}$, so that these matrices can be considered as the $n^{th}$ roots of the matrices $A_r$ and $B_r^{-1}$, respectively.

Therefore, the range of non-equivalent representations of $T$ is reduced to $\varphi_1, \varphi_2 \in [0, \frac{2\pi}{r})$. This fact will be of extreme importance for the determination of the good operators in the fractional case.

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