Rauzy fractals of random substitutions

Gohlke, Philipp; Mitchell, Andrew; Rust, Dan; Samuel, Tony

DOI:
10.48550/arXiv.2401.06732

RAUZY FRACTALS OF RANDOM SUBSTITUTIONS

P. GOHLKE, A. MITCHELL, D. RUST, AND T. SAMUEL

Abstract. We develop a theory of Rauzy fractals for random substitutions, which are a generalisation of deterministic substitutions where the substituted image of a letter is determined by a Markov process. We show that a Rauzy fractal can be associated with a given random substitution in a canonical manner, under natural assumptions on the random substitution. Further, we show the existence of a natural measure supported on the Rauzy fractal, which we call the Rauzy measure, that captures geometric and dynamical information. We provide several different constructions for the Rauzy fractal and Rauzy measure, which we show coincide, and ascertain various analytic, dynamical and geometric properties. While the Rauzy fractal is independent of the choice of (non-degenerate) probabilities assigned to a given random substitution, the Rauzy measure captures the explicit choice of probabilities. Moreover, Rauzy measures vary continuously with the choice of probabilities, thus provide a natural means of interpolating between Rauzy fractals of deterministic substitutions.

1. Introduction

Rauzy fractals are geometric objects that can be associated with sequences arising from substitutions. They first appeared in 1982 in the work of Rauzy [33], who constructed a domain exchange of a compact subset of $\mathbb{R}^2$ that reflects the action of the *tribonacci substitution*. Questions regarding diffraction spectra and dynamical properties of substitution sequences can often be reframed in terms of the geometry and topology of the associated Rauzy fractal. Perhaps most notably, the Pisot substitution conjecture can be reformulated in terms of tiling properties of Rauzy fractals—see for instance [3, 38].

1.1. **Substitutions.** Sequences arising from (deterministic) substitutions are the prototypical examples of mathematical quasicrysals. A substitution is a rule that replaces each symbol from a finite alphabet with a concatenation of symbols from the same alphabet. For example, the *tribonacci substitution* θ_t is defined over the three-letter alphabet $\mathcal{A} = \{a, b, c\}$ by the rule

$$\theta_t \colon \begin{cases} a \mapsto ab, \\ b \mapsto ac, \\ c \mapsto a, \end{cases} \tag{1.1}$$

and the *twisted tribonacci substitution* $\tilde{\theta}_t$ is defined over the same alphabet by the rule

$$\tilde{\theta}_t \colon \begin{cases} a \mapsto ba, \\ b \mapsto ac, \\ c \mapsto a. \end{cases} \tag{1.2}$$

Note that the only difference between the tribonacci substitution and the twisted version is that the order of letters in the image of a has been reversed. The action of a substitution extends naturally to finite words and infinite sequences, by applying the substitution to each letter in turn and concatenating the result in the order prescribed by the initial word (or sequence).

To a given substitution, a subshift can be associated in a natural way. Questions concerning characterisation of diffraction and dynamical spectra, order versus disorder, and the topological and dynamical features of substitution subshifts are central questions in the field of aperiodic order, which have been extensively investigated; see [2, 6, 38, 39] and the references therein.

To a given substitution over an alphabet $\mathcal{A} = \{a_1, \ldots, a_N\}$, we associate an $N \times N$-matrix M_θ, called the *substitution matrix of* θ, defined by $(M_\theta)_{i,j} = |\theta(a_j)|_{a_i}$. The matrix M_θ encodes how the number

(P. Gohlke) Centre for Mathematical Sciences, Lund University, 221 00 Lund, Box 118, Sweden
(A. Mitchell & T. Samuel) School of Mathematics, University of Birmingham, Edgbaston, B15 2TT, UK
(D. Rust) School of Mathematics and Statistics, The Open University, Milton Keynes, MK7 6AA, UK
2020 *Mathematics Subject Classification.* 52C23, 28A80, 37A50, 37B10.
Key words and phrases. Aperiodic sequence; random substitution; Rauzy fractal; graph directed iterated function systems.

of occurrences of a given letter in a word changes under the substitution action. If M_θ is a primitive matrix, then we say that the substitution θ is *primitive*. In this case, the Perron–Frobenius theorem gives that M_θ has a unique largest (real) eigenvalue $\lambda > 1$ and corresponding left and right eigenvectors with positive entries, denoted by $\mathbf{L}$ and $\mathbf{R}$ respectively. We take $\mathbf{L}$ and $\mathbf{R}$ to be normalised such that the entries of $\mathbf{R}$ sum to 1 and $\mathbf{L} \cdot \mathbf{R} = 1$, and refer to the triple $(\lambda, \mathbf{L}, \mathbf{R})$ as the *Perron-–Frobenius data* of θ.

If λ is a Pisot number, that is, $\lambda > 1$ and all of its Galois conjugates have modulus strictly less than 1, then we call θ a *Pisot* substitution. If, in addition, the characteristic polynomial of M_θ is irreducible over $\mathbb{Z}$, then we say θ is *irreducible Pisot*. If $|\det(M_\theta)| = 1$, then we say that θ is *unimodular*.

Observe that both the tribonacci and twisted tribonacci substitutions θ_t and $\tilde{\theta}_t$ defined in (1.1) and (1.2) have the same substitution matrix, given by

$$M = M_{\theta_t} = M_{\tilde{\theta}_t} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which is primitive since all entries of M^3 are positive. The Perron–Frobenius eigenvalue of M is the tribonacci constant $t \approx 1.83929$; in particular, t is the unique real number satisfying $t^3 = t^2 + t + 1$. Since t is a degree-three Pisot number and the matrix M has determinant one, it follows that θ_t and $\tilde{\theta}_t$ are both unimodular, irreducible Pisot substitutions.

1.2. **Rauzy fractals of substitutions.** To an irreducible Pisot substitution θ, one can associate a compact set $\mathcal{R}_\theta$ called a *Rauzy fractal*. Many questions relating to Pisot substitutions can be reformulated in terms of the geometry of the associated Rauzy fractal.

Let $(\lambda, \mathbf{L}, \mathbf{R})$ be the Perron–Frobenius data of an irreducible Pisot substitution θ. By irreducibility, the entries of $\mathbf{L}$ and $\mathbf{R}$ are rational functions in λ. We let $\lambda_2, \dots, \lambda_N$ denote the other eigenvalues of M_θ. Replacing λ with λ_i, for $i \in \{2, \dots, N\}$, in $\mathbf{L}$ and $\mathbf{R}$ gives eigenvectors $\mathbf{L}_i$ and $\mathbf{R}_i$ corresponding to the eigenvalue λ_i, with the same normalisation properties as $\mathbf{L}$ and $\mathbf{R}$.

Let $\mathbb{H}$ denote the contracting hyperplane of M_θ, namely, the $(d-1)$-dimensional subspace of $\mathbb{R}^d$ spanned by the right eigenvectors corresponding to the non-dominant eigenvalues. Let h_θ denote the action of M_θ restricted to $\mathbb{H}$. Since θ is irreducible Pisot, every eigenvalue of M_θ, other than the Perron–Frobenius eigenvalue, has absolute value strictly less than 1. It will sometimes be necessary to define a metric on the hyperplane $\mathbb{H}$. For our purposes, it will be convenient to choose a metric for which the action of h_θ is a contraction. Such a metric always exists as a consequence of the matrix M_θ being diagonalisable, which follows from the fact that θ is irreducible Pisot.

Every primitive substitution has a power that admits a substitution-fixed point, that is, a right-infinite sequence x such that $\theta^k(x) = x$ for some $k \geqslant 1$; see [6, 32] for further details. If θ is Pisot, then this sequence has the additional property of being *C-balanced* [1], namely, the abelianisation of any finite subword of x lies within a uniformly bounded distance of some fixed vector depending on n. In particular, this vector is n-times the right Perron–Frobenius eigenvector of the substitution matrix.

If x denotes a substitution-fixed point of a Pisot substitution θ, then the *broken line* or *staircase* $\mathcal{S}(x)$ associated with the sequence x is the subset of $\mathbb{Z}^d$ consisting of the abelianisation vectors of the finite words obtained by truncating x after n letters, for all positive integers n. Here, by abelianisation vector of a given finite word, we mean the vector whose entries consists of the number of occurrences of each letter in the word. For instance, the abelianisation vector of the word *abacaba* is $(4, 2, 1)^T$.

Let π denote the projection along $\mathbf{R}$ onto $\mathbb{H}$. Since x is C-balanced, the points in the staircase of x lie a bounded distance from (a multiple of) the Perron–Frobenius eigenvector $\mathbf{R}$. Thus, it follows that the set $\mathcal{R}^*(x) = \pi(\mathcal{S}(x))$ is bounded. The *Rauzy fractal* of θ is then defined by $\mathcal{R}(x) = \overline{\mathcal{R}^*(x)}$. Every substitution-fixed point x gives rise to the same set $\mathcal{R}(x)$—see [38] for more details. As such, we write $\mathcal{R}_\theta$ for this common set. The Rauzy fractals of the tribonacci and twisted tribonacci substitutions are plotted in Figure 1, together with a visualisation of the construction. In each of these images, the Rauzy fractals have been decomposed into measure-disjoint subtiles corresponding to each letter.

The key topological properties of Rauzy fractals associated with substitutions are stated in the following.

Proposition 1.1 ([38, Theorem 2.6])**.** *Let θ be a unimodular irreducible Pisot substitution over a d-letter alphabet, where $d \geq 2$. The Rauzy fractal $\mathcal{R}_\theta$ is a compact subset of $\mathbb{R}^{d-1}$ with non-empty interior.*

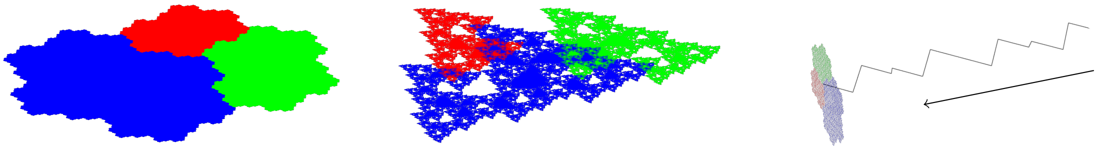

FIGURE 1. Rauzy fractals for the tribonacci (left) and twisted tribonacci (center) substitutions. Staircase and projection (right).

Moreover, each subtile is the closure of its interior and contains an open ball, thus has full Hausdorff dimension.

The Rauzy fractal associated with a substitution over a d-letter alphabet can be decomposed in a natural way into d regions called *subtiles*, which are measure-disjoint. These subtiles are related via a graph-directed iterated function system that arises naturally from the substitution action. For the tribonacci substitution, the subtiles $\mathcal{R}_a$, $\mathcal{R}_b$ and $\mathcal{R}_c$ are the unique non-empty compact sets satisfying

$$\mathcal{R}_a = h(\mathcal{R}_a) \cup h(\mathcal{R}_b) \cup h(\mathcal{R}_c), \quad \mathcal{R}_b = h(\mathcal{R}_a) + \mathbf{v}, \quad \mathcal{R}_c = h(\mathcal{R}_b) + \mathbf{v},$$

where h denotes the action of the substitution matrix on the contracting plane and $\mathbf{v} \in \mathbb{H}$. In Figure 1, the subtiles are highlighted for the Rauzy fractals of the tribonacci and twisted tribonacci substitutions, where the blue subtile corresponds to the letter a, the green subtile corresponds to the letter b and the red subtile corresponds to the letter c.

1.3. **Random substitutions.** Random substitutions are a generalisation of deterministic substitutions where the substituted image of a letter is chosen from a fixed finite set with respect to a probability distribution. For example, given $p \in (0,1)$, a random analogue of the tribonacci substitution can be defined by

$$\vartheta_{\mathbf{P}} \colon \begin{cases} a \mapsto \begin{cases} ab & \text{with probability } p, \\ ba & \text{with probability } 1-p, \end{cases} \\ b \mapsto ac \quad \text{with probability } 1, \\ c \mapsto a \quad \text{with probability } 1. \end{cases}$$

The action of a random substitution extends naturally to finite words, by applying the random substitution *independently* to each letter in turn and concatenating the result in the order prescribed by the initial word. Similarly to the deterministic setting, a subshift can be associated with a given random substitution in a natural way. Moreover, a substitution matrix can be associated in a similar manner and the notion of primitivity extends naturally to random substitutions – we give the precise definitions in Section 2. However, in stark contrast to deterministic substitutions, subshifts associated with random substitutions typically have positive topological entropy. In fact, in the primitive setting, a random substitution subshift has zero topological entropy if and only if it is the subshift of a deterministic substitution [28]. Despite the presence of positive entropy, the corresponding diffraction measure can still admit a non-trivial pure point component [8], indicating long-range correlations. To a given primitive random substitution, an ergodic measure can be associated in a natural way, which we call the *frequency measure* corresponding to the random substitution. While the subshift of a random substitution is blind to the choice of (non-degenerate) probabilities, the frequency measure captures the explicit choice of probabilities.

The theory of random substitutions has developed significantly in recent years. For example, dynamical and diffraction spectra have been studied for several classes of examples [8, 15, 30]; a systematic approach to topological and measure theoretic entropy has been developed [16, 18]; and sufficient conditions under which a random substitution subshift is topologically mixing have been ascertained [13, 27].

In the present paper, we develop a theory of Rauzy fractals for random substitutions. Since subshifts of deterministic primitive substitutions are minimal, the Rauzy fractal obtained by projecting any sequence from the subshift is a translation of the Rauzy fractal associated with a substitution-fixed point.

However, subshifts of random substitutions are typically not minimal, and there is no direct analogue of a substitution-fixed point for random substitutions. Consequently, the Rauzy fractals obtained by projecting sequences from random substitution subshifts are no longer uniform (up to translation). Nevertheless, there is a maximal Rauzy fractal, that emerges from every transitive point of the subshift and is therefore typical for every fully supported ergodic measure.

Another difference to the deterministic setting is that the points in the projected staircase construction can no longer be expected to be uniformly distributed on the Rauzy fractal. Rather, they typically follow a specific distribution that is determined by the probability parameters. We refer to this distribution as the *Rauzy measure* and we show that the Rauzy measure is an almost-sure object with respect to several natural measures on the random substitution subshift. It can further be obtained by following a typical succession of randomly created inflation words, which makes it accessible to numerical approximation. A natural application for Rauzy measures is the explicit computation of diffraction measures, as has been demonstrated in [8] for the example of the random Fibonacci substitution.

1.4. **Outline and overview of main results.** We begin by presenting the preliminaries on random substitutions, in Section 2. Then, in Section 3, we develop a general theory of Rauzy fractals for C-balanced sequences, which we utilise in the construction of the Rauzy fractal associated with a random substitution. While we have developed this framework with applications to random substitutions in mind, we believe that the results will be of independent interest. These include (semi-)continuity properties with respect to the underlying sequence and a dynamical interpretation via generic factors. We also show that Rauzy measures satisfy the *Lebesgue covering property*. Namely, that repeating the Rauzy measure along an appropriately chosen lattice combine to a constant multiple of Lebesgue measure.

In Section 4, we return our attention to random substitutions. Utilising the framework of Rauzy fractals for C-balanced sequences developed in Section 3, we provide two methods for constructing a canonical Rauzy fractal associated with an irreducible Pisot random substitution, which we prove coincide. Using our construction, we can show that Rauzy fractals of random substitutions are the closure of their interior and contain an open ball, thus have full Hausdorff dimension, analogous to the deterministic setting. Moreover, the subtiles of the Rauzy fractal associated with a random substitution can be obtained as the attractors of a graph-directed iterated function system (GIFS). However, we highlight that in contrast to the deterministic setting, this GIFS may not satisfy the open set condition.

Section 5 concerns Rauzy measures associated with random substitutions. While the Rauzy fractal is blind to the choice of (non-degenerate) probabilities assigned to a random substitution, the Rauzy measure captures the explicit choice of probabilities. We show that this measure can be constructed both via the invariant distribution of the random substitution and via the Dirac masses associated with the projection of points in the staircase of a typical element of the subshift. Further, we show that similarly to Rauzy fractals themselves, Rauzy measures are self-similar objects with respect to an appropriately chosen GIFS. Harvesting this interpretation, we obtain that the Rauzy measure depends continuously on the probability parameters. As a consequence, the Rauzy measures of random substitutions provide a convenient tool to interpolate between two (or more) deterministic substitutions that share the same substitution matrix. Here, it is worth pointing out that every irreducible Pisot substitution (up to taking higher powers) shares its substitution matrix with a substitution that satisfies the Pisot substitution conjecture [9]. It follows from the results in Section 3 that Rauzy measures associated with random substitutions satisfy the Lebesgue covering property. Consequently, the Rauzy measure is absolutely continuous with respect to Lebesgue measure. As a byproduct of our proof, we demonstrate the absolute continuity of self-similar measures for a certain class of GIFS, which may be of independent interest.

Finally, in Section 6, we present several examples illustrating the main results.

2. Preliminaries

Throughout, we let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a given set B, we write $\#B$ for the cardinality of B and let $\mathcal{F}(B)$ be the set of non-empty finite subsets of B.

An alphabet $\mathcal{A} = \{a_1, \ldots, a_d\}$ is a finite set of symbols, which we call *letters*. A *word* with letters in $\mathcal{A}$ is a finite concatenation of letters in $\mathcal{A}$. We write $|u| = n$ for the *length* of a given word u and, for $m \in \mathbb{N}$, we let $\mathcal{A}^m$ denote the set of all words of length m with letters in $\mathcal{A}$. We let ε denote the empty word, which has length zero by convention. We write $\mathcal{A}^+ = \bigcup_{m \in \mathbb{N}} \mathcal{A}^m$ for the set of all non-empty finite words with letters in $\mathcal{A}$ and $\mathcal{A}^* = \mathcal{A}^+ \cup \{\varepsilon\}$. Further, we let $\mathcal{A}^{\mathbb{N}} = \{a_{i_1} a_{i_2} \cdots : a_{i_j} \in \mathcal{A} \text{ for all } j \in \mathbb{N}\}$ denote the set of all infinite sequences with elements in $\mathcal{A}$ and endow $\mathcal{A}^{\mathbb{N}}$ with the discrete product topology. With this topology, the space $\mathcal{A}^{\mathbb{N}}$ is compact and metrisable. We let S denote the (*left*) *shift map*, defined by $S(w)_i = w_{i+1}$ for $w \in \mathcal{A}^{\mathbb{N}}$, and call a closed subset X of $\mathcal{A}^{\mathbb{N}}$ a *subshift* if it is shift-invariant, namely $S(X) = X$.

If $i, j \in \mathbb{N}$, and $x = x_1 x_2 \cdots \in \mathcal{A}^{\mathbb{N}}$, then we let $x_{[i,j]} = x_i x_{i+1} \cdots x_j$ if $i \leqslant j$, and $x_{[i,j]} = \varepsilon$ if $j < i$. We use the same notation if $v \in \mathcal{A}^+$ and $1 \leqslant i \leqslant j \leqslant |v|$. For $u, v \in \mathcal{A}^+$ (or $v \in \mathcal{A}^{\mathbb{N}}$), we write $u \triangleleft v$ if u is a subword of v, namely if there exist $i, j \in \mathbb{N}$ with $i \leqslant j$ so that $u = v_{[i,j]}$. For $u, v \in \mathcal{A}^+$, we write $|v|_u$ for the number of (possibly overlapping) occurrences of u as a subword of v.

If $u = a_{i_1} \cdots a_{i_n}, v = a_{j_1} \cdots a_{j_m} \in \mathcal{A}^*$, for some $n, m \in \mathbb{N}_0$, we write uv for the concatenation of u and v, that is, $uv = a_{i_1} \cdots a_{i_n} a_{j_1} \cdots a_{j_m} \in \mathcal{A}^{n+m}$. The *abelianisation* of a word $u \in \mathcal{A}^*$ is the vector $\psi(u) \in \mathbb{N}_0^d$, defined by $\psi(u)_i = |u|_{a_i}$ for all $i \in \{1, \ldots, d\}$.

2.1. **Random substitutions.** We define a random substitution via the data that is required to determine its action on letters. In the second step we extend it to a random map on words.

Definition 2.1. Let $\mathcal{A} = \{a_1, \ldots, a_d\}$ be a finite alphabet. A random substitution $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ is a finite-set-valued function $\vartheta \colon \mathcal{A} \to \mathcal{F}(\mathcal{A}^+)$ together with a set of non-degenerate probability vectors

$$\mathbf{P} = \left\{ \mathbf{p}_i = (p_{i,1}, \ldots, p_{i,r_i}) : r_i = \#\vartheta(a_i),\, \mathbf{p}_i \in (0,1]^{r_i} \text{ and } \sum_{j=1}^{r_i} p_{i,j} = 1 \text{ for all } 1 \leqslant i \leqslant d \right\},$$

such that

$$\vartheta_{\mathbf{P}} \colon a_i \mapsto \begin{cases} s^{(i,1)} & \text{with probability } p_{i,1}, \\ \vdots & \qquad \vdots \\ s^{(i,r_i)} & \text{with probability } p_{i,r_i}, \end{cases}$$

for every $1 \leqslant i \leqslant d$, where $\vartheta(a_i) = \{s^{(i,j)}\}_{1 \leqslant j \leqslant r_i}$. We call each $s^{(i,j)}$ a *realisation* of $\vartheta_{\mathbf{P}}(a_i)$. A *marginal* of $\vartheta_{\mathbf{P}}$ is a deterministic substitution θ such that $\theta(a_i)$ is a realisation of $\vartheta_{\mathbf{P}}(a_i)$ for all $1 \leqslant i \leqslant d$.

Example 2.2 (Random tribonacci)**.** Let $p \in (0,1)$. The *random tribonacci substitution* $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ is the random substitution defined over the alphabet $\mathcal{A} = \{a, b, c\}$ by

$$\vartheta_{\mathbf{P}} \colon \begin{cases} a \mapsto \begin{cases} ab & \text{with probability } p, \\ ba & \text{with probability } 1-p, \end{cases} \\ b \mapsto ac \quad \text{with probability } 1, \\ c \mapsto a \quad \text{with probability } 1, \end{cases}$$

with defining data $r_a = 2$, $r_b = r_c = 1$, $s^{(a,1)} = ab$, $s^{(a,2)} = ba$, $s^{(b,1)} = ac$, $s^{(c,1)} = a$, $\mathbf{P} = \{\mathbf{p}_a = (p, 1-p), \mathbf{p}_b = (1), \mathbf{p}_c = (1)\}$, and corresponding set-valued substitution $\vartheta \colon a \mapsto \{ab, ba\}, b \mapsto \{ac\}, c \mapsto a$. It is a local mixture of the tribonacci and twisted tribonacci substitutions introduced in Section 1.1.

We now describe how a random substitution $\vartheta_{\mathbf{P}}$ determines a (countable state) Markov matrix Q, indexed by $\mathcal{A}^+ \times \mathcal{A}^+$. We view the entry $Q_{u,v}$ of Q as the probability to map u to v by the random substitution. Formally, $Q_{a_i, s^{(i,j)}} = p_{i,j}$ for $j \in \{1, \ldots, r_i\}$ and $Q_{a_i, v} = 0$ if $v \notin \vartheta(a_i)$. We extend the action of $\vartheta_{\mathbf{P}}$ to finite words by *independently* mapping each letter to one of its realisations. Specifically, given $n \in \mathbb{N}$, $u = a_{i_1} \cdots a_{i_n} \in \mathcal{A}^n$ and $v \in \mathcal{A}^+$, we let

$$\mathcal{D}_n(v) = \{(v^{(1)}, \ldots, v^{(n)}) \in (\mathcal{A}^+)^n : v^{(1)} \cdots v^{(n)} = v\}$$

be the set of all decompositions of v into n individual words and let

$$Q_{u,v} = \sum_{(v^{(1)}, \ldots, v^{(n)}) \in \mathcal{D}_n(v)} \prod_{j=1}^{n} Q_{a_{i_j}, v^{(j)}}.$$

Namely, $\vartheta_{\mathbf{P}}(u) = v$ with probability $Q_{u,v}$.

For $u \in \mathcal{A}^+$, let $(\vartheta_{\mathbf{P}}^n(u))_{n \in \mathbb{N}}$ be a stationary Markov chain on a given probability space $(\Omega_u, \mathcal{F}_u, \mathbb{P}_u)$, with transition matrix Q, that is

$$\mathbb{P}_u[\vartheta_{\mathbf{P}}^{n+1}(u) = w \mid \vartheta_{\mathbf{P}}^n(u) = v] = \mathbb{P}_v[\vartheta_{\mathbf{P}}(v) = w] = Q_{v,w},$$

for all $v, w \in \mathcal{A}^+$ and $n \in \mathbb{N}$. In particular, $\mathbb{P}_u[\vartheta_{\mathbf{P}}^n(u) = v] = (Q^n)_{u,v}$, for all $u, v \in \mathcal{A}^+$, and $n \in \mathbb{N}$. Likewise, in case that u is a random word (that is, a word-valued random variable), we let $(\vartheta_{\mathbf{P}}^n(u))_{n\in\mathbb{N}}$ be a stationary Markov chain, induced by the transition matrix Q as outlined above. We typically write $\mathbb{P}$ for $\mathbb{P}_u$ if the initial (random) word is understood. In this case, we also write $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$. As above, we say that v is a *realisation* of $\vartheta_{\mathbf{P}}^n(u)$ if $(Q^n)_{u,v} > 0$ and set

$$\vartheta^n(u) = \{v \in \mathcal{A}^+ : (Q^n)_{u,v} > 0\}$$

to be the set of all realisations of $\vartheta_{\mathbf{P}}^n(u)$. Conversely, $\vartheta_{\mathbf{P}}^n(u)$ may be regarded as the set $\vartheta^n(u)$, equipped with the additional structure of a probability vector. If $u = a \in \mathcal{A}$ is a letter, then we call a word $v \in \vartheta^k(a)$ a *(level-k) inflation word*. The approach of defining a random substitution in terms of a Markov chain can be traced back to work of Peyrière [31] and was pursued further by Denker and Koslicki [23, 24].

Given a random substitution $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ over an alphabet $\mathcal{A} = \{a_1, \ldots, a_d\}$, with cardinality $d \in \mathbb{N}$, we define the *substitution matrix* $M = M_{\vartheta_{\mathbf{P}}} \in \mathbb{R}^{d\times d}$ of $\vartheta_{\mathbf{P}}$ by

$$M_{i,j} = \mathbb{E}\left[|\vartheta_{\mathbf{P}}(a_j)|_{a_i}\right] = \sum_{k=1}^{r_j} p_{j,k} |s^{(j,k)}|_{a_i}.$$

Since M has only non-negative entries, its spectral radius is also a real eigenvalue of maximal modulus, denoted by λ. By construction, $\lambda \geqslant 1$, where $\lambda = 1$ occurs if and only if M is column-stochastic. This corresponds to the trivial case of a non-expanding random substitution, which we discard in the following. If M is primitive (that is, if there exists a $k \in \mathbb{N}_0$ so that all the entries of M^k are positive), Perron–Frobenius theory implies that λ is a simple eigenvalue and that the corresponding left and right eigenvectors $\mathbf{L} = (L_1, \ldots, L_d)^T$ and $\mathbf{R} = (R_1, \ldots, R_d)^T$ may be chosen to have strictly positive entries. We normalise these eigenvectors such that $\|\mathbf{R}\|_1 = 1 = \mathbf{L}^T\mathbf{R}$. In this situation, we call λ the *Perron–Frobenius eigenvalue* of $\vartheta_{\mathbf{P}}$, and $\mathbf{L}$ and $\mathbf{R}$ the *left* and *right Perron–Frobenius eigenvectors* of $\vartheta_{\mathbf{P}}$, respectively.

Definition 2.3. We say that $\vartheta_{\mathbf{P}}$ is *primitive* if $M = M_{\vartheta_{\mathbf{P}}}$ is primitive and its Perron–Frobenius eigenvalue satisfies $\lambda > 1$.

We emphasise that for a random substitution $\vartheta_{\mathbf{P}}$, being primitive is in fact independent of the (non-degenerate) data $\mathbf{P}$. In this sense, primitivity is a property of ϑ rather than $\vartheta_{\mathbf{P}}$.

Another standard assumption in the study of random substitutions is *compatibility*, see for example [13, 14, 16, 27, 34]. In the following, recall that we denote the abelianisation of a finite word u by $\psi(u)$.

Definition 2.4. We say that a random substitution $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ is *compatible* if for all $a \in \mathcal{A}$ and $u, v \in \vartheta(a)$, we have $\psi(u) = \psi(v)$.

Observe that compatibility is independent of the choice of probabilities, and that a random substitution $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ is compatible if and only if for all $u \in \mathcal{A}^+$, we have that $|s|_a = |t|_a$ for all s and $t \in \vartheta(u)$, and $a \in \mathcal{A}$. We write $|\vartheta(u)|_a$ to denote this common value, and let $|\vartheta(u)|$ denote the common length of words in $\vartheta(u)$. In which case, letting $M = M_{\vartheta_{\mathbf{P}}}$ denote the substitution matrix of $\vartheta_{\mathbf{P}}$, we have that $M_{i,j} = |\vartheta(a_j)|_{a_i}$ for all $a_i, a_j \in \mathcal{A}$. Note that the random tribonacci substitution defined in Example 2.2 is compatible, since $\psi(ab) = \psi(ba) = (1, 1, 0)^T$. It is also primitive, since the cube of its substitution matrix is positive. Since the matrix of a compatible random substitution is independent of $\mathbf{P}$, we often drop the explicit dependence on $\mathbf{P}$ in the notion, and write M_ϑ for the matrix of $\vartheta_{\mathbf{P}}$.

For random substitutions which are both primitive and compatible, the notions of unimodular and (irreducible) Pisot extend naturally from the deterministic setting.

Definition 2.5. We say that a compatible primitive random substitution $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ is *Pisot* if the largest eigenvalue of the substitution matrix M_ϑ is a Pisot number. If, in addition, the characteristic polynomial of M_ϑ is irreducible over $\mathbb{Z}$, then we say that $\vartheta_{\mathbf{P}}$ is *irreducible Pisot*. If $|\det(M_\vartheta)| = 1$, then we say that $\vartheta_{\mathbf{P}}$ is *unimodular*.

Definition 2.6. Given a random substitution $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$, a word $u \in \mathcal{A}^+$ is called *(ϑ-) legal* if there exists an $a_i \in \mathcal{A}$ and $k \in \mathbb{N}$ such that u appears as a subword of some word in $\vartheta^k(a_i)$. We define the *language* of ϑ by $\mathcal{L}_\vartheta = \{u \in \mathcal{A}^+ : u \text{ is } \vartheta\text{-legal}\}$.

The *random substitution subshift* associated with $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ is the system (X_ϑ, S), where S is the usual shift map, and

$$X_\vartheta = \{w \in \mathcal{A}^{\mathbb{N}} : \text{every subword of } w \text{ is } \vartheta\text{-legal}\}.$$

If $\vartheta_{\mathbf{P}}$ is primitive and compatible, the corresponding sequence space X_ϑ is always non-empty [35]. The notation X_ϑ mirrors the fact that the subshift of a random substitution does not depend on the choice of $\mathbf{P}$. We endow X_ϑ with the subspace topology inherited from $\mathcal{A}^{\mathbb{N}}$, and since X_ϑ is defined in terms of a language, it is a compact S-invariant subspace of $\mathcal{A}^{\mathbb{N}}$. Hence, X_ϑ is a subshift. For $n \in \mathbb{N}$, we write $\mathcal{L}_\vartheta^n = \mathcal{L}_\vartheta \cap \mathcal{A}^n$ to denote the subset of $\mathcal{L}_\vartheta$ consisting of words of length n. We also note that, when $\vartheta_{\mathbf{P}}$ is primitive, $X_{\vartheta^k} = X_\vartheta$ for all $k \in \mathbb{N}$. For primitive random substitutions, the associated subshift is topologically transitive.

Proposition 2.7 ([35, Prop. 13])**.** Let $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ be a primitive random substitution. Then, the associated subshift X_ϑ is topologically transitive.

For compatible random substitutions, every element in the associated subshift has well-defined letter frequencies. Under the additional assumption of irreducible Pisot, it was shown in [27] that every element is in fact C-balanced.

Theorem 2.8 ([27, Theorem 33])**.** Let $\vartheta_{\mathbf{P}}$ be a compatible and irreducible Pisot random substitution. Then, there exists a $C \geq 1$ such that every element of X_ϑ is C-balanced.

The set-valued function ϑ naturally extends to X_ϑ, where for $w = w_1 w_2 \cdots \in X_\vartheta$ we let $\vartheta(w)$ denote the (infinite) set of sequences of the form $v = v_1 v_2 \cdots$, with $v_j \in \vartheta(w_j)$ for all $j \in \mathbb{N}$. By definition, it is easily verified that $\vartheta(X_\vartheta) \subseteq X_\vartheta$. Some properties of ϑ are reminiscent of continuous functions, although ϑ itself is *not* a function.

Lemma 2.9 ([18, Lemma 2.5])**.** If $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ is a random substitution and $X \subseteq \mathcal{A}^{\mathbb{N}}$ is compact, then $\vartheta(X)$ is compact.

2.2. **Frequency measures.** The choice of the probability parameters $\mathbf{P}$ induces a probabilistic structure on X_ϑ that is reflected by appropriate choices of probability measures. In analogy to the notion of a substitution-fixed point in the deterministic setting, we introduce the concept of an invariant measure.

Definition 2.10. A probability measure ν on X_ϑ is *invariant* under $\vartheta_{\mathbf{P}}$, if for all $w \in \mathcal{L}_\vartheta$,

$$\nu([w]) = \sum_{v \in \mathcal{L}_\vartheta^{|w|}} \nu([v]) \, \mathbb{P}[\vartheta_{\mathbf{P}}(v)_{[1,|w|]} = w].$$

In many cases, the probability $\mathbb{P}[\vartheta_{\mathbf{P}}(v)_{[1,|w|]} = w]$ depends only on a prefix of v. More precisely, if $m \in \mathbb{N}$ is large enough to guarantee $|\vartheta(v)| \geqslant |w|$ for all $v \in \mathcal{L}_\vartheta^m$, we get

$$\nu([w]) = \sum_{v \in \mathcal{L}_\vartheta^{m}} \nu([v]) \, \mathbb{P}[\vartheta_{\mathbf{P}}(v)_{[1,|w|]} = w]. \tag{2.1}$$

It is not difficult to verify that every primitive compatible random substitution $\vartheta_{\mathbf{P}}$ permits an invariant measure (possibly up to replacing $\vartheta_{\mathbf{P}}$ by a higher power); we refer to [17] for details. A $\vartheta_{\mathbf{P}}$-invariant measure ν is generally not shift-invariant. However, ν is intimately connected to a natural ergodic measure on (X_ϑ, S). Recall that a point $x \in X_\vartheta$ is *generic* for a measure ϱ if

$$\varrho = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^k x},$$

in the weak topology. The following was shown in [17, 19] in the context of bi-infinite sequence spaces; the proof carries over immediately to the one-sided setting.

Theorem 2.11. Let $\vartheta_{\mathbf{P}}$ be a primitive random substitution. There is an S-invariant, ergodic measure ϱ on X_ϑ such that for every $\vartheta_{\mathbf{P}}$-invariant measure ν, we have that ν-almost every $x \in X_\vartheta$ is generic for ϱ.

We call ϱ the *frequency measure* corresponding to $\vartheta_{\mathbf{P}}$ because the value $\varrho([w])$ is precisely the frequency of w in x for ν-almost every (and ϱ-almost every) $x \in X_\vartheta$. In fact, this frequency is also observed in large inflation words in the sense that the relative frequency of w in the random word $\vartheta_{\mathbf{P}}^n(a)$ converges $\mathbb{P}$-almost surely to $\varrho([w])$—see [19] for more details. Probabilistic aspects of the subshift X_ϑ are naturally studied in terms of its frequency measures, including measure theoretic entropy [18] and L^q-spectra [29].

3. Rauzy fractals of C-balanced sequences

A sequence x is called *C-balanced* if there exists a constant $C \geqslant 1$ such that for all $j, k, n \in \mathbb{N}$ and $a \in \mathcal{A}$, we have $\big||x_{[j,j+n-1]}|_a - |x_{[k,k+n-1]}|_a\big| < C$. We also extend the definition of C-balanced to subshifts as follows: a subshift X is C-balanced if every sequence $x \in X$ is C-balanced. This property is central to the construction of the Rauzy fractal associated with an irreducible Pisot substitution. More generally, the same procedure allows a Rauzy fractal to be defined for any C-balanced sequence and, further, for any topologically transitive C-balanced subshift. In this section, we outline this more general construction. We prove several topological and analytic properties that we will utilise in the construction of Rauzy fractals of random substitutions in Section 4, and which we believe are of interest in their own right.

We first provide an alternative characterisation of the C-balanced property, which will be useful for our purposes. A sequence $w \in \mathcal{A}^{\mathbb{N}}$ has *uniformly well-defined letter frequencies* if, for all $a_i \in \mathcal{A}$ and all sequences $(j_n)_n$ of positive integers, the limit

$$r_i = \lim_{n\to\infty} \frac{1}{n} \left|w_{[j_n, j_n+n-1]}\right|_{a_i} \tag{3.1}$$

exists and is independent of the sequence $(j_n)_n$. We call the vector $\mathbf{r} = \mathbf{r}(w)$ the *letter frequency vector of w*. Observe that $\mathbf{r}$ is a probability vector. The following characterisation of the C-balanced property was proved in [11], and motivates why a Rauzy fractal can be associated with any C-balanced sequence.

Lemma 3.1 ([11, Proposition 2.4])**.** A sequence $w \in \mathcal{A}^{\mathbb{N}}$ is C-balanced if and only if w has uniformly well-defined letter frequencies and there exists another constant B such that for any finite subword u of w and $a_i \in \mathcal{A}$, we have $||u|_{a_i} - |u|r_i| < B$, where r_i is the entry of the letter frequency vector corresponding to a_i.

For topologically transitive C-balanced subshifts, there exists a uniform letter frequency vector.

Proposition 3.2. If X is a topologically transitive C-balanced subshift, then there is a probability vector $\mathbf{r}$ such that for every $x \in X$, $\mathbf{r}$ is the letter frequency vector of x.

Proof. Since X is C-balanced, every element of X has uniformly well-defined letter frequencies by Lemma 3.1. Let $\mathbf{r}(x)$ denote the letter frequency vector of a given element $x \in X$. By transitivity, there exists an element w with dense-shift orbit in X. Thus, for every $x \in X$, there is a sequence $(n_k)_k$ of positive integers such that $S^{n_k}(w) \to x$ as $k \to \infty$. In particular, $\mathbf{r}(x) = \mathbf{r}(w)$. □

3.1. Rauzy fractals of C-balanced sequences.

Let w be a C-balanced sequence over a finite alphabet $\mathcal{A} = \{a_1, \dots, a_d\}$, with letter frequency vector $\mathbf{v}$, and let $\mathbb{H}$ be a $(d-1)$-dimensional hyperplane passing through the origin, such that every vector in $\mathbb{H}$ is linearly independent to $\mathbf{v}$. Let $\pi \colon \mathbb{R}^d \to \mathbb{R}^d$ denote the linear projection along $\mathbf{v}$ onto $\mathbb{H}$. In the following, we use the convention $w_{[1,0]} = \varepsilon$. For each $a \in \mathcal{A}$, let

$$\mathcal{S}_a(w) = \{\psi(w_{[1,n]}) : n \in \mathbb{N}_0 \text{ and } w_{n+1} = a\}.$$

Setting $\mathcal{R}_a^*(w) = \pi(S_a(w))$ and $\mathcal{R}_a = \overline{\mathcal{R}_a^*}$, we define the Rauzy fractal $\mathcal{R}(w)$ of w by

$$\mathcal{R}(w) = \bigcup_{a\in\mathcal{A}} \mathcal{R}_a(w).$$

Since w is C-balanced, $\mathcal{R}(w)$ is bounded, and since $\mathcal{R}(w)$ is a finite union of closed sets, it is compact. We use the same notation if w a finite word, with the observation that for finite words $\mathcal{R}(w)$ is a finite set and thus bounded.

The following describes how the action of the shift map on a C-balanced sequence corresponds to a translation in the Rauzy fractal.

Proposition 3.3. Let $w \in \mathcal{A}^{\mathbb{N}}$ be a C-balanced sequence. Then, for all $a \in \mathcal{A}$ and $k \geqslant 0$,

$$\mathcal{R}_a^*(S^k(w)) \subseteq \mathcal{R}_a^*(w) - \pi(\psi(w_{[1,k]})) \quad \text{and} \quad \mathcal{R}_a(S^k(w)) \subseteq \mathcal{R}_a(w) - \pi(\psi(w_{[1,k]})).$$

Proof. If $x \in \mathcal{R}_a^*(S^k(w))$, then there exists an $m \geqslant 0$ with $x = \pi(\psi(S^k(w)_{[1,m]}))$ and $S^k(w)_{m+1} = a$ (that is, $w_{m+k+1} = a$). From this and by linearity of the projection map π, we have

$$\pi(\psi(S^k(w)_{[1,m]})) = \pi(\psi(w_{[k+1,k+m]})) = \pi(\psi(w_{[1,k+m]})) - \pi(\psi(w_{[1,k]})); \tag{3.2}$$

hence, $x \in \mathcal{R}_a^*(w) - \pi(\psi(w_{[1,k]}))$. Thus, $\mathcal{R}_a^*(S^k(w)) \subseteq \mathcal{R}_a^*(w) - \pi(\psi(w_{[1,k]}))$, and taking closure gives $\mathcal{R}_a(S^k(w)) \subseteq \mathcal{R}_a(w) - \pi(\psi(w_{[1,k]}))$. □

3.2. **Analytic properties.** In this section we look at limiting behaviours, with respect to the Hausdorff metric, of Rauzy fractals of C-balanced sequences. The following characterisation of convergence in the Hausdorff metric will be most convenient for our purposes.

Definition 3.4. For a sequence $(A_n)_n$ of compact sets in a metric space (X, d), the *Kuratowski limit inferior* is defined by

$$\operatorname*{Li}_{n\to\infty} A_n = \left\{ x : \limsup_{n\to\infty} d(x, A_n) = 0 \right\},$$

where $d(x, A_n)$ is the distance between the point x and the set A_n. Analogously, the *Kuratowski limit superior* is defined by

$$\operatorname*{Ls}_{n\to\infty} A_n = \left\{ x : \liminf_{n\to\infty} d(x, A_n) = 0 \right\}.$$

If the Kuratowski limit inferior and superior agree, then the common set is called the *Kuratowski limit* of A_n and is denoted by $\operatorname*{Lim}_{n\to\infty} A_n$.

If A_n converges to A in the Hausdorff metric, then A is the Kuratowski limit of of $(A_n)_n$. Conversely, if for all but a finite number of $n \in \mathbb{N}$, the set A_n is compact, then Kuratowski convergence is equivalent to convergence in the Hausdorff metric [10]. Further, observe that

$$\operatorname*{Li}_{n\to\infty} A_n \subseteq \operatorname*{Ls}_{n\to\infty} A_n.$$

For more details on Kuratowski convergence and its implications, we refer the reader to [10].

It is possible for a sequence $(w^n)_{n\in\mathbb{N}}$ of C-balanced sequences to converge to a C-balanced sequence w, but for $\mathcal{R}(w^n)$ not to converge to $\mathcal{R}(w)$ in the Hausdorff metric. However, the following lower semi-continuity always holds.

Lemma 3.5. If $(w^n)_{n\in\mathbb{N}}$ is a sequence of C-balanced sequences converging to a C-balanced sequence w, then

$$\mathcal{R}_a(w) \subseteq \operatorname*{Li}_{n\to\infty} \mathcal{R}_a(w^n).$$

Proof. Let $v \in \mathcal{R}_a(w)$ and let $\varepsilon > 0$. There exists an $m = m(\varepsilon) \geqslant 0$ with $|\pi(\psi(w_{[1,m]})) - v| < \varepsilon$ and $w_{m+1} = a$. By assumption, there is an n_0 such that $w^n_{[1,m+1]} = w_{[1,m+1]}$ for all $n \geqslant n_0$, and therefore $\pi(\psi(w_{[1,m]})) \in \mathcal{R}_a(w^n)$. It follows that $|v - \mathcal{R}_a(w^n)| < \varepsilon$ for all $n \geqslant n_0$. Hence, $|v - \mathcal{R}_a(w^n)| \to 0$ as $n \to \infty$, so v lies in $\operatorname{Li}_{n\to\infty} \mathcal{R}_a(w^n)$. Since $v \in \mathcal{R}_a(w)$ was chosen arbitrarily, the assertion follows. □

It follows from the above and Proposition 3.3 that for every element in the orbit closure of a C-balanced sequence w, the associated Rauzy fractal is contained in a translate of the Rauzy fractal of w.

Lemma 3.6. Let w be a C-balanced sequence. For every x in the shift-orbit closure of w, there is a vector $t \in \mathcal{R}(w)$ with $\mathcal{R}_a(x) \subseteq \mathcal{R}_a(w) - t$, for all $a \in \mathcal{A}$.

Proof. By assumption, there exists a sequence $(n_k)_k$ with

$$\lim_{k\to\infty} S^{n_k}(w) = x.$$

Let $\mathcal{T}$ be the set of accumulation points of the sequence $\pi(\psi(w_{[1,n_k]}))$. Note that the set $\mathcal{T}$ is non-empty since the sequence is bounded. For $t \in \mathcal{T}$ there exists a subsequence $(m_k)_k$ such that $\pi(\psi(w_{[1,m_k]})) \to t$ and $S^{m_k}(w) \to x$ as $k \to \infty$. Hence, by Lemma 3.5 and Proposition 3.3, we obtain $\mathcal{R}_a(x) \subseteq \mathcal{R}_a(w) - t$. Since $0 \in \mathcal{R}(x)$, it follows that $t \in \mathcal{R}(w)$. In fact, since the above holds for every $t \in \mathcal{T}$, we have

$$\mathcal{R}_a(x) \subseteq \bigcap_{t\in\mathcal{T}} \mathcal{R}_a(w) - t.$$ □

Corollary 3.7. Let X be a subshift and suppose that $w, x \in X$ are C-balanced and have dense shift-orbit in X. Then, there exists a vector $t \in \mathcal{R}(w)$ such that $\mathcal{R}_a(x) = \mathcal{R}_a(w) - t$, for all $a \in \mathcal{A}$.

Proof. By Lemma 3.6, there exist vectors $s, t \in \mathcal{R}(w)$ with $\mathcal{R}_a(x) \subseteq \mathcal{R}_a(w) - t \subseteq \mathcal{R}_a(x) - s - t$, for all $a \in \mathcal{A}$. Since $\mathcal{R}_a(x)$ is bounded, this is only possible if all the subset relations are in fact equalities and $s = -t$. Thus, $\mathcal{R}_a(x) = \mathcal{R}_a(w) - t$ for all $a \in \mathcal{A}$. □

As a consequence of Lemma 3.5 and Lemma 3.6, if a sequence $(w^n)_n$ of C-balanced words converges to a sequence $w \in X$ with dense shift-orbit, $(\mathcal{R}(w^n))_n$ converges to $\mathcal{R}(w)$ in the Hausdorff metric.

Proposition 3.8. Let X be a C-balanced subshift. If $(w^n)_n$ is a sequence in X that converges to an element $w \in X$ with dense shift-orbit, then $\mathcal{R}_a(w^n) \to \mathcal{R}_a(w)$ in the Hausdorff metric as $n \to \infty$, for all $a \in \mathcal{A}$.

Proof. By Lemma 3.6, for every w^n there exists a vector t_n in $\mathcal{R}(w)$ such that $\mathcal{R}_a(w^n) \subseteq \mathcal{R}_a(w) - t_n$. Let $(n_k)_k$ be a subsequence such that t_{n_k} converges to some $t \in \mathcal{R}(w)$ as $k \to \infty$. This implies that

$$\lim_{k\to\infty} \mathcal{R}_a(w) - t_{n_k} = \mathcal{R}_a(w) - t$$

in the Hausdorff distance. Using Lemma 3.5, we obtain that

$$\mathcal{R}_a(w) \subseteq \operatorname*{Li}_{n\to\infty} \mathcal{R}_a(w^n) \subseteq \operatorname*{Li}_{k\to\infty} \mathcal{R}_a(w^{n_k}) \subseteq \operatorname*{Li}_{k\to\infty} \mathcal{R}_a(w) - t_{n_k} = \mathcal{R}_a(w) - t.$$

It follows that $t = 0$ and that each of the subset relations is in fact an equality. Since every subsequence of t_n converges to 0, we obtain that $\lim_{n\to\infty} t_n = 0$, therefore

$$\mathcal{R}_a(w) \subseteq \operatorname*{Li}_{n\to\infty} \mathcal{R}_a(w^n) \subseteq \operatorname*{Ls}_{n\to\infty} \mathcal{R}_a(w^n) \subseteq \operatorname*{Ls}_{n\to\infty} \mathcal{R}_a(w) - t_n = \mathcal{R}_a(w).$$

Hence,

$$\operatorname*{Li}_{n\to\infty} \mathcal{R}_a(w^n) = \operatorname*{Ls}_{n\to\infty} \mathcal{R}_a(w^n) = \mathcal{R}_a(w),$$

implying that the Kuratowski limit exists and is given by $\mathcal{R}_a(w)$. Since all the sets $\mathcal{R}_a(w^n)$ are contained in the bounded Minkowski difference $\mathcal{R}(w) - \mathcal{R}(w)$, the convergence in Hausdorff distance follows. $\square$

3.3. **Rauzy fractals as generic factors.** For the remainder of this section, we assume X to be a topologically transitive C-balanced subshift. Let us fix a sequence $w \in X$ with dense shift-orbit as a reference point. We can define a mapping $\phi\colon X' \to \mathcal{R}(w)$ on the set X' of points with dense shift-orbit by $\phi(x) = t$, where t is the unique vector such that $\mathcal{R}(x) = \mathcal{R}(w) - t$.

Proposition 3.9. The map $\phi\colon X' \to \mathcal{R}(w)$ is continuous.

Proof. Let $x \in X'$ and $x^n \in X'$ for all $n \in \mathbb{N}$ such that $x^n \to x$. By the definition of ϕ, we have that $\mathcal{R}(x) = \mathcal{R}(w) - \phi(x)$ and $\mathcal{R}(x^n) = \mathcal{R}(w) - \phi(x^n)$. Further, by Proposition 3.8,

$$\mathcal{R}(w) - \phi(x) = \mathcal{R}(x) = \lim_{n\to\infty} \mathcal{R}(x^n) = \lim_{n\to\infty} \mathcal{R}(w) - \phi(x^n)$$

in the Hausdorff distance. This implies that $\lim_{n\to\infty} \phi(x^n) = \phi(x)$. $\square$

This result is particularly useful if the set of transitive sequences X' is invariant under the shift map. This is the case if some (equivalently every) sequence $x \in X'$ is *recurrent*. That is, every subword of x appears in x infinitely often. We take this as a standing assumption for the remainder of this subsection.

Proposition 3.9 has a convenient dynamical interpretation, as it helps to construct an explicit *generic factor* of the subshift (X, S). Following [21], we say that a transitive dynamical system (Y, T) is a generic factor of (X, S) if there is a continuous map φ from the set of transitive points X' of X to the set of transitive points Y' of Y such that $\varphi \circ S = T \circ \varphi$ on X'. Generic factors are a convenient classification tool in cases when the maximal equicontinuous factor is trivial and the system is not uniquely ergodic, such that *a priori* there is not a unique choice for the Kronecker factor. For details on the concept of a *maximal equicontinuous generic factor* (MEGF) and some of its applications in the context of aperiodic order we refer the reader to [22].

In order to interpret ϕ as a factor map, we need to equip $\mathcal{R}(w)$ with an action that corresponds to the shift action on X'. It is readily verified from Corollary 3.7 and Proposition 3.3 that

$$\phi(Sx) = \phi(x) + \pi(\psi(x_1)), \tag{3.3}$$

for all $x \in X'$. For this translation to be independent of x, we wish to identify the vectors $\pi(\mathbf{e_i})$ for all $1 \leqslant i \leqslant d$. That is, we consider $(\mathbf{v}_i)_{i=2}^d$, with $\mathbf{v}_i = \pi(\mathbf{e_i} - \mathbf{e_1})$, and the lattice spanned by these vectors

$$\mathcal{J} = \left\{ \sum_{i=2}^{d} z_i \mathbf{v_i} : z_i \in \mathbb{Z} \text{ for } i \in \{1, 2, \dots, d\} \right\}, \tag{3.4}$$

where $(\mathbf{e}_i)_{i=1}^d$ denotes the standard basis of $\mathbb{R}^d$. With $\pi_{\mathcal{J}}$ the natural projection from $\mathbb{H}$ to the $d-1$ dimensional torus $\mathbb{H}/\mathcal{J}$, we obtain the following consequence of Proposition 3.9.

Corollary 3.10. Let X be the orbit closure of a recurrent, C-balanced sequence. The map $\pi_{\mathcal{J}} \circ \phi$ is a generic factor map from (X, S) to the equicontinuous dynamical system (G, T), where T is the torus rotation

$$T\colon \mathbb{H}/\mathcal{J} \to \mathbb{H}/\mathcal{J}, \quad x \mapsto x + \pi(\mathbf{e_1}) \mod \mathcal{J},$$

and $G = \pi_{\mathcal{J}}(\mathcal{R}(w))$ is the subgroup of $\mathbb{H}/\mathcal{J}$ generated by the rotation T.

Proof. The map $\varphi = \pi_{\mathcal{J}} \circ \phi$ is clearly continuous as it is a composition of continuous maps; compare Proposition 3.9. Projecting the relation in (3.3) to the torus yields

$$\varphi(Sx) = \varphi(x) + \pi(\mathbf{e}_1) \mod \mathcal{J} = T(\varphi(x)),$$

for all $x \in X'$. By construction, the set $\{\psi(x_{[1,n]})\}_{n\in\mathbb{N}}$ lies dense in $\mathcal{R}(x)$ and consequently the points

$$\phi(S^n x) = \phi(x) + \pi(\psi(x_{[1,n]})),$$

with $n \in \mathbb{N}$, lie dense in $\mathcal{R}(w) = \mathcal{R}(x) + \phi(x)$. This holds in particular for $\mathcal{R}^*(w) = \{\phi(S^n w)\}_{n\in\mathbb{N}_0}$, whose projection is given by

$$\pi_{\mathcal{J}}(\mathcal{R}^*(w)) = \{\varphi(S^n w)\}_{n\in\mathbb{N}_0} = \{T^n(0)\}_{n\in\mathbb{N}_0},$$

the (forward) orbit of 0 under T. Due to the compactness of $\mathcal{R}(w)$, this yields

$$G := \overline{\{T^n(0)\}_{n\in\mathbb{N}}} = \pi_{\mathcal{J}}(\mathcal{R}(w)).$$

Similarly, we obtain that $\{\varphi(S^n x) = T^n\varphi(x)\}_{n\in\mathbb{N}}$ is dense in $\pi_{\mathcal{J}}(\mathcal{R}(w))$ for every $x \in X'$. That is, $\varphi(x)$ has a dense orbit in (G, T) for every $x \in X'$. □

Remark 3.11. If (X, S) is the subshift of a unimodular and irreducible Pisot substitution, we have that $X' = X$ and (G, T) is in fact a factor. Under the assumption that the projection $\pi_{\mathcal{J}}\colon \mathcal{R}(w) \to \mathbb{H}/\mathcal{J}$ is almost surely 1-1, the map φ is even a measurable isomorphism between (X, S) and (G, T), where both are endowed with their unique ergodic measure. In this case, (X, S) has pure point dynamical spectrum and (G, T) is the maximal equicontinuous factor [37]. For random substitutions, the spectrum is generally richer and eigenfunctions do not necessarily have a continuous representative. In fact, it was shown for the random Fibonacci substitution in a geometric setting that the MEF is trivial and a natural analogue of φ provides the MEGF instead [8].

3.4. **Measures and the Lebesgue covering property.** For a C-balanced sequence w and $n \in \mathbb{N}_0$, let

$$\mu(w_{[1,n]}) = \sum_{i=1}^{n} \delta_{\pi\circ\psi(w_{[1,i]})} \quad \text{and} \quad \mu(w) = \lim_{n\to\infty} n^{-1}\mu(w_{[1,n]})$$

whenever this latter weak limit exists. Let us emphasize that $\mu(w)$ is a measure associated with a sequence w and not to be confused with the evaluation of a measure at a singleton set. We call the measure $\mu(w)$, when it exists, a *Rauzy measure*. Note, in any case, that there always exists a limit point, under the topology of weak convergence, of the sequence $(n^{-1}\mu(w_{[1,n]}))_n$, by the Banach–Alaoglu theorem. However, this limit point may not be unique.

Note that the Rauzy fractal $\mathcal{R}(w)$ always contains the support $\operatorname{supp}\mu(w)$ of the Rauzy measure.

In this section, we show that covering the stable subspace $\mathbb{H}$ with Rauzy measures (according to the lattice $\mathcal{J}$) creates a multiple of Lebesgue measure. Recall the definition of $\mathcal{J}$ from (3.4). For $\mathbf{j} \in \mathcal{J}$, we define a corresponding translation $t_{\mathbf{j}}\colon \mathbb{H} \to \mathbb{H}$ by $t_{\mathbf{j}}(\mathbf{x}) = \mathbf{x} + \mathbf{j}$, which is a bijection on $\mathbb{H}$.

Theorem 3.12. If $w \in \mathcal{A}^{\mathbb{N}}$ is an infinite, C-balanced word with totally irrational frequency vector $\mathbf{r}$, and if the weak limit $\mu(w)$ exists, then

$$\mu_{\mathcal{J}}(w) = \sum_{\mathbf{j}\in\mathcal{J}} \mu(w) \circ t_{\mathbf{j}}^{-1} = D\,\mathrm{Leb},$$

where D is the density of points in $\mathcal{J}$ and Leb denotes the Lebesgue measure on $\mathbb{H}$.

As an intermediate step, instead of $\mathbb{H}$ we consider the plane orthogonal to $\mathbf{v_0} = (1, \dots, 1)$, that is, $E_0 = \{\mathbf{v} \in \mathbb{R}^d : \mathbf{v} \cdot \mathbf{v_0} = 0\}$. Observe that the intersection $V_0 = E_0 \cap \mathbb{Z}^d$ is a lattice.

Lemma 3.13. V_0 is the integer span of the vectors $(\mathbf{w_i})_{i=2}^d$, with $\mathbf{w_i} = (\mathbf{e_i} - \mathbf{e_1})$.

Proof. Notice that each of the vectors $\mathbf{w_i}$ lies in V_0 and so does each of their linear combinations with integer coefficients. Conversely, if $\mathbf{u} = (u_1, \ldots, u_d) \in V_0$, then $u_1 = -\sum_{i=2}^d u_i$ and so $\mathbf{u} = \sum_{i=2}^d u_i \mathbf{w_i}$. $\square$

Analogously to the definition of E_0 and V_0, for an integer $k \geqslant 0$, we let $E_k = \{\mathbf{v} \in \mathbb{R}^d : \mathbf{v} \cdot \mathbf{v_0} = k\}$, $E_{[0,k]} = \{\mathbf{v} \in \mathbb{R}^d : \mathbf{v} \cdot \mathbf{v_0} \in [0, k]\}$, $V_k = E_k \cap \mathbb{Z}^d$, and $V_{[0,k]} = E_{[0,k]} \cap \mathbb{Z}^d$.

Since the scalar product of a vector in $\mathbb{Z}^d$ with $\mathbf{v_0}$ is always an integer, each element of $\mathbb{Z}^d$ is contained in precisely one of the affine spaces E_k. Hence,

$$V_{[0,n]} = \bigsqcup_{k=0}^{n} V_k. \tag{3.5}$$

This observation will be useful for the following decomposition result, and for which we recall that $\mathcal{S}(w) = \{\psi(w_{[1,k]}) : k = 0, 1, 2, \ldots\}$.

Lemma 3.14. If $w \in \mathcal{A}^n$, then

$$V_{[0,n]} = \bigsqcup_{\mathbf{v} \in V_0} \mathcal{S}(w) + \mathbf{v}.$$

Proof. Note that $\psi(w_{[1,n]}) \cdot \mathbf{v_0} = n$ for all $n \in \mathbb{N}_0$. Hence, the intersection of $\mathcal{S}(w)$ with each of the affine spaces E_k is a singleton $\mathbf{s_k} \in V_k$. Since $V_k = V_0 + \mathbf{s_k}$, we obtain from (3.5) that

$$V_{[0,n]} = \bigsqcup_{k=0}^{n} V_0 + \mathbf{s_k} = \bigsqcup_{k=0}^{n} \bigsqcup_{\mathbf{v} \in V_0} \mathbf{s_k} + \mathbf{v} = \bigsqcup_{\mathbf{v} \in V_0} \mathcal{S}(w) + \mathbf{v}. \qquad \square$$

We now give the proof of Theorem 3.12.

Proof of Theorem 3.12. It suffices to show that $\mu_{\mathcal{J}}(w)(B) = D\,\mathrm{Leb}(B)$ for every open ball B in $\mathbb{H}$ (e.g. with respect to the maximum-norm, due to the π–λ theorem). By construction, the restriction of π to V_0 is a bijection onto its image $\mathcal{J}$. Hence, for each $\mathbf{j} \in \mathcal{J}$ there is precisely one $\mathbf{v} = \mathbf{v_j} \in V_0$ with $\pi(\mathbf{v}) = \mathbf{j}$. Given an arbitrary bounded and continuous function f on $\mathbb{H}$, we obtain

$$\begin{aligned}\mu(w) \circ t_{\mathbf{j}}^{-1}(f) = \mu(w)(f \circ t_{\mathbf{j}}) &= \lim_{n\to\infty} n^{-1} \mu(w_{[1,n]})(f \circ t_{\mathbf{j}}) \\ &= \lim_{n\to\infty} n^{-1} \sum_{\mathbf{y} \in \mathcal{S}(w_{[1,n]})} f(\mathbf{j} + \pi(\mathbf{y})) = \lim_{n\to\infty} n^{-1} \sum_{\mathbf{z} \in \mathcal{S}(w_{[1,n]}) + \mathbf{v_j}} f(\pi(\mathbf{z})),\end{aligned}$$

where $S(w_{[1,n]}) = \{\psi(w_{[1,i]}) : i \in \{0, 1, 2, \ldots, n\}\}$, for $n \in \mathbb{N}_0$. This chain of equalities implies, by the Portmanteau theorem, in the sense of weak convergence, that

$$\mu(w) \circ t_{\mathbf{j}}^{-1} = \lim_{n\to\infty} n^{-1} \sum_{\mathbf{z} \in \mathcal{S}(w_{[1,n]}) + \mathbf{v_j}} \delta_{\pi(\mathbf{z})}.$$

Due to the C-balancedness of w, each $\mu(w) \circ t_{\mathbf{j}}^{-1}$ has compact support and hence, given an arbitrary open ball B, only finitely many of the expressions $\mu(w) \circ t_{\mathbf{j}}^{-1}(B)$ are non-zero. This together with Lemma 3.14 and the characterisation of weak convergence in terms of open sets, implies

$$\begin{aligned}\mu_{\mathcal{J}}(w)(B) = \sum_{\mathbf{j} \in \mathcal{J}} \mu(w) \circ t_{\mathbf{j}}^{-1}(B) &\leqslant \lim_{n\to\infty} n^{-1} \sum_{\mathbf{j} \in \mathcal{J}} \sum_{\mathbf{z} \in \mathcal{S}(w_{[1,n]}) + \mathbf{v_j}} \delta_{\pi(\mathbf{z})}(B). \\ &= \lim_{n\to\infty} n^{-1} \sum_{\mathbf{x} \in V_{[0,n]}} \delta_{\pi(\mathbf{x})}(B) = \lim_{n\to\infty} n^{-1} \#(\pi^{-1}(B) \cap \mathbb{Z}^d \cap E_{[0,n]}).\end{aligned}$$

By standard results on model sets (see for example [20, Proposition 4.2] or [36, Proposition 2.1]), the intersection of $\mathbb{Z}^d$ with $\pi^{-1}(B)$ has well-defined frequency 1 (along appropriate averaging sequences). Hence, the limit does not change if the cardinality in the last expression is replaced by the volume of the corresponding tube T_n, given by $T_n = \pi^{-1}(B) \cap E_{[0,n]}$. Since $\mathbb{H}$ is not necessarily perpendicular to $\mathbf{r}$, it is convenient to project B to $\mathbf{r}^{\perp}$. More precisely, we let $\pi_{\mathbf{r}}$ be the projection along $\mathbf{r}$ onto $\mathbf{r}^{\perp}$ and observe

$$\pi^{-1}(B) = \pi^{-1}(\pi_{\mathbf{r}}(B)) = \pi_{\mathbf{r}}(B) + \mathbb{R}\mathbf{r}.$$

For all $\mathbf{v} \in \mathbb{R}^d$ the intersection $(\mathbf{v}+\mathbb{R}\mathbf{r}) \cap E_{[0,n]}$ has the same length r_n. Indeed, let $\mathbf{w}_0$ and $\mathbf{w}_n = c_n\mathbf{r}+\mathbf{w}_0$ be the intersection points of $\mathbf{v}+\mathbb{R}\mathbf{r}$ with E_0 and E_n, respectively. Then, $n = \mathbf{w}_n \cdot \mathbf{v}_0 = c_n\mathbf{r} \cdot \mathbf{v}_0 = c_n$, and hence $r_n = \|\mathbf{w}_n - \mathbf{w}_0\| = n\|\mathbf{r}\|$, which is independent of $\mathbf{v}$. The volume of T_n is therefore given by

$$r_n \operatorname{Leb}(\pi_{\mathbf{r}}(B)) = n\|\mathbf{r}\| \operatorname{Leb}(\pi_{\mathbf{r}}(B)),$$

yielding

$$\mu_{\mathcal{J}}(w)(B) \leqslant \|\mathbf{r}\| \operatorname{Leb}(\pi_{\mathbf{r}}(B)).$$

A parallel argument (using the characterisation of weak convergence in terms of closed sets) shows that

$$\mu_{\mathcal{J}}(w)(\overline{B}) \geqslant \|\mathbf{r}\| \operatorname{Leb}(\pi_{\mathbf{r}}(\overline{B}))$$

for every closed ball $\overline{B}$ in $\mathbb{H}$. Using the (inner) regularity of Lebesgue measure, this implies that in fact $\mu_{\mathcal{J}}(w)(B) = \|\mathbf{r}\| \operatorname{Leb}(\pi_{\mathbf{r}}(B))$ for every open ball B. Hence, we obtain

$$\mu_{\mathcal{J}}(w) = \|\mathbf{r}\| \operatorname{Leb} \circ \pi_{\mathbf{r}}. \tag{3.6}$$

Since $\pi_{\mathbf{r}}$ restricts to a linear isomorphism from $\mathbb{H}$ to $\mathbf{r}^{\perp}$, there is a constant c such that $\mu_{\mathcal{J}}(w) = c \operatorname{Leb}$. We can therefore determine the normalisation by considering the fundamental domain J, spanned by $(\mathbf{w}_i)_{i=2}^d$. Its image $J' = \pi_{\mathbf{r}}(J)$ is then spanned by the vectors $(\mathbf{u}_i)_{i=2}^d$, with $\mathbf{u}_i = \pi_{\mathbf{r}}(\mathbf{e}_i - \mathbf{e}_1)$. Since $\mathbf{r}$ is perpendicular to each of the $\mathbf{u}_i$ the Lebesgue measure of J' can be expressed as

$$\frac{1}{\|\mathbf{r}\|} |\det(\mathbf{r}, \mathbf{u}_2, \ldots, \mathbf{u}_d)| = \frac{1}{\|\mathbf{r}\|} |\det(\mathbf{r}, \mathbf{e}_2 - \mathbf{e}_1, \ldots, \mathbf{e}_d - \mathbf{e}_1)| = \frac{1}{\|\mathbf{r}\|} \sum_{j=1}^{d} r_j = \frac{1}{\|\mathbf{r}\|},$$

where the first step follows from the fact that $\mathbf{u}_i$ and $\mathbf{e}_i - \mathbf{e}_1$ only differ by a multiple of $\mathbf{r}$ and the second step follows by a straightforward calculation, for example by expanding the determinant along the first column. It follows from (3.6) that $\mu_{\mathcal{J}}(w)(J) = \|\mathbf{r}\| \operatorname{Leb}(J') = 1$, and therefore $c = (\operatorname{Leb}(J))^{-1}$, which is precisely the density of points in $\mathcal{J}$. □

The statement in Theorem 3.12 holds for every weak accumulation point of the sequence $(\mu(w_{[1,n]}))_n$, even if the limit does not exist. This can be verified by restricting to the corresponding subsequence in the proof provided above.

3.5. **Rauzy measures as pushforward measures.** Recall that we assume X to be a topologically transitive, C-balanced subshift, with X' denoting the set of transitive points. Let ϱ be a fully supported ergodic measure on (X, S). Then, X' is a set of full measure for ϱ, and ϱ-almost every $x \in X'$ is generic. For such points, the Rauzy measure is (up to translation) given by the pushforward of ϱ under the map ϕ, defined in Section 3.3.

Proposition 3.15. For every ϱ-generic point x, we have $\mu(x) \circ t_{\phi(x)}^{-1} = \varrho \circ \phi^{-1}$, where $t_{\phi(x)} \colon x \mapsto x + \phi(x)$.

Proof. We regard ϱ as an ergodic measure on the (non-compact) dynamical system (X', S). Let x be a ϱ-generic point and define

$$\varrho_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^k x},$$

for all $n \in \mathbb{N}$. Note that the weak convergence of ϱ_n persists under the restriction to the (full measure) set X'. Since ϕ is continuous on X', this implies that $\varrho_n \circ \phi^{-1}$ converges to $\varrho \circ \phi^{-1}$ in the weak topology. Since $\phi(S^k x) = \phi(x) + \pi(\psi(x_{[1,k]}))$, we obtain by definition of the Rauzy measure

$$\mu(x) \circ t_{\phi(x)}^{-1} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\phi(S^k x)} = \lim_{n\to\infty} \varrho_n \circ \phi^{-1} = \varrho \circ \phi^{-1},$$

and the claim follows. □

Remark 3.16. For generic points x, this offers an additional interpretation of the Lebesgue covering property. In fact, the restriction of $\mu_{\mathcal{J}}(x)$ to a fundamental domain coincides with the natural projection of $\mu(x)$ to $\mathbb{H}/\mathcal{J}$ under $\pi_{\mathcal{J}}$. Since $\varphi = \pi_{\mathcal{J}} \circ \phi$, this is (up to translation) the same as $\mu \circ \varphi^{-1}$ by Proposition 3.15. On the other hand, by Corollary 3.10, $\mu \circ \varphi^{-1}$ is the unique T-invariant probability measure on the group G. Because we assumed that the entries of the right eigenvector are rationally independent, the group G is in fact the full $d-1$ dimensional torus, and the Haar measure is an appropriate multiple of Lebesgue measure.

4. Rauzy fractals of random substitutions

Here, we present two methods of canonically associating a Rauzy fractal with a given compatible irreducible Pisot random substitution. We provide a construction in Section 4.1 via the set of *level-∞ inflation words*. The elements of this set can be viewed as analogues of substitution-fixed points for random substitutions. Utilising the results proved in Section 3 on Rauzy fractals of C-balanced sequences, we show that every element in this set with dense shift-orbit gives rise to the same Rauzy fractal. In Section 4.2, we provide an alternative construction, via the *prefix language* of the random substitution. We prove that these two constructions coincide in Section 4.3.

Notation. The Rauzy fractal associated with a (compatible, irreducible Pisot) random substitution $(\vartheta, \mathbf{P})$ does not depend on the explicit choice of (non-degenerate) probabilities $\mathbf{P}$. As such, we suppress the dependence on $\mathbf{P}$ throughout this section, and simply refer to ϑ as a random substitution. To further simplify our notation, we let $(\lambda, \mathbf{L}, \mathbf{R})$ denote the Perron–Frobenius data of ϑ (which is independent of $\mathbf{P}$ by compatibility) and let π denote the projection along $\mathbf{R}$ onto $\mathbb{H}$. Recall from Theorem 2.8 that the assumptions on ϑ guarantee that every sequence in X_ϑ is C-balanced, such that the results from Section 3 are applicable in the present setting.

4.1. Construction via sequences with dense shift-orbit.

Given a random substitution ϑ with associated subshift X_ϑ, we define the *level-∞ inflation set* by

$$X_\vartheta^\infty = \bigcap_{n \in \mathbb{N}_0} \vartheta^n(X_\vartheta).$$

It follows from Cantor's intersection theorem that the set X_ϑ^∞ is always non-empty, noting that $\vartheta^n(X_\vartheta) \subseteq \vartheta^{n-1}(X_\vartheta)$ for all $n \in \mathbb{N}$ and that the sets $\vartheta^n(X_\vartheta)$ are compact by Lemma 2.9.

As ϑ is a compatible irreducible Pisot random substitution, any marginal is an irreducible Pisot substitution, and hence the substitution-fixed point of any marginal belongs to X_ϑ^∞. Moreover, one can construct a point in X_ϑ^∞ with a dense shift-orbit, see for instance [35, Proposition 13].

We define $\mathcal{R}_{\vartheta,a} = \mathcal{R}_a(w)$, where $w \in X_\vartheta^\infty$ is any element with dense shift-orbit in X_ϑ. The *Rauzy fractal associated with the random substitution* ϑ is the set $\mathcal{R}_\vartheta = \cup_{a \in \mathcal{A}} \mathcal{R}_{\vartheta,a}$. It will soon be shown in Section 4.3 that $\mathcal{R}_{\vartheta,a}$ (and therefore $\mathcal{R}_\vartheta$) is independent of the choice of element w with this property.

4.2. Construction via the prefix language.

An alternative method of associating a Rauzy fractal with a random substitution is via the *prefix language* of the random substitution. It will be shown in Section 4.3 that this construction is equivalent to the one given in Section 4.1. For now, we differentiate the two constructions by placing a hat $\widehat{\ }$ above any alternate versions.

In what follows, we let $\mathcal{A}_1$ denote the set of *eventually first* letters: namely, $a \in \mathcal{A}_1$ if and only if there is a letter $b \in \mathcal{A}$ such that a is the first letter of some realisation of $\vartheta^n(b)$ for infinitely many $n \in \mathbb{N}$.

Definition 4.1. Let ϑ be a random substitution. We define the *prefix language* of ϑ by

$$\mathcal{L}_{\vartheta,p} = \{v \in \mathcal{A}^+ : v \text{ is a prefix of some } w \in \vartheta^k(a),\, k \in \mathbb{N}_0,\, a \in \mathcal{A}_1\}.$$

The Rauzy fractal associated with the random substitution ϑ is defined from the projection of the set of all possible abelianisation vectors arising from words in $\mathcal{L}_{\vartheta,p}$. For each $a \in \mathcal{A}$, let

$$\mathcal{S}_a(\mathcal{L}_{\vartheta,p}) = \{\psi(v) : va \in \mathcal{L}_{\vartheta,p}\},$$

and define $\widehat{\mathcal{R}}^*_{\vartheta,a} = \pi(\mathcal{S}_a(\mathcal{L}_{\vartheta,p}))$ and $\widehat{\mathcal{R}}_{\vartheta,a} = \overline{\widehat{\mathcal{R}}^*_{\vartheta,a}}$. The *(prefix) Rauzy fractal associated with the random substitution* ϑ is the set $\widehat{\mathcal{R}}_\vartheta = \cup_{a \in \mathcal{A}} \widehat{\mathcal{R}}_{\vartheta,a}$.

4.3. Equivalence of the two constructions.

If $x \in X_\vartheta^\infty$, then $x_{[1,n]} \in \mathcal{L}_{\vartheta,p}$ for all $n \in \mathbb{N}_0$, and hence

$$\mathcal{S}_a(x) \subseteq \mathcal{S}_a(\mathcal{L}_{\vartheta,p}). \tag{4.1}$$

Projecting this identity gives that $\mathcal{R}_a(x) \subseteq \widehat{\mathcal{R}}_{\vartheta,a}$. Thus, the Rauzy fractal corresponding to any sequence in X_ϑ^∞ is contained in the Rauzy fractal of ϑ defined via the prefix language. If the sequence is additionally assumed to have a dense shift-orbit, then this inclusion is an equality.

Theorem 4.2. Let ϑ be a compatible, irreducible Pisot random substitution and let $w \in X_\vartheta^\infty$ be an element with dense shift-orbit in X_ϑ. Then, for all $a \in \mathcal{A}$, we have $\mathcal{R}_a(w) = \widehat{\mathcal{R}}_{\vartheta,a}$.

Proof. We first observe that there is a point $\widetilde{w} \in X_\vartheta^\infty$ with the following properties.

- For each $n \in \mathbb{N}$, $\widetilde{w}$ can be decomposed as a concatenation of level-n inflation words.
- In each of these decompositions, every word $u \in \vartheta^n(a)$ appears, for all $a \in \mathcal{A}$.
- The shift-orbit of $\widetilde{w}$ is dense.

The construction of such a point is given in [35, Proposition 13], where we also observe that, although it is not explicitly stated, the point constructed belongs to X_ϑ^∞.

Our goal now is to show that $\mathcal{R}_a(\tilde{w}) = \widehat{\mathcal{R}}_{\vartheta,a}$. By (4.1), we have that $\mathcal{R}_a(\widetilde{w}) \subset \widehat{\mathcal{R}}_{\vartheta,a}$ for all $a \in \mathcal{A}$. So, if we can show that $\mathcal{R}_a^*(\widetilde{w})$ is dense in $\widehat{\mathcal{R}}_{\vartheta,a}$, then we can conclude that $\mathcal{R}_a(\widetilde{w}) = \widehat{\mathcal{R}}_{\vartheta,a}$. To this end, let $q \in \widehat{\mathcal{R}}_{\vartheta,a}^*$ and $\varepsilon > 0$. There exists a word $va \in \mathcal{L}_{\vartheta,p}$ such that $q = \pi \circ \psi(v)$, that is, va is a prefix of some word $u \in \vartheta^k(b)$, for some $k \in \mathbb{N}$ and $b \in \mathcal{A}$. In fact, we can choose k arbitrarily large. Let $k \in \mathbb{N}$ be such that $h_\vartheta^k(\widehat{\mathcal{R}}_{\vartheta,a})$ is contained in a ball of radius ε centered at the origin. This is possible because h_ϑ is a contractive linear transformation. Since u appears in the level-k inflation word decomposition of $\widetilde{w}$, there exists a word $v' \in \mathcal{L}_{\vartheta,p}$ such that $\widetilde{w} = \vartheta^k(v')va\cdots$. Hence, $M^k\psi(v') + \psi(v) \in \mathcal{S}_a(\widetilde{w})$ and thereby, $h_\vartheta^k(\pi \circ \psi(v')) + q \in \mathcal{R}_a^*(\widetilde{w})$. By the assumption on k, this point lies in the ε-neighborhood of $q \in \widehat{\mathcal{R}}_{\vartheta,a}^*$. Since q was arbitrary, this shows that $\mathcal{R}_a^*(\widetilde{w})$ lies dense in $\widehat{\mathcal{R}}_{\vartheta,a}$.

Now, let $w \in X_\vartheta^\infty$. Since w and $\widetilde{w}$ both have dense shift-orbit in X_ϑ, it follows by Corollary 3.7 that there exists a vector t such that $\mathcal{R}_a(w) = \mathcal{R}_a(\widetilde{w}) - t$ for all $a \in \mathcal{A}$. But since $w \in X_\vartheta^\infty$, we have $\mathcal{R}_a(w) \subseteq \widehat{\mathcal{R}}_{\vartheta,a} = \mathcal{R}_a(\widetilde{w})$, so we must have $t = 0$. This completes the proof. $\square$

Therefore, we may now refer to both constructions of the Rauzy fractal by $\mathcal{R}_\vartheta$ without ambiguity.

4.4. **Topological properties.** Using the characterisation given by Theorem 4.2, together with the analytic properties of Rauzy fractals corresponding to C-balanced sequences proved in Section 3, we can obtain several key topological properties of Rauzy fractals of random substitutions.

Proposition 4.3. Let ϑ be a compatible, irreducible Pisot random substitution. If θ is a marginal of ϑ^k for some $k \in \mathbb{N}$, then $\mathcal{R}_{\theta,a} \subseteq \mathcal{R}_{\vartheta,a}$ for all $a \in \mathcal{A}$.

Proof. Let $w \in X_\vartheta^\infty$ be an element with a dense shift-orbit in X_θ (for example, take a θ-fixed point). By Theorem 4.2, we have that $\mathcal{R}_a(w) = \mathcal{R}_{\theta,a}$ for all $a \in \mathcal{A}$, and projecting the identity in (4.1) gives $\mathcal{R}_a(w) \subseteq \mathcal{R}_{\vartheta,a}$, so the result follows. $\square$

Proposition 4.4. Let ϑ be a compatible, irreducible Pisot random substitution. There exists a sequence of marginals θ_n (of powers of ϑ) such that $\mathcal{R}_{\theta_n} \to \mathcal{R}_\vartheta$ in the Hausdorff metric.

Proof. Let $w \in X_\vartheta^\infty$ be an element with dense shift-orbit. For each $n \in \mathbb{N}$, let a^n be a letter such that $w_{[1,|\vartheta^n|]} \in \vartheta^n(a^n)$. Such a letter always exists since $w \in X_\vartheta^\infty$. Thus, there exists a marginal θ_n of ϑ^n such that $\theta_n(a^n) = w_{[1,|\vartheta^n(a^n)|]}$. In particular, there exists a sequence of words $(w^n)_n$ such that $w^n \to w$, for which $w^n \in X_{\theta_n}$ for all $n \in \mathbb{N}$. Hence, it follows by Lemma 3.6 and Theorem 4.2 that, for each $n \in \mathbb{N}$, there is a vector $t_n \in \mathbb{R}^d$ such that

$$\mathcal{R}_a(w^n) \subseteq \mathcal{R}_{\theta_n,a} + t_n \subseteq \mathcal{R}_{\vartheta,a} + t_n,$$

where the second inclusion follows by (4.1). Since $w^n \to w$ and w has dense shift-orbit in X_ϑ, Proposition 3.8 gives that $\mathcal{R}_a(w^n) \to \mathcal{R}_a(w) = \mathcal{R}_{\vartheta,a}$ as $n \to \infty$. Therefore, we must have that $t_n \to 0$, hence, we conclude that $\mathcal{R}_{\theta_n,a} \to \mathcal{R}_{\vartheta,a}$ as $n \to \infty$. $\square$

As a consequence of these results, some topological properties of Rauzy fractals for deterministic substitutions transfer to the random setting. In the following two results, we apply Proposition 1.1 and therefore also assume that our random substitutions are unimodular.

Corollary 4.5. Let ϑ be a compatible, unimodular, irreducible Pisot random substitution. The Rauzy fractal $\mathcal{R}_\vartheta$ is the closure of its interior and contains an open ball.

Proof. Let θ be a marginal of ϑ. Since θ is an irreducible Pisot deterministic substitution, its Rauzy fractal contains an open ball by Proposition 1.1. Thus, by Proposition 4.3, so does $\mathcal{R}_\vartheta$.

We now prove that $\mathcal{R}_\vartheta$ is the closure of its interior. Let $\varepsilon > 0$ and let $x \in \mathcal{R}_\vartheta$. By Proposition 4.4, there exists a sequence of marginals θ_n of ϑ such that $\mathcal{R}_{\theta_n} \to \mathcal{R}_\vartheta$ in the Hausdorff metric. Hence, there exists an $n \in \mathbb{N}$ and $y \in \mathcal{R}_{\theta_n}$ such that $|x-y| < \varepsilon/2$. Since $\mathcal{R}_{\theta_n}$ is the closure of its interior, there is a $z \in \operatorname{int}(\mathcal{R}_{\theta_n})$ such that $|y-z| < \varepsilon/2$; hence, $|x-z| < \varepsilon$. By Proposition 4.3, we have $\mathcal{R}_{\theta_n} \subseteq \mathcal{R}_\vartheta$, so it follows that $z \in \operatorname{int}(\mathcal{R}_\vartheta)$ and we conclude that $\mathcal{R}_\vartheta = \overline{\operatorname{int}(\mathcal{R}_\vartheta)}$. □

Corollary 4.6. Let ϑ be a compatible, unimodular, irreducible Pisot random substitution. Then the Rauzy fractal $\mathcal{R}_\vartheta$ has full Hausdorff dimension. Specifically, $\dim_{\mathrm{H}} \mathcal{R}_\vartheta = d-1$.

Proof. This follows immediately from the fact that $\mathcal{R}_\vartheta$ contains an open ball. □

4.5. **GIFS.** In a similar manner to Rauzy fractals of deterministic substitutions, the subtiles of the Rauzy fractal of a random substitution are the attractors of a graph-directed iterated function system (GIFS) that reflects the action of the random substitution.

Definition 4.7. Let ϑ be a random substitution and let $\mathcal{P}_\vartheta$ be the finite set defined by

$$\mathcal{P}_\vartheta = \{(p,a,s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^* : \text{ there exists } b \in \mathcal{A} \text{ such that } pas \in \vartheta(b)\}.$$

The *prefix-suffix graph* of ϑ is the finite directed graph Γ_ϑ with vertex set $\mathcal{A}$ such that there is an edge labelled by $(p,a,s) \in \mathcal{P}_\vartheta$ from a to b if $pas \in \vartheta(b)$.

Theorem 4.8. Let ϑ be a compatible, irreducible Pisot random substitution. For each $a \in \mathcal{A}$, the subtile $\mathcal{R}_{\vartheta,a}$ is the unique (non-empty) compact solution of the self-consistency relation

$$\mathcal{R}_{\vartheta,a} = \bigcup_{b\in\mathcal{A}} \bigcup_{\substack{(p,a,s)\in\mathcal{P}_\vartheta \\ pas\in\vartheta(b)}} h(\mathcal{R}_{\vartheta,b}) + \pi(\psi(p)). \tag{4.2}$$

Proof. For each $a \in \mathcal{A}$, the vectors $q \in \mathcal{S}_a(\mathcal{L}_{\vartheta,p})$ are precisely the vectors of the form $q = Mr + \psi(p)$, where $r \in \mathcal{S}_b(\mathcal{L}_{\vartheta,p})$ for some $b \in \mathcal{A}$ and $pas \in \vartheta(b)$. Thus,

$$\mathcal{S}_a(\mathcal{L}_{\vartheta,p}) = \bigcup_{b\in\mathcal{A}} \bigcup_{\substack{(p,b,s)\in\mathcal{P}_\vartheta \\ pas\in\vartheta(b)}} M(\mathcal{S}_b(\mathcal{L}_{\vartheta,p})) + \psi(p),$$

and projecting to $\mathbb{H}$ gives

$$\mathcal{R}^*_{\vartheta,a} = \bigcup_{b\in\mathcal{A}} \bigcup_{\substack{(p,b,s)\in\mathcal{P}_\vartheta \\ pas\in\vartheta(b)}} \pi \circ M(\mathcal{S}_b(\mathcal{L}_{\vartheta,p})) + \pi(\psi(p)). \tag{4.3}$$

Since $\pi \circ M = h \circ \pi$, it follows from (4.3) that

$$\mathcal{R}^*_{\vartheta,a} = \bigcup_{b\in\mathcal{A}} \bigcup_{\substack{(p,b,s)\in\mathcal{P}_\vartheta \\ pas\in\vartheta(b)}} h(\mathcal{R}^*_{\vartheta,b}) + \pi(\psi(p)).$$

Taking closure gives (4.2).

The relation in (4.2) is in fact the self-consistency relation for a self-similar set $(\mathcal{R}_{\vartheta,a})_{a\in\mathcal{A}}$ of a GIFS $\mathcal{G} = (G, (X_a)_{a\in\mathcal{A}}, (f_e)_{e\in E})$. Here, $G = \Gamma_\vartheta$, the prefix-suffix graph, and $X_a = \mathbb{H}$ for all $a \in \mathcal{A}$. For each edge $e = (p,b,s)$ in Γ_ϑ, the corresponding contraction is given by $f_e = f_p$, where $f_p \colon x \mapsto h(x) + \pi \circ \psi(p)$. Thus, it follows that the sets $(\mathcal{R}_a)_{a\in\mathcal{A}}$ are in fact the unique, non-empty compact sets satisfying (4.2)—see, for instance, [26]. □

It should be emphasised that, in contrast to the case for deterministic substitutions, the sets $\mathcal{R}_{\vartheta,a}$ are in general no longer measure-disjoint for different choices of $a \in \mathcal{A}$.

5. Measures on Rauzy fractals of random substitutions

In this section, we regard $\vartheta_{\mathbf{P}}$ properly as a random substitution (equipped with probabilities). We equip the (typed) Rauzy fractal $\mathcal{R}_{\vartheta,a}$ with a natural measure, called the (typed) *Rauzy measure associated with* ϑ. This object can be defined in several different ways, which we show all coincide.

Given a finite word w, we associate with the set $\mathcal{R}_a^*(w)$ a corresponding counting measure

$$\mu_a(w) = \sum_{x \in \mathcal{R}_a^*(w)} \delta_x = \sum_{y \in \mathcal{S}_a(w)} \delta_{\pi(y)}.$$

Here the last step holds because π restricted to the integer lattice is injective, and here $\mathcal{R}_a^*(w)$ is as defined in Section 3.1. This is consistent with the notation for $\mu(w)$ employed in Section 3.4 in the sense that $\mu(w) = \sum_{a \in \mathcal{A}} \mu_a(w)$. This relation also carries over to an infinite sequence w if we set $\mu_a(w) = \lim_{n\to\infty} n^{-1} \mu_a(w_{[1,n]})$, provided that the limit exists. Note, if w is a *random* word (or random sequence), the expression $\mu_a(w)$ is to be interpreted as a random measure. In general, it will depend on the choice of $w \in X_\vartheta$, but once a set of probability parameters $\mathbf{P}$ is fixed, almost all sequences will essentially give rise to the same measure. Recall that ν denotes the $\vartheta_{\mathbf{P}}$-invariant measure on X_ϑ and ϱ is the corresponding (ergodic) frequency measure. Also, we let ϕ be the map from the set of transitive points to the Rauzy fractal, defined in Section 3.3. This is well-defined up to a reference point w' with $\phi(w') = 0$. We summarise the main results of this section in the following.

Theorem 5.1. There is a vector of measures $\overline{\mu} = (\overline{\mu}_a)_{a \in \mathcal{A}}$, with the following properties for all $a \in \mathcal{A}$.

1. $\overline{\mu}_a$ is supported on $\mathcal{R}_{\vartheta,a}$ and $\overline{\mu}_a(\mathcal{R}_{\vartheta,a}) = R_a$.
2. $\overline{\mu}_a = \mu_a(w)$ for ν-almost all $w \in X_\vartheta$ (in particular, the limit $\mu_a(w)$ exists).
3. $\overline{\mu}_a = \lim_{n\to\infty} |\vartheta^n(v)|^{-1} \mu_a(\vartheta_{\mathbf{P}}^n(v))$ holds $\mathbb{P}$-almost surely for all $v \in \mathcal{A}^+$.
4. $\overline{\mu}_a$ is absolutely continuous relative to Lebesgue measure, with bounded density.
5. $\overline{\mu}_a$ depends (weakly-)continuously on the probability parameters $\mathbf{P}$.
6. $(R_a^{-1} \overline{\mu}_a)_{a \in \mathcal{A}}$ is self-similar for the GIFS in Section 4.5, equipped with appropriate probabilities.

Given an appropriate reference point $w' \in X_\vartheta$ with $\phi(w') = 0$, we further have

1. $\mu = \sum_{a \in \mathcal{A}} \overline{\mu}_a$ coincides with $\mu(w)$ for ϱ-almost every $w \in X_\vartheta$, up to a translation by $\phi(w)$.
2. μ is the push-forward of ϱ under ϕ.

The first item is an immediate consequence of the following two, which, in turn, will be proved in Proposition 5.12 and Proposition 5.11, respectively. We obtain the absolute continuity of the measures from the Lebesgue covering property discussed in Section 5.6. The interpretation in terms of a GIFS is provided in Proposition 5.8 and we will use this to obtain the continuous dependence on the probability parameters. The last two items follow from the general framework on balanced sequences as we will show in Section 5.5.

Our first aim is to determine how $\mu_a(w)$ changes as we apply $\vartheta_{\mathbf{P}}$. We develop in parallel the prerequisites to show that there is a Rauzy measure that emerges both by following the expected measure as well as by following a generic path of the random variables $(\vartheta_{\mathbf{P}}^n(v))_{n \in \mathbb{N}}$. This framework follows a similar inflation word formalism established in [16] for topological entropy of random substitutions and expanded in [18] for measure theoretic entropy, whereby we will establish an "inflation word version" of the Rauzy measure and subsequently show this to be the same as the one obtained as the Rauzy measure (as described in Section 3.4) of a generic infinite sequence.

Lemma 5.2. For every $w \in \mathcal{A}^*$, we have

$$\mu_a(\vartheta_{\mathbf{P}}(w)) = \sum_{b \in \mathcal{A}} \sum_{x \in \mathcal{R}_b^*(w)} \mu_a(\vartheta_{\mathbf{P}}(b)) * \delta_{hx},$$

where the occurrence of $\mu_a(\vartheta_{\mathbf{P}}(b))$ is to be understood as an independent random measure in each summand.

Proof. For $w = w_1 \cdots w_n$ we obtain $\vartheta_{\mathbf{P}}(w) = \vartheta_{\mathbf{P}}(w_1) \cdots \vartheta_{\mathbf{P}}(w_n)$, and hence

$$\mathcal{S}_a(\vartheta_{\mathbf{P}}(w)) = \bigcup_{k=1}^{n} \mathcal{S}_a(\vartheta_{\mathbf{P}}(w_k)) + \psi(\vartheta_{\mathbf{P}}(w_{[1,k-1]})),$$

to be understood as a union of (independent) random sets. Recall that $\psi(\vartheta_{\mathbf{P}}(w_{[1,k-1]})) = M\psi(w_{[1,k-1]})$, where M is the matrix of $\vartheta_{\mathbf{P}}$, and therefore

$$\mathcal{R}_a^*(\vartheta_{\mathbf{P}}(w)) = \bigcup_{k=1}^{n} \mathcal{R}_a^*(\vartheta_{\mathbf{P}}(w_k)) + h(\pi \circ \psi(w_{[1,k-1]})).$$

This gives

$$\mu_a(\vartheta_{\mathbf{P}}(w)) = \sum_{k=1}^{n} \mu_a(\vartheta_{\mathbf{P}}(w_k)) * \delta_{h(\pi\circ\psi(w_{[1,k-1]}))}.$$

Recall that $w_k = b$ precisely if $\pi \circ \psi(w_{[1,k-1]}) \in \mathcal{R}_b^*(w)$. We can therefore reformulate the last equation as

$$\mu_a(\vartheta_{\mathbf{P}}(w)) = \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(w)} \mu_a(\vartheta_{\mathbf{P}}(b)) * \delta_{hx}. \qquad \square$$

In order to formulate the random GIFS later on, we need one more ingredient.

Lemma 5.3. For every $w \in \mathcal{A}^+$,

$$\mu_a(\vartheta_{\mathbf{P}}(w)) = \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(w)} \sum_{p,\, pas\in\vartheta(b)} \delta_x \circ f_p^{-1}.$$

Proof. We can be more explicit about $\mu_a(\vartheta_{\mathbf{P}}(b))$, which can be rewritten as

$$\mu_a(\vartheta_{\mathbf{P}}(b)) = \sum_{p,\, pas=\vartheta(b)} \delta_{\pi\circ\psi(p)},$$

yielding

$$\mu_a(\vartheta_{\mathbf{P}}(w)) = \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(w)} \sum_{p,\, pas=\vartheta(b)} \delta_{hx+\pi\circ\psi(p)}.$$

With the definition of f_p as above, we obtain

$$\mu_a(\vartheta_{\mathbf{P}}(w)) = \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(w)} \sum_{p,\, pas=\vartheta(b)} \delta_x \circ f_p^{-1}. \qquad \square$$

5.1. **Expected values.** Recall that by random word, we mean a random variable, whose co-domain is $\mathcal{A}^*$. In the following, for a random word w, we set $\overline{\mu}_a(w) := \mathbb{E}[\mu_a(w)]$.

Corollary 5.4. For every random word w of fixed length, we have

$$\overline{\mu}_a(\vartheta_{\mathbf{P}}(w)) = \sum_{b\in\mathcal{A}} \overline{\mu}_a(\vartheta_{\mathbf{P}}(b)) * \left(\overline{\mu}_b(w) \circ h^{-1}\right).$$

Proof. Taking expectation values in Lemma 5.2 and using the Markov property of the substitution action,

$$\mathbb{E}[\mu_a(\vartheta_{\mathbf{P}}(w))] = \sum_{b\in\mathcal{A}} \overline{\mu}_a(\vartheta_{\mathbf{P}}(b)) * \mathbb{E}\left[\sum_{x\in\mathcal{R}_b^*(w)} \delta_{h_\vartheta x}\right] = \sum_{b\in\mathcal{A}} \overline{\mu}_a(\vartheta_{\mathbf{P}}(b)) * \left(\overline{\mu}_b(w) \circ h^{-1}\right). \qquad \square$$

The following should be compared with the set valued recursion in Theorem 4.8. Up to a normalisation factor, this will be our measure analogue of the GIFS.

Corollary 5.5. For every random word w of fixed length, we have

$$\overline{\mu}_a(\vartheta_{\mathbf{P}}(w)) = \sum_{(p,b,s)\in\Gamma_\theta} \mathbb{P}[\vartheta_{\mathbf{P}}(b) = pas]\, \overline{\mu}_b(w) \circ f_p^{-1}.$$

Proof. This result follows from Lemma 5.3 and taking expectation. $\square$

Note that the recursion in Corollary 5.4 has the structure of a matrix convolution with a vector. Following this interpretation and starting from a fixed word $v \in \mathcal{A}^*$, we define the measure-valued vector

$$\overline{\mu}^n(v) := (\overline{\mu}_a(\vartheta_{\mathbf{P}}^n(v)))_{a\in\mathcal{A}},$$

for all $n \in \mathbb{N}_0$. By construction, for each $a \in \mathcal{A}$, the measures

$$\frac{1}{|\vartheta_{\mathbf{P}}^n(v)|_a}\overline{\mu}_a^n(v)$$

are probability measures on a compact space for each $n \in \mathbb{N}$ and hence have a weak accumulation point. In the following, we show that we in fact get convergence. Let us outline the general idea for the proof. Taking Fourier transforms in Corollary 5.4, we obtain a self-consistency relation that involves a matrix multiplication. Iterating this relation leads to a matrix cocycle that converges compactly.

As a first step, we introduce some notation. First, fix some $v \in \mathcal{A}^*$ and let us denote by $\widehat{\mu}_a^n$ the Fourier transform of $\overline{\mu}_a^n(v)$. Similarly, we set $\widehat{\mu}^n = (\widehat{\mu}_a^n)_{a\in\mathcal{A}}$. Since $\overline{\mu}_a(v)$ is a finite linear combination of Dirac measures on $\mathbb{H}$, its Fourier transform is represented by a continuous function, taking values in the dual vector space $\mathbb{H}^*$. The latter can be identified with $\mathbb{R}^{d-1}$. A straightforward calculation yields

$$\mathrm{FT}[\overline{\mu}_a(v) \circ h^{-1}](k) = \mathrm{FT}[\overline{\mu}_a(v)](gk),$$

where $g = h^*$ is the dual of h, given by

$$\langle gk, x\rangle = \langle k, hx\rangle,$$

for all $x \in \mathbb{H}$ and $k \in \mathbb{H}^*$. We also let $B = (B_{ab})_{a,b\in\mathcal{A}}$ be the (function-valued) matrix given by

$$B_{ab} = \mathrm{FT}[\overline{\mu}_a(\vartheta_{\mathbf{P}}(b))],$$

for all $a, b \in \mathcal{A}$. Then, applying the Fourier transform to the relation in Corollary 5.4, with $w = \vartheta_{\mathbf{P}}^n(v)$, we obtain

$$\widehat{\mu}_a^{n+1}(k) = \sum_{b\in\mathcal{A}} B(k)_{ab}\widehat{\mu}_b^n(gk).$$

In a more compact manner, this can be written as a cocycle relation over the dynamical system $k \mapsto gk$, given by

$$\widehat{\mu}^{n+1}(k) = B(k)\,\widehat{\mu}^n(gk).$$

Iterating this relation gives

$$\widehat{\mu}^n(k) = B^{(n)}(k)\,\widehat{\mu}^0(g^n k), \quad B^{(n)}(k) = B(k)B(gk)\cdots B(g^{n-1}k). \tag{5.1}$$

By construction, the assignments $k \mapsto B(k)$ and $k \mapsto \widehat{\mu}^0(k)$ are both Lipshitz-continuous. Since g is a contraction, the iteration $k \mapsto gk$ approaches the fixed point 0 exponentially fast. Since all entries of B and $\widehat{\mu}^0$ are defined via the Fourier transform of a measure, evaluation at 0 gives the total mass of the corresponding measure. Hence,

$$B(0) = M, \quad \widehat{\mu}^0(0) = \psi(v).$$

Following the line of argument in [7], we quickly arrive at the following; compare [7, Theorem 4.6] and the discussion thereafter.

Lemma 5.6. The sequence $(\lambda^{-n}B^{(n)}(k))_{n\to\infty}$ converges compactly on $\mathbb{H}^*$ to a matrix valued function $C(k)$. The limit is of the form

$$C(k) = c(k)\mathbf{L}^T,$$

where $k \mapsto c(k)$ is a continuous vector valued function with $c(0) = \mathbf{R}$.

From this, convergence of the measure valued vectors $\lambda^{-n}\overline{\mu}^n(v)$ is obtained as a corollary. More precisely, we get the following.

Proposition 5.7. There exists a vector of finite measures $\overline{\mu} = (\overline{\mu}_a)_{a\in\mathcal{A}}$ such that

$$\lim_{n\to\infty} \frac{1}{|\vartheta^n(v)|}\overline{\mu}^n(v) := \lim_{n\to\infty} \frac{1}{|\vartheta^n(v)|}(\overline{\mu}_a(\vartheta_{\mathbf{P}}^n(v)))_{a\in\mathcal{A}} = \overline{\mu}$$

in the sense of weak convergence, for all $v \in \mathcal{A}^+$. Further $\|\overline{\mu}\| = \mathbf{R}$, where $\|\overline{\mu}\| = (\|\overline{\mu}_a\|)_{a\in\mathcal{A}}$ and $\|\overline{\mu}_a\| = \overline{\mu}_a(\mathbb{H})$ denotes the total mass of the measure $\overline{\mu}_a$.

Proof. Let us note that

$$\lim_{n\to\infty} \lambda^{-n} |\vartheta^n(v)| = \lim_{n\to\infty} \lambda^{-n} \sum_{a\in\mathcal{A}} (M^n \psi(v))_a = \mathbf{L}^T \psi(v) =: L_v.$$

Hence, when taking the limit of a product, we may replace $|\vartheta^n(v)|$ by $\lambda^n L_v$. Recall that we denote the Fourier transform of $\overline{\mu}^n(v)$ by $\widehat{\mu}^n$. By (5.1), we obtain via Lemma 5.6

$$\lim_{n\to\infty} (L_v \lambda^n)^{-1} \widehat{\mu}^n(k) = L_v^{-1} c(k) \mathbf{L}^T \psi(v) = c(k),$$

for all $k \in \mathbb{H}^*$. Since $k \mapsto c(k)$ is continuous, Levy's continuity theorem implies that $c(k)$ is the Fourier Transform of some vector valued measure $\overline{\mu}$ such that

$$\lim_{n\to\infty} \frac{1}{|\vartheta^n(v)|} \overline{\mu}^n(v) = \lim_{n\to\infty} (L_v \lambda^n)^{-1} \overline{\mu}^n(v) = \overline{\mu}.$$

In particular, $\|\overline{\mu}\| = c(0) = \mathbf{R}$. □

There is an explicit representation of $\overline{\mu}$ as an infinite convolution product, involving measure valued matrices. Indeed, we may rewrite the relation in Corollary 5.4 as

$$\overline{\mu}^{n+1}(v) = \mathcal{M} * (\overline{\mu}^n(v) \circ h^{-1}),$$

where $\mathcal{M} = (\overline{\mu}_a(\vartheta_{\mathbf{P}}(b)))_{a,b\in\mathcal{A}}$. From this, it is straightforward to verify that $\overline{\mu}$ satisfies the self-similarity relation

$$\overline{\mu} = \frac{1}{\lambda} \mathcal{M} * (\overline{\mu} \circ h^{-1}).$$

Iterating this relation leads to

$$\overline{\mu} = \mathcal{M}^\infty \mathbf{R}, \quad \mathcal{M}^\infty = \mathop{\Large *}_{n=0}^{\infty} \frac{1}{\lambda} \mathcal{M} \circ h^{-n}.$$

Note, the Fourier transform of $\mathcal{M}^\infty$ is given by the matrix valued function $C(k)$ in Lemma 5.6. This result also guarantees the existence of the infinite convolution as a limit (via Levy's continuity theorem). From Lemma 5.6, we may also infer that $\mathcal{M}^\infty = m^\infty \mathbf{L}^T$, for some measure valued vector m^∞. Combining this with the fact that $\overline{\mu} = \mathcal{M}^\infty \mathbf{R}$, we have $m^\infty = \overline{\mu}$ and hence

$$\mathcal{M}^\infty = \overline{\mu} \mathbf{L}^T.$$

It is worth noting that sometimes the analysis of the matrix valued convolution $\mathcal{M}^\infty$ can be reduced to a scalar Bernoulli-like convolution. This is the case for the random Fibonacci substitution, see [8].

The following is finally the measure GIFS relation.

Proposition 5.8. The vector of measures $\overline{\mu}$ satisfies the consistency relation

$$\overline{\mu}_a = \frac{1}{\lambda} \sum_{(p,b,s)} \mathbb{P}[\vartheta_{\mathbf{P}}(b) = pas]\, \overline{\mu}_b \circ f_p^{-1}. \tag{5.2}$$

Proof. Due to Corollary 5.5, we obtain

$$\frac{1}{|\vartheta^{n+1}(v)|} \overline{\mu}_a^{n+1} = \frac{|\vartheta^n(v)|}{|\vartheta^{n+1}(v)|} \sum_{(p,b,s)} \mathbb{P}[\vartheta_{\mathbf{P}}(b) = pas]\, \frac{1}{|\vartheta^n(v)|} \overline{\mu}_b^n \circ f_p^{-1}.$$

Sending $n \to \infty$ gives the desired relation due to Proposition 5.7. □

As observed above, we have $\|\overline{\mu}_a\| = R_a$ for all $a \in \mathcal{A}$. Renormalising via $\widetilde{\mu}_a = R_a^{-1} \overline{\mu}_a$, we therefore obtain a vector $(\widetilde{\mu}_a)_{a\in\mathcal{A}}$ of probability measures. By (5.2), these measures satisfy the relation

$$\widetilde{\mu}_a = \sum_{(p,b,s)} p^a_{(p,b,s)} \widetilde{\mu}_b \circ f_p^{-1}, \quad p^a_{(p,b,s)} = \frac{1}{\lambda} \mathbb{P}[\vartheta_{\mathbf{P}}(b) = pas] \frac{R_b}{R_a}.$$

Recall from Section 4.5 that the parameters (p, b, s) label the edges E of an appropriate GIFS $\mathcal{G}$. In fact, one can verify that $p^a = (p^a_e)_{e\in E(a)}$ defines a probability vector on $E(a)$ for each $a \in \mathcal{A}$. Indeed,

$$\sum_{(p,b,s)} p^a_{(p,b,s)} = \frac{1}{\lambda R_a} \sum_{b\in\mathcal{A}} R_b \sum_{(p,s)} \mathbb{P}[\vartheta_{\mathbf{P}}(b) = pas],$$

where

$$\sum_{(p,s)} \mathbb{P}[\vartheta_{\mathbf{P}}(b) = pas] = \sum_{v \in \vartheta(b)} \mathbb{P}[\vartheta_{\mathbf{P}}(b) = v]|v|_a = \mathbb{E}[|\vartheta_{\mathbf{P}}(b)|_a] = M_{ab},$$

and hence,

$$\sum_{(p,b,s)} p^a_{(p,b,s)} = \frac{1}{\lambda R_a} \sum_{b \in \mathcal{A}} M_{ab} R_b = 1.$$

Using these probabilities, we may hence extend $\mathcal{G}$ to a GIFS $\mathcal{G}' = (G, (X_a)_{a\in\mathcal{A}}, (f_e)_{e\in E}, (p^a)_{a\in\mathcal{A}})$ and obtain that $(\widetilde{\mu}_a)_{a\in\mathcal{A}}$ is the unique self-similar measure vector for $\mathcal{G}'$. It directly follows that $(\overline{\mu}_a)_{a\in\mathcal{A}}$ is the unique solution to (5.2). Furthermore, this solution is (globally) attractive in the weak topology and, moreover, the support of the Rauzy measure $\overline{\mu}_a$ is the Rauzy fractal $\mathcal{R}_a$.

5.2. **Continuity of the invariant measure.** Let d_{MK} denote the *Monge–Kantorovich metric* on the space of probability measures on a given metric space X, defined by

$$d_{\mathrm{MK}}(\mu, \nu) = \sup_{g \in \mathrm{Lip}_1(X,\mathbb{R})} \left| \int g \, d\mu - \int g \, d\nu \right|,$$

where $\mathrm{Lip}_1(X, \mathbb{R})$ denotes the set of Lipschitz functions from X to $\mathbb{R}$ with Lipschitz constant at most 1.

In the context of iterated function systems, it is known that the corresponding invariant measure depends continuously on the probability parameters that are assigned to the edges, see for instance [12, Theorem 3.4]. The proof is analogous in the graph-directed case and is considered folklore. Since we are not aware of a direct reference, we present a proof below.

Proposition 5.9. The self-similar measures $(\mu_a)_{a\in\mathcal{A}}$ of a GIFS $\mathcal{G} = (G, (X_a)_{a\in\mathcal{A}}, (f_e)_{e\in E}, (p^a)_{a\in\mathcal{A}})$ depend Lipshitz continuously on the probability parameters (with respect to the Monge–Kantorovich metric).

Proof. We denote the terminal vertex of an edge e by $t(e)$. Let $(\mu_a)_{a\in\mathcal{A}}$ and $(\mu'_a)_{a\in\mathcal{A}}$ be the self-similar measures associated with the probability parameters $(p^a)_{a\in\mathcal{A}}$ and $(p'^a)_{a\in\mathcal{A}}$ respectively, and assume that $|p^a_e - p'^a_e| \leqslant \varepsilon$ for all $a \in \mathcal{A}$ and $e \in E(a)$. Note that the definition of the Monge–Kantorovich metric does not change if we take the supremum over the Lipshitz continuous functions g with 0 in the image of g, denoted by $\mathrm{Lip}_{1,0}$. We therefore obtain

$$\begin{aligned}
d_{\mathrm{MK}}(\mu_a, \mu'_a) &= \sup_{g \in \mathrm{Lip}_{1,0}(X_a,\mathbb{R})} \left| \int g \, d\Big(\sum_{e\in E(a)} p^a_e \mu_{t(e)} \circ f_e^{-1} \Big) - \int g \, d\Big(\sum_{e\in E(a)} p'^a_e \mu'_{t(e)} \circ f_e^{-1} \Big) \right| \\
&\leqslant \sum_{e\in E(a)} \sup_{g \in \mathrm{Lip}_{1,0}(X_a,\mathbb{R})} \left| \int g\circ f_e \, d(p^a_e \mu_{t(a)}) - \int g \circ f_e \, d(p'^a_e \mu'_{t(e)}) \right| \\
&\leqslant \sum_{e\in E(a)} \sup_{g \in \mathrm{Lip}_{1,0}(X_a,\mathbb{R})} \left| \int g\circ f_e \, d(p^a_e \mu_{t(e)}) - \int g \circ f_e \, d(p^a_e \mu'_{t(e)}) \right| \\
&+ \sum_{e\in E(a)} |p^a_e - p'^a_e| \sup_{g \in \mathrm{Lip}_{1,0}(X_a,\mathbb{R})} \left| \int g \circ f_e \, d\mu'_{t(e)} \right| \\
&\leqslant r \sum_{e\in E(a)} p^a_e \sup_{g \in \mathrm{Lip}_{1,0}(X_a,\mathbb{R})} \left| \int r^{-1} g\circ f_e \, d\mu_{t(e)} - \int r^{-1} g \circ f_e \, d\mu'_{t(e)} \right| + c\varepsilon,
\end{aligned}$$

where $c = \mathrm{diam}(X_a) \sum_{e\in E(a)} \mu_{t(e)}(X_{t(e)})$ and r is the common contraction ratio of the maps $(f_e)_{e\in E}$. Since every f_e has contraction ratio r, it follows that $r^{-1} g \circ f_e \in \mathrm{Lip}_1(X_{t(e)}, \mathbb{R})$. Hence, we obtain from the above that

$$d_{\mathrm{MK}}(\mu_a, \mu'_a) \leqslant r \sum_{e\in E(a)} p^a_e d_{\mathrm{MK}}(\mu_{t(e)}, \mu'_{t(e)}) + c\varepsilon \leqslant r \max_{b\in\mathcal{A}} d_{\mathrm{MK}}(\mu_b, \mu'_b) + c\varepsilon.$$

Taking the maximum and reorganising yields

$$\max_{a\in\mathcal{A}} d_{\mathrm{MK}}(\mu_a, \mu'_a) \leqslant \frac{c\varepsilon}{1-r}. \qquad \square$$

Corollary 5.10. With respect to the Monge–Kantorovich metric, the vector of probability measures $\overline{\mu}$ depends continuously on the probability parameters of the underlying random substitution.

Since convergence in the Monge–Kantorovich metric implies weak convergence, the same holds with respect to the topology of weak convergence.

We note that, although Rauzy measures for random substitutions have been shown above to vary continuously with respect to the choice of probabilities, this is not strictly true in the case of the Rauzy fractals themselves (say with respect to the Hausdorff metric). However, the distinction only becomes apparent at the extremal cases; that is, when the probabilities become degenerate. Indeed, for all non-degenerate probabilities (those for which all $p_{i,j} \neq 0$), the Rauzy fractal is independent of the probabilities. This follows from the fact that, for non-degenerate random substitutions, the subtiles $\mathcal{R}_a$ satisfy the GIFS relations (4.2) of Theorem 4.8, and these relations are independent of the generating probabilities.

On the other hand, for a degenerate choice of probabilities, the subtiles $\mathcal{R}_a$ will in general only satisfy a GIFS relation for a random substitution with fewer choices of images (that is, the one given by removing the choice for any images of letters with probability $p_{i,j} = 0$). This corresponds to removing edgings from the defining graph of the GIFS. Therefore, the support of the measure may become smaller at extremal points. This is observed in the case of the random tribonacci substitution in Figure 4.

5.3. **Almost-sure results.**

Proposition 5.11. Let $v \in \mathcal{A}^+$. For all $a \in \mathcal{A}$,

$$\lim_{n\to\infty} \frac{1}{|\vartheta^n(v)|} \mu_a(\vartheta_{\mathbf{P}}^n(v)) = \overline{\mu}_a$$

almost surely.

Proof. Applying the random substitution $\vartheta_{\mathbf{P}}^m$ to the random word $\vartheta_{\mathbf{P}}^n(v)$, we obtain from Lemma 5.2

$$\mu_a(\vartheta_{\mathbf{P}}^{n+m}(v)) = \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v))} \mu_a(\vartheta_{\mathbf{P}}^m(b)) * \delta_{h_\vartheta m\, x}. \tag{5.3}$$

For the Fourier transforms

$$g_a^n := \lambda^{-n}\, \mathrm{FT}[\mu_a(\vartheta_{\mathbf{P}}^n(v))]$$

we obtain from (5.3) that

$$g_a^{n+m}(k) = \lambda^{-n} \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v))} f_{x,n}^{a,m,b}(k) \mathrm{e}^{-2\pi \mathrm{i} k\cdot h_\vartheta^m x}, \tag{5.4}$$

where for each $a, b \in \mathcal{A}$, $m \in \mathbb{N}$ and $k \in \mathbb{H}^*$, the family of random variables $\{f_{x,n}^{a,m,b}\}_{x,n}$ is independent and identically distributed as

$$f_{x,n}^{a,m,b}(k) \sim \lambda^{-m}\, \mathrm{FT}[\mu_a(\vartheta_{\mathbf{P}}^m(b))](k),$$

where $\sim$ denotes equality of distributions. Since the language $\mathcal{L}_\vartheta$ is C-balanced, there is a radius $r > 0$ such that $\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v))$ is contained in the ball $B_r(0)$ for all $n \in \mathbb{N}$. Hence, for all k in a fixed compact set $K \subset \mathbb{H}^*$ there is a uniform constant $c = c(K)$ with $0 < c < 1$ and $d > 0$ such that

$$\mathrm{e}^{-2\pi \mathrm{i} k\cdot h_\vartheta^m x} = 1 + r_m(x,k), \quad |r_m(x,k)| < dc^m.$$

In the next step, we estimate the error that we obtain by replacing the exponential by 1 in (5.4) . To this end, observe that

$$|f_{x,n}^{a,m,b}(k)| < f_{x,n}^{a,m,b}(0) = \lambda^{-m}|\vartheta^m(b)|_a = \lambda^{-m}(M^m)_{ab},$$

and $\#\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v)) = |\vartheta_{\mathbf{P}}^n(v)|_b = (M^n\psi(v))_b$, yielding

$$\begin{aligned} \left| \lambda^{-n} \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v))} f_{x,n}^{a,m,b}(k) r_m(x,k) \right| &\leqslant \lambda^{-n-m} \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v))} dc^m |\vartheta^m(b)|_a \\ &= dc^m \lambda^{-n-m} \sum_{b\in\mathcal{A}} (M^m)_{ab}(M^n\psi(v))_b = \mathcal{O}(c^m), \end{aligned}$$

since the entries of $\lambda^{-n}M^n$ are uniformly bounded for all $n \in \mathbb{N}$. Combining this with (5.4) yields

$$g_a^{n+m}(k) = \mathcal{O}(c^m) + \lambda^{-n} \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v))} f_{x,n}^{a,m,b}(k).$$

An appropriate version of the strong law of large numbers (compare for example [25, Lemma 3]) yields that for each $b \in \mathcal{A}$ and every $k \in K$,

$$\lim_{n\to\infty} \frac{1}{|\vartheta^n(v)|_b} \sum_{x\in\mathcal{R}_b^*(\vartheta_{\mathbf{P}}^n(v))} f_{x,n}^{a,m,b}(k) = \mathbb{E}[f_{x,n}^{a,m,b}(k)] = \lambda^{-m}\,\mathrm{FT}[\overline{\mu}_a(\vartheta_{\mathbf{P}}^m(b))](k) = \lambda^{-m} B_{ab}^{(m)}(k) \tag{5.5}$$

holds almost surely.

Our next goal is to get the same statement with a changed order of the quantifiers: we require that for a set of full probability the convergence holds for all $k \in K$. Taking a countable intersection of full measure sets, we easily obtain that almost-surely convergence holds on a dense countable subset of K. Due to the Arzelà-Ascoli theorem, it suffices to show (for each possible realisation) the equicontinuity of the (uniformly bounded) sequence of functions in (5.5) in order to get uniform convergence on K almost-surely. In fact, since any uniform modulus of convergence is preserved under taking averages, it is enough to show the equicontinuity of the family $\{f_{x,n}^{a,m,b}\}_{x,n}$ for all possible realisations. Since the random variables are iid, with only finitely many possible realisations, equicontinuity follows from the fact that each of these realisations is a continuous function on K. Hence, there exists a set $\Omega(m, K)$ of full $\mathbb{P}$-measure such that the convergence in (5.5) is uniform on K for all realisations in $\Omega(m, K)$. Since $\mathbb{H}^*$ is σ-compact, there is a countable subset $\mathcal{T}$ such that $\mathbb{H}^* = \cup_{t\in\mathcal{T}} K + t$. Hence, for all realisations in the full-measure set

$$\Omega = \bigcap_{t\in\mathcal{T}} \bigcap_{m\in\mathbb{N}} \Omega(m, K+t),$$

we obtain compact convergence in (5.5) for all $m \in \mathbb{N}$. Note that

$$\lim_{n\to\infty} \lambda^{-n}|\vartheta^n(v)|_b = \lim_{n\to\infty} \lambda^{-n}(M\psi(v))_b = R_b L_v.$$

Thus, for all realisations in Ω, we get

$$\lim_{n\to\infty} \lambda^{-n} \sum_{b\in\mathcal{A}} \sum_{x\in\mathcal{R}_b^*(\vartheta^n(v))} f_{x,n}^{a,m,b}(k) = L_v \sum_{b\in\mathcal{A}} \lambda^{-m} B^{(m)}(k)_{ab} R_b.$$

Due to Lemma 5.6, we get

$$\lim_{m\to\infty} L_v \sum_{b\in\mathcal{A}} \lambda^{-m} B^{(m)}(k)_{ab} R_b = L_v \sum_{b\in\mathcal{A}} c(k)_a L_b R_b = L_v c(k)_a,$$

uniformly on compact sets. Hence, for all realisations in Ω, we get that

$$\limsup_{n\to\infty} |g_a^n(k) - L_v c(k)_a| = \limsup_{n\to\infty} |g_a^{n+m}(k) - L_v c(k)_a| = o(1) \xrightarrow{m\to\infty} 0,$$

implying

$$\lim_{n\to\infty} g_a^n(k) = L_v c(k)_a,$$

uniformly on compact sets. Since $c(k)_a$ is the Fourier transform of $\overline{\mu}_a$, this implies by Levy's continuity theorem that

$$\lim_{n\to\infty} \lambda^{-n}\mu_a(\vartheta_{\mathbf{P}}^n(v)) = L_v \overline{\mu}_a,$$

in the weak topology for all realisations in Ω. Up to a renormalisation, this is precisely what we intended to show. □

5.4. **The substitution-invariant distribution.** Let ν be a $\vartheta_{\mathbf{P}}$-invariant probability measure as introduced in Section 2. By construction, $1 = \nu(X_\vartheta) = \nu(\vartheta^n(X_\vartheta))$ for all $n \in \mathbb{N}$ and hence X_ϑ^∞ has full measure for ν. Using similar ideas as in the previous section, we obtain the analogue of Proposition 5.11 for the invariant distribution ν.

Proposition 5.12. For ν-almost every w, we have

$$\mu_a(w) := \lim_{n\to\infty} \frac{1}{n}\mu_a(w_{[1,n]}) = \overline{\mu}_a,$$

in the sense of weak convergence, for all $a \in \mathcal{A}$.

First, we need some preparation. As a first step, we control the size change of a word under a random substitution.

Lemma 5.13. There exists a positive integer $r \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}$, $m \in \mathbb{N}$ and all $v \in \mathcal{L}_{\vartheta}^{n}$ with

$$n \geqslant n^{+}(\ell, m) = \lceil \lambda^{-m} \ell \rceil + r$$

we have $|\vartheta^m(v)| \geqslant \ell$, and for all $v \in \mathcal{L}^n$ with

$$n \leqslant n^{-}(\ell, m) = \lfloor \lambda^{-m} \ell \rfloor - r$$

we have $|\vartheta^m(v)| \leqslant \ell$.

Proof. This is a consequence of the C-balancedness of the language. Indeed, it guarantees the existence of some $c > 0$ such that for all $n \in \mathbb{N}$ and $v \in \mathcal{L}_{\vartheta}^{n}$, we obtain

$$||v|_a - nR_a| < c$$

for all $a \in \mathcal{A}$ and hence for all $m \in \mathbb{N}$,

$$|\vartheta^m(v)| = \sum_{a \in \mathcal{A}} (M^m \psi(v))_a \leqslant n\lambda^m + \sum_{a,b \in \mathcal{A}} c(M^m)_{ab} \leqslant \lambda^m (n + r),$$

for some $r \in \mathbb{N}$ because the entries of $\lambda^{-m} M^m$ are bounded in m. Similarly, we get

$$|\vartheta^m(v)| \geqslant \lambda^m (n - r).$$

The statement of the Lemma is just a reformulation of these two observations. □

In a similar manner as before we wish to obtain weak convergence via pointwise convergence of the Fourier transforms. We therefore define for every $a \in \mathcal{A}$ a function on $\mathcal{L}_{\theta}$ via

$$\widehat{\mu}_a \colon w \mapsto \mathrm{FT}[\mu_a(w)].$$

Lemma 5.14. Let $\ell, m \in \mathbb{N}$ and $n \geqslant n^{+}(\ell, m)$. For every $k \in \mathbb{H}^*$, $a \in \mathcal{A}$ and $c \in \mathbb{C}$, we have

$$\nu\{w : \widehat{\mu}_a(w_{[1,\ell]})(k) = c\} = \sum_{v \in \mathcal{L}_{\vartheta}^{n}} \nu[v] \mathbb{P}[\widehat{\mu}_a(\vartheta_{\mathbf{P}}^{m}(v)_{[1,\ell]})(k) = c].$$

Proof. Let $f \colon \mathcal{L}_{\vartheta} \to \mathbb{C}$, be defined by $f(w) = \widehat{\mu}_a(w)(k)$, and let $f_\ell(w) = f(w_{[1,\ell]})$. By (2.1),

$$\nu[f_\ell = c] = \sum_{u \in \mathcal{L}_{\vartheta}^{\ell}, f(u) = c} \nu[u] = \sum_{u \in \mathcal{L}_{\vartheta}^{\ell}, f(u) = c} \sum_{v \in \mathcal{L}_{\vartheta}^{n}} \nu[v] \, \mathbb{P}[\vartheta_{\mathbf{P}}^{m}(v)_{[1,\ell]} = u] = \sum_{v \in \mathcal{L}_{\vartheta}^{n}} \nu[v] \, \mathbb{P}[f(\vartheta_{\mathbf{P}}^{m}(v)_{[1,\ell]}) = c],$$

which yields the desired assertion. □

Lemma 5.15. For every $k \in \mathbb{H}^*$, there exists a set $\Omega_k \subset X_\vartheta$ with $\nu(\Omega_k) = 1$ such that for all $w \in \Omega_k$, we have

$$\lim_{\ell \to \infty} \ell^{-1} \, \widehat{\mu}_a(w_{[1,\ell]})(k) = c(k)_a.$$

Proof. We show this via an application of the Borel–Cantelli lemma. Let $\varepsilon > 0$ be arbitrary. With the same notation as in the proof of Lemma 5.14 we set

$$A_\ell = \{w : |\ell^{-1} f_\ell(w) - c(k)_a| > \varepsilon\}, \quad \widetilde{A}_\ell = \{w \in \mathcal{L}_{\vartheta}^{\ell} : |\ell^{-1} f(w) - c(k)_a| > \varepsilon\}.$$

Then, for arbitrary $m \in \mathbb{N}$, by Lemma 5.14 we get for $n = n^{+}(\ell, m)$,

$$\nu(A_\ell) = \sum_{v \in \mathcal{L}_{\vartheta}^{n}} \nu[v] \, \mathbb{P}[\vartheta_{\mathbf{P}}^{m}(v)_{[1,\ell]} \in \widetilde{A}_\ell]. \tag{5.6}$$

By the Borel–Cantelli lemma, the desired almost-sure convergence follows as soon as we have shown that

$$\sum_{\ell \in \mathbb{N}} \nu(A_\ell) < \infty.$$

We therefore aim to obtain an appropriate upper bound for $\mathbb{P}[\vartheta_{\mathbf{P}}^{m}(v)_{[1,\ell]} \in \widetilde{A}_\ell]$. By the definition of $n = n^{+}(\ell, m)$, it follows that $\ell \leqslant |\vartheta^m(v)| \leqslant \ell + (2r+1)\lambda^m$. Since every letter in a word w

contributes a summand on the complex unit circle to $f(w)$, this implies, for all realisation of $\vartheta_{\mathbf{P}}(v)$, that $|f(\vartheta_{\mathbf{P}}(v)) - f(\vartheta_{\mathbf{P}}(v)_{[1,\ell]})| \leqslant (2r+1)\lambda^m$. Choosing $\ell_0 = \ell_0(m,\varepsilon)$ large enough so that

$$\frac{(2r+1)\lambda^m}{\ell_0} < \frac{\varepsilon}{2}$$

we obtain that for all $\ell \geqslant \ell_0$, the condition $\vartheta_{\mathbf{P}}(v)_{[1,\ell]} \in \widetilde{A}_\ell$ implies that

$$|\ell^{-1} f(\vartheta_{\mathbf{P}}(v)) - c(k)_a| > \frac{\varepsilon}{2}. \tag{5.7}$$

Similar to the proof of Proposition 5.11, we have

$$\begin{aligned}
\ell^{-1} f(\vartheta_{\mathbf{P}}(v)) &= \mathcal{O}(c^m) + \frac{\lambda^m}{\ell} \sum_{b\in\mathcal{A}} \sum_{x \in \mathcal{R}_b^*(v)} f_x^{a,m,b}(k) \\
&= \mathcal{O}(c^m) + \sum_{b\in\mathcal{A}} R_b \frac{\lambda^m |v|}{\ell} \frac{|v|_b}{R_b|v|} \frac{1}{|v|_b} \sum_{x\in\mathcal{R}_b^*(v)} f_x^{a,m,b}(k),
\end{aligned}$$

where $|c| < 1$ and $\{f_x^{a,m,b}(k)\}_x$ are iid, with distribution given by $\lambda^{-m}\,\mathrm{FT}[\mu_a(\vartheta_{\mathbf{P}}^m(b))](k)$. In particular

$$\mathbb{E}[f_x^{a,m,b}(k)] = \lambda^{-m} B_{ab}^{(m)}(k). \tag{5.8}$$

Similarly to above, we have that $\ell^{-1}\lambda^m|v|$ differs from 1 by at most $\ell^{-1}\lambda^m(r+1)$. Furthermore, the C-balanced property implies that

$$1 - \frac{|v|_b}{R_b|v|} = \mathcal{O}(|v|^{-1}) = \mathcal{O}(\ell^{-1}).$$

Since $|f_x^{a,m,b}(k)|$ is uniformly bounded by a constant C, we obtain that

$$\left| \ell^{-1} f(\vartheta_{\mathbf{P}}(v)) - \sum_{b\in\mathcal{A}} R_b \frac{1}{|v|_b} \sum_{x\in\mathcal{R}_b^*(v)} f_x^{a,m,b}(k) \right| = \mathcal{O}(c^m) + \mathcal{O}(\ell^{-1}\lambda^m) + \mathcal{O}(\ell^{-1}). \tag{5.9}$$

We may choose m large enough and then $\ell_1 = \ell_1(m,\varepsilon) \geqslant \ell_0(m,\varepsilon)$ large enough such that for $\ell \geqslant \ell_1$ the sum of the error terms on the right hand side is smaller than $\varepsilon/4$. Due to Lemma 5.6, we further know that

$$\lim_{m\to\infty} \sum_{b\in\mathcal{A}} R_b \lambda^{-m} B_{ab}^{(m)}(k) = \sum_{b\in\mathcal{A}} c(k)_a R_b L_b = c(k)_a$$

and hence we may assume that m has been chosen large enough to ensure that

$$\left| \sum_{b\in\mathcal{A}} R_b \lambda^{-m} B_{ab}^{(m)}(k) - c(k)_a \right| \leqslant \frac{\varepsilon}{8}.$$

Together with (5.7) and (5.9), this yields that for all $\ell \geqslant \ell_1$, we have

$$\frac{\varepsilon}{8} < \sum_{b\in\mathcal{A}} R_b \left| \frac{1}{|v|_b} \sum_{x\in\mathcal{R}_b^*(v)} f_x^{a,m,b}(k) - \lambda^{-m} B_{ab}^{(m)}(k) \right|.$$

Due to (5.8) and since $(R_b)_{b\in\mathcal{A}}$ is a probability vector, this requires that for some $b \in \mathcal{A}$, we have that

$$\frac{\varepsilon}{8} < \left| \frac{1}{|v|_b} \sum_{x\in\mathcal{R}_b^*(v)} \left(f_x^{a,m,b}(k) - \mathbb{E}[f_x^{a,m,b}(k)] \right) \right|. \tag{5.10}$$

By Cramér's theorem on large deviations, there exists a constant $r = r(a,m,b,k) > 0$ such that (given $|v|_b$ is large enough) the probability for this event is bounded by $\mathrm{e}^{-r|v|_b} \leqslant \mathrm{e}^{-r'\ell}$, for some $r' > 0$. In summary, there exists a number $m \in \mathbb{N}$ and some $\ell_2 \in \mathbb{N}$ such that for all $\ell \geqslant \ell_2$ the condition $\vartheta_{\mathbf{P}}^m(v)_{[1,\ell]} \in \widetilde{A}_\ell$ implies that (5.10) holds for some $b \in \mathcal{A}$ and

$$\mathbb{P}[\vartheta_{\mathbf{P}}^m(v)_{[1,\ell]} \in \widetilde{A}_\ell] \leqslant d\mathrm{e}^{-r'\ell}.$$

Since the choice of r' is independent of both ℓ and $v \in \mathcal{L}_\vartheta^n$ we obtain, via (5.6), that $\nu(A_\ell) \leqslant d\mathrm{e}^{-r'\ell}$, which is summable over ℓ. The desired result follows by an application of the Borel-Cantelli lemma. □

We are now equipped to prove the main result of this subsection.

Proof of Proposition 5.12. Given $k \in \mathbb{H}^*$ we know by Lemma 5.15 that for w in a set Ω_k of full measure, we get

$$\lim_{\ell \to \infty} \ell^{-1} \widehat{\mu}_a(w_{[1,\ell]})(k) = c(k)_a. \tag{5.11}$$

The rest follows as in the proof of Proposition 5.11. Given a compact set K, there is a dense countable subset $E \subset K$ and a set $\Omega(E)$ of full measure such that for all $w \in \Omega(E)$ the convergence holds on E. Since each $\ell^{-1}\widehat{\mu}_a(w_{[1,\ell]})$ is the average of functions with a uniformly bounded Lipshitz constant, the family $\{\ell^{-1}\widehat{\mu}_a(w_{[1,\ell]})\}_{\ell \in \mathbb{N}}$ is equicontinuous. By the Arzelà-Ascoli theorem, this implies uniform convergence on K for all $w \in \Omega(E)$. We can hence find a set Ω with $\nu(\Omega) = 1$ such that for all $w \in \Omega$ the convergence in (5.11) holds uniformly on all compact subsets of $\mathbb{H}^*$, and in particular pointwise for all $k \in \mathbb{H}^*$. Since $c(k)_a$ is the Fourier transform of $\overline{\mu}_a$, the claim follows by an application of Levy's continuity theorem. □

Corollary 5.16. Let $\overline{\mu}_a^\nu(w_{[1,n]}) = \mathbb{E}_\nu[\mu_a(w_{[1,n]})]$. Then,

$$\lim_{n \to \infty} \frac{1}{n} \overline{\mu}_a^\nu(w_{[1,n]}) = \overline{\mu}_a,$$

for all $a \in \mathcal{A}$, in the sense of weak convergence.

Proof. Let f be an arbitrary bounded and uniformly continuous function on $\mathbb{H}$. Then, since each of $n^{-1}\overline{\mu}^\nu(w_{[1,n]})$ is a probability measure, we obtain $|n^{-1}\overline{\mu}_a^\nu(w_{[1,n]})(f)| \leqslant \|f\|_\infty$. By Lebesgue's dominated convergence theorem, we may therefore conclude from Proposition 5.12,

$$\lim_{n \to \infty} \frac{1}{n} \mu_a^\nu(w_{[1,n]})(f) = \lim_{n \to \infty} \int_{X_\vartheta} \frac{1}{n} \mu_a(w_{[1,n]})(f) \,\mathrm{d}\nu(w) = \overline{\mu}_a(f).$$

Since f was arbitrary, this shows the required weak convergence of probability measures. □

5.5. **The frequency measure.** Let ϱ be the frequency measure associated with the random substitution $\vartheta_{\mathbf{P}}$. Since ϱ is ergodic, we obtain from Proposition 3.15 an additional interpretation of the Rauzy measure as a pushforward of ϱ under the map $\phi\colon X'_\vartheta \to \mathcal{R}_\vartheta$ in Section 3.3, where X'_ϑ is the set of transitive points of X_ϑ. For ϕ to be well-defined we need to pick a reference point $w \in X'_\vartheta$ and declare $\phi(w) = 0$. To be specific, let us choose $w \in X_\vartheta^\infty \cap X'_\vartheta$ to be one of the ν-typical points with the property that $\mu_a(w) = \overline{\mu}_a$ for all $a \in \mathcal{A}$ (Proposition 5.12) and that w is ϱ-generic (Theorem 2.11).

Corollary 5.17. The Rauzy measure $\mu = \sum_{a \in \mathcal{A}} \overline{\mu}_a$ coincides with $\varrho \circ \phi^{-1}$. Further, $\mu(x) \circ t_{\phi(x)}^{-1} = \mu$ holds for ϱ-almost every $x \in X_\vartheta$.

Proof. Since we have assumed that w is ϱ-generic, its Rauzy measure $\mu = \sum_{a \in \mathcal{A}} \overline{\mu}_a$ coincides with $\varrho \circ \phi^{-1}$ (up to a shift by $\phi(w) = 0$), due to Proposition 3.15. Likewise, since ϱ is ergodic by Theorem 2.11, ϱ-almost every point is generic, and therefore satisfies $\mu(x) \circ t_{\phi(x)}^{-1} = \varrho \circ \phi^{-1} = \mu$, again by Proposition 3.15. □

In particular, the Rauzy measure is the same for ϱ-almost every sequence, up to a rigid translation that is detmined by the image under ϕ. Note that every $\vartheta_{\mathbf{P}}$-invariant measure ν assigns full measure to the fibre $\phi^{-1}(0)$.

5.6. **Lebesgue covering property.** Finally, we transfer the Lebesgue covering property from the context of C-balanced subshifts to the Rauzy measures associated with random substitutions. Recall that the lattice $\mathcal{J}$ is the integer span of the vectors $\pi(\mathbf{e}_i - \mathbf{e}_1)$ with $2 \leqslant i \leqslant d$ and D denotes the density of points in $\mathcal{J}$.

Corollary 5.18. Let $\overline{\mu}$ be the self-similar measure vector corresponding to a compatible irreducible Pisot random substitution $\vartheta_{\mathbf{P}}$. Then,

$$\sum_{j \in \mathcal{J}} \sum_{a \in \mathcal{A}} \overline{\mu}_a \circ t_j^{-1} = D \,\mathrm{Leb}\,.$$

Proof. By Proposition 5.12, we have for ν-almost every w that

$$\sum_{a \in \mathcal{A}} \overline{\mu}_a = \lim_{n \to \infty} \frac{1}{n} \sum_{a \in \mathcal{A}} \mu_a(w_{[1,n]}) = \lim_{n \to \infty} \frac{1}{n} \mu(w_{[1,n]}) = \mu(w),$$

and hence the claim follows immediately from Theorem 3.12. □

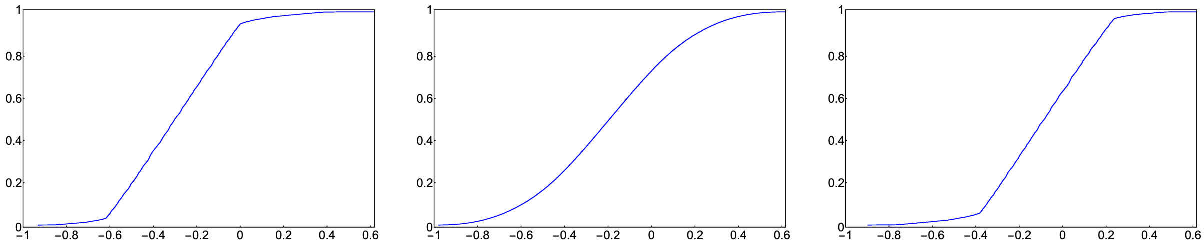

FIGURE 2. Computer simulations of the distribution functions for the measure $\mu_{a,p}$ for random Fibonacci, for $p = 1/16$ (left), $p = 1/2$ (middle) and $p = 15/16$ (right).

Corollary 5.19. Under the requirements of Corollary 5.18, each of the measures $\overline{\mu}_a$ with $a \in \mathcal{A}$ is absolutely continuous with respect to Lebesgue measure, with density bounded above by D.

Proof. This follows from the fact that $\overline{\mu}_a$ is dominated by D times Leb due to Corollary 5.18. $\square$

6. Examples

The Fibonacci substitution $\theta\colon a \mapsto ab$, $b \mapsto a$ is the prototypical example of a unimodular, irreducible Pisot substitution over a two-letter alphabet. The Rauzy fractal associated with θ is an interval. In particular, $\mathcal{R}_{\theta,a} = [-\tau^{-2}, \tau^{-3}]$ and $\mathcal{R}_{\theta,b} = [\tau^{-3}, \tau^{-1}]$. A broad class of Rauzy fractals in one dimension can be obtained by considering "reshuffled" versions of the Fibonacci substitution. Namely, by changing the ordering of the letters in realisations of θ and its powers. By locally mixing two or more of these substitutions (of the same power), we obtain a broad class of compatible random substitutions, for which the properties of unimodularity and irreducible Pisot are inherited. In this section, we consider several random substitutions constructed in this manner. First, we consider the classical random Fibonacci substitution, which is obtained by locally mixing θ with the substitution $\tilde{\theta}\colon a \mapsto ba$, $b \mapsto a$, which gives rise to the Rauzy fractal with tiles $\mathcal{R}_{\tilde{\theta},a} = [-\tau^{-1}, 0]$, $\mathcal{R}_{\tilde{\theta},b} = [0, \tau^{-2}]$.

Example 6.1. Given $p \in (0,1)$, let $\vartheta_{\mathbf{P}}$ be the random Fibonacci substitution, namely the random substitution

$$\vartheta_{\mathbf{P}}\colon \begin{cases} a \mapsto \begin{cases} ab & \text{with probability } p, \\ ba & \text{with probability } 1-p, \end{cases} \\ b \mapsto a \quad \text{with probability } 1. \end{cases}$$

By Theorem 4.8, the tiles of the Rauzy fractal associated with $\vartheta_{\mathbf{P}}$ are the unique, non-empty, compact sets $\mathcal{R}_a$ and $\mathcal{R}_b$ satisfying

$$\begin{aligned} \mathcal{R}_a &= \left(-\frac{1}{\tau}\mathcal{R}_a\right) \cup \left(-\frac{1}{\tau}\mathcal{R}_a - \frac{1}{\tau}\right) \cup \left(-\frac{1}{\tau}\mathcal{R}_b\right) \\ \mathcal{R}_b &= \left(-\frac{1}{\tau}\mathcal{R}_a\right) \cup \left(-\frac{1}{\tau}\mathcal{R}_a + \frac{1}{\tau^2}\right); \end{aligned}$$

in particular, $\mathcal{R}_a = [-1, \tau^{-1}]$ and $\mathcal{R}_b = [-\tau^{-2}, 1]$. We highlight that there exist points in these intervals which do not appear in the corresponding intervals of the Rauzy fractal of either marginal of $\vartheta_{\mathbf{P}}$. Further, note that $\mathcal{R}_a$ and $\mathcal{R}_b$ intersect on a set of positive Lebesgue measure. The above graph-directed iterated function system can be rewritten as an ordinary iterated function system for $\mathcal{R}_a$:

$$\mathcal{R}_a = \left(-\frac{1}{\tau}\mathcal{R}_a\right) \cup \left(-\frac{1}{\tau}\mathcal{R}_a - \frac{1}{\tau}\right) \cup \left(\frac{1}{\tau^2}\mathcal{R}_a\right) \cup \left(\frac{1}{\tau^2}\mathcal{R}_a - \frac{1}{\tau^3}\right).$$

This system does not satisfy the open set condition, since the similarity dimension does not equal 1 and so Hutchinson's formula is not satisfied. It follows that the graph-directed iterated function system also does not satisfy the open set condition, again in contrast to the deterministic setting.

Let $\mu_{a,p}$ and $\mu_{b,p}$ be the Rauzy measures on $\mathcal{R}_a$ and $\mathcal{R}_b$, respectively, induced by the random substitution $\vartheta_{\mathbf{P}}$. By Proposition 5.9, the measures $\mu_{a,p}$ and $\mu_{b,p}$ are continuous with respect to the probability parameters. The mass distribution function of the measure $\mu_{a,p}$ is plotted in Figure 2 for several values of p. We note that when $p = 0$ or $p = 1$, the measure $\mu_{a,p}$ is Lebesgue measure supported on the a tile of the respective marginal. However, when $0 < p < 1$, $\mu_{a,p}$ is supported on the interval $\mathcal{R}_a = [-1, \tau^{-1}]$.

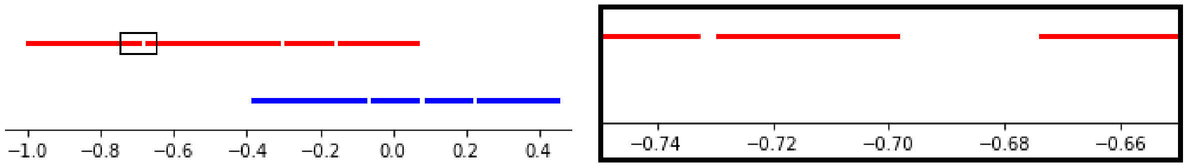

FIGURE 3. Subtiles $\mathcal{R}_a$ (red) and $\mathcal{R}_b$ (blue) of the Rauzy fractal associated with any non-degenerate random substitution defined over the set-valued substitution $\vartheta\colon a \mapsto \{abbaabaa\}$, $b \mapsto \{abaab, baaab\}$. We remark that within each of the subtiles $\mathcal{R}_a$ and $\mathcal{R}_b$ there exist infinitely many gaps, the lengths of which decay exponentially fast.

There are six different deterministic substitutions which have the same substitution matrix as the square of the Fibonacci substitution. The Rauzy fractals associated with four of these substitutions are intervals. However, the substitutions $\theta_1\colon a \mapsto aab$, $b \mapsto ba$ and $\theta_2\colon a \mapsto baa$, $b \mapsto ab$, give rise to Rauzy fractals which are the infinite union of a disjoint collection of closed intervals. In fact, the boundary of these Rauzy fractals has Hausdorff dimension equal to $\log(1+\sqrt{2})/2\log(\tau) \approx 0.915785$—see [4, 5] for more details. In the following, we consider the Rauzy fractals of random substitutions obtained by locally mixing two or more substitutions with the same substitution matrix as the square of Fibonacci.

Example 6.2. Let $\vartheta_{\mathbf{P}}$ be any (non-degenerate) random substitution defined over the set-valued substitution $\vartheta\colon a \mapsto \{aab\}$, $b \mapsto \{ab, ba\}$, obtained by locally mixing a substitution whose Rauzy fractal is an interval and one whose Rauzy fractal has boundary with positive Hausdorff dimension. The Rauzy fractal of the random substitution $\vartheta_{\mathbf{P}}$ is an interval, with overlapping subtiles $\mathcal{R}_a = [-\tau^{-1}, \tau^{-1}]$ and $\mathcal{R}_b = [0, 1]$. In fact, it can be verified that all random substitutions with the same substitution matrix as the square of Fibonacci give rise to Rauzy fractals which are intervals, with the exception of the enantiomorphic pair of deterministic substitutions whose intervals have fractal boundary.

Now, let $\vartheta'_{\mathbf{P}}$ be a (non-degenerate) random substitution defined over the set-valued substitution $\vartheta'\colon a \mapsto \{aab, aba\}$, $b \mapsto \{ab, ba\}$. The Rauzy fractal associated with $\vartheta'_{\mathbf{P}}$ is the same as the Rauzy fractal associated with the random substitution $\vartheta_{\mathbf{P}}$. This highlights that removing a realisation from a random substitution need not alter the associated Rauzy fractal.

Example 6.3. Let $\vartheta_{\mathbf{P}}$ be a (non-degenerate) random substitution defined over the set-valued substitution $\vartheta\colon a \mapsto \{abbaabaa\}$, $b \mapsto \{abaab, baaab\}$. The Rauzy fractal associated with $\vartheta_{\mathbf{P}}$ consists of an infinite union of disjoint subintervals. A plot of the subtiles $\mathcal{R}_a$ and $\mathcal{R}_b$ is provided in Figure 3.

We conclude by presenting some consequences of our results for the Rauzy fractal of the random tribonacci substitution. Since the random tribonacci substitution is defined over a three-letter alphabet, it gives rise to a Rauzy fractal with Hausdorff dimension two.

Example 6.4. Given $p \in (0,1)$, let $\vartheta_{\mathbf{P}} = (\vartheta, \mathbf{P})$ be the the random tribonacci substitution, namely, the random substitution given by

$$\vartheta_{\mathbf{P}}\colon \begin{cases} a \mapsto \begin{cases} ab & \text{with probability } p, \\ ba & \text{with probability } 1-p, \end{cases} \\ b \mapsto ac \quad \text{with probability } 1, \\ c \mapsto a \quad \text{with probability } 1. \end{cases}$$

Let $h\colon \mathbb{H} \mapsto \mathbb{H}$ denote the action of the substitution matrix on the contracting plane and, for each $a_i \in \mathcal{A}$, let $h_{a_i} = h(x) + \pi(\psi(a_i))$. By Theorem 4.8, the subtiles $\mathcal{R}_a$, $\mathcal{R}_b$ and $\mathcal{R}_c$ of the Rauzy fractal of $\vartheta_{\mathbf{P}}$ are the unique, non-empty, compact sets satisfying the graph-directed iterated function system

$$\begin{aligned} \mathcal{R}_a &= h(\mathcal{R}_a) \cup h_b(\mathcal{R}_a) \cup h(\mathcal{R}_b) \cup h(\mathcal{R}_c) \\ \mathcal{R}_b &= h(\mathcal{R}_a) \cup h_a(\mathcal{R}_a) \\ \mathcal{R}_c &= h_a(\mathcal{R}_b). \end{aligned}$$

We note that the map h, and hence the maps h_a and h_b, are affine transformations, but are not similarities. In an analogous manner to random Fibonacci, the above system can be rewritten as an ordinary iterated function system for $\mathcal{R}_a$:

$$\mathcal{R}_a = h(\mathcal{R}_a) \cup h_b(\mathcal{R}_a) \cup h \circ h(\mathcal{R}_a) \cup h \circ h_a(\mathcal{R}_a) \cup h \circ h_a \circ h(\mathcal{R}_a) \cup h \circ h_a \circ h_a(\mathcal{R}_a).$$

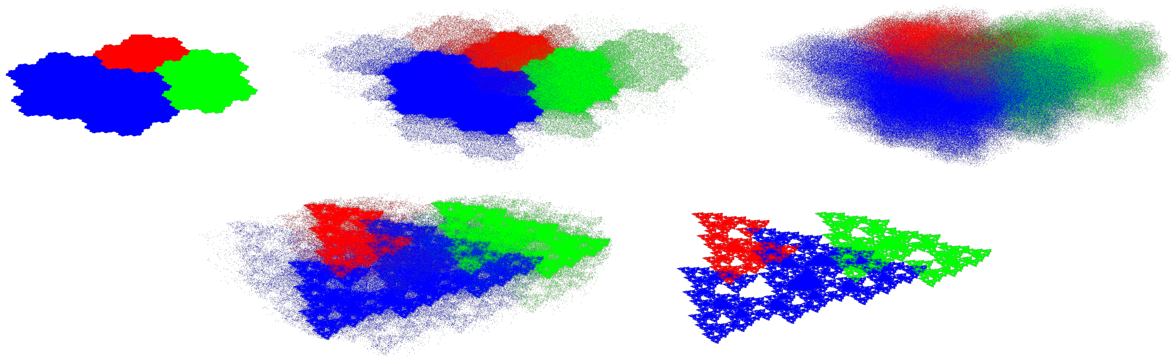

FIGURE 4. Evolution of Rauzy measures for the random tribonacci substitution as p goes from 1 to 0. From top-left to bottom-right, the values of p are 1, 15/16, 1/2, 1/16 and 0 respectively. Points were generated experimentally (justified by Proposition 5.11) according to the generating probabilities in such a way that point-densities approximate the mass distributions of each measure.

Now, let $\mu_{a,p}$, $\mu_{b,p}$ and $\mu_{c,p}$ denote the Rauzy measures associated with the letters a, b and c, respectively. By Proposition 5.9, these measures are weakly continuous with respect to p. Their mass distributions are plotted for a selection of p in Figure 4.

Acknowledgements

PG acknowledges support from DFG grant GO 3794/1-1. AM extends thanks to EPSRC DTP and the University of Birmingham for financial support, and SFB 1283 and the Universität Bielefeld for funding research visits to Bielefeld, where part of this work was undertaken. AM and TS would also like to thank the Birmingham–Leiden collaboration fund of the University of Birmingham and EPSRC grant EP/Y023358/1, which also supported this work.

References

[1] B. Adamczewski. Balances for fixed points of primitive substitutions. *Theoret. Comput. Sci.*, 307:47–75, 2003.

[2] S. Akiyama and P. Arnoux. *Substitution and Tiling Dynamics: Introduction to Self-inducing Structures*. Lecture Notes in Mathematics. Springer Cham, 2017.

[3] S. Akiyama, M. Barge, V. Berthé, J.-Y. Lee, and A. Siegel. On the pisot substitution conjecture. In J. Kellendonk, D. Lenz, and J. Savinien, editors, *Mathematics of Aperiodic Order*, pages 33–72. Springer Basel, 2015.

[4] M. Baake, F. Gähler, and J. Mazác. On the Fibonacci tiling and its modern ramifications. To appear in *Proceedings of the International Conference on Quasicrystals (ICQ15)*, 2024.

[5] M. Baake, A. Gorodetski, and J. Mazác. A naturally appearing family of cantorvals. Preprint. Available at arXiv:2401.05372.

[6] M. Baake and U. Grimm. *Aperiodic Order. Vol. 1: A Mathematical Invitation*. Cambridge University Press, 2013.

[7] M. Baake and U. Grimm. Fourier transform of Rauzy fractals and point spectrum of 1D Pisot inflation tilings. *Doc. Math.*, 25:2303–2337, 2020.

[8] M. Baake, T. Spindeler, and N. Strungaru. Diffraction of compatible random substitutions in one dimension. *Indag. Math. (N.S.)*, 29:1031–1071, 2018.

[9] M. Barge. Pure discrete spectrum for a class of one-dimensional substitution tiling systems. *Discrete Contin. Dyn. Syst.*, 36:1159–1173, 2016.

[10] G. Beer. *Topologies on Closed and Closed Convex Sets*. Kluwer Academic Publishers, 1993.

[11] V. Berthé and V. Delecroix. Beyond substitutive dynamical systems: S-adic expansions. *Numeration and substitution 2012*, pages 81–123, 2014.

[12] P. Centore and E. Vrscay. Continuity of attractors and invariant measures for iterated function systems. *Canadian Mathematical Bulletin*, 37(3):315–329, 1994.

[13] G. B. Escolano, N. Mañibo, and E. D. P. Miro. Mixing properties and entropy bounds of a family of Pisot random substitutions. *Indag. Math. (N.S.)*, 33:965–991, 2022.

[14] R. Fokkink, D. Rust, and V. Salo. Automorphism groups of random substitution subshifts. To appear in *Indag. Math.*, 2024.

[15] C. Godrèche and J. Luck. Quasiperiodicity and randomness in tilings of the plane. *J. Stat. Phys.*, 55:1–28, 1989.

[16] P. Gohlke. Inflation word entropy for semi-compatible random substitutions. *Monatsh. Math.*, 192:93–110, 2020.

[17] P. Gohlke. *Aperiodic Order and Singular Spectra*. PhD thesis, Bielefeld University, 2021.

[18] P. Gohlke, A. Mitchell, D. Rust, and T. Samuel. Measure theoretic entropy of random substitution subshifts. *Ann. Henri Poincaré*, 2022.

[19] P. Gohlke and T. Spindeler. Ergodic frequency measures for random substitutions. *Studia Math.*, 255:265–301, 2020.

[20] A. Hof. Uniform distribution and the projection method. *Quasicrystals and discrete geometry*, pages 201–206, 1998.

[21] W. Huang and X. Ye. Generic eigenvalues, generic factors and weak disjointness. In *Dynamical systems and group actions*, volume 567 of *Contemp. Math.*, pages 119–142. Amer. Math. Soc., Providence, RI, 2012.

[22] G. Keller. Maximal equicontinuous generic factors and weak model sets. *Discrete Contin. Dyn. Syst.*, 40(12):6855–6875, 2020.

[23] D. Koslicki. *Substitution Markov chains with applications to molecular evolution.* PhD thesis, Pennsylvania State Univ., 2012.

[24] D. Koslicki and M. Denker. Substitution Markov chains and Martin boundaries. *Rocky Mountain J. Math*, 46:1963–1985, 2016.

[25] T. Kurtz, R. Lyons, R. Pemantle, and Y. Peres. A conceptual proof of the Kesten-Stigum theorem for multi-type branching processes. *Classical and modern branching processes*, pages 181–185, 1997.

[26] R. D. Mauldin and M. Urbanski. *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*. Cambridge Tracts in Mathematics. Cambridge University Press, 2003.

[27] E. D. P. Miro, D. Rust, L. Sadun, and G. S. Tadeo. Topologiocal mixing of random substitutions. *Israel J. Math.*, 255:123–153, 2022.

[28] A. Mitchell. On word complexity and topological entropy of random substitution subshifts. Preprint. Available at arXiv:2305.04817.

[29] A. Mitchell and A. Rutar. Multifractal analysis of measures arising from random substitutions. To appear in *Commun. Math. Phys.*, 2024.

[30] M. Moll. Diffraction of random noble means words. *J. Stat. Phys.*, 156:1221–1236, 2014.

[31] J. Peyrière. Substitutions aléatoires itérées. *Séminaire de théorie des nombres de Bordeaux*, 80–81:1701–1709, 1981.

[32] M. Queffélec. *Substitution dynamical systems: Spectral analysis*. Springer, 2nd edition, 2010.

[33] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[34] D. Rust. Periodic points in random substitution subshifts. *Monatsh. Math*, 193:683–704, 2020.

[35] D. Rust and T. Spindeler. Dynamical systems arising from random substitutions. *Indag. Math. (N.S.)*, 29:1131–1155, 2018.

[36] M. Schlottmann. Cut-and-project sets in locally compact abelian groups. *Quasicrystals and Discrete Geometry*, pages 247–264, 1998.

[37] A. Siegel. Spectral theory and geometric representation of substitutions. In *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Math.*, pages 199–252. Springer, Berlin, 2002.

[38] A. Siegel and J. M. Thuswaldner. *Topological properties of Rauzy fractals*. Mémoires de la Société Mathématique de France. Société mathématique de France, 2009.

[39] B. Solomyak. Dynamics of self-similar tilings. *Ergod. Dyn. Sys.*, 17:695–738, 1997.