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Towards a Holographic-Type Perspective in the Analysis of Complex-System Dynamics

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Abstract: By operating with the Scale Relativity Theory by means of two scenarios (Schrödinger and Madelung-type scenarios) in the dynamics of complex systems, we can achieve a description of these complex systems through a holographic-type perspective. Then, a gauge invariance of the Riccati type becomes functional in complex-system dynamics, which implies several consequences: conservation laws (in particular, for dynamics, the kinetic momentum conservation law), simultaneity and synchronization among the structural units' (belonging to a complex system) dynamics, and temporal patterns through harmonic mappings. Finally, an economic case analysis is highlighted.

Keywords: Scale Relativity Theory; Schrödinger-type scenario; Madelung-type scenario; SL(2R) group; harmonic mappings; fractal; multi-fractal; Riccati-type gauge

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1. Introduction

Determinism is not an indicator of regular (periodic motions, self-structures, etc.) or predictable behaviors in the dynamics of complex systems (for example, gene networks which interact through activation and inhibition, and a set of activated genes defines a tissue; the dynamics of a cell is made up of proteins which are in chemical reactions, and its evolution allows an adaptation to the environment; the brain is a collection of neurons that transmit electrical and chemical impulses to each other; the stock market implies brokers making trades that create global phenomena such as bubbles or crashes; a company is a set of people and organizations that interact with each other and which also interacts with its environment through stakeholders, etc.) [1,2]. In linear analysis, on which a complex system relies almost exclusively, unbounded predictability has been an intrinsic characteristic of complex-system dynamics. The evolution of non-linear analysis and the explaining of the laws governing chaos have shown that the reductionist method of analysis, which is applicable to complex systems, has been hitherto restricted, but its applicability has also been restricted. Unrestricted predictability is not a feature of complex systems; rather, it is a fundamental effect of their reduction by the linear approach. Only a handful of such behaviors—where non-linearity and chaos emphasize regular behaviors (superstructure, flammability, etc.), specifically, a generality in the laws dictating the dynamics of complex systems [3,4]—can be found.

The behaviors of complex systems are determined by the conditions in which they function and the restrictions to which they are subjected. The non-linearity and chaoticity of any complex system are determined by the interactions between its structural units. In such a situation, the generality of the dynamic laws of complex systems becomes functional and must be found in the employed mathematical procedures. There are authors who progressively discuss the holographic implementation involved in the description of complex-system dynamics (fractal/multifractal paradigm of nature) [5–9].

On the other hand, the standard physical models applied in the characterization of the dynamics of complex systems are founded on an otherwise unbased hypothesis of the differentiability of the physical quantities utilized to explain their evolution. Then, the positive outcome of these models must be understood progressively [5]. Differentiable and integrable mathematical procedures fail when they are intended to illustrate the dynamics of complex systems involving only non-linearity and chaoticity. For the purpose of describing the dynamics of these systems, it is required to expressly introduce the scale resolution concept in the physical variables and, moreover, into the fundamental equations of these dynamics [7–9].

It can be inferred that any physical variable, dependent on the usual mathematical procedures on space–time co-ordinates, will also depend on a scale resolution. As a consequence, any physical variable used to characterize the dynamics of complex systems will function as the limit of a family of mathematical functions, the function being non-differentiable for a zero-scale resolution and differentiable for a non-zero-scale resolution [7–9].

Therefore, there is a need for the evolution of both new geometric structures and physical theories, for which the laws of motion, invariant to time co-ordinate transformations, are also invariant to transformations with respect to the scale resolution. In our opinion, such an approach can be one based on the concept of the fractal/multifractal and the corresponding physical model depicted in the Scale Relativity Theory (SRT) [7–9]. From this perspective, the holographic implementation in the portrayal of complex-system dynamics will be made explicit based on the depiction of the dynamics of the structural units of any complex-system fractal/multifractal curves.

Some consequences are evident:

(i) Motions with constraints on the continuous and differentiable curves in a Euclidean space are replaced by free motions on the fractal/multifractal curves in a fractal/multifractal space;
(ii) Motion curves act both as the geodesics of a fractal/multifractal space and as the flow lines of a fractal/multifractal fluid;
(iii) The structural units of any complex system are replaced by their own geodesics, any external constraint being rendered as a choice of geodesics on the basis of local–global/whole–part compatibility, etc.;
(iv) For large time scales, relative to the inverse of the largest value of the Lyapunov exponent, deterministic trajectories can be substituted by families of potential trajectories, so that the concept of position will be replaced by that of probability density [3,4];
(v) Taking into account that the portrayal of complex-system dynamics is given by fractal/multifractal curves and that these curves have the self-similarity property, the "holographic" description of the complex-system dynamics become functional.

Then, two scenarios in the portrayal of complex-system dynamics become operational: one scenario is explained through the multifractal Schrödinger equation (Schrödinger-type scenario) and another through the multifractal hydrodynamic equation system (Madelung-type scenario). The two scenarios for describing complex-system dynamics are not mutually exclusive; moreover, they are complementary. For more details regarding the various mathematical procedures in complex-system descriptions, see [10–16].

Taking into account the fact that, in any of the scenarios, the symmetries are highlighted, in the current paper, several symmetries will be explained and their consequences will be discussed.

2. Gauge Invariances of the Riccati Type and Conservation Laws in Complex-System Dynamics

It is well-known that the dynamics of complex systems in the SRT [7–9] can be described through a Schrödinger-type scenario explained through the differential equation (multifractal Schrödinger equation)

$$\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \partial_l \partial^l \Psi + i\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial_t \Psi = 0, \quad (1)$$

where

$$\partial_t = \frac{\partial}{\partial t}, \ \partial_l = \frac{\partial}{\partial x^l}, \ \partial_l \partial^l = \frac{\partial^2}{\partial x_l^2} \quad (2)$$

In the above relations:

- Ψ is the states function;
- dt is the scale resolution;
- x^l is the multifractal spatial co-ordinate;
- t is the non-multifractal temporal co-ordinate with the role of an affine parameter;
- λ is a parameter associated with the fractal/multifractal-non-fractal/non-multifractal scale transition;
- $f(\alpha)$ is the singularity spectrum with a singularity index of order $\alpha = \alpha(D_F)$;
- D_F is the fractal dimension of the motion curves [3,4,6].

There are many modes of the definition of fractal dimensions: the Kolmogorov fractal dimension, the Hausdorff–Besikovitch fractal dimension, etc. [3,4,6]. It is noted that, by selecting one of these definitions and operating with it, the fractal dimension must be constant and arbitrary for the entirety of the dynamical analysis. Moreover, operating with $f(\alpha)$, the following possibilities emerge in the complex-system dynamics:

- It is possible to identify the "areas" of the complex-system dynamics that are characterized by a certain fractal dimension (monofractal complex-system dynamics);
- It is possible to identify the number of "areas" whose fractal dimensions are situated within an interval of values (multifractal complex-system dynamics);
- It is possible to identify the classes of universality in the complex-system dynamic laws, even when the regular or strange attractors have different aspects.

In such a context, through a convenient choice of the singularity spectrum $f(\alpha)$, concerted with the scale resolution, various functional dynamics regimes can be highlighted—period-doubling regimes, damping-oscillation regimes, intermittence regimes, and quasi-periodicity regimes, as we will show in the following (in particular, a specific pattern will be given).

The multifractal Schrödinger equation admits, by an extension of [10], an extra set of symmetries, such as a SL(2,R)-type group [11]. A general discussion on the SL(2,R)-type groups and their implications in quantum mechanics and general relativity can be found in [3,4,10]. It is mentioned that, in the standard Schrödinger equation, special symmetries and their implications are given in [12–14].

For what follows, it is necessary to construct the finite equations of the specific SL(2,R) group, according to [15,16]. Operating with the variables (t, r) as above, the finite equations are given by:

$$t \to \frac{\alpha t + \beta}{\gamma t + \delta}; \ r \to \frac{r}{\gamma t + \delta} \quad (3)$$

Then, the infinitesimal generators are given by:

$$X_1 = \frac{\partial}{\partial t}, \ \ X_2 = t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \ \ X_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} \quad (4)$$

and satisfy the structural equations:

$$[X_1, X_2] = X_1,\ \ [X_2, X_3] = X_3,\ \ [X_3, X_1] = -2X_2 \tag{5}$$

The group possesses an invariant function, extracted as the solution of a partial differential equation:

$$\begin{gathered}(cX_1 + 2bX_2 + aX_3)f(t, r) = 0 \\ \left(at^2 + 2bt + c\right)\frac{\partial f}{\partial t} + (at + b)r\frac{\partial f}{\partial r} = 0\end{gathered} \tag{6}$$

The general solution of this equation is

$$\frac{r^2}{at^2 + 2bt + c} \tag{7}$$

The ratio found in relation (7) is constant and, moreover, it depicts the various paths of transitivity of the action given by (4).

We must notice that the content of time in Equation (3) is not classical anymore, at least not in general, being a ratio of co-ordinates representing two uniform motions. The second Equation (3) can be taken as representing the content of a spatial co-ordinate of the motion in terms of the classical co-ordinate of the uniform motion. For more details, see [10,15,16].

On the other hand, by choosing Ψ of the form (the Madelung's choosing)

$$\Psi = \sqrt{\rho}e^{is} \tag{8}$$

where $\sqrt{\rho}$ is the amplitude and s is the phase, and introducing the real velocity fields (V_D^i—differentiable velocity field, V_F^i—non-differentiable velocity field):

$$V_D^i = 2\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial^i s \tag{9}$$

$$V_F^i = i\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial^i \ln\rho \tag{10}$$

The multifractal Schrödinger equation is reduced to the multifractal hydrodynamic equation system:

$$\partial_t V_D^i + V_D^l \partial_l V_D^i = -\partial^i Q \tag{11}$$

$$\partial_t \rho + \partial_l\left(\rho V_D^l\right) = 0 \tag{12}$$

with Q as the multifractal-specific potential:

$$Q = -2\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\frac{\partial^l\partial_l\sqrt{\rho}}{\sqrt{\rho}} = -V_F^i V_F^i - \frac{1}{2}\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial_l V_F^l \tag{13}$$

Equation (11) defines the multifractal-specific momentum conservation law, while Equation (12) defines the multifractal-state density conservation law. The multifractal-specific potential (13) induces the multifractal-specific force:

$$F^i = -\partial^i Q = -2\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\partial^i\frac{\partial^l\partial_l\sqrt{\rho}}{\sqrt{\rho}} \tag{14}$$

Relation (14) is a measure of the multifractality of the motion curves of the dynamics.

The main conclusions which result from Equations (11)–(13) are given in [8,9]. Now, using the tensor:

$$\hat{\tau}^{il} = 2\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\rho\partial^i\partial^l \ln\rho \tag{15}$$

Equation (14) takes the form:

$$\rho\partial^i Q = \partial_l\hat{\tau}^{il} \tag{16}$$

Moreover, since the tensor $\hat{\tau}^{il}$ can also be expressed as:

$$\hat{\tau}^{il} = \eta\left(\partial_l V_F^i + \partial_i V_F^l\right) \tag{17}$$

with

$$\eta = \lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\rho \tag{18}$$

In these conditions, a multifractal linear constitutive equation for a multifractal "viscous fluid" becomes functional, offering, at the same time, the reason for an original interpretation of the coefficient η as a multifractal dynamic viscosity of the multifractal fluid.

In order to have a better grasp of these mathematical facts, it is necessary to return to the transformation (3) and approach it from the point of view of multifractal physics.

Firstly, let us observe that the multifractal-specific force (14) is defined with the help of a gradient. As an immediate consequence, if 'r' denotes the distance of the moving complex-system structural unit from the center of the multifractal-specific force, then

$$r^2 d\theta = \dot{a} dt \qquad \dot{a}\frac{dt}{d\theta} = r^2 \qquad \dot{a} = const. \tag{19}$$

where θ is the central angle of the position vector of the moving complex-system structural unit with respect to the center of the multifractal-specific force and $\dot{a}$ is the constant of the multifractal kinetic moment.

Now, since according to (7),

$$\frac{r^2}{at^2 + 2bt + c} = L = const.\,, \tag{20}$$

Then, from (20) and (19), through the substitutions

$$\frac{dt}{d\theta} = \dot{w},\ \ \frac{Lat^2}{\dot{a}} = \frac{1}{M}w^2,\ \ \frac{2Lbt}{\dot{a}} = -2\frac{R}{M}w,\ \ \frac{Lc}{\dot{a}} = K \tag{21}$$

the following Riccati-type differential equation is satisfied (i.e., we operate here with a Riccati-type gauge):

$$\dot{w} - \frac{1}{M}w^2 + 2\frac{R}{M}w - K = 0. \tag{22}$$

For obviously physical reasons, it is necessary to seek out the most general solution of that equation. José Cariñena et al. offer us a pass in a short but modern and pertinent review of the integrability of Riccati's equation [17]. For the current needs, it should be noted that the complex numbers

$$w_0 \equiv R + iM\Omega,\ \ w_0^* \equiv R - iM\Omega;\ \Omega^2 = \frac{K}{M} - \left(\frac{R}{M}\right)^2 \tag{23}$$

The roots of the quadratic polynomial on the right side of Equation (22) are the two solutions (constants) of the equation: being constants, their derivative is zero; being roots of the right-hand polynomial, they cancel. Therefore, first, we perform the homographic transformation:

$$z = \frac{w - w_0}{w - w_0^*} \tag{24}$$

and now it can easily be seen by a direct calculation that z is a solution of the linear and homogeneous equation of the first order:

$$\dot{z} = 2i\Omega z \therefore z(t) = z(0)e^{2i\Omega t} \tag{25}$$

Therefore, if we conveniently express the initial condition $z(0)$, we can give the general solution of Equation (22) by simply inverting the transformation (24), with the result

$$w = \frac{w_0 + re^{2i\Omega(t-t_r)}w_0^*}{1 + re^{2i\Omega(t-t_r)}} \tag{26}$$

where r and t_r are the two real constants that characterize the solution. Using Equation (23), we can put this solution in real terms, i.e.,

$$\begin{aligned} z = \quad & R + M\Omega\Big(\tfrac{2r\sin[2\Omega(t-t_r)]}{1+r^2+2r\cos[2\Omega(t-t_r)]} \\ & + i\tfrac{1-r^2}{1+r^2+2r\cos[2\Omega(t-t_r)]}\Big) \end{aligned} \tag{27}$$

which highlights a frequency modulation through what we would call a Stoler transformation [15,16] which leads us to a complex form of this parameter. More than that, if we make the notation

$$r \equiv \cot h\tau, \tag{28}$$

Equation (27) becomes

$$z = R + M\Omega h \tag{29}$$

where h is given by

$$h = -i\frac{\cos h\tau - e^{-2i\Omega(t-t_m)}\sin h\tau}{\cos h\tau + e^{-2i\Omega(t-t_m)}\sin h\tau}. \tag{30}$$

The meaning of this complex parameter will be clear a little later. For the moment, let us note that any dynamic process appears here as a frequency-modulation process by means of a gauge invariance of the Riccati type, imposed through the multifractal kinetic momentum conservation law.

3. Gauge Invariances of the Riccati Type and Simultaneity in Complex-System Dynamics

Consider a complex system revolving in the multifractal-specific force. It can be viewed as a swarm of structural units, such as a swarm highlighting multifractal laws (for a classical case, see [18]), as long as it is viewed as a swarm of free structural units. In the first part of Equation (3), this means that the structural units are considered simultaneously. Then, each structural unit can be identified in the swarm by α, β, γ, δ, or by three non-homogeneous co-ordinates, according to [9]. The simultaneity in the motion of the swarm of structural units can be differentially characterized, giving a Riccati-type equation:

$$d\frac{\alpha t + \beta}{\gamma t + \delta} = 0, \;\; dt = \omega^1 t^2 + \omega^2 t + \omega^3 \tag{31}$$

Thus, it is sufficient to define a coframe of the SL(2R) algebra:

$$\omega^1 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\delta - \beta\gamma};\; \omega^2 = \frac{\alpha d\delta - \delta d\alpha + \beta d\gamma - \gamma d\beta}{\alpha\delta - \beta\gamma};\; \omega^3 = \frac{\beta d\delta - \delta d\beta}{\alpha\delta - \beta\gamma} \tag{32}$$

Such an algebra can be verified through the Maurer–Cartan equations:

$$\begin{gathered} d \wedge \omega^1 - \omega^1 \wedge \omega^2 = 0 \\ d \wedge \omega^2 + 2\left(\omega^3 \wedge \omega^1\right) = 0 \\ d \wedge \omega^3 - \omega^2 \wedge \omega^3 = 0 \end{gathered} \tag{33}$$

which describe the coframe.

According to [19], the right member of (31) is an integral. The Cartan–Killing metric of this coframe is given by the quadratic form $(\omega^2/2)^2 - \omega^1\omega^3$, which defines a Riemannian three-dimensional space. The geodesics of such a space are:

$$\omega^1 = a^1(d\theta);\ \omega^2 = 2a^2(d\theta);\ \omega^3 = a^3(d\theta) \tag{34}$$

where $a^{1,2,3}$ are constants and θ is the affine parameter of the geodesics. Taking into account the geodesics, (31) is an ordinary differential equation of the Riccati type:

$$\frac{dt}{d\theta} = a^1 t^2 + 2a^2 t + a^3 \tag{35}$$

This equation can be identified with (19). Mathematically, this requires an ensemble generated by a harmonic mapping between the positions in space and the structural units. Following the same line of thought previously presented, a solution of the same type as (30) can be highlighted.

4. Gauge Invariances of the Riccati Type and Synchronizations in Complex-System Dynamics

According to the meanings of the state function Ψ from the SRT, a physical significance is only attached to the density of the state $\rho = \Psi\,\overline{\Psi}$. In such a context, if $\Psi = a + ib$, then the constant density of states can be localized inside the unity radius circle

$$x^2 + y^2 = 1 \tag{36}$$

where

$$\frac{a^2}{\rho} = x^2,\ \frac{b^2}{\rho} = y^2 \tag{37}$$

Now, the Lobachevsky plane metric can be generated as a Caylean metric of a Euclidean plane, for which the absoluteness is the circle (36). In such a context, by following the method described in [15,16], the following is obtained:

$$-\frac{ds^2}{k^2} = \frac{\Omega(dX, dX)}{\Omega(X, X)} - \frac{\Omega^2(X, dX)}{\Omega^2(X, X)}, \tag{38}$$

In the above relation, $\Omega(X, Y)$ is the duplication of $\Omega(X, X)$ and k is a constant connected to the space curvature. Therefore:

$$\frac{ds^2}{k^2} = \frac{(1-y^2)dx^2 + 2xydxdy + (1-x^2)dy^2}{(1-x^2-y^2)^2}, \tag{39}$$

where

$$\begin{aligned} \Omega(X, X) &= 1 - x^2 - y^2 \\ \Omega(X, dX) &= -xdx - ydy \\ \Omega(dX, dX) &= -dx^2 - dy^2 \end{aligned} \tag{40}$$

Now, by means of substitutions:

$$x = \frac{h\overline{h} - 1}{h\overline{h} + 1}, \qquad y = \frac{h + \overline{h}}{h\overline{h} + 1} \tag{41}$$

with

$$h = u + iv, \qquad \overline{h} = u - iv \tag{42}$$

The metric (39) takes the form of a Poincaré metric of the superior complex plane:

$$\frac{ds^2}{k^2} = 4\frac{dhd\overline{h}}{\left(h-\overline{h}\right)^2} = \frac{du^2+dv^2}{v^2}, \tag{43}$$

The metric (43) induces the simply transitive group in the quantities h and $\overline{h}$, whose actions are:

$$\begin{aligned} h &\leftrightarrow \tfrac{ah+b}{ch+d}, \\ \overline{h} &\leftrightarrow \tfrac{a\overline{h}+b}{c\overline{h}+d}, \end{aligned} \tag{44}$$

The structure of this group is typical of SL(2R), i.e.,

$$\begin{aligned} \left[B^1, B^2\right] &= B^1, \\ \left[B^2, B^3\right] &= B^3, \\ \left[B^3, B^1\right] &= -2B^2 \end{aligned} \tag{45}$$

where B^l are the infinitesimal generators of the group:

$$\begin{aligned} B^1 &= \tfrac{\partial}{\partial h} + \tfrac{\partial}{\partial \overline{h}} \\ B^2 &= h\tfrac{\partial}{\partial h} + \overline{h}\tfrac{\partial}{\partial \overline{h}} \\ B^3 &= h^2\tfrac{\partial}{\partial h} + \overline{h}^2\tfrac{\partial}{\partial \overline{h}} \end{aligned} \tag{46}$$

and admits the two-form (43).

Since (43) is invariant with respect to the group $SL(2R)$ [9,15,16], this group can be assimilated with a "synchronization" group between the different structural units of the complex system.

5. Harmonic Mappings and Working Regimes in Complex-System Dynamics

In what follows, complex-system dynamics will be generated through harmonic mappings. Let it be assumed that the complex-system dynamics can be depicted through the variables $\left(Y^j\right)$ for which the multifractal metric below applies:

$$h_{ij}dY^i dY^j \tag{47}$$

In a usual space of the multifractal metric:

$$\gamma_{\alpha\beta}dX^\alpha dX^\beta \tag{48}$$

In these conditions, the field equations are generated by the multifractal Lagrangian:

$$L = \gamma^{\alpha\beta}h_{ij}\frac{dY^i dY^j}{\partial X^\alpha \partial X^\beta} \tag{49}$$

In the present case, (47) is defined through (43). Therefore, if the variational principle

$$\delta\int L\sqrt{\gamma}d^3x = 0 \tag{50}$$

holds true for $\gamma = \left|\gamma_{\alpha\beta}\right|$, then the multifractal Euler equations corresponding to the variational principle (45) become:

$$\begin{aligned} \left(h-\overline{h}\right)\nabla(\nabla h) &= 2(\nabla h)^2 \\ \left(h-\overline{h}\right)\nabla\left(\nabla\overline{h}\right) &= 2\left(\nabla\overline{h}\right)^2 \end{aligned} \tag{51}$$

with the solution:

$$h = \frac{\cosh(\Phi/2) - \sinh(\Phi/2)e^{-i\alpha}}{\cosh(\Phi/2) + \sinh(\Phi/2)e^{-i\alpha}},\ \alpha \in \mathbb{R} \tag{52}$$

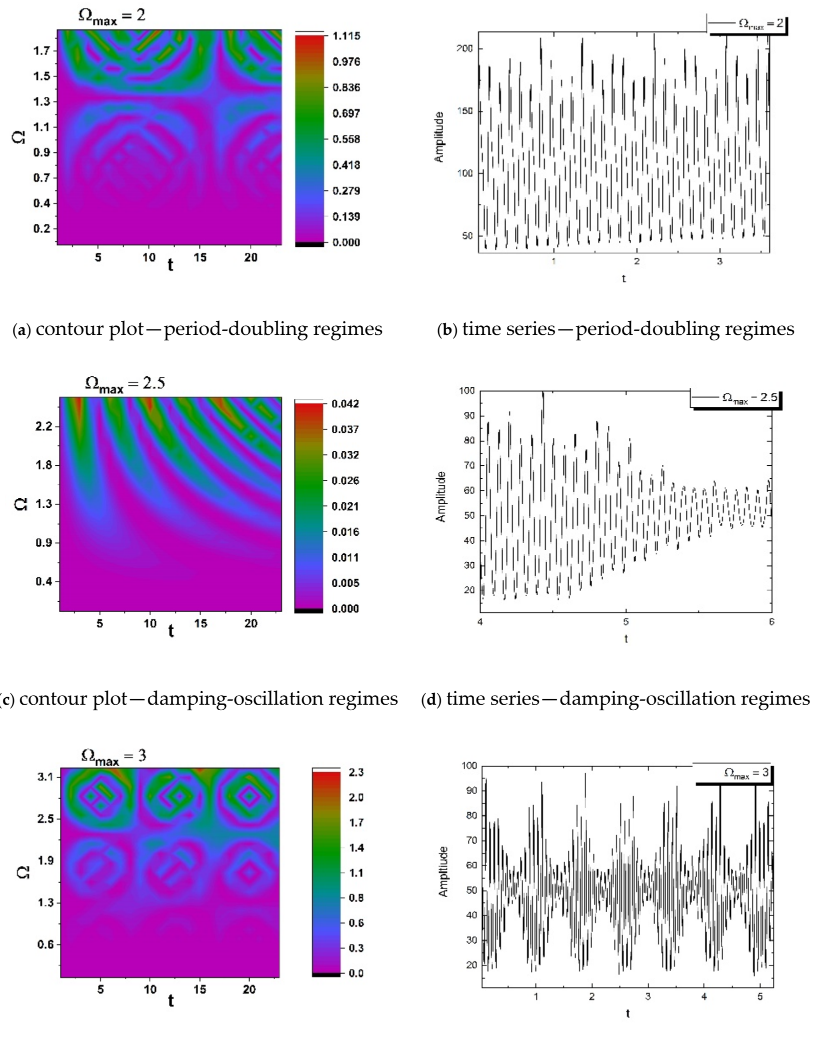
In the above relation, α is real and arbitrary, as long as $(\Phi/2)$ is the solution of a multifractal Laplace equation for the free space, such that

$$\nabla^2(\Phi/2) = 0 \tag{53}$$

It should be noted that, up to the factor $(-\mathrm{i})$, (30) can be identified with (52) through the identifications

$$\Phi/2 \equiv \tau,\ \alpha = -2\mathrm{i}\Omega(\mathrm{t} - \mathrm{t_m})$$

Now, the significance of this complex parameter must be linked to the harmonic mappings between the Euclidean space (i.e., measurement space) and the hyperbolic space (i.e., phase space, in which the "self-structuring" manifests). In Figure 1a–h, the natural transition of the complex-system dynamics is to progress from the normal period-doubling state towards the damped oscillating and a strong modulated dynamics. The complex-system dynamics never reach a chaotic state but they permanently evolve towards that state.

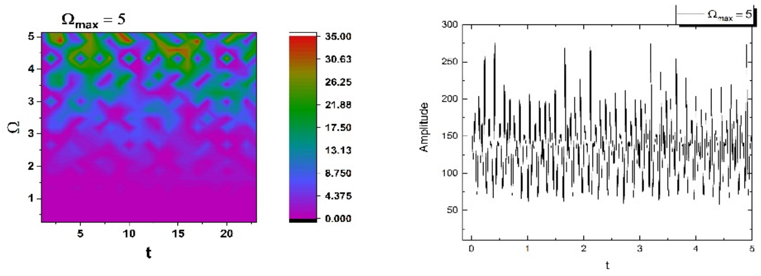


(**a**) contour plot—period-doubling regimes (**b**) time series—period-doubling regimes

(**c**) contour plot—damping-oscillation regimes (**d**) time series—damping-oscillation regimes

(**e**) contour plot—quasi-periodicity regimes (**f**) time series—quasi-periodicity regimes

Figure 1. *Cont.*



(**g**) contour plot—intermittence regimes (**h**) time series—intermittence regimes

Figure 1. (**a**–**h**)—Various work regimes of complex-system dynamics (contour plot and time series) as a function of scale resolution chosen by Ω_{max}: period-doubling regimes for $\Omega_{max} = 2$ (**a**,**b**); damping-oscillation regimes for $\Omega_{max} = 2.5$ (**c**,**d**); quasi-periodicity regimes for $\Omega_{max} = 3$ (**e**,**f**); and intermittence regimes for $\Omega_{max} = 5$ (**g**,**h**).

A periodicity can be identified, regarding the entire series of transitions. The evolution of the complex systems sees a "jump" into a period-doubling oscillation state and the transition resumes towards a quasi-chaotic state.

We present in Figure 2 a specific pattern (reconstituted attractor) for the period-doubling regimen.

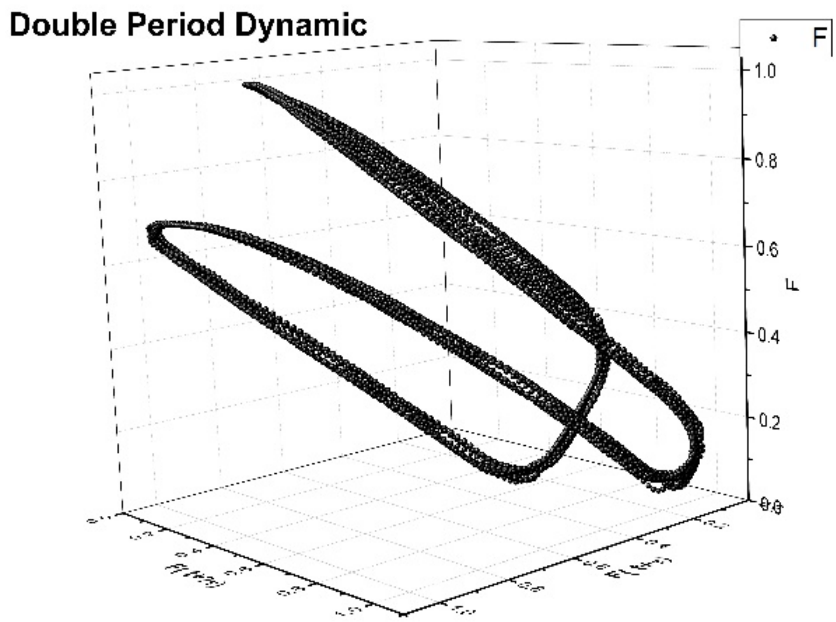


Figure 2. A specific pattern (reconstituted attractor) for the period-doubling regimen.

The bifurcation map is presented in Figure 3 where it is observed that the complex-system dynamics start from a steady state (period-doubling regimes) and evolve towards a chaotic one ($\Omega_{max} = 2, 2.5, 3, 5$), but they never reach that state. We note that such work regimes are manifested both in biological and biocompatible material structures (period-doubling regimes [20–23], damping-oscillation regimes [24–27], quasi-periodicity regimes [28,29], and intermittence regimes [30–32]).

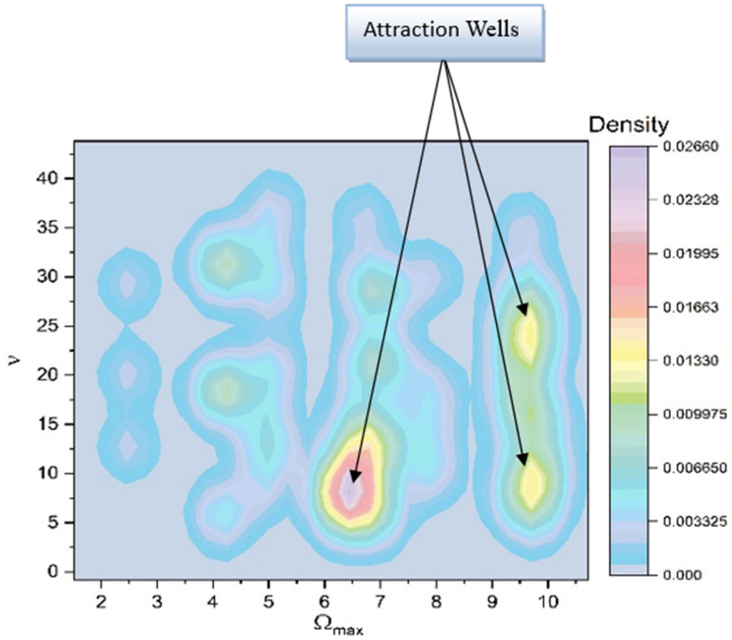


Figure 3. Bifurcation map by means of oscillation frequency of the complex-system dynamics as a function of scale resolution chosen by Ω_{max}.

6. An Economic Case Analysis

Initially applied in the consumer goods market, the fractal/multi-fractal theory has encompassed other economics domains such as: price changes in the open market, the distribution of income of the companies and the scaling relation of thet company's size fluctuations [33,34].

The financial markets, through their inherently speculative character, can constitute a trans- and interdisciplinary domain. The globalization of financial markets and the application of new informational technologies emphasize the complexity degree of these markets.

The 2008 economic crisis has challenged the mainstream economic theory. In such a context, speculative bubbles, an increasing phenomenon encountered in these markets, are the result of the individual reaction to market signals through the price at that particular moment (an aspect which can be assimilated into the digital signals) in the context of an optimistic state induced by the behaviors of large investors and of the State through banking or monetary policies (analogical signals) [35,36]. The results of contemporary empirical studies prove that the market dynamics and the evolution of economic indicators are not a random phenomenon.

The markets have a fractal/multi-fractal structure in the long term, being characterized by a self-memory. In such a context, the individual reaction to the market signal can be associated with period doubling, damping oscillations, quasi-periodicity, and intermittences (i.e., to the digital signals), while the behaviors of large investors and of the State through banking or monetary policies can be associated with the "world finite effect" [6] (i.e., analogical signals). The economical structures emphasize the fluctuations but they never reach the chaos state (see Figure 2).

7. Conclusions

In the Schrödinger-type and Madelung-type scenarios for the description of complex-system dynamics, the SL(2R) symmetries are highlighted. The existence of such symmetries has several consequences for the aforementioned dynamics: conservation laws as gauge invariances of the Riccati type (in particular, for SRT dynamics, the kinetic momentum conservation law); simultaneity as gauge invariances of the Riccati type; synchronization as gauge invariances of the Riccati type; and working regimes through harmonic mappings.

Moreover, the existence of such symmetries implies, through a Poincaré-type metric of the hyperbolic space, that holography can be associated with deep learning.

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