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Character sums over elements of extensions of finite fields with restricted coordinates

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ABSTRACT

We obtain nontrivial bounds for character sums with multiplicative and additive characters over finite fields over elements with restricted coordinate expansion. In particular, we obtain a nontrivial estimate for such a sum over a finite field analogue of the Cantor set.

Contents

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1. Introduction

Let $(\vartheta_1, \ldots, \vartheta_r)$ be a basis of the finite field

$$\mathbb{F}_{q^r} = \{a_1\vartheta_1 + \ldots + a_r\vartheta_r : \ a_1, \ldots, a_r \in \mathbb{F}_q\}$$

of q^r elements over the finite field $\mathbb{F}_q$ of q elements.

Motivated by a series of recent outstanding results on integers with restricted digital expansion in a given basis, there has also been very significant progress in studying elements $\omega \in \mathbb{F}_{q^r}$ with various restrictions on their coordinates $(a_1, \ldots, a_r)$ in the expansion

$$\omega = a_1\vartheta_1 + \ldots + a_r\vartheta_r \in \mathbb{F}_{q^r},$$

we refer to [4] for a brief outline of such results (both settings on integers and finite fields), some new results and further references, in particular on bounds of various character sums over such field elements.

Here, given a set $\mathcal{A} \subseteq \mathbb{F}_q$, we consider the set

$$\mathcal{S}_r(\mathcal{A}) = \{a_1\vartheta_1 + \ldots + a_r\vartheta_r : \ a_1, \ldots, a_r \in \mathcal{A}\} \tag{1.1}$$

that is the set of $u \in \mathbb{F}_{q^r}$ whose coordinates are restricted to the set $\mathcal{A}$.

In particular, one of the natural examples is the case of $q = 3$ and $\mathcal{A} = \{0, 2\}$ which leads to a Cantor-like set $\mathcal{S}_r(\mathcal{A}) \subseteq \mathbb{F}_{3^r}$.

The main goal of this paper is to estimate mixed character sums

$$S_r(\mathcal{A}; \chi, \psi; f_1, f_2) = \sum_{\omega \in \mathcal{S}_r(\mathcal{A})} \chi(f_1(\omega))\, \psi(f_2(\omega)),$$

with rational functions $f_1(X), f_2(X) \in \mathbb{F}_{q^r}(X)$, of degrees d_1 and d_2, respectively, and where χ and ψ are a fixed multiplicative and additive character of $\mathbb{F}_{q^r}$, respectively (with the natural conventions that the poles of $f_1(X)$ and $f_2(X)$ are excluded from summation).

We are especially interested in the case when $\mathcal{A}$ is of cardinality $\#\mathcal{A}$ relatively small compared to q. In particular, we are interested in obtaining nontrivial bounds in the case of small values of the parameter

$$\rho = \frac{\log \#\mathcal{A}}{\log q}.$$

It is well known that such bounds can be used to study, for example, the distribution of primitive elements in the values of polynomials on elements from $\mathcal{S}_r(\mathcal{A})$ or their pseudorandom properties. Since these applications are quite standard, we do not present them here.

2. Notation and conventions

Throughout the paper, we fix the size q of the ground field, and thus also its characteristic p while the parameter r is allowed to grow.

We also fix an additive character ψ and a multiplicative character χ of $\mathbb{F}_{q^r}$ which are not both principal.

As usual, we use $\overline{\mathbb{F}_q}$ to denote the algebraic closure of $\mathbb{F}_q$. It is useful to recall that $\overline{\mathbb{F}_q} \subseteq \overline{\mathbb{F}_p}$.

For a finite set $\mathcal{S}$, we use $\#\mathcal{S}$ to denote its cardinality.

We denote by $\log_2 x$ the binary logarithm of $x > 0$.

We adopt the Vinogradov symbol $\ll$, that is, for any quantities A and B we have the following equivalent definitions:

$$A \ll B \iff A = O(B) \iff |A| \leqslant cB$$

for some constant $c > 0$, which throughout the paper is allowed to depend on the degrees d, d_1, d_2, the ground field size q and the integer parameter $s \geqslant 1$ (but not on the main parameter r).

For a rational function $g(X) \in \overline{\mathbb{F}_p}(X)$ and an element $w \in \overline{\mathbb{F}_p}$ we define $\mathrm{ord}_w\, g$ to be the unique integer so that $(X - w)^{\mathrm{ord}_w(g)} g$ extends to a rational function which has no zero or pole at w.

We also write

$$\mathbf{e}_p(z) = \exp(2\pi i z/p).$$

Finally, we also recall our convention that the poles of functions in the arguments of multiplicative and additive characters are always excluded from summation.

3. Main results

We define the following sets of rational functions.

Definition 3.1. *For integers $d \geqslant 0$ and $n \geqslant 2$,*

- *let $\mathcal{Q}_{d,n}$ be the set of rational functions $g(X) \in \mathbb{F}_{q^r}(X)$ of degree at most d, which are not an n-th power of some rational function in $\overline{\mathbb{F}_p}(X)$.*
- *let $\mathcal{R}_d$ be the set of rational functions $f(X) \in \mathbb{F}_{q^r}(X)$ of degree at most d, which have at least one pole of order that is not a multiple of p.*

We note that we allow $d = 0$ in Definition 3.1, that is non-zero constant functions, in which case $\mathcal{Q}_{d,n} = \mathcal{R}_d = \emptyset$.

We are now ready to present our main result. We recall our convention that implied constants are allowed to depend on the integer parameters d_1, d_2, q and s.

For an integer $s \geqslant 1$ we define

$$\kappa_s(\rho) = \frac{s\rho(2\rho - 1) + \rho - 1}{4s(s\rho + 1)}. \tag{3.1}$$

Theorem 3.2. *Let χ and ψ be a multiplicative and additive character, respectively, and let $f_1(X), f_2(X) \in \mathbb{F}_{q^r}(X)$. Assume that at least one of the following conditions holds*

(i) *χ is nonprincipal of order n and $f_1(X) \in \mathcal{Q}_{d,n}$,*
(ii) *ψ is nonprincipal and $f_2(X) \in \mathcal{R}_d$.*

Then for any fixed integer $s \geqslant 1$, we have

$$S_r(\mathcal{A}; \chi, \psi; f_1, f_2) \ll (\#\mathcal{A})^r \, q^{-r\kappa_s(\rho)}.$$

Clearly for any $\rho > 1/2$ we have $\kappa_s(\rho) > 0$ for a sufficiently large s.

In particular, with

$$\rho = \frac{\log 2}{\log 3}$$

taking $s = 5$ in Theorem 3.2 we have the following nontrivial bound for a "Cantor-like" set in finite fields.

Corollary 3.3. *Let $q = 3$ and $\mathcal{A} = \{0, 2\}$. Under the conditions of Theorem 3.2 we have*

$$S_r(\mathcal{A}; \chi, \psi; f_1, f_2) \ll 2^{\gamma r}$$

where

$$\gamma = 1 - \frac{\log 3}{\log 2} \cdot \kappa_5\left(\frac{\log 2}{\log 3}\right) = 0.99128\ldots .$$

We remark that both Theorem 3.2 and Corollary 3.3 apply to *Kloosterman sums*

$$\sum_{\omega \in \mathcal{S}_r(\mathcal{A})} \psi\left(a\omega + b\omega^{-1}\right), \qquad (a, b) \in \mathbb{F}_{q^r} \times \mathbb{F}_{q^r}^*,$$

over elements of $\mathcal{S}_r(\mathcal{A})$.

We note that unfortunately Theorem 3.2 does not apply to polynomials f_2 if either χ is principal or $f_1 \notin \mathcal{Q}_{d,n}$. Hence, we introduce another class of functions which actually originates from [4].

Definition 3.4. *Let $\mathcal{P}_d$ be the set of rational functions $f(X) \in \mathbb{F}_{q^r}(X)$ of degree d such that for any $\omega \in \mathbb{F}_{q^r}^*$ the function*

$$f_\omega(X) = f(X+\omega) - f(X)$$

is not of the form

$$f_\omega(X) = \alpha\left(g(X)^p - g(X)\right) + \beta X$$

for some rational function $g(X) \in \overline{\mathbb{F}_q}(X)$ and $\alpha, \beta \in \overline{\mathbb{F}_q}$.

We refer to [4] for examples of functions from $\mathcal{P}_d$.

For a function $f_2 \in \mathcal{P}_d$, we are only able to obtain a version of Theorem 3.2 with $s = 1$, and hence we save only

$$\kappa_1(\rho) = \frac{2\rho^2 - 1}{4(\rho+1)}.$$

Theorem 3.5. *Let χ and ψ be a multiplicative and additive character, respectively, and let $f_1(X), f_2(X) \in \mathbb{F}_{q^r}(X)$. Assume that ψ is nonprincipal and $f_2(X) \in \mathcal{P}_d$. Then, we have*

$$S_r(\mathcal{A}; \chi, \psi; f_1, f_2) \ll (\#\mathcal{A})^r\, q^{-r\kappa_1(\rho)}.$$

Note that $\kappa_1(\rho) > 0$ only for

$$\rho > 2^{-1/2} = 0.707106\ldots > \frac{\log 2}{\log 3} = 0.63092\ldots,$$

and hence unfortunately Theorem 3.5 does not apply to the setting of Corollary 3.3.

4. Ratios and linear combinations of shifts of rational functions

Various versions of the following results have been well-known, see, for example, the proof of [1, Theorem 1] or of [5, Theorem 1].

It is convenient to introduce the following notation. Given a vector $\mathbf{v} = (v_1, \ldots, v_{2s}) \in \overline{\mathbb{F}_p}^{2s}$ and a rational function $f \in \overline{\mathbb{F}_p}(X)$, we set

$$P_{\mathbf{v},f}(X) = \prod_{i=1}^{s} \frac{f\left(X + v_i\right)}{f\left(X + v_{s+i}\right)}. \tag{4.1}$$

The implied constants in this section may depend only on $d = \deg f$ and s, but are uniform with respect to other parameters, including q, and most importantly n, r and V.

Lemma 4.1. *Let $f(X) \in \mathcal{Q}_{d,n}$ for some integers $d \geqslant 1$ and $n \geqslant 2$. For any set $\mathcal{V} \subseteq \overline{\mathbb{F}_p}$ of cardinality V, for each integer $s \geqslant 1$ we have*

$$\#\left\{\mathbf{v} = (v_1, \ldots, v_{2s}) \in \mathcal{V}^{2s} :\ P_{\mathbf{v},f}(X) \notin \mathcal{Q}_{2ds,n}\right\} \ll V^s.$$

Proof. Without loss of generality, we can assume that all zeros and poles of f are of order less than n, that is,

$$f(X) = \prod_{j=1}^{h} (X - \alpha_j)^{u_j},$$

where $\alpha_j \in \overline{\mathbb{F}_p}$ are pairwise distinct and $u_j \in \{\pm 1, \ldots, \pm(n-1)\}$, $j = 1, \ldots, h$.

If $v_1, \ldots, v_{2s} \in \mathcal{V}$ are chosen so that there exist some integers k and ℓ with $1 \leqslant k \leqslant 2s$ and $1 \leqslant \ell \leqslant h$ so that

$$v_k - \alpha_\ell \neq v_i - \alpha_j$$

for all $(i, j) \neq (k, \ell)$ then for $\beta = \alpha_\ell - v_k$ we have

$$\operatorname{ord}_\beta \prod_{i=1}^{s} f\left(X + v_i\right) / f\left(X + v_{s+i}\right) \equiv u_\ell \not\equiv 0 \pmod n$$

and thus, the above rational function is not an n-th power.

Let E be the number of $\mathbf{v} = (v_1, \ldots, v_{2s}) \in \mathcal{V}^{2s}$ for which $P_{\mathbf{v},f}(X) \notin \mathcal{Q}_{2ds,n}$. Then for each choice of $1 \leqslant i \leqslant 2s$ there is some index $k \neq i$, $1 \leqslant k \leqslant 2s$, such that $v_i - v_k$ belongs to the difference set of the set $\{\alpha_1, \ldots, \alpha_h\}$ and thus can take at most $h(h-1) + 1 \leqslant d^2$ values. In particular, the components of $\mathbf{v}$ can be partitioned into at most s groups such that differences of elements within each group belong to the above difference set. This immediately implies that $E \ll V^s$ and concludes the proof. $\square$

We use Lemma 4.1 to control sums of multiplicative characters. To control sums of additive characters we need its appropriate analogue for linear combinations instead of products as in (4.1). Namely, given a vector $\mathbf{v} = (v_1, \ldots, v_{2s}) \in \overline{\mathbb{F}_p}^{2s}$ and a rational function $f \in \overline{\mathbb{F}_p}(X)$, we set

$$L_{\mathbf{v},f}(X) = \sum_{i=1}^{s} \left(f\left(X + v_i\right) - f\left(X + v_{s+i}\right)\right). \tag{4.2}$$

Definition 4.2. *We define the set $\mathcal{E}$ of exceptional rational functions as the set of rational functions $f(X) \in \mathbb{F}_{p^r}(X)$ such that there exists $\alpha, \beta \in \overline{\mathbb{F}_p}$ and $h(X) \in \overline{\mathbb{F}_p}(X)$ so that $f(X) = \alpha(h(X)^p - h(X)) + \beta X$.*

Then we have the following additive analogue of Lemma 4.1.

Lemma 4.3. *Let $f(X) \in \mathcal{R}_d$ for some integers $d \geqslant 1$. For any set $\mathcal{V} \subseteq \overline{\mathbb{F}_p}$ of cardinality V, for each integer $s \geqslant 1$ we have*

$$\#\left\{\mathbf{v} = (v_1, \ldots, v_{2s}) \in \mathcal{V}^{2s} :\ L_{\mathbf{v},f}(X) \in \mathcal{E}\right\} \ll V^s.$$

Proof. Clearly, all functions from $\mathcal{E}$ have a pole of order that is a multiple of p. It is also clear that if $f_1, \ldots, f_n \in \overline{\mathbb{F}_p}(X)$ are such that f_1 has a pole at $\alpha \in \overline{\mathbb{F}_p}$ of order $u \geqslant 1$ and $f_2, \ldots, f_n$ have no poles at w then $f_1 + \ldots + f_n$ has a pole at α of the same order u.

This implies that if $L_{\mathbf{v},f}(X) \in \mathcal{E}$ then for each choice of $1 \leqslant i \leqslant 2s$ there is some index $k \neq i$, $1 \leqslant k \leqslant 2s$, such that

$$v_i - v_k \in \{\alpha - \gamma :\ \gamma \text{ is a pole of } f\}.$$

Indeed, otherwise, that is, for other choices of $(v_1, \ldots, v_{2s}) \in \mathcal{V}^{2s}$, if α is a pole $f \in \overline{\mathbb{F}_p}(X)$ of order $\operatorname{ord}_\alpha f \not\equiv 0 \pmod p$, then $f(X + v_i)$ has a pole $\beta = \alpha - v_i$ of the same order and which is not a pole of any other function involved in $L_{\mathbf{v},f}$. Hence

$$\operatorname{ord}_\beta L_{\mathbf{v},f} = \operatorname{ord}_\alpha f \not\equiv 0 \pmod p,$$

and therefore $L_{\mathbf{v},f} \notin \mathcal{E}$. We see that the number of such choices of $(v_1, \ldots, v_{2s}) \in \mathcal{V}^{2s}$ is at most

$$2^s \binom{2s}{s} d^s V^s \ll V^s$$

and the result now follows. $\square$

5. Character sums over linear subspaces

We need is [4, Lemma 3.2] which follows instantly from the Weil bound for mixed character sums with rational functions due to Castro and Moreno [2] (see also more general results of Fu and Wan [3, Theorem 5.6]) and the orthogonality of additive characters.

Lemma 5.1. *Let χ and ψ be a multiplicative and additive character, respectively, and let $g_1(X), g_2(X) \in \mathbb{F}_{q^r}(X)$ be rational functions of degrees at most d. Assume that at least one of the following conditions holds*

(i) *χ is nonprincipal of order e and $g_1(X) \in \mathcal{Q}_{d,e}$,*
(ii) *ψ is nonprincipal and $g_2(X) \notin \mathcal{E}$.*

Then for any affine subspace $\mathcal{L} \subseteq \mathbb{F}_{q^r}$ we have

$$\sum_{\lambda \in \mathcal{L}} \chi(g_1(\lambda))\psi(g_2(\lambda)) \ll q^{r/2}.$$

The following result is our main technical tool.

Lemma 5.2. *Let χ and ψ be a multiplicative and additive character, respectively, and let $g_1(X), g_2(X) \in \mathbb{F}_{q^r}(X)$ be rational functions of degrees at most d. Assume that at least one of the following conditions holds*

(i) *χ is nonprincipal of order e and $g_1(X) \in \mathcal{Q}_{d,e}$,*
(ii) *ψ is nonprincipal and $g_2(X) \notin \mathcal{E}$.*

Then for a linear space $\mathcal{L} \subseteq \mathbb{F}_{p^r}$ of dimension t and arbitrary set $\mathcal{U} \subseteq \mathcal{L}$ and $\mathcal{V} \subseteq \mathbb{F}_{p^r}$ of cardinalities U and V, respectively, for each fixed integer $s \geqslant 1$ we have

$$\sum_{u\in\mathcal{U}} \left|\sum_{v\in\mathcal{V}} \chi(g_1(\lambda+v))\psi(g_2(\lambda+v))\right| \ll U^{1-1/(2s)} \left(q^{t/(2s)}V^{1/2} + q^{r/(4s)}V\right).$$

Proof. Let

$$S = \sum_{\lambda\in\mathcal{U}} \left|\sum_{v\in\mathcal{V}} \chi(g_1(u+v))\psi(g_2(u+v))\right|.$$

Applying the Hölder inequality, we derive

$$\begin{aligned} S^{2s} &\leqslant U^{2s-1} \sum_{\lambda\in\mathcal{U}} \left|\sum_{v\in\mathcal{V}} \chi(g_1(u+v))\psi(g_2(u+v))\right|^{2s} \\ &\leqslant U^{2s-1} \sum_{\lambda\in\mathcal{L}} \left|\sum_{v\in\mathcal{V}} \chi(g_1(\lambda+v))\psi(g_2(\lambda+v))\right|^{2s} \\ &= U^{2s-1} \sum_{\lambda\in\mathcal{L}} \sum_{\mathbf{v}=(v_1,\ldots,v_{2s})\in\mathcal{V}^{2s}} \chi\left(P_{\mathbf{v},f}(\lambda)\right)\psi\left(L_{\mathbf{v},f}(\lambda)\right) \\ &= U^{2s-1} \sum_{\mathbf{v}=(v_1,\ldots,v_{2s})\in\mathcal{V}^{2s}} \sum_{\lambda\in\mathcal{L}} \chi\left(P_{\mathbf{v},f}(\lambda)\right)\psi\left(L_{\mathbf{v},f}(\lambda)\right), \end{aligned}$$

where $P_{\mathbf{v},f}(X)$ and $L_{\mathbf{v},f}(X)$ are defined by (4.1) and (4.2), respectively.

We now see that if at least one of the above conditions (i) or (ii) holds that by either Lemma 4.1 or Lemma 4.3 we can apply Lemma 5.1 to the inner sum over the linear space $\mathcal{L}$ for all but $O(V^s)$ vectors $\mathbf{v} \in \mathcal{V}$, for which we estimate the inner sum trivially as q^t. Hence

$$S^{2s} \ll U^{2s-1}\left(q^tV^s + q^{r/2}V^{2s}\right),$$

and the result follows. □

6. Proof of Theorems 3.2 and 3.5

We recall the definition (1.1) of the set $\mathcal{S}_r(\mathcal{A})$, and some real positive parameter $\tau \in [0,1]$ and set $t = \lfloor \tau r \rfloor$, to be chosen later, we define the sets

$$\begin{aligned}
\mathcal{U} &= \{a_1\vartheta_1 + \ldots + a_t\vartheta_t : \ a_1, \ldots, a_t \in \mathcal{A}\}, \\
\mathcal{L} &= \{a_1\vartheta_1 + \ldots + a_t\vartheta_t : \ a_1, \ldots, a_t \in \mathbb{F}_q\}, \\
\mathcal{V} &= \{a_{t+1}\vartheta_{t+1} + \ldots + a_r\vartheta_r : \ a_{t+1}, \ldots, a_r \in \mathcal{A}\},
\end{aligned}$$

of cardinalities

$$U = q^{\rho t} \ll q^{\rho\tau r}, \qquad L = q^t \ll q^{\tau r}, \qquad V = q^{\rho(r-t)} \ll q^{\rho(1-\tau)r},$$

respectively. We can now write

$$\begin{aligned}
S_r(\mathcal{A}; \chi, \psi; f_1, f_2) &= \sum_{\omega \in \mathcal{S}_r(\mathcal{A})} \chi(f_1(\omega))\, \psi(f_2(\omega)) \\
&= \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \chi(f_1(u+v))\, \psi(f_2(u+v)).
\end{aligned}$$

Thus

$$|S_r(\mathcal{A}; \chi, \psi; f_1, f_2)| \leqslant \sum_{u \in \mathcal{U}} \left| \sum_{v \in \mathcal{V}} \chi(f_1(u+v))\, \psi(f_2(u+v)) \right|.$$

Under the conditions of Theorem 3.2, by Lemma 5.2 and the above cardinality estimates we have

$$\begin{aligned}
S_r(\mathcal{A}; \chi, \psi; f_1, f_2) &\ll U^{1-1/(2s)} \left(q^{t/(2s)} V^{1/2} + q^{r/(4s)} V \right) \\
&\ll q^{\rho\tau r(1-1/(2s)) + \tau r/(2s) + \rho(1-\tau)r/2} + q^{\rho\tau r(1-1/(2s)) + r/(4s) + \rho(1-\tau)r} \\
&= q^{r(\tau\rho(1-1/(2s)) + \tau(1/(2s) - \rho/2) + \rho/2)} + q^{r(\tau\rho(1-1/(2s)) + \rho(1-\tau) + 1/(4s))}.
\end{aligned}$$

Hence we have

$$S_r(\mathcal{A}; \chi, \psi; f_1, f_2) \ll q^{r\Delta_{s,\rho}(\tau)} \tag{6.1}$$

where

$$\Delta_{s,\rho}(\tau) = \left(1 - \frac{1}{2s}\right)\rho\tau + \max\left\{ \frac{\tau}{2s} + \frac{\rho(1-\tau)}{2}, \frac{1}{4s} + \rho(1-\tau) \right\}.$$

To minimise $\Delta_s(\tau)$ we choose

$$\tau_0 = \frac{2s\rho + 1}{2(s\rho + 1)}$$

to equalise the terms inside of the above maximum and compute

$$\begin{aligned}\Delta_{s,\rho}(\tau_0) &= \left(1 - \frac{1}{2s}\right)\rho\tau_0 + \frac{1}{4s} + \rho(1 - \tau_0) \\ &= \rho + \frac{1}{4s} - \frac{1}{2s}\rho\tau_0 = \rho - \frac{1}{4s}(2\rho\tau_0 - 1) = \rho - \kappa_s(\rho),\end{aligned}$$

where $\kappa_s(\rho)$ is given by (3.1), which together with (6.1) concludes the proof of Theorem 3.2.

To prove Theorem 3.5, we note that for $s = 1$, the above argument still applies for $f_2 \in \mathcal{P}_d$, and the result follows.

Data availability

No data was used for the research described in the article.

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