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The Sierpiński gasket as the Martin boundary of a non-isotropic Markov chain

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Abstract
In 2012 Lau and Ngai, motivated by the work of Denker and Sato, gave an example of an isotropic Markov chain on the set of finite words over a three letter alphabet, whose Martin boundary is homeomorphic to the Sierpiński gasket. Here, we extend the results of Lau and Ngai to a class of non-isotropic Markov chains.

Setting and construction of the Markov chain

Let ϑ denote the empty word, $\Sigma^0 :=$ and $\Sigma := \{1,2,3\}$. We consider $\Sigma^* := \bigcup_{n\in\mathbb{N}_0} \Sigma^n$, the set of all finite words over the alphabet Σ. Analogously, let Σ^∞ be the space of all infinite words. Furthermore, for $n \in \mathbb{N}$ let $V^n := \{1^n, 2^n, 3^n\}$ and $\tilde{\Sigma}^n := \Sigma^n \setminus V^n$.

Let $\mathcal{K}$ denote the Sierpiński gasket. The *standard projection* $\pi\colon \Sigma^\infty \to \mathcal{K}$ is defined by $\pi(\mathbf{x}) = \lim_{n\to\infty} S_{i_1} \circ \cdots \circ S_{i_n}(x_0)$ for $\mathbf{x} = i_1 i_2 \ldots \in \Sigma^\infty$, $x_0 \in \mathbb{R}^2$ arbitrary. Two states $\mathbf{x}, \mathbf{y} \in \Sigma^\infty$ are called *π-equivalent*, denoted by $\mathbf{x} \sim_\pi \mathbf{y}$, if $\pi(\mathbf{x}) = \pi(\mathbf{y})$.

Let $p \in (0, 1/2)$ and $q := 1 - 2p$. We define the transition matrix $P\colon \Sigma^* \times \Sigma^* \to [0,1]$ by

$$P(u,v) := \begin{cases} p & \text{if } u = \omega i j^{n-k} \in \tilde{\Sigma}^n, v \in \Sigma^n \\ & \text{and } u \sim_\pi v \text{ or } v = \omega i j^{n-k-1} i \text{ for distinct } i,j \in \Sigma, \\ q & \text{if } u = \omega i j^{n-k} \in \tilde{\Sigma}^n, v \in \Sigma^n \\ & \text{and } v = \omega i j^{n-k-1} l \text{ for pairwise distinct } i,j,l \in \Sigma, \\ 1/3 & \text{if } u \in V^n \text{ and } v = ui \text{ for } i \in \Sigma, \\ 0 & \text{otherwise}. \end{cases}$$

We denote by $(X_n)_{n\in\mathbb{N}_0}$ the Markov chain with origin ϑ, state space Σ^* and transition probability matrix P. For $p = 1/3$, the above Markov chain coincides with the one in [5].

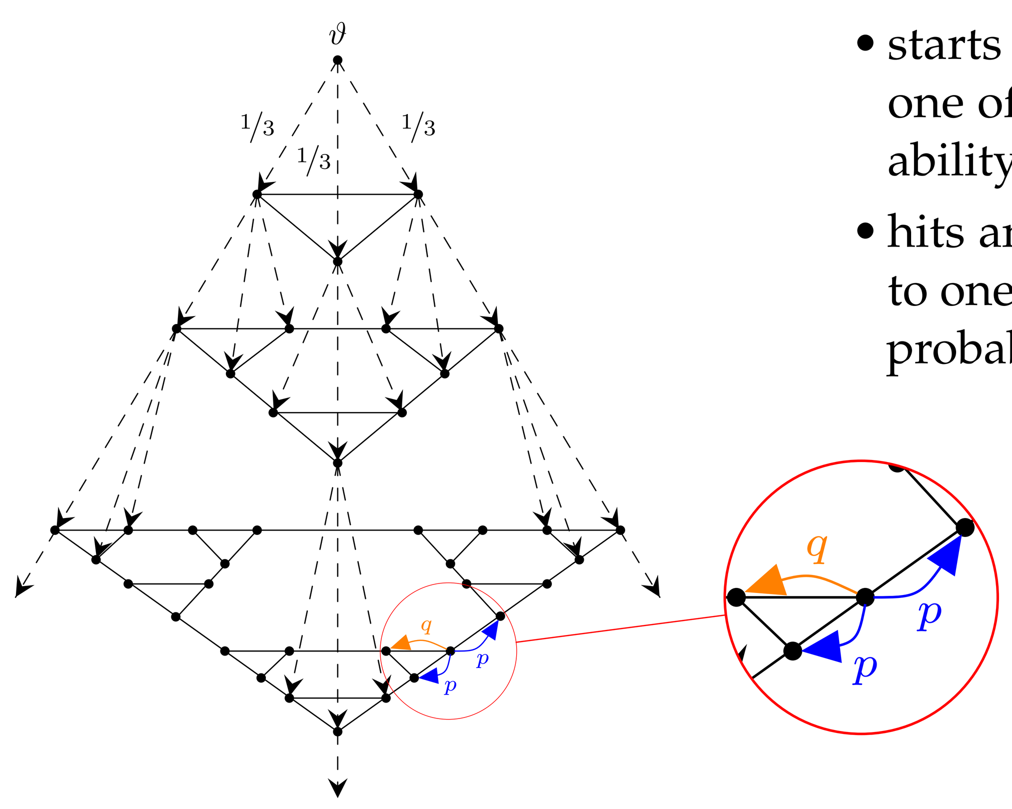

Figure 1: Transition probabilities of the Markov chain.

If the chain

- starts at a word in $\tilde{\Sigma}^n$, then it walks to one of its three neighbours with probability p or q.
- hits an element $u \in V^n$, then it moves to one of its kids on the next level with probability $1/3$.

Hitting probabilities

We denote the probability, conditioned on starting at a state $x \in \Sigma^*$, to eventually arrive at a state $y \in \Sigma^*$ by

$$\rho(x,y) := \mathbb{P}(\exists\, k \in \mathbb{N}_0 : X_k = y \,|\, X_0 = x).$$

We are concerned with computing the probability to be absorbed by i^n, $i \in \Sigma$, when starting at some $x \in \Sigma^n$. To this end we define $\boldsymbol{\rho}\colon \Sigma^* \to [0,1]^3$ by

$$\boldsymbol{\rho}(x) := [\rho(x,1^n), \rho(x,2^n), \rho(x,3^n)].$$

For $n \geq 2$ define

$$\begin{aligned} \alpha_n &:= \rho(12^{n-1}, 1^n), \\ \beta_n &:= \rho(12^{n-1}, 2^n), \\ \gamma_n &:= \rho(12^{n-1}, 3^n), \end{aligned} \qquad \text{and} \qquad \begin{aligned} a_n &:= \rho(1^{n-1}2, 1^n), \\ b_n &:= \rho(1^{n-1}2, 2^n), \\ c_n &:= \rho(1^{n-1}2, 3^n). \end{aligned}$$

Figure 2 shows the above hitting probabilities for $n = 2$ and $n = 3$.

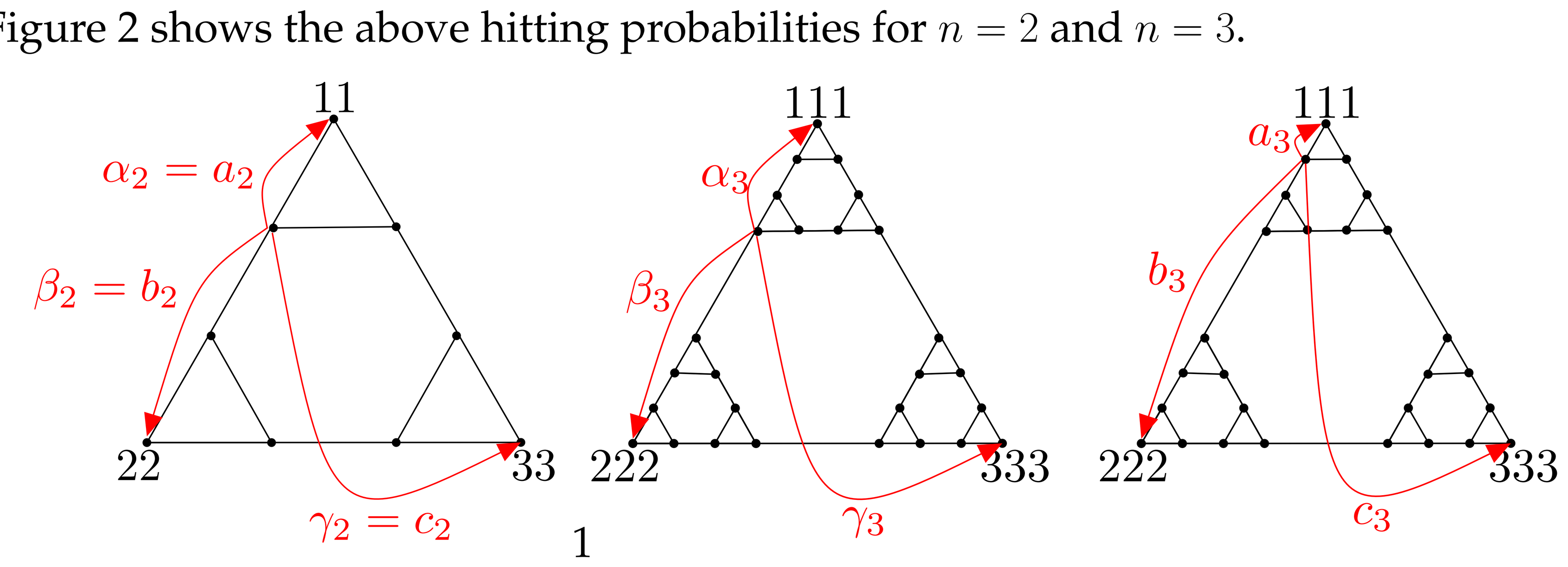

Figure 2: Hitting probabilities a_n, b_n, c_n and $\alpha_n, \beta_n, \gamma_n$ for $n = 2$ and $n = 3$.

Furthermore, define

$$A_n^{(1)} := \begin{bmatrix} 1 & 0 & 0 \\ \alpha_n & \beta_n & \gamma_n \\ \alpha_n & \gamma_n & \beta_n \end{bmatrix}, \quad A_n^{(2)} := \begin{bmatrix} \beta_n & \alpha_n & \gamma_n \\ 0 & 1 & 0 \\ \gamma_n & \alpha_n & \beta_n \end{bmatrix}, \quad A_n^{(3)} := \begin{bmatrix} \beta_n & \gamma_n & \alpha_n \\ \gamma_n & \beta_n & \alpha_n \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix $A_n^{(i)}$ contains the probabilities that the process, starting in one of the three vertices of one of the three subgraphs of Σ^n, reaches V^n.

Denote the standard i-th row unit vector of $\mathbb{R}^3$ by $\boldsymbol{e}_i$ and let $x = i_1 \ldots i_n \in \Sigma^n$. With the above, we can express the hitting probability vector $\boldsymbol{\rho}(x)$ as a matrix product, namely,

$$\boldsymbol{\rho}(x) = \boldsymbol{e}_{i_n} A_2^{(i_{n-1})} \cdots A_n^{(i_1)}. \tag{1}$$

Theorem 1: Limits of the hitting probabilities, [5, 3]

$$\lim_{n\to\infty} (\alpha_n, \beta_n, \gamma_n) = (2/5, 2/5, 1/5) \quad \text{and} \quad \lim_{n\to\infty} (a_n, b_n, c_n) = (1, 0, 0).$$

Interestingly, the limits of these sequences are independent of the chosen parameter $p \in (0, 1/2)$ and are equal to the ones obtained in the isotropic case considered in [5].

With these limits we can

- prove that the random matrix product in (1) converges.
- use a representation of the Martin kernel in terms of these hitting probabilities to extend the kernel to the infinite words.
- prove that the Martin metric can also be extended to the infinite words.
- find an analogue of the $(1/5)$-$(2/5)$-rule for the P-harmonic functions.

Main results

The main results are the following.

Theorem 2: Sierpiński gasket as Martin boundary, [5, 3]

The Martin boundary $\mathcal{M}$ of $(X_n)_{n\in\mathbb{N}_0}$ is homeomorphic to the Sierpiński gasket $\mathcal{K}$.

Theorem 3: Minimal Martin boundary, [5, 3]

The minimal Martin boundary $\mathcal{M}_{\text{min}}$ of $(X_n)_{n\in\mathbb{N}_0}$ is homeomorphic to $\{1^\infty, 2^\infty, 3^\infty\}$.

Theorem 4: Space of P-harmonic functions, [5, 3]

The P-harmonic functions on the Martin boundary coincide with the canonical harmonic functions of [4, 6], the space of P-harmonic functions on the Sierpińki gasket $\mathcal{K}$ is three-dimensional.

Future work

It would be of interest to investigate

- what happens if one rotates the directions of the transition probabilities p and q.
- to prefer a clockwise or anti-clockwise direction in the subgraphs on each level.
- to choose different probabilities for each direction.
- a modified trasition operator such that the minimal Martin boundary can be an arbitrary Borel subset.

Simulations indicate that for the first three points the same results as in Theorem 1 may hold.

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