Structure Constants for New Infinite-Dimensional Lie Algebras of $U(N_+, N_-)$ Tensor Operators and Applications

M. Calixto*

Department of Physics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, U.K.
and
Instituto Carlos I de Física Teórica y Computacional, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, Granada 18002, Spain.

Abstract

The structure constants for Moyal brackets of an infinite basis of functions on the algebraic manifolds M of pseudo-unitary groups $U(N_+, N_-)$ are provided. They generalize the Virasoro and $\mathcal{W}_\infty$ algebras to higher dimensions. The connection with volume-preserving diffeomorphisms on M, higher generalized-spin and tensor operator algebras of $U(N_+, N_-)$ is discussed. These centrally-extended, infinite-dimensional Lie-algebras provide also the arena for non-linear integrable field theories in higher dimensions, residual gauge symmetries of higher-extended objects in the light-cone gauge and C^*-algebras for tractable non-commutative versions of symmetric curved spaces.

*E-mail: pymc@swansea.ac.uk / calixto@ugr.es

The general study of infinite-dimensional algebras and groups, their quantum deformations (in particular, central extensions) and representation theory has not progressed very far, except for some important achievements in one- and two-dimensional systems, and there can be no doubt that a breakthrough in the subject would provide new insights into the two central problems of modern physics: unification of all interactions and exact solvability in QFT and statistics.

The aforementioned achievements refer mainly to Virasoro and Kac-Moody symmetries (see e.g. [1, 2]), which have played a fundamental role in the analysis and formulation of conformally-invariant (quantum and statistical) field theories in one and two dimensions, and systems in higher dimensions which in some essential respects are one- or two-dimensional (e.g. String Theory). Generalizations of the Virasoro symmetry, as the algebra $\mathrm{diff}(S^1)$ of reparametrisations of the circle, lead to the infinite-dimensional Lie algebras of area-preserving diffeomorphisms $\mathrm{sdiff}(\Sigma)$ of two-dimensional surfaces Σ. These algebras naturally appear as a residual gauge symmetry in the theory of relativistic membranes [3], which exhibits an intriguing connection with the quantum mechanics of space constant (e.g. vacuum configurations) $SU(N)$ Yang-Mills potentials in the limit $N \to \infty$ [4]; the argument that the internal symmetry space of the $U(\infty)$ pure Yang-Mills theory must be a functional space, actually the space of configurations of a string, was pointed out in Ref. [5]. Moreover, the $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$ algebras of area-preserving diffeomorphisms of the cylinder [6] generalize the underlying Virasoro gauged symmetry of the light-cone two-dimensional induced gravity discovered by Polyakov [7] by including all positive conformal-spin currents [8], and induced actions for these $\mathcal{W}$-gravity theories have been proposed [9, 10]. Also, the $\mathcal{W}_{1+\infty}$ (dynamical) symmetry has been identified by [11] as the set of canonical transformations that leave invariant the Hamiltonian of a two-dimensional electron gas in a perpendicular magnetic field, and appears to be relevant in the classification of all the universality classes of *incompressible quantum fluids* and the identification of the quantum numbers of the excitations in the Quantum Hall Effect. Higher-spin symmetry algebras where introduced in [12] and could provide a guiding principle towards the still unknown "M-theory".

It is remarkable that area-preserving diffeomorphisms, higher-spin and $\mathcal{W}$ algebras can be seen as distinct members of a one-parameter family $\mathcal{L}_\mu(su(2))$ —or the non-compact version $\mathcal{L}_\mu(su(1,1))$— of non-isomorphic [13] infinite-dimensional Lie-algebras of $SU(2)$ —and $SU(1,1)$— tensor operators, more precisely, the factor algebra $\mathcal{L}_\mu(su(2)) = \mathcal{U}(su(2))/\mathcal{I}_\mu$ of the universal enveloping algebra $\mathcal{U}(su(2))$ by the ideal $\mathcal{I}_\mu = (\hat{C} - \hbar^2\mu)\mathcal{U}(su(2))$ generated by the Casimir operator $\hat{C}$ of $su(2)$ (μ denotes an arbitrary complex number). The structure constants for $\mathcal{L}_\mu(su(2))$ and $\mathcal{L}_\mu(su(1,1))$ are well known for the Racah-Wigner basis of tensor operators [14], and they can be written in terms of Clebsch-Gordan and (generalized) $6j$-symbols [3, 8, 15]. Another interesting feature of $\mathcal{L}_\mu(su(2))$ is that, when μ coincides with the eigenvalue of $\hat{C}$ in an irrep D_j of $SU(2)$, that is $\mu = j(j+1)$, there exists and ideal χ in $\mathcal{L}_\mu(su(2))$ such that the quotient $\mathcal{L}_\mu(su(2))/\chi \simeq sl(2j+1, C)$ or $su(2j+1)$, by taking a compact real form of the complex Lie algebra [16]. That is, for $\mu = j(j+1)$ the infinite-dimensional algebra $\mathcal{L}_\mu(su(2))$ collapses to a finite-dimensional one. This fact was used in [3] to approximate $\lim_{\substack{\mu\to\infty \\ \hbar\to 0}} \mathcal{L}_\mu(su(2)) \simeq \mathrm{sdiff}(S^2)$ by $su(N)|_{N\to\infty}$ ("large number of colors").

The generalization of these constructions to general unitary groups proves to be quite unwieldy, and a canonical classification of $U(N)$-tensor operators has, so far, been proven to exist only for $U(2)$ and $U(3)$ (see [14] and references therein). Tensor labeling is provided in these cases by the Gel'fand-Weyl pattern for vectors in the carrier space of the irreps of $U(N)$.

In this letter, a quite appropriate basis of operators for $\mathcal{L}_{\vec{\mu}}(u(N_+,N_-)), \vec{\mu}=(\mu_1,\dots,\mu_N)$, $N\equiv N_+ + N_-$, is provided and the structure constants, for the particular case of the boson realization of quantum associative operatorial algebras on algebraic manifolds $M_{N_+N_-} = U(N_+,N_-)/U(1)^N$, are calculated. The particular set of operators in $\mathcal{U}(u(N_+,N_-))$ is the following:

$$
\begin{aligned}
\hat{L}^I_{|m|} &\equiv \prod_\alpha (\hat{G}_{\alpha\alpha})^{I_\alpha - (\sum_{\beta>\alpha}|m_{\alpha\beta}| + \sum_{\beta<\alpha}|m_{\beta\alpha}|)/2} \prod_{\alpha<\beta} (\hat{G}_{\alpha\beta})^{|m_{\alpha\beta}|} \\
\hat{L}^I_{-|m|} &\equiv \prod_\alpha (\hat{G}_{\alpha\alpha})^{I_\alpha - (\sum_{\beta>\alpha}|m_{\alpha\beta}| + \sum_{\beta<\alpha}|m_{\beta\alpha}|)/2} \prod_{\alpha<\beta} (\hat{G}_{\beta\alpha})^{|m_{\alpha\beta}|}
\end{aligned} \tag{1}
$$

where $\hat{G}_{\alpha\beta}$, $\alpha,\beta=1,\dots,N$, are the $U(N_+,N_-)$ Lie-algebra (step) generators with commutation relations:

$$
\left[\hat{G}_{\alpha_1\beta_1}, \hat{G}_{\alpha_2\beta_2}\right] = \hbar(\eta_{\alpha_1\beta_2}\hat{G}_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1}\hat{G}_{\alpha_1\beta_2})\,, \tag{2}
$$

and $\eta=\mathrm{diag}(1,\overset{N_+}{\dots},1,-1,\overset{N_-}{\dots},-1)$ is used to raise and lower indices; the upper (generalized spin) index $I\equiv(I_1,\dots,I_N)$ of $\hat{L}$ in (1) represents a N-dimensional vector which, for the present, is taken to lie on an half-integral lattice; the lower index m symbolizes a integral upper-triangular $N\times N$ matrix, and $|m|$ means absolute value of all its entries. Thus, the operators $\hat{L}^I_m$ are labeled by $N+N(N-1)/2 = N(N+1)/2$ indices, in the same way as wave functions ψ^I_m in the carrier space of irreps of $U(N)$. An implicit quotient by the ideal $\mathcal{I}_{\vec{\mu}} = \prod_{j=1}^N(\hat{C}_j - \hbar^j\mu_j)\mathcal{U}(u(N_+,N_-))$ generated by the Casimir operators

$$
\hat{C}_1 = \hat{G}^\alpha_\alpha = \hbar\mu_1\,, \quad \hat{C}_2 = \hat{G}^\beta_\alpha\hat{G}^\alpha_\beta = \hbar^2\mu_2\,, \dots \tag{3}
$$

is understood. The manifest expression of the structure constants f for the commutators

$$
\left[\hat{L}^I_m, \hat{L}^J_n\right] = \hat{L}^I_m\hat{L}^J_n - \hat{L}^J_n\hat{L}^I_m = f^{IJl}_{mnK}[\vec{\mu}]\hat{L}^K_l \tag{4}
$$

of a pair of operators (1) of $\mathcal{L}_{\vec{\mu}}(u(N_+,N_-))$ entails an unpleasant and difficult computation, because of inherent ordering problems. However, the essence of the full quantum algebra $\mathcal{L}_{\vec{\mu}}(u(N_+,N_-))$ can be still captured in a classical construction by extending the Poisson-Lie bracket

$$
\left\{L^I_m, L^J_n\right\}_{\mathrm{PL}} = (\eta_{\alpha_1\beta_2}G_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1}G_{\alpha_1\beta_2})\frac{\partial L^I_m}{\partial G_{\alpha_1\beta_1}}\frac{\partial L^J_n}{\partial G_{\alpha_2\beta_2}} \tag{5}
$$

of a pair of functions L^I_m, L^J_n on the commuting coordinates $G_{\alpha\beta}$ to its deformed version, in the sense of Ref. [17]. To perform calculations with (5) is still rather complicated because of non-canonical brackets for the generating elements $G_{\alpha\beta}$. Nevertheless, there is a standard boson operator realization $G_{\alpha\beta}\equiv a_\alpha\bar{a}_\beta$ of the generators of $u(N_+,N_-)$ for which things simplify greatly. Indeed, we shall understand that the quotient by the ideal generated by polynomials $G_{\alpha_1\beta_1}G_{\alpha_2\beta_2} - G_{\alpha_1\beta_2}G_{\alpha_2\beta_1}$ is taken, so that the Poisson-Lie bracket (5) coincides with the standard Poisson bracket

$$
\left\{L^I_m, L^J_n\right\}_{\mathrm{P}} = \eta_{\alpha\beta}\left(\frac{\partial L^I_m}{\partial a_\alpha}\frac{\partial L^J_n}{\partial \bar{a}_\beta} - \frac{\partial L^I_m}{\partial \bar{a}_\beta}\frac{\partial L^J_n}{\partial a_\alpha}\right) \tag{6}
$$

for the Heisenberg-Weyl algebra. There is basically only one possible deformation of the bracket (6) —corresponding to a normal ordering— that fulfills the Jacobi identities [17], which is the

Moyal bracket [18]:

$$\left\{L_m^I, L_n^J\right\}_{\rm M} = L_m^I * L_n^J - L_n^J * L_m^I = \sum_{r=0}^{\infty} 2\frac{(\hbar/2)^{2r+1}}{(2r+1)!} P^{2r+1}(L_m^I, L_n^J)\,, \tag{7}$$

where $L * L' \equiv \exp(\frac{\hbar}{2}P)(L, L')$ is an invariant associative $*$-product and

$$P^r(L, L') \equiv \Upsilon_{\imath_1\jmath_1}\dots\Upsilon_{\imath_r\jmath_r}\frac{\partial^r L}{\partial x_{\imath_1}\dots\partial x_{\imath_r}}\frac{\partial^r L'}{\partial x_{\jmath_1}\dots\partial x_{\jmath_r}}\,, \tag{8}$$

with $x \equiv (a, \bar{a})$ and $\Upsilon_{2N\times 2N} \equiv \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}$. We set $P^0(L, L') \equiv LL'$; see also that $P^1(L, L') = \{L, L'\}_{\rm P}$. Note the resemblance between the Moyal bracket (7) for *covariant symbols* L_m^I and the standard commutator (4) for operators $\hat{L}_m^I$. It is worthwhile mentioning that Moyal brackets where identified as the primary quantum deformation $\mathcal{W}_\infty$ of the classical algebra w_∞ of area-preserving diffeomorphysms of the cylinder (see Ref. [19]).

With this information at hand, the manifest expression of the structure constants f for the Moyal bracket (7) is the following:

$$\begin{aligned}
\left\{L_m^I, L_n^J\right\}_{\rm M} &= \sum_{r=0}^{\infty} 2\frac{(\hbar/2)^{2r+1}}{(2r+1)!}\eta^{\alpha_0\alpha_0}\dots\eta^{\alpha_{2r}\alpha_{2r}} f_{mn}^{IJ}(\alpha_0,\dots,\alpha_{2r}) L_{m+n}^{I+J-\sum_{j=0}^{2r}\delta_{\alpha_j}}\,,\\
f_{mn}^{IJ}(\alpha_0,\dots,\alpha_{2r}) &= \sum_{\wp\in\Pi_2^{(2r+1)}} (-1)^{\ell_\wp+1}\prod_{s=0}^{2r} f_\wp(I^{(s)}_{\alpha_{\wp(s)}}, m) f_\wp(J^{(s)}_{\alpha_{\wp(s)}}, -n)\,,\\
f_\wp(I^{(s)}_{\alpha_{\wp(s)}}, m) &= I^{(s)}_{\alpha_{\wp(s)}} + (-1)^{\theta(s-\ell_\wp)}(\sum_{\beta>\alpha_{\wp(s)}} m_{\alpha_{\wp(s)}\beta} - \sum_{\beta<\alpha_{\wp(s)}} m_{\beta\alpha_{\wp(s)}})/2\,,\\
I^{(s)}_{\alpha_{\wp(s)}} &= I_{\alpha_{\wp(s)}} - \sum_{t=(\ell_\wp+1)\theta(s-\ell_\wp)}^{s-1}\delta_{\alpha_{\wp(t)},\alpha_{\wp(s)}}\,,\quad I^{(0)} = I^{(\ell_\wp+1)} \equiv I\,,\\
\theta(s-\ell_\wp) &= \left\{\begin{array}{l} 0\,,\ \ s\le\ell_\wp \\ 1\,,\ \ s>\ell_\wp \end{array}\right.\,,\qquad \delta_{\alpha_j} = (\delta_{1,\alpha_j},\dots,\delta_{N,\alpha_j})\,,
\end{aligned} \tag{9}$$

where $\Pi_2^{(2r+1)}$ denotes the set of all possible partitions $\wp$ of a string $(\alpha_0,\dots,\alpha_{2r})$ of length $2r+1$ into two substrings

$$(\overbrace{\alpha_{\wp(0)},\dots,\alpha_{\wp(\ell)}}^{\ell_\wp})(\overbrace{\alpha_{\wp(\ell+1)},\dots,\alpha_{\wp(2r)}}^{2r+1-\ell_\wp}) \tag{10}$$

of length $\ell_\wp$ and $2r+1-\ell_\wp$, respectively. The number of elements $\wp$ in $\Pi_2^{(2r+1)}$ is clearly $\dim(\Pi_2^{(2r+1)}) = \sum_{\ell=0}^{2r+1}\frac{(2r+1)!}{(2r+1-\ell)!\ell!} = 2^{2r+1}$.

For $r = 0$, there are just 2 partitions: $(\alpha)(\cdot)$, $(\cdot)(\alpha)$, and the leading (classical, $\hbar\to 0$) structure constants are, for example:

$$f_{mn}^{IJ}(\alpha) = J_\alpha(\sum_{\beta>\alpha} m_{\alpha\beta} - \sum_{\beta<\alpha} m_{\beta\alpha}) - I_\alpha(\sum_{\beta>\alpha} n_{\alpha\beta} - \sum_{\beta<\alpha} n_{\beta\alpha})\,. \tag{11}$$

They reproduce in this limit the Virasoro commutation relations for the particular generators $V_k^{(\alpha\beta)} \equiv L_{ke_{\alpha\beta}}^{\delta_\alpha}$, where $k\in\mathcal{Z}$ and $e_{\alpha\beta}$ denotes an upper-triangular matrix with zero entries except for 1 at $(\alpha\beta)$-position, if $\alpha < \beta$, or 1 at the $(\beta\alpha)$-position if $\alpha > \beta$. Indeed, there are $N(N-1)/2$ *non-commuting* Virasoro sectors in (9), corresponding to each positive root in $SU(N_+, N_-)$, with classical commutation relations:

$$\left\{V_k^{(\alpha\beta)}, V_l^{(\alpha\beta)}\right\}_{\rm P} = \eta^{\alpha\alpha}\mathrm{sign}(\beta-\alpha)\,(k-l)V_{k+l}^{(\alpha\beta)}\,. \tag{12}$$

For large r, we can benefit from the use of algebraic-computing programs like [20] to deal with the high number of partitions.

Our boson operator realization $G_{\alpha\beta} \equiv a_\alpha \bar{a}_\beta$ of the $u(N_+, N_-)$ generators corresponds to the particular case of $\vec{\mu}_0 = (N, 0, \ldots, 0)$ for the Casimir eigenvalues, so that the commutation relations (9) are related to the particular algebra $\mathcal{L}_{\vec{\mu}_0}(u(N_+, N_-))$ (see below for more general cases). A different (minimal) realization of $\mathcal{L}_\mu(su(1,1))$ in terms of a single boson $(a, \bar{a})$, which corresponds to $\mu_c = \bar{s}_c(\bar{s}_c - 1) = -3/16$ for the critical value $\bar{s}_c = 3/4$ of the *symplin* degree of freedom $\bar{s}$, was given in [21]; this case is also related to the *symplecton* algebra of [14]. Note the close resemblance between the algebra (9) —and the leading structure constants (11)— and the quantum deformation $\mathcal{W}_\infty \simeq \mathcal{L}_0(su(1,1))$ of the algebra of area-preserving diffeomorphisms of the cylinder [8, 19], although we recognize that the case discussed in this letter is far richer.

If the analyticity of the symbols L_m^I of (1) is taken into account, then one should worry about a restriction of the range of the indices $I_\alpha, m_{\alpha\beta}$. The subalgebra $\mathcal{L}^\Lambda_{\vec{\mu}_0}(u(N_+, N_-)) \equiv \{L_m^I \mid \Lambda_\alpha = I_\alpha - (\sum_{\beta>\alpha}|m_{\alpha\beta}| + \sum_{\beta<\alpha}|m_{\beta\alpha}|)/2 \in \mathcal{N}\}^\dagger$ of polynomial functions on $G_{\alpha\beta}$, the structure constants $f_{mn}^{IJ}(\alpha_0, \ldots, \alpha_{2r})$ of which are zero for $r > (\sum_\alpha(I_\alpha + J_\alpha) - 1)/2$, can be extended beyond the "wedge" $\Lambda \geq 0$ by analytic continuation, that is, by revoking this restriction to $\Lambda \in \mathcal{Z}/2$. The aforementioned "extension beyond the wedge" (see [8, 12] for similar concepts) makes possible the existence of conjugated pairs $(L_m^I, L_{-m}^{I'})$, with $\sum_\alpha I_\alpha + I'_\alpha = 2r+1$ and $I_\alpha + I'_\alpha \equiv r_\alpha \in \mathcal{N}$, that give rise to central terms under commutation:

$$\Xi(L_m^I, L_n^{I'}) = \frac{\hbar^{2r+1}2^{-2r}(-1)^{\sum_{\alpha=N_++1}^N r_\alpha}}{\prod_{\alpha=1}^N(2r+1-r_\alpha)!} f_{m,-m}^{II'}(1^{(r_1)}, \ldots, N^{(r_N)})\delta_{m+n,0}\hat{1}\,, \tag{13}$$

where $(1^{(r_1)}, \ldots, N^{(r_N)})$ is a string of length $\sum_{\alpha=1}^N r_\alpha = 2r+1$ and $\alpha^{(r_\alpha)}$ denotes a substring made of r_α-times α, for each α. The generator $\hat{1} \equiv L_0^0$ is central (commutes with everything) and the Lie algebra two-cocycle (13) defines a non-trivial central extension of $\mathcal{L}_{\vec{\mu}_0}(u(N_+, N_-))$ by $U(1)$.

A thorough study of the Lie-algebra cohomology of $\mathcal{L}_{\vec{\mu}}(u(N_+, N_-))$ and its irreps still remains to be acoomplished; it requires a separate attention and shall be left for future works. Two-cocycles like (13) provide the essential ingredient to construct invariant geometric action functionals on coadjoint orbits of $\mathcal{L}_{\vec{\mu}}(u(N_+, N_-))$ —see e.g. [10] for the derivation of the WZNW action of $D = 2$ matter fields coupled to chiral $\mathcal{W}_\infty$ gravity background from $\mathcal{W}_\infty \simeq \mathcal{L}_0(su(1,1))$).

In order to deduce the structure constants for general $\mathcal{L}_{\vec{\mu}}(u(N_+, N_-))$ from $\mathcal{L}_{\vec{\mu}_0}(u(N_+, N_-))$, a procedure similar to that of Ref. [12], for the particular case of $\mathcal{U}(sl(2,\Re))$, can be applied. Special attention must be paid to the limit $\lim_{\substack{\vec{\mu}\to\infty \\ \hbar\to 0}} \mathcal{L}_{\vec{\mu}}(u(N_+, N_-)) \simeq \mathcal{P}_{\mathcal{C}}(M_{N_+N_-})$, which coincides with the Poisson algebra of complex (wave) functions $\psi^I_{|m|}$, $\psi^I_{-|m|} \equiv \bar{\psi}^I_{|m|}$ on algebraic manifolds (coadjoint orbits [22]) $M_{N_+N_-} \simeq U(N_+, N_-)/U(1)^N$. ‡ It is well known that there

$^\dagger\mathcal{N}$, $\mathcal{Z}$, $\Re$ and $\mathcal{C}$ denote the set of natural, integer, real and complex numbers, respectively

‡For $N_- \neq 0$, other cases could be also contemplated (e.g. continuous series of $SU(1,1)$)

exists a natural symplectic structure $(M_{N_+N_-}, \Omega)$, which defines the Poisson algebra

$$\left\{\psi^I_m, \psi^J_n\right\} = \Omega^{\alpha_1\beta_1;\alpha_2\beta_2} \frac{\partial \psi^I_m}{\partial g^{\alpha_1\beta_1}} \frac{\partial \psi^J_n}{\partial g^{\alpha_2\beta_2}} \tag{14}$$

and an invariant symmetric bilinear form $\langle\psi^I_m|\psi^J_n\rangle = \int v(g)\bar{\psi}^I_m(g)\psi^J_n(g)$ given by the natural invariant measure $v(g) \sim \Omega^{N(N-1)/2}$ on $U(N_+, N_-)$, where $g^{\alpha\beta} = \bar{g}^{\beta\alpha} \in \mathcal{C}$, $\alpha \neq \beta$, is a (local) system of complex coordinates on $M_{N_+N_-}$. The structure constants for (14) can be obtained through $f^{IJl}_{mnK} = \langle\psi^K_l|\{\psi^I_m, \psi^J_n\}\rangle$. Also, an associative $\star$-product can be defined through the convolution of two functions $(\psi^I_m \star \psi^J_n)(g') \equiv \int v(g)\psi^I_m(g)\psi^J_n(g^{-1} \bullet g')$, which gives the algebra $\mathcal{P}_{\mathcal{C}}(M_{N_+N_-})$ a non-commutative character — $g \bullet g'$ denotes the group composition law of $U(N_+, N_-)$. A manifest expression for all these structures is still in progress [23].

Taking advantage of all these geometrical tools, action functionals for $\mathcal{L}_\infty(u(N_+, N_-))$ Yang-Mills gauge theories in D dimensions could be built as:

$$\begin{aligned} S &= \int \mathrm{d}^D x \langle F_{\nu\gamma}(x,g)|F^{\nu\gamma}(x,g)\rangle \,, \\ F_{\nu\gamma} &= \partial_\nu A_\gamma - \partial_\gamma A_\nu + \{A_\nu, A_\gamma\} \,, \\ A_\nu(x,g) &= A^m_{\nu I}(x)\psi^I_m(g) \,, \quad \nu,\gamma = 1,\ldots,D \,, \end{aligned} \tag{15}$$

the 'vacuum configurations' (spacetime-constant potentials $X_\nu(g) \equiv A_\nu(0,g)$) of which, define the action for higher-extended objects: $N(N-1)$-'branes', in the usual nomenclature. Here, $\mathcal{L}_\infty(u(N_+, N_-))$ plays the role of gauge symplectic (volume-preserving) diffeomorphisms $L_\psi \equiv \{\psi, \cdot\}$ on the $N(N-1)$-brane $M_{N_+N_-}$. A particularly interesting case might be $SU(2,2) = U(2,2)/U(1)$: the conformal group in $3+1$ (or the AdS group in $4+1$) dimensions, in an attempt to construct 'conformal gravities' in realistic dimensions. The infinite-dimensional algebra $\mathcal{L}_\mu(u(2,2))$ might be seen as the *generalization of the Virasoro (two-dimensional) conformal symmetry to* $3+1$ *dimensions*.

Finally, let me comment on the potential relevance of the C^*-algebras $\mathcal{L}_{\vec{\mu}}(\mathcal{G})$ on tractable non-commutative versions [24] of symmetric curved spaces $M = G/H$, where the notion of a pure state ψ^I_m replaces that of a point. The possibility of describing phase-space physics in terms of the quantum analog of the algebra of functions (the covariant symbols L^I_m), and the absence of localization expressed by the Heisenberg uncertainty relation, was noticed a long time ago by Dirac [25]. Just as the standard differential geometry of M can be described by using the algebra $C^\infty(M)$ of smooth complex functions ψ on M (that is, $\lim_{\substack{\vec{\mu}\to\infty \\ \hbar\to 0}} \mathcal{L}_{\vec{\mu}}(\mathcal{G})$, when considered as an associative, commutative algebra), a non-commutative geometry for M can be described by using the algebra $\mathcal{L}_{\vec{\mu}}(\mathcal{G})$, seen as an associative algebra with a non-commutative $*$-product like (7,8). The appealing feature of a non-commutative space M is that a G-invariant 'lattice structure' can be constructed in a natural way, a desirable property as regards finite models of quantum gravity (see e.g. [26] and Refs. therein). Indeed, as already mentioned, $\mathcal{L}_{\vec{\mu}}(\mathcal{G})$ collapses to $\mathrm{Mat}_d(\mathcal{C})$ (the full matrix algebra of $d \times d$ complex matrices) whenever μ_α coincides with the eigenvalue of $\hat{C}_\alpha$ in a d-dimensional irrep $D_{\vec{\mu}}$ of G. This fact provides a finite (d-points) 'fuzzy' or 'cellular' description of the non-commutative space M, the classical (commutative) case being recovered in the limit $\vec{\mu} \to \infty$. The notion of space itself could be the collection of all of them, enclosed in a single irrep of $\mathcal{L}_{\vec{\mu}}(\mathcal{G})$ for general $\vec{\mu}$, with different multiplicities, as it actually happens with the reduction of an irrep of the centrally-extended

Virasoro group under its $SL(2,\Re)$ subgroup [27]; The multiplicity should increase with $\vec{\mu}$ ('the density of points'), so that classical-like spaces are more abundant. It is also a very important feature of $\mathcal{L}_{\vec{\mu}}(u(N_+,N_-))$ that the quantization deformation scheme (7) does not affect the maximal finite-dimensional subalgebra $su(N_+,N_-)$ ('good observables' or preferred coordinates [17]) of non-commuting 'position operators'

$$\begin{aligned} y_{\alpha\beta} = \tfrac{\lambda}{2\hbar}(\hat{G}_{\alpha\beta}+\hat{G}_{\beta\alpha})\,,\ \ y_{\beta\alpha} = \tfrac{i\lambda}{2\hbar}(\hat{G}_{\alpha\beta}-\hat{G}_{\beta\alpha})\,, \quad \alpha<\beta\,, \\ y_{\alpha} = \tfrac{\lambda}{\hbar}(\eta_{\alpha\alpha}\hat{G}_{\alpha\alpha}-\eta_{\alpha+1,\alpha+1}\hat{G}_{\alpha+1,\alpha+1})\,, \end{aligned} \tag{16}$$

on the algebraic manifold $M_{N_+N_-}$, where λ denotes a parameter that gives y dimensions of length (e.g., the square root of the Planck area $\hbar G$). The 'volume' v_j of the $N-1$ submanifolds M_j of the *flag manifold* $M_{N_+N_-}=M_N\supset\ldots\supset M_2$ (see e.g. [28] for a definition of flag manifolds) is proportional to the eigenvalue μ_j of the $su(N_+,N_-)$ Casimir operator $\hat{C}_j$ in those coordinates: $v_j=\lambda^j\mu_j$. Large volumes (flat-like spaces) correspond to a high density of points (large μ). In the classical limit $\lambda\to 0$, $\mu\to\infty$, the coordinates y commute.

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