Algebraic Quantization on the Torus and Modular Invariance

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1 Algebraic Quantization

The aim of the Algebraic Quantization is the quantum description of a physical system by means of the unitary and irreducible representations of its symmetry group. Two cases have to be considered, corresponding to systems without constraints and to those with constraints, respectively.

In the simplest case, the group $\widetilde{G}$ of quantum symmetries will be a central extension by $U(1)$ of the group G of classical symmetries. Then the starting point is a Lie group $\widetilde{G}$ which is a principal bundle with fiber $U(1)$ and base G. The group law has the generic form:

$$g'' = g' * g\,, \qquad \zeta'' = \zeta'\zeta \exp[i\xi(g', g)]\,, \tag{1}$$

where $g'', g', g \in G$, $\zeta'', \zeta', \zeta \in U(1)$, and $\xi : G \times G \to R$ is a 2-cocycle. The representation is built by means of the left action of the elements of $\widetilde{G}$ on complex functions (wave functions) on the group manifold, $\hat{g}\Psi(g') = \Psi(g * g')$, verifying the $U(1)$-function condition (phase invariance of Quantum Mechanics): $\Psi(\zeta * g) = \zeta\Psi(g)\,, \ \ \forall g \in \widetilde{G}\ ,\ \forall \zeta \in U(1)$.

However, this representation is reducible (all right transformations commute with it). To reduce it, we have to impose certain restrictions on the wave functions in order to trivialize this right action. Some new concepts are needed for this purpose. We call a subgroup $A \subset \widetilde{G}$ horizontal if $A \cap U(1) = \mathbf{1}_{\widetilde{G}}$ (which implies that the restriction of ξ to A is a coboundary, that is, $\xi(a_1, a_2) = \eta(a_1 * a_2) - \eta(a_1) - \eta(a_2)$, for some funtion η on A). Taking into account that the group commutator is $[g', g] = g' * g * g'^{-1} * g^{-1}$, we define the characteristic subgroup G_C as the maximal horizontal subgroup such that the commutator group $[G_C, \widetilde{G}]$ is also horizontal. G_C contains those transformations which do not possess dynamical (symplectic) content (such as time evolution, rotations, gauge symmetries).

The following step is to introduce the concept of polarization subgroup $G_{\mathcal{P}}$, as a maximal horizontal subgroup which includes the characteristic subgroup, $G_C \subset$

$G_{\mathcal{P}}$. To reduce the representation we impose the polarization condition on the wave funtions:

$$\Psi(g * G_{\mathcal{P}}) = \Psi(g)\,. \tag{2}$$

In this way, we obtain a unitary irreducible representation of the group $\widetilde{G}$ on polarized $U(1)$-function on the group by means of its left action.

The second case is a non-trivial generalization of the formalism consisting in substituting the structure group $U(1)$ of phase invariance by a bigger group T (to account for "gauge" invariance, constraints, etc., see []).

With this generalization, the group $\widetilde{G}$ becomes a principal fibre bundle with structure group T. The group T itself will be a (non-trivial, in general) central extension by $U(1)$, and will be, in general, non-abelian.

Let's consider a UIR D of T on a complex vector space E. If T is non-abelian, its representations can have dimension greater than 1. The Hilbert space $\mathcal{H}(\widetilde{G}\,)$ is made out of those E-valued functions on the group $\widetilde{G}$ polarized (as in the case of structure group $U(1)$) verifying the T-function condition from the left:

$$\Psi(g_T * g) = D(g_T)\Psi(g)\,. \tag{3}$$

It must be stressed that the construction of the Hilbert space $\mathcal{H}(\widetilde{G}\,)$ depends on the particular choice of the UIR D^α of T, where α is an index characterizing the representation. Therefore, we have *non-equivalent quantizations* for each choice of non-equivalent representation D^α of T (in the sense of superselection sectors).

The quantum operators are defined as before. However, in general not all the quantum operators preserve the Hilbert space $\mathcal{H}^\alpha(\widetilde{G}\,)$, i.e. not all the (left) transformations of $\widetilde{G}$ are compatible with the T-function condition (which is also imposed from the left). Therefore, we define the subgroup of *good operators*, $G_{\mathcal{H}}$, as those preserving the Hilbert space $\mathcal{H}^\alpha(\widetilde{G}\,)$. This subgroup can be characterized by the condition:

$$[G_{\mathcal{H}}, T] \subset \text{Ker } D^\alpha(T)\,. \tag{4}$$

The rest of quantum operators, those not preserving the condition above, are *bad operators*. Among them, there may be operators which are not so bad, in the sense that they can be interpreted as quantization-changing operators, taking the whole Hilbert space $\mathcal{H}^\alpha(\widetilde{G}\,)$ to another $\mathcal{H}^{\alpha'}(\widetilde{\mathcal{G}})$, where α and α' label non-equivalent representations.

Special care should be taken if the structure group possesses dynamical (symplectic) content, i.e. the 2-cocycle ξ is not a coboundary when restricted to T, and we cannot impose, in general, the whole group T in the T-function condition (it would lead to inconsistencies). He have to choose a polarization subgroup T_p in T, and impose the condition: $\Psi(g_{T_B} * g) = D(g_{T_B})\Psi(g)$, where $T_B = T_p \cup U(1)$ and D is a representation of T_B. Then we proceed in the same way as before, simply changing T for T_B everywhere.

2 Quantization of the Heisenberg-Weyl group on the torus

Now, as a direct application of the formalism introduced in the previous section, let us consider the problem of the quantization of the torus as a symplectic manifold. We can perform it considering $\widetilde{G}$ as the Heisenberg-Weyl (H-W) group, with group law:

$$\begin{aligned} \vec{x}\,'' &= \vec{x}\,' + \vec{x} \\ \zeta'' &= \zeta'\zeta \exp\{\frac{i}{\hbar} m\omega[(1+\lambda)x_1' x_2 + \lambda x_1 x_2']\}\,, \end{aligned} \tag{5}$$

and T a fibre bundle with base $\Gamma_{\vec{L}} \equiv \left\{e_{\vec{k}},\, \vec{k} \in Z \times Z\right\}$ and fibre $U(1)$, where $e_{\vec{k}}$ are translations of $\vec{x}$ by an amount of $\vec{L}_{\vec{k}} \equiv (k_1 L_1, k_2 L_2)$ (therefore $\widetilde{G}\,/T \sim T^2$). λ parametrizes different (equivalent) 2-cocycles. The fibration of T by $U(1)$ depends on the values of m, ω, L_1 and L_2, and is, in general, non-trivial (see [] for a detailed discussion). Two cases have to be considered:

When $\frac{m\omega L_1 L_2}{2\pi\hbar} = n \in N$, the structure group is $T = \Gamma_{\vec{L}} \times U(1)$, and the T-function condition reads $\Psi(g_T * g) = \mathcal{D}(g_T)\Psi(g)$, with $\mathcal{D}(e_{\vec{k}}, \zeta) = \zeta D(e_{\vec{k}})$. $D(e_{\vec{k}})$ is a representation of the group $\Gamma_{\vec{L}} \approx Z \times Z$. We shall restrict ourselves to the trivial representation $D^0(e_{\vec{k}}) = 1$ for the time being, and the non-trivial ones will be obtained later on.

The T-function condition is written as $e^{\frac{i}{\hbar} m\omega[(1+\lambda)k_1 L_1 x_2 + \lambda k_2 L_2 x_1]}\Psi^0(\vec{x} + \vec{L}_{\vec{k}}, \zeta) = \Psi^0(\vec{x}, \zeta)$. This restriction on the wave functions has severe consecuences: (a) There exist only two possible polarizations[1] leading to $\Phi^0(x_1)$ and $\Phi^0(x_2)$ respectively, (b) The wave function is distributional, with support on discrete, equally spaced values, and (c) The dimension of the representations (and of the Hilbert space) is n.

Explicitly, the allowed values for the coordinates are $x_2 = \frac{k}{n} L_2$ or $x_1 = \frac{k}{n} L_1$, $k \in Z$, depending on the polarization we choose. The wave functions have the form $\Phi^0(x_2) = \sum_{k\in Z} a_k \delta(x_2 - \frac{k}{n} L_2)$, with periodicity in the coefficients a_k, $a_k = a_{k+n}, \forall k \in Z$, that allow to write it on a more compact form: $\Phi^0(x_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^0(x_2)$, where

$$\Lambda_k^0(x_2) \equiv \sum_{k_2 \in Z} \delta(x_2^{(k)} - k_2 L_2) = \frac{1}{L_2} \sum_{q \in Z} e^{i2\pi q x_2^{(k)}/L_2}\,, \tag{6}$$

with $x_2^{(k)} \equiv x_2 - \frac{k}{n} L_2$.

The subgroup of good operators is: $G_{\mathcal{H}} = \left\{\zeta(\hat{\eta}_1)^{\frac{k_1}{n}}(\hat{\eta}_2)^{\frac{k_2}{n}},\ \ k_1, k_2 \in Z,\ \zeta \in U(1)\right\}$, with $\hat{\eta}_1 \equiv e_{(1,0)}$ and $\hat{\eta}_2 \equiv e_{(0,1)}$.

We can obtain the whole set of non-equivalent quantization acting with the bad operators (those operators of $\widetilde{G}$ that are not in $G_{\mathcal{H}}$, see []): $\Phi^{\vec{\alpha}}(x_2) = \hat{\eta}_1^{\alpha_1}\hat{\eta}_2^{\alpha_2}\Phi^0(x_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^{\vec{\alpha}}(x_2)$.

[1] We are considering only real polarizations. There exist also a complex polarization leading to holomorphic wave functions.

Bad operators, in this simple case, can be interpreted as quantization-changing operators. The range of inequivalent quantizations is given by $\alpha_1 \in [0, \frac{L_1}{n})$ and $\alpha_2 \in [0, \frac{L_2}{n})$.

When[2] $\frac{m\omega L_1 L_2}{2\pi\hbar} = \frac{n}{r} \in Q$, the structure group T possesses dynamical (symplectic) content and we have to choose a polarization subgroup T_p. Since T has a characteristic subgroup $G_C = \{r\vec{L}_{\vec{k}}, \vec{k} \in Z \times Z\}$, $T_p = G_C \cup \{k\vec{L}_{\vec{k}_p}, k \in Z\}$, where $\vec{k}_p = (1,0)$ or $(0,1)$.

The T-function condition is $\Psi(g_{T_B} * g) = \mathcal{D}(g_{T_B})\Psi(g)$, where $T_B \equiv T_p \cup U(1)$. Let's consider (for simplicity) the trivial representation $\mathcal{D}^0(g_{T_p}, \zeta) = \zeta$ of T_p. The two possible choices of $\vec{k}_p$ lead to non-equivalent representations, of dimension n:

For $\vec{k}_p = (0,1)$ the wave functions $\Phi^0_\perp(x_2)$ are the same as in the integer case. The difference is in the good operators, $G^\perp_{\mathcal{H}} = \left\{(\hat{\eta}_1)^{r\frac{k_1}{n}}, (\hat{\eta}_2)^{\frac{k_2}{n}}, \;\; k_1, k_2 = 0, ..., n-1\right\}$.

For $\vec{k}_p = (1,0)$ the wave functions have support in the values $x_2 = k\frac{r}{n}L_2$, $k \in Z$, satisfy $\Phi^0_\parallel(x_2 + rk_2L_2) = \Phi^0_\parallel(x_2)$, and have the form $\Phi^0_\parallel(x_2) = \sum_{k=0}^{n-1} a_k \Lambda^{r,0}_k(x_2)$, where $\Lambda^{r,0}_k(x_2) \equiv \frac{1}{rL_2}\sum_{q\in Z} e^{i2\pi q x_2^{r,(k)}/(rL_2)}$, with $x_2^{r,(k)} \equiv x_2 - \frac{k}{n}rL_2$. The good operators are: $G^\parallel_{\mathcal{H}} = \left\{(\hat{\eta}_1)^{\frac{k_1}{n}}, (\hat{\eta}_2)^{r\frac{k_2}{n}}, \;\; k_1, k_2 = 0, ..., n-1\right\}$.

The nontrivial representations of T_B can be obtained as before, with the action of the bad operators (see []). It should be stressed that, although $\frac{m\omega L_1 L_2}{2\pi\hbar} = \frac{n}{r}$, the dimension of the quantum representation is n, as in the integer case. These representations can be interpreted in terms of a torus r times bigger in one direction, i.e. the area of the effective torus is rL_1L_2, and then $\frac{m\omega(rL_1L_2)}{2\pi\hbar} = n$. Therefore, the same results as in the integer case apply, but changing L_2 by rL_2 if $\vec{k}_p = (1,0)$ or L_1 by rL_1 if $\vec{k}_p = (0,1)$. Another possibility is to interprete the wave functions as multi-valued (r-valued) functions on the original torus, therefore building a vector representation.

In conclusion, we can say that in the fractional case we have to substitute the traditional $U(1)$-bundle over the torus by a vector bundle of rank r and Chern class n. The operators of T_B will act in a diagonal way but those of T that are not in T_B will mix the different component of the vector-valued function, building in this way a (r-dimensional) representation of the whole group T.

In this particular case, and because the representations of T are of finite dimension (despite it has dynamical content), we could have considered the whole group T and its representations, without resorting to its subgroup T_B for imposing the constraints.

See [] for applications of this study to the Fractional Quantum Hall Effect.

[2]The irrational case requires techniques from noncommutative geometry and will not be considered here.

2.1 Modular invariance

We consider as starting group $\widetilde{G}$ the Schrödinger group (or $WSp(2,R)$ group). This group contains $Sp(2,R) \approx SL(2,R)$ as a subgroup, representing the (linear) symplectic transformations of the plane as a phase space. We repeat the same procedure as before, with T the same structure group, and one obtains, essentially, the same wave functions, although new polarizations are now allowed, related by modular transformations (see [] for the explicit computations).

The condition for good operators, $[G_{\mathcal{H}}, T] \subset \text{Ker } D^{\vec{\alpha}}(T)$, gives, besides the ones obtained before, new good operators coming from the $Sp(2,R)$ subgroup. The result $SL(2,Z) \subset G_{\mathcal{H}}$ would be expected (statement of modular invariance of the quantum theory), but the final result depends on the particular representation $D^{\vec{\alpha}}(T)$ chosen, and on the value of $\frac{m\omega L_1 L_2}{2\pi\hbar} = \frac{n}{r}$. For the integer case ($r = 1$) we obtain the result: $SL(2,Z) \subset G_{\mathcal{H}} \Leftrightarrow \vec{\alpha} = 0$ (i.e. trivial rep.), and n even. If n is odd, only a subgroup of modular transformations preserves the Hilbert space $\mathcal{H}^0(\widetilde{G})$ (see [] for details). If a non-trivial representation $D^{\vec{\alpha}}(T)$ is chosen, the result depends on the rational or irrational character of $\frac{na_1}{L_1}$ and $\frac{na_2}{L_2}$. If any of them is irrational then the whole $SL(2,Z)$ is bad. When they are rationals only a proper subgroup of $SL(2,Z)$ survives as good operators. Particularly interesting is the case $\frac{na_1}{L_1} = \frac{na_2}{L_2} = \frac{1}{2}$, corresponding to antiperiodic boundary conditions, and for which the same results as in the trivial representation and n odd is obtained, for all values of n. For the fractional case ($r \neq 1$), only a proper subgroup of $SL(2,Z)$ remains as good operators.

These results agree with those of Bos and Nair [] (although they consider only the integer case), but are in disagreement with the results of Iengo and Lechner [], who obtain no constraint on the values of n and the representation $D^{\vec{\alpha}}(T)$ corresponds to that of antiperiodic wave functions, i.e. $\frac{n\alpha_1}{rL_1} = \frac{n\alpha_2}{rL_2} = \frac{1}{2}$.

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