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Article

New Results on Boas–Bellman-Type Inequalities in Semi-Hilbert Spaces with Applications

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Abstract: In this article, we investigate new findings on Boas–Bellman-type inequalities in semi-Hilbert spaces. These spaces are generated by semi-inner products induced by positive and positive semidefinite operators. Our objective is to reveal significant properties of such spaces and apply these results to the field of multivariable operator theory. Specifically, we derive new inequalities that relate to the joint A-numerical radius, the joint operator A-seminorm, and the Euclidean A-seminorm of tuples of semi-Hilbert space operators. We assume that A is a nonzero positive operator. Our discoveries provide insights into the structure of semi-Hilbert spaces and have implications for a broad range of mathematical applications and beyond.

Keywords: Boas–Bellman inequality; Bessel's inequality; joint A-numerical radius; Euclidean A-seminorm

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1. Introduction

Inequalities play a crucial role in analysis and find applications in various areas of mathematics (see [1–7] and related sources). Among these, Bessel's inequality and the Boas–Bellman inequality hold significant importance and are widely used in the study of operators on Hilbert spaces.

Recently, there has been more interest in studying positive semidefinite inner product spaces that are induced by positive semidefinite operators. These spaces, also called semi-Hilbert spaces, are a bit different from Hilbert spaces because they may not always be complete, but they still have certain rules that make them useful. There is a growing body of literature on the subject of semi-Hilbert spaces (see [8–14] and other related works) that explore their properties and potential applications. In this paper, we focus on a specific positive semidefinite inner product space that is created by a positive semidefinite operator called A. We call this space $(\mathcal{F}, \langle\cdot,\cdot\rangle_A)$.

Semi-Hilbert spaces are useful in studying different mathematical problems. The objective of our research paper is to present novel inequalities in semi-Hilbert spaces, specifically of the Boas–Bellman type. These inequalities serve as valuable tools in enhancing our comprehension of operator properties. We start by explaining the notation and definitions of semi-Hilbert spaces and then present our main findings.

Throughout this paper, we focus on a complex Hilbert space $\mathcal{F}$ with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We use $\mathbb{L}(\mathcal{F})$ to denote the set of all bounded linear operators on $\mathcal{F}$

and T^* to represent the adjoint of a bounded linear operator T on $\mathcal{F}$. An operator $T \in \mathbb{L}(\mathcal{F})$ is considered positive, denoted as $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{F}$. In this work, when we use the term "operator", we specifically refer to an element of the set $\mathbb{L}(\mathcal{F})$, and we assume that A is a nonzero positive operator. For any such A, we define a positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle_A : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ as $\langle x, y\rangle_A = \langle Ax, y\rangle$ for all $x, y \in \mathcal{F}$. The seminorm induced by $\langle \cdot, \cdot \rangle_A$ is denoted by $\| \cdot \|_A$. For any vector $x \in \mathcal{F}$, this seminorm is defined as $\|x\|_A = \sqrt{\langle x, x\rangle_A}$. It is worth noting that the seminorm $\| \cdot \|_A$ vanishes on a vector $x \in \mathcal{F}$ if and only if x belongs to the null space $\mathcal{N}(A)$ of A. Moreover, the seminorm $\| \cdot \|_A$ induces a norm on $\mathcal{F}$ if and only if A is one-to-one. Consequently, the semi-Hilbert space $(\mathcal{F}, \| \cdot \|_A)$ is complete if and only if the range $\mathcal{R}(A)$ of A is closed in $\mathcal{F}$.

First, let us mention some well-established inequalities that apply to both real and complex inner product spaces. However, in the context of this paper, we can assume that $\mathcal{F}$ is a complex Hilbert space without losing any generality. To begin our discussion, we introduce Bessel's inequality (refer to ([15], p. 391) for more information), which asserts that if we have a set of orthonormal vectors $e_1, e_2, \cdots, e_d$ in $\mathcal{F}$, meaning that they satisfy $\langle e_i, e_j\rangle = \delta_{ij}$ (where δ_{ij} is the Kronecker delta symbol) for all $i, j \in \{1, \ldots, d\}$, then the following inequality holds for any vector $x \in \mathcal{F}$:

$$\sum_{i=1}^{d} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

Additional findings linked to Bessel's inequality can be found in references [16] through [17], while Chapter XV in the book [15] also provides valuable insights.

In 1941, R.P. Boas [18] and R. Bellman [19] (independently, in 1944) established a generalized form of Bessel's inequality, as documented in ([15], p. 392). Specifically, if x and $y_1, \ldots, y_d$ belong to $\mathcal{F}$, then the subsequent inequality holds:

$$\sum_{i=1}^{d} |\langle x, y_i\rangle|^2 \leq \|x\|^2 \left[\max_{1\leq i\leq d} \|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq d} |\langle y_i, y_j\rangle|^2 \right)^{\frac{1}{2}} \right]. \tag{1}$$

Mitrinović–Pečarić–Fink proved a recent extension of the Boas–Bellman result, as detailed in ([15], p. 392). Specifically, they established an inequality that holds for elements x and $y_1, \ldots, y_d$ in $\mathcal{F}$ and complex numbers $\theta_1, \ldots, \theta_d \in \mathbb{C}$. The inequality is as follows:

$$\left| \sum_{i=1}^{d} \theta_i \langle x, y_i\rangle \right|^2 \leq \|x\|^2 \sum_{i=1}^{d} |\theta_i|^2 \left[\max_{1\leq i\leq d} \|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq d} |\langle y_i, y_j\rangle|^2 \right)^{\frac{1}{2}} \right]. \tag{2}$$

Furthermore, the authors observed that choosing $\theta_i = \overline{\langle x, y_i\rangle}$ in (2) leads to the Boas–Bellman inequality (1). Other related results on the Boas–Bellman inequality can be found in [20].

This paper introduces new discoveries that expand the Mitrinović–Pečarić–Fink and Boas–Bellman inequalities to the realm of semi-Hilbert spaces. The research is relevant to multivariable operator theory, and it presents novel inequalities that relate to tuples of operators in semi-Hilbert spaces. Specifically, we investigate the joint A-numerical radius, joint operator A-seminorm, and Euclidean A-seminorm and establish novel connections among these concepts. These Boas–Bellman-type inequalities offer several advantages, enhancing our understanding of semi-Hilbert spaces and their applications in multivariable operator theory. They provide valuable insights into the relationships between different numerical measures, such as the joint A-numerical radius and the joint operator A-seminorm. Additionally, these findings have broad implications and can be applied to various mathematical and scientific contexts. Overall, the Boas–Bellman-type inequalities significantly contribute to the progress of mathematics and related fields.

2. Preliminary Results

To establish our main theorem, we introduce a lemma that not only serves as a valuable tool in our proof but also stands out in its own right. This lemma provides nine upper bounds for the quantity $\left\|\sum_{i=1}^{d}\mu_i X_i\right\|_A$, where $X_j \in \mathcal{F}$ and $\mu_j \in \mathbb{C}$ for all $j \in \{1,\dots,d\}$.

Lemma 1. *If $X_1,\dots,X_d \in \mathcal{F}$ and $\mu_1,\dots,\mu_d \in \mathbb{C}$, the following inequality holds:*

$$\left\|\sum_{i=1}^{d}\mu_i X_i\right\|_A^2 \le \begin{cases} \max\limits_{1\le i\ne j\le d}\{|\mu_i\mu_j|\}\sum\limits_{1\le i\ne j\le d}\left|\langle X_i, X_j\rangle_A\right|; \\ \\ \left[\left(\sum\limits_{i=1}^{d}|\mu_i|^{\alpha}\right)^2-\left(\sum\limits_{i=1}^{d}|\mu_i|^{2\alpha}\right)\right]^{\frac{1}{\alpha}}\left(\sum\limits_{1\le i\ne j\le d}\left|\langle X_i, X_j\rangle_A\right|^{\beta}\right)^{\frac{1}{\beta}}, \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \left[\left(\sum\limits_{i=1}^{d}|\mu_i|\right)^2-\sum\limits_{i=1}^{d}|\mu_i|^2\right]\max\limits_{1\le i\ne j\le d}\left|\langle X_i, X_j\rangle_A\right|. \end{cases}$$

$$+\begin{cases} \max\limits_{1\le i\le d}|\mu_i|^2\sum\limits_{i=1}^{d}\|X_i\|_A^2, \\ \\ \left(\sum\limits_{i=1}^{d}|\mu_i|^{2\mu}\right)^{\frac{1}{\mu}}\left(\sum\limits_{i=1}^{d}\|X_i\|_A^{2\nu}\right)^{\frac{1}{\nu}},\ \text{ where } \mu>1, \frac{1}{\mu}+\frac{1}{\nu}=1; \\ \\ \sum\limits_{i=1}^{d}|\mu_i|^2\max\limits_{1\le i\le d}\|X_i\|_A^2, \end{cases}$$

Proof. Let $X_j \in \mathcal{F}$ and $\mu_j \in \mathbb{C}$ for all $j \in \{1,\dots,d\}$. It can be observed that

$$\begin{aligned} \left\|\sum_{i=1}^{d}\mu_i X_i\right\|_A^2 &= \left|\langle \sum_{i=1}^{d}\mu_i X_i, \sum_{j=1}^{d}\mu_j X_j\rangle_A\right| \\ &= \left|\sum_{i=1}^{d}\sum_{j=1}^{d}\mu_i\overline{\mu_j}\langle X_i, X_j\rangle_A\right| \\ &\le \sum_{i=1}^{d}\sum_{j=1}^{d}|\mu_i||\overline{\mu_j}|\left|\langle X_i, X_j\rangle_A\right|. \end{aligned}$$

Thus, it follows that

$$\left\|\sum_{i=1}^{d}\mu_i X_i\right\|_A^2 \le \sum_{i=1}^{d}|\mu_i|^2\|X_i\|_A^2+\sum_{1\le i\ne j\le d}|\mu_i||\mu_j|\left|\langle X_i, X_j\rangle_A\right|. \tag{3}$$

On the other hand, by applying Hölder's inequality, we can express that

$$\sum_{i=1}^{d}|\mu_i|^2\|X_i\|_A^2 \le \begin{cases} \max\limits_{1\le i\le d}|\mu_i|^2\sum\limits_{i=1}^{d}\|X_i\|_A^2; \\ \\ \left(\sum\limits_{i=1}^{d}|\mu_i|^{2\mu}\right)^{\frac{1}{\mu}}\left(\sum\limits_{i=1}^{d}\|X_i\|_A^{2\nu}\right)^{\frac{1}{\nu}},\ \text{ where } \mu>1, \frac{1}{\mu}+\frac{1}{\nu}=1; \\ \\ \sum\limits_{i=1}^{d}|\mu_i|^2\max\limits_{1\le i\le d}\|X_i\|^2. \end{cases}$$

Using Hölder's inequality for double sums, we can further obtain

$$\sum_{1\le i\ne j\le d}|\mu_i||\mu_j|\left|\langle X_i,X_j\rangle_A\right| \le \begin{cases} \max\limits_{1\le i\ne j\le d}|\mu_i\mu_j| \sum\limits_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|; \\[2ex] \left(\sum\limits_{1\le i\ne j\le d}|\mu_i|^\alpha|\mu_j|^\alpha\right)^{\frac{1}{\alpha}}\left(\sum\limits_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|^\beta\right)^{\frac{1}{\beta}}, \text{ where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] \sum\limits_{1\le i\ne j\le d}|\mu_i||\mu_j| \max\limits_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|, \end{cases}$$

$$= \begin{cases} \max\limits_{1\le i\ne j\le d}\{|\mu_i\mu_j|\} \sum\limits_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|; \\[2ex] \left[\left(\sum\limits_{i=1}^d|\mu_i|^\alpha\right)^2-\left(\sum\limits_{i=1}^d|\mu_i|^{2\alpha}\right)\right]^{\frac{1}{\alpha}}\left(\sum\limits_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|^\beta\right)^{\frac{1}{\beta}}, \text{ where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] \left[\left(\sum\limits_{i=1}^d|\mu_i|\right)^2-\sum\limits_{i=1}^d|\mu_i|^2\right] \max\limits_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|. \end{cases}$$

The desired result can be deduced by incorporating (3). □

Remark 1. *The set of inequalities contained in Lemma 1 actually consists of 9 variations that can be obtained by combining the first 3 with the last 3.*

A particular case that may be related to the Boas–Bellman result is embodied in the following inequality.

Corollary 1. *With the assumptions in Lemma 1, we have*

$$\begin{aligned} \left\|\sum_{i=1}^d \mu_i X_i\right\|_A^2 &\le \sum_{i=1}^d|\mu_i|^2\left\{\max_{1\le i\le d}\|X_i\|_A^2+\Theta\right\} \\ &\le \sum_{i=1}^d|\mu_i|^2\left\{\max_{1\le i\le d}\|X_i\|_A^2+\left(\sum_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|^2\right)^{\frac{1}{2}}\right\}, \end{aligned}$$

where

$$\Theta=\frac{\left[\left(\sum_{i=1}^d|\mu_i|^2\right)^2-\sum_{i=1}^d|\mu_i|^4\right]^{\frac{1}{2}}}{\sum_{i=1}^d|\mu_i|^2}\left(\sum_{1\le i\ne j\le d}\left|\langle X_i,X_j\rangle_A\right|^2\right)^{\frac{1}{2}}.$$

Proof. The first inequality can be obtained by utilizing the second branch in the first curly bracket for $\alpha=\beta=2$ in combination with the third branch in the second curly bracket.

To prove the second inequality in the corollary, we can rely on the fact that

$$\left[\left(\sum_{i=1}^d|\mu_i|^2\right)^2-\sum_{i=1}^d|\mu_i|^4\right]^{\frac{1}{2}}\le\sum_{i=1}^d|\mu_i|^2.$$

From here, it is clear that the proof is complete. □

In the following, we present coarser upper bounds for $\left\|\sum_{i=1}^d \mu_i X_i\right\|_A^2$ that may be of practical interest in various applications.

Corollary 2. *The inequalities below hold under the assumptions of Lemma 1:*

$$\left\|\sum_{i=1}^{d}\mu_i X_i\right\|_A^2 \leq \begin{cases} \max\limits_{1\leq i\leq d}|\mu_i|^2 \sum\limits_{i=1}^{d}\|X_i\|_A^2; \\ \left(\sum\limits_{i=1}^{d}|\mu_i|^{2\mu}\right)^{\frac{1}{\mu}}\left(\sum\limits_{i=1}^{d}\|X_i\|_A^{2\nu}\right)^{\frac{1}{\nu}}, \text{ where } \mu>1, \frac{1}{\mu}+\frac{1}{\nu}=1; \\ \sum\limits_{i=1}^{d}|\mu_i|^2 \max\limits_{1\leq i\leq d}\|X_i\|_A^2, \end{cases}$$

$$+\begin{cases} \max\limits_{1\leq i\leq d}|\mu_i|^2 \sum\limits_{1\leq i\neq j\leq d}\left|\langle X_i, X_j\rangle_A\right|; \\ (d-1)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{d}|\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{1\leq i\neq j\leq d}\left|\langle X_i, X_j\rangle_A\right|^{\beta}\right)^{\frac{1}{\beta}}, \text{ where } \alpha>1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (d-1)\sum\limits_{i=1}^{d}|\mu_i|^2 \max\limits_{1\leq i\neq j\leq d}\left|\langle X_i, X_j\rangle_A\right|. \end{cases}$$

Proof. Using the Cauchy–Bunyakovsky–Schwarz-type inequality given below,

$$\left(\sum_{i=1}^{d}a_i\right)^2 \leq d\sum_{i=1}^{d}a_i^2, \quad a_i\in\mathbb{R}^+, \ 1\leq i\leq d,$$

we can rewrite the inequalities as follows:

$$\left(\sum_{i=1}^{d}|\mu_i|^{\alpha}\right)^2 - \sum_{i=1}^{d}|\mu_i|^{2\alpha} \leq (d-1)\sum_{i=1}^{d}|\mu_i|^{2\alpha} \qquad (d\geq 1)$$

and

$$\left(\sum_{i=1}^{d}|\mu_i|\right)^2 - \sum_{i=1}^{d}|\mu_i|^2 \leq (d-1)\sum_{i=1}^{d}|\mu_i|^2 \qquad (d\geq 1).$$

Furthermore, it is clear that

$$\max_{1\leq i\neq j\leq d}\left\{\left|\mu_i\mu_j\right|\right\} \leq \max_{1\leq i\leq d}|\mu_i|^2.$$

Therefore, taking Lemma 1 into account, we obtain the desired result. □

Remark 2. *Corollary 2 incorporates the following noteworthy inequalities:*

$$\left\|\sum_{i=1}^{d}\mu_i X_i\right\|_A^2 \leq \max_{1\leq i\leq d}|\mu_i|^2\left[\sum_{i=1}^{d}\|X_i\|_A^2 + \sum_{1\leq i\neq j\leq d}\left|\langle X_i, X_j\rangle_A\right|\right].$$

Furthermore, when $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have:

$$\left\|\sum_{i=1}^{d}\mu_i X_i\right\|_A^2 \leq \left(\sum_{i=1}^{d}|\mu_i|^{2p}\right)^{\frac{1}{p}}\left[\left(\sum_{i=1}^{d}\|X_i\|^{2q}\right)^{\frac{1}{q}} + (d-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq d}\left|\langle X_i, X_j\rangle_A\right|^{q}\right)^{\frac{1}{q}}\right].$$

In addition, we have the following inequality:

$$\left\| \sum_{i=1}^{d} \mu_i X_i \right\|_A^2 \le \sum_{i=1}^{d} |\mu_i|^2 \left[\max_{1\le i\le d} \|X_i\|_A^2 + (d-1) \max_{1\le i\neq j\le d} \left| \langle X_i, X_j \rangle_A \right| \right].$$

Clearly, when $p = q = 2$, we obtain

$$\left\| \sum_{i=1}^{d} \mu_i X_i \right\|_A^2 \le \left(\sum_{i=1}^{d} |\mu_i|^4 \right)^{\frac{1}{2}} \left[\left(\sum_{i=1}^{d} \|X_i\|_A^4 \right)^{\frac{1}{2}} + (d-1)^{\frac{1}{2}} \left(\sum_{1\le i\neq j\le d} \left| \langle X_i, X_j \rangle_A \right|^2 \right)^{\frac{1}{2}} \right].$$

We can now present an additional result that complements the inequality (2) originally introduced by Mitrinović, Pečarić, and Fink in ([15], p. 392).

Theorem 1. *Consider vectors $x, y_1, \ldots, y_d \in \mathcal{F}$ and complex numbers $\theta_1, \ldots, \theta_d \in \mathbb{C}$. Then, the following inequalities hold:*

$$\begin{aligned} \left| \sum_{i=1}^{d} \theta_i \langle x, y_i \rangle_A \right|^2 \le \|x\|_A^2 \times & \begin{cases} \max\limits_{1\le i\le d} |\theta_i|^2 \sum\limits_{i=1}^{d} \|y_i\|_A^2; \\ \left(\sum\limits_{i=1}^{d} |\theta_i|^{2\mu} \right)^{\frac{1}{\mu}} \left(\sum\limits_{i=1}^{d} \|y_i\|_A^{2\nu} \right)^{\frac{1}{\nu}}, \text{ where } \mu > 1, \frac{1}{\mu} + \frac{1}{\nu} = 1; \\ \sum\limits_{i=1}^{d} |\theta_i|^2 \max\limits_{1\le i\le d} \|y_i\|_A^2, \end{cases} \\ + \|x\|_A^2 \times & \begin{cases} \max\limits_{1\le i\neq j\le d} \{ |\theta_i \theta_j| \} \sum\limits_{1\le i\neq j\le d} \left| \langle y_i, y_j \rangle_A \right|; \\ \left[\left(\sum\limits_{i=1}^{d} |\theta_i|^{\alpha} \right)^2 - \left(\sum\limits_{i=1}^{d} |\theta_i|^{2\alpha} \right) \right]^{\frac{1}{\alpha}} \left(\sum\limits_{1\le i\neq j\le d} \left| \langle y_i, y_j \rangle_A \right|^{\beta} \right)^{\frac{1}{\beta}}, \\ \text{where } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left[\left(\sum\limits_{i=1}^{d} |\theta_i| \right)^2 - \sum\limits_{i=1}^{d} |\theta_i|^2 \right] \max\limits_{1\le i\neq j\le d} \left| \langle y_i, y_j \rangle_A \right|. \end{cases} \end{aligned}$$

Proof. Consider the vectors $x, y_1, \ldots, y_d$ in $\mathcal{F}$ and $\theta_1, \ldots, \theta_d \in \mathbb{C}$. We observe that

$$\sum_{i=1}^{d} \theta_i \langle x, y_i \rangle_A = \langle x, \sum_{i=1}^{d} \overline{\theta_i} y_i \rangle_A.$$

By applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \left| \sum_{i=1}^{d} \theta_i \langle x, y_i \rangle_A \right|^2 &= \left| \langle x, \sum_{i=1}^{d} \overline{\theta_i} y_i \rangle_A \right|^2 \\ &\le \|x\|_A^2 \left\| \sum_{i=1}^{d} \overline{\theta_i} y_i \right\|_A^2. \end{aligned}$$

Using Lemma 1 with $\mu_i = \overline{\theta_i}$, $X_i = y_i$ for all $i \in \{1, \ldots, d\}$, we can obtain the desired result. □

The following specific inequalities are valid.

Corollary 3. *Considering the assumptions in Theorem 1, the following inequalities hold:*

$$\left|\sum_{i=1}^{d}\theta_i\langle x,y_i\rangle_A\right|^2 \leq \begin{cases} \|x\|_A^2\sum_{i=1}^{d}|\theta_i|^2\left\{\max_{1\leq i\leq d}\|y_i\|_A^2+\left(\sum_{1\leq i\neq j\leq d}\left|\langle y_i,y_j\rangle_A\right|^2\right)^{\frac{1}{2}}\right\}; \\ \|x\|_A^2\max_{1\leq i\leq d}|\theta_i|^2\left\{\sum_{i=1}^{d}\|y_i\|_A^2+\sum_{1\leq i\neq j\leq d}\left|\langle y_i,y_j\rangle_A\right|\right\} \\ \|x\|_A^2\left(\sum_{i=1}^{d}|\theta_i|^{2p}\right)^{\frac{1}{p}}\left\{\left(\sum_{i=1}^{d}\|y_i\|_A^{2q}\right)^{\frac{1}{q}}+(d-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq d}\left|\langle y_i,y_j\rangle_A\right|^q\right)^{\frac{1}{q}}\right\} \\ \text{where } \; p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \|x\|_A^2\sum_{i=1}^{d}|\theta_i|^2\left\{\max_{1\leq i\leq d}\|y_i\|_A^2+(d-1)\max_{1\leq i\neq j\leq d}\left|\langle y_i,y_j\rangle_A\right|\right\}. \end{cases}$$

Remark 3. *It should be noted that the initial inequality presented in Corollary 3 is a generalization of a finding originally established by Mitrinović–Pečarić–Fink in [15]. Meanwhile, the remaining three inequalities offer similar inequalities with regard to the p-norms of the vector* $\left(|\theta_1|^2,\ldots,|\theta_d|^2\right)$.

3. Some Extensions of Boas–Bellman-Type Inequalities

In this section, our objective is to utilize the outcomes from the prior section to derive various Boas–Bellman-Type inequalities in the context of semi-Hilbert spaces. It is worth noting that by substituting x with $A^{1/2}x$ and y_i with $A^{1/2}y_i$ in (1), we can obtain the following result:

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \leq \|x\|_A^2\left[\max_{1\leq i\leq n}\|y_i\|_A^2+\left(\sum_{1\leq i\neq j\leq n}\left|\langle x,y_i\rangle_A\right|^2\right)^{\frac{1}{2}}\right] \tag{4}$$

for $x,y_1,\ldots,y_d\in\mathcal{F}$.

By choosing $\theta_i=\overline{\langle x,y_i\rangle_A}$ $(i\in\{1,\ldots,d\})$ in Theorem 1, 9 different inequalities can be obtained. However, we only consider the inequalities that can be derived from Corollary 3.

By applying the second inequality in Corollary 3 with $\theta_i=\overline{\langle x,y_i\rangle_A}$, we obtain

$$\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2\right)^2 \leq \|x\|_A^2\max_{1\leq i\leq d}\left|\langle x,y_i\rangle_A\right|^2\left\{\sum_{i=1}^{d}\|y_i\|_A^2+\sum_{1\leq i\neq j\leq d}\left|\langle y_i,y_j\rangle_A\right|\right\}.$$

By taking the square root of this inequality, we obtain

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \leq \|x\|_A\max_{1\leq i\leq d}\left|\langle x,y_i\rangle_A\right|\left\{\sum_{i=1}^{d}\|y_i\|_A^2+\sum_{1\leq i\neq j\leq n}\left|\langle y_i,y_j\rangle_A\right|\right\}^{\frac{1}{2}}, \tag{5}$$

where $x,y_1,\ldots,y_d$ are vectors in $\mathcal{F}$.

Assuming that $(e_i)_{1\leq i\leq d}$ forms an A-orthonormal family in $\mathcal{F}$ (meaning that $\langle e_i,e_j\rangle_A=\delta_{ij}$ for all $i,j\in\{1,\ldots,d\}$, where δ_{ij} denotes the Kronecker symbol), we can use (5) to obtain

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \leq \sqrt{d}\|x\|_A\max_{1\leq i\leq d}\left|\langle x,e_i\rangle_A\right|, \quad x\in\mathcal{F}.$$

By applying the third inequality in Corollary 3 with $\theta_i = \overline{\langle x, y_i \rangle_A}$, we can infer

$$\left(\sum_{i=1}^{d} \left|\langle x, y_i \rangle_A\right|^2\right)^2 \leq \|x\|_A^2 \left(\sum_{i=1}^{d} \left|\langle x, y_i \rangle_A\right|^{2p}\right)^{\frac{1}{p}} \left\{ \left(\sum_{i=1}^{d} \|y_i\|_A^{2q}\right)^{\frac{1}{q}} + (d-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq d} \left|\langle y_i, y_j \rangle_A\right|^q\right)^{\frac{1}{q}} \right\},$$

for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Upon taking the square root of this inequality, we arrive at the following expression:

$$\sum_{i=1}^{d} \left|\langle x, y_i \rangle_A\right|^2 \leq \|x\|_A \left(\sum_{i=1}^{d} \left|\langle x, y_i \rangle_A\right|^{2p}\right)^{\frac{1}{2p}} \times \left\{ \left(\sum_{i=1}^{d} \|y_i\|_A^{2q}\right)^{\frac{1}{q}} + (d-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq d} \left|\langle y_i, y_j \rangle_A\right|^q\right)^{\frac{1}{q}} \right\}^{\frac{1}{2}}, \tag{6}$$

for all $x, y_1, \ldots, y_d \in \mathcal{F}$, $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$.

The above inequality (6) becomes, for an A-orthornormal family $(e_i)_{1 \leq i \leq d}$,

$$\sum_{i=1}^{d} \left|\langle x, e_i \rangle_A\right|^2 \leq d^{\frac{1}{q}} \|x\|_A \left(\sum_{i=1}^{d} \left|\langle x, e_i \rangle_A\right|^{2p}\right)^{\frac{1}{2p}}, \quad x \in \mathcal{F}.$$

Substituting $\theta_i = \overline{\langle x, y_i \rangle_A}$ for $i \in \{1, \ldots, d\}$ into the last inequality of Corollary 3 yields

$$\left(\sum_{i=1}^{d} \left|\langle x, y_i \rangle_A\right|^2\right)^2 \leq \|x\|_A^2 \sum_{i=1}^{d} \left|\langle x, y_i \rangle_A\right|^2 \left\{ \max_{1 \leq i \leq d} \|y_i\|_A^2 + (d-1) \max_{1 \leq i \neq j \leq d} \left|\langle y_i, y_j \rangle_A\right| \right\}.$$

So, we obtain the following generalized Boas–Bellman-type inequality:

$$\sum_{i=1}^{d} \left|\langle x, y_i \rangle_A\right|^2 \leq \|x\|_A^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|_A^2 + (d-1) \max_{1 \leq i \neq j \leq d} \left|\langle y_i, y_j \rangle_A\right| \right\}, \tag{7}$$

for all vectors $x, y_1, \ldots, y_d \in \mathcal{F}$.

Remark 4. *For A-orthonormal families of vectors, it is clear that (7) provides an extension of the Bessel inequality in the context of semi-Hilbert spaces.*

Remark 5. *To compare (4), which represents the Boas–Bellman result in the context of semi-Hilbert spaces, with (7), we can examine the following quantities.*

$$\Delta := \left(\sum_{1 \leq i \neq j \leq d} \left|\langle y_i, y_j \rangle_A\right|^2\right)^{\frac{1}{2}}$$

and

$$\Gamma := (d-1) \max_{1 \leq i \neq j \leq d} \left|\langle y_i, y_j \rangle_A\right|.$$

If we consider A to be the identity operator and use the same example as in [21], we find that Δ and Γ are not comparable. Therefore, in general, it is not possible to compare (4), which represents the Boas–Bellman result in the context of semi-Hilbert spaces, with (7).

4. Inequalities for Operators

In this section, we utilize the inequalities derived in the previous section to establish various inequalities for operators that act on semi-Hilbert spaces. We specifically use Bombieri-type inequalities in the context of semi-Hilbert spaces to derive bounds for the joint A-numerical radius and the Euclidean A-seminorm of operator tuples.

We begin by introducing some concepts and definitions related to operator theory in semi-Hilbert spaces. Firstly, we define the A-adjoint of an operator $T \in \mathbb{L}(\mathcal{F})$ as an operator $R \in \mathbb{L}(\mathcal{F})$ such that for every $x, y \in \mathcal{F}$, $\langle Tx, y\rangle_A = \langle x, Ry\rangle_A$, or equivalently, $AR = T^*A$ (see [22]). It is important to note that not every operator has an A-adjoint. The set of operators that admit A-adjoints is denoted by $\mathbb{L}_A(\mathcal{F})$. According to Douglas' theorem [23], an operator T belongs to $\mathbb{L}_A(\mathcal{F})$ if and only if $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$. In the case that $T \in \mathbb{L}_A(\mathcal{F})$, the "reduced" solution of the equation $AX = T^*A$ is called the distinguished A-adjoint operator of T, denoted by $T^{\sharp_A}$. Additionally, $T^{\sharp_A}$ is also in $\mathbb{L}_A(\mathcal{F})$, and $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$, where $P_{\overline{\mathcal{R}(A)}}$ is the orthogonal projection onto the closure of the range of A.

In the context of operator theory in semi-Hilbert spaces, an important result, known as the Douglas theorem, states that if T is an operator in $\mathbb{L}_A(\mathcal{F})$, then $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$ is a necessary and sufficient condition for T to belong to the space $\mathbb{L}_A(\mathcal{F})$. Another consequence of the Douglas theorem is that operators in $\mathbb{L}_{A^{1/2}}(\mathcal{F})$, referred to as A-bounded operators, can be identified by the existence of a constant $c > 0$ such that $\|Tx\|_A < c\|x\|_A$ for all $x \in \mathcal{F}$. It is worth noting that $\mathbb{L}_A(\mathcal{F})$ and $\mathbb{L}_{A^{1/2}}(\mathcal{F})$ are subalgebras of $\mathbb{L}(\mathcal{F})$, but they are neither closed nor dense in $\mathbb{L}(\mathcal{F})$. The inclusions $\mathbb{L}_A(\mathcal{F}) \subseteq \mathbb{L}_{A^{1/2}}(\mathcal{F}) \subseteq \mathbb{L}(\mathcal{F})$ are generally strict, although if A is one-to-one and has a closed range, these inclusions hold with equality. For further information on results related to operator theory in semi-Hilbert spaces, a number of references are recommended, including [8,12,22,24].

For the following discussion, we define $\mathbb{L}(\mathcal{F})^d$ as the set of all d-tuples of operators. Let $\mathbf{S} = (S_1, \ldots, S_d) \in \mathbb{L}(\mathcal{F})^d$ be a d-tuple of operators. The two quantities $\omega_A(\mathbf{S})$ and $\|\mathbf{S}\|_A$ are introduced in [25]. Specifically, for a d-tuple of operators $\mathbf{S} = (S_1, \ldots, S_d) \in \mathbb{L}(\mathcal{F})^d$, we define

$$\omega_A(\mathbf{S}) := \sup_{x \in \mathbb{S}^A_{\mathcal{F}}} \sqrt{\sum_{k=1}^d \left|\langle S_k x, x\rangle_A\right|^2} \quad \text{and} \quad \|\mathbf{S}\|_A = \sup_{x \in \mathbb{S}^A_{\mathcal{F}}} \sqrt{\sum_{k=1}^d \|S_k x\|_A^2}, \tag{8}$$

where $\mathbb{S}^A_{\mathcal{F}}$ denotes the unit sphere of $\mathcal{F}$ with respect to the norm $\|\cdot\|_A$. That is, $\mathbb{S}^A_{\mathcal{F}}$ is the set of all vectors $x \in \mathcal{F}$ such that $\|x\|_A = 1$.

It should be noted that the definitions of $\omega_A(\mathbf{S})$ and $\|\mathbf{S}\|_A$, which were introduced in [25], can result in infinity even when $d = 1$, as pointed out in various sources, such as [26]. However, if $\mathbf{S} = (S_1, \ldots, S_d) \in \mathbb{L}_{A^{1/2}}(\mathcal{F})^d$, then they become two equivalent seminorms, as shown in [25]. In this case, $\omega_A(\mathbf{S})$ is known as the joint A-numerical radius of $\mathbf{S}$, while $\|\mathbf{S}\|_A$ is referred to as the joint operator A-seminorm of $\mathbf{S}$.

When $T \in \mathbb{L}_{A^{1/2}}(\mathcal{F})$, we can define the A-numerical radius and the operator A-seminorm of T by substituting $d = 1$ in (8). The A-numerical radius of T is the supremum of $|\langle Tx, x\rangle_A|$ over all $x \in \mathbb{S}^A_{\mathcal{F}}$, while the operator A-seminorm of T is the supremum of $\|Tx\|_A$ over all $x \in \mathbb{S}^A_{\mathcal{F}}$. These quantities have been extensively studied in the literature, as evidenced by various works, such as [9,25] and their references.

The open unit ball $\mathbf{B}_d$ in $\mathbb{C}^d$ is defined as

$$\mathbf{B}_d := \left\{\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{C}^d\,;\ \|\eta\|_2^2 := \sum_{k=1}^d |\eta_k|^2 < 1\right\}.$$

An alternative joint A-seminorm for $\mathbf{S} = (S_1, \ldots, S_d) \in \mathbb{L}_{A^{1/2}}(\mathcal{F})^d$ was introduced in [27]. This joint A-seminorm is called the Euclidean A-seminorm. It is denoted by $\|\mathbf{S}\|_A$ and defined as follows:

$$\|\mathbf{S}\|_{e,A} = \sup_{(\eta_1, \ldots, \eta_d) \in \mathbf{B}_d} \|\eta_1 S_1 + \ldots + \eta_d S_d\|_A.$$

Below is a description of the first result we obtained in this section.

Theorem 2. *For any* $\mathbf{S} = (S_1, \ldots, S_d) \in \mathbb{L}_{A^{1/2}}(\mathcal{F})^d$ *and* $\mu_1, \ldots, \mu_d \in \mathbb{C}$, *the following result holds:*

$$\left\|\sum_{i=1}^d \mu_i S_i\right\|_A^2 \le \begin{cases} \max\limits_{1\le i\le d}|\mu_i|^2\|\mathbf{S}\|_A^2; \\[2ex] \left(\sum\limits_{i=1}^d |\mu_i|^{2\mu}\right)^{\frac{1}{\mu}}\left(\sum\limits_{i=1}^d \|S_i\|_A^{2\nu}\right)^{\frac{1}{\nu}}, \text{ where } \mu>1, \frac{1}{\mu}+\frac{1}{\nu}=1; \\[2ex] \sum\limits_{i=1}^d |\mu_i|^2 \max\limits_{1\le i\le d}\|S_i\|_A^2, \end{cases}$$
$$+\begin{cases} \max\limits_{1\le i\le d}|\mu_i|^2 \sum\limits_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right); \\[2ex] (d-1)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^d |\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{1\le i\ne j\le d}\omega_A^\beta\left(S_j^{\sharp_A}S_i\right)\right)^{\frac{1}{\beta}}, \\ \text{where } \alpha>1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] (d-1)\sum\limits_{i=1}^d |\mu_i|^2 \max\limits_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right). \end{cases}$$

Proof. By applying Corollary 2 to $X_i = S_i x$, we obtain the following result:

$$\left\|\sum_{i=1}^d \mu_i S_i x\right\|_A^2 \le \begin{cases} \max\limits_{1\le i\le d}|\mu_i|^2 \sum\limits_{i=1}^d \|S_i x\|_A^2; \\[2ex] \left(\sum\limits_{i=1}^d |\mu_i|^{2\mu}\right)^{\frac{1}{\mu}}\left(\sum\limits_{i=1}^d \|S_i x\|_A^{2\nu}\right)^{\frac{1}{\nu}}, \text{ where } \mu>1, \frac{1}{\mu}+\frac{1}{\nu}=1; \\[2ex] \sum\limits_{i=1}^d |\mu_i|^2 \max\limits_{1\le i\le d}\|S_i x\|_A^2, \end{cases} \tag{9}$$
$$+\begin{cases} \max\limits_{1\le i\le d}|\mu_i|^2 \sum\limits_{1\le i\ne j\le d}\left|\left\langle S_j^{\sharp_A}S_i x, x\right\rangle_A\right|; \\[2ex] (d-1)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^d |\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{1\le i\ne j\le d}\left|\left\langle S_j^{\sharp_A}S_i x, x\right\rangle_A\right|^\beta\right)^{\frac{1}{\beta}}, \\ \text{where } \alpha>1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] (d-1)\sum\limits_{i=1}^d |\mu_i|^2 \max\limits_{1\le i\ne j\le d}\left|\left\langle S_j^{\sharp_A}S_i x, x\right\rangle_A\right| \end{cases}$$

for $x \in \mathcal{F}$.

Keep in mind that

$$\sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\left\|\sum_{i=1}^d \mu_i S_i x\right\|_A^2 = \left\|\sum_{i=1}^d \mu_i S_i\right\|_A^2,$$

$$\sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\sum_{i=1}^d \|S_i x\|_A^2 = \|\mathbf{S}\|_A^2,$$

$$\sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\left(\sum_{i=1}^d \|S_i x\|_A^{2\nu}\right)^{\frac{1}{\nu}} \le \left(\sum_{i=1}^d \|S_i\|_A^{2\nu}\right)^{\frac{1}{\nu}}$$

and

$$\sup_{x\in\mathbb{S}_{\mathcal{F}}^{A}}\max_{1\le i\le d}\|S_ix\|_A^2=\max_{1\le i\le d}\|S_i\|_A^2.$$

Moreover, it is clear that

$$\sup_{x\in\mathbb{S}_{\mathcal{F}}^{A}}\sum_{1\le i\ne j\le d}\left|\left\langle S_j^{\sharp_A}S_ix,x\right\rangle_A\right|\le\sum_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right),$$

$$\sup_{x\in\mathbb{S}_{\mathcal{F}}^{A}}\left(\sum_{1\le i\ne j\le d}\left|\left\langle S_j^{\sharp_A}S_ix,x\right\rangle_A\right|^{\beta}\right)^{\frac{1}{\beta}}\le\left(\sum_{1\le i\ne j\le d}\omega_A^{\beta}\left(S_j^{\sharp_A}S_i\right)\right)^{\frac{1}{\beta}}$$

and

$$\sup_{x\in\mathbb{S}_{\mathcal{F}}^{A}}\max_{1\le i\ne j\le d}\left|\left\langle S_j^{\sharp_A}S_ix,x\right\rangle_A\right|=\max_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right).$$

By computing the supremum over all $x\in\mathbb{S}_{\mathcal{F}}^{A}$ in the inequality (9) and then utilizing its subadditivity property, we are able to obtain the desired result. □

Corollary 4. *Let* $\mathbf{S}=(S_1,\dots,S_d)\in\mathbb{L}_{A^{1/2}}(\mathcal{F})^d$. *Then*

$$\|\mathbf{S}\|_{e,A}^2\le\max_{1\le i\le d}\|S_i\|_A^2+(d-1)\max_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right).\tag{10}$$

Proof. We obtain from Theorem 2:

$$\left\|\sum_{i=1}^{d}\mu_iS_i\right\|_A^2\le\sum_{i=1}^{d}|\mu_i|^2\left[\max_{1\le i\le d}\|S_i\|_A^2+(d-1)\max_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right)\right].$$

Taking the supremum over the set $\mathbf{B}_d$ for $(\mu_1,\dots,\mu_d)$ yields that

$$\begin{aligned}\|\mathbf{S}\|_{e,A}^2&=\sup_{(\mu_1,\dots,\mu_d)\in\mathbf{B}_d}\left\|\sum_{i=1}^{d}\mu_iS_i\right\|_A^2\\&\le\sup_{(\mu_1,\dots,\mu_d)\in\mathbf{B}_d}\sum_{i=1}^{d}|\mu_i|^2\left[\max_{1\le i\le d}\|S_i\|_A^2+(d-1)\max_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right)\right]\\&=\max_{1\le i\le d}\|S_i\|_A^2+(d-1)\max_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right).\end{aligned}$$

The desired result (10) is thereby proven. □

Theorem 3. *Suppose that* $\mathbf{S}=(S_1,\dots,S_d)\in\mathbb{L}_{A^{1/2}}(\mathcal{F})^d$. *Then,*

$$\omega_A^2(\mathbf{S})\le\max_{1\le i\le d}\{\omega_A(S_i)\}\left\{\|\mathbf{S}\|_A^2+\sum_{1\le i\ne j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right)\right\}^{\frac{1}{2}}.\tag{11}$$

Moreover, we have

$$\omega_A^2(\mathbf{S})\le\left(\sum_{i=1}^{d}\omega_A^{2p}(S_i)\right)^{\frac{1}{2p}}\left\{\left(\sum_{i=1}^{d}\|S_i\|_A^{2q}\right)^{\frac{1}{q}}+(d-1)^{\frac{1}{p}}\left(\sum_{1\le i\ne j\le d}\omega_A^{q}\left(S_j^{\sharp_A}S_i\right)\right)^{\frac{1}{q}}\right\}^{\frac{1}{2}}$$

for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Based on inequality (5), we can conclude that for $y_i = S_i x$, the following estimate is satisfied:

$$\sum_{i=1}^{d}\left|\left\langle x,S_i x\right\rangle_A\right|^2 \le \|x\|_A \max_{1\le i\le d}\left|\left\langle x,S_i x\right\rangle_A\right|\left\{\sum_{i=1}^{d}\|S_i x\|_A^2+\sum_{1\le i\neq j\le d}\left|\left\langle S_j^{\sharp_A}S_i x,x\right\rangle_A\right|\right\}^{\frac{1}{2}},$$

for $x \in \mathcal{F}$.

If we take the supremum over $x \in \mathbb{S}_{\mathcal{F}}^A$, then we obtain

$$\begin{aligned}
\omega_A^2(\mathbf{S}) &= \sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\left(\sum_{i=1}^{d}\left|\left\langle x,S_i x\right\rangle_A\right|^2\right)\\
&\le \sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\left(\|x\|_A \max_{1\le i\le d}\left|\left\langle x,S_i x\right\rangle_A\right|\left\{\sum_{i=1}^{d}\|S_i x\|_A^2+\sum_{1\le i\neq j\le d}\left|\left\langle S_j^{\sharp_A}S_i x,x\right\rangle_A\right|\right\}^{\frac{1}{2}}\right)\\
&\le \sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\left(\max_{1\le i\le d}\left|\left\langle x,S_i x\right\rangle_A\right|\right)\sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\left\{\sum_{i=1}^{d}\|S_i x\|_A^2+\sum_{1\le i\neq j\le d}\left|\left\langle S_j^{\sharp_A}S_i x,x\right\rangle_A\right|\right\}^{\frac{1}{2}}\\
&\le \max_{1\le i\le d}\{\omega_A(S_i)\}\left\{\sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\sum_{i=1}^{d}\|S_i x\|_A^2+\sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\sum_{1\le i\neq j\le d}\left|\left\langle S_j^{\sharp_A}S_i x,x\right\rangle_A\right|\right\}^{\frac{1}{2}}\\
&\le \max_{1\le i\le d}\{\omega_A(S_i)\}\left\{\|\mathbf{S}\|_A^2+\sum_{1\le i\neq j\le d}\sup_{x\in\mathbb{S}_{\mathcal{F}}^A}\left|\left\langle S_j^{\sharp_A}S_i x,x\right\rangle_A\right|\right\}^{\frac{1}{2}}\\
&= \max_{1\le i\le d}\{\omega_A(S_i)\}\left\{\|\mathbf{S}\|_A^2+\sum_{1\le i\neq j\le d}\omega_A\left(S_j^{\sharp_A}S_i\right)\right\}^{\frac{1}{2}},
\end{aligned}$$

which proves (11).

By employing (6), we can deduce that for $y_i = S_i x$, the following inequality holds:

$$\sum_{i=1}^{d}\left|\left\langle x,S_i x\right\rangle_A\right|^2 \le \|x\|_A\left(\sum_{i=1}^{d}\left|\left\langle x,S_i x\right\rangle_A\right|^{2p}\right)^{\frac{1}{2p}}\times\left\{\left(\sum_{i=1}^{d}\|S_i x\|_A^{2q}\right)^{\frac{1}{q}}+(d-1)^{\frac{1}{p}}\left(\sum_{1\le i\neq j\le d}\left|\left\langle S_j^{\sharp_A}S_i x,x\right\rangle_A\right|^{q}\right)^{\frac{1}{q}}\right\}^{\frac{1}{2}},$$

for $x \in \mathcal{F}$ and $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$. So, the second inequality in Theorem 3 can be obtained by taking the supremum over x belonging to $\mathbb{S}_{\mathcal{F}}^A$. $\square$

Remark 6. *For* $p = q = 2$, *in Theorem 3, we obtain*

$$\omega_A^2(\mathbf{S}) \le \left(\sum_{i=1}^{d}\omega_A^4(S_i)\right)^{\frac{1}{4}}\left\{\left(\sum_{i=1}^{d}\|S_i\|_A^4\right)^{\frac{1}{2}}+(d-1)^{\frac{1}{2}}\left(\sum_{1\le i\neq j\le d}\omega_A^2\left(S_j^{\sharp_A}S_i\right)\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

Theorem 4. *Assuming the conditions of Theorem 2, we obtain the following:*

$$
\|\mathbf{S}\|_A^4 \leq \begin{cases} \sum_{i=1}^{d} \|S_i\|_A^4; \\ \\ d^{\frac{1}{\mu}} \left(\sum_{i=1}^{d} \|S_i\|_A^{4\nu} \right)^{\frac{1}{\nu}}, \text{ where } \mu > 1, \frac{1}{\mu} + \frac{1}{\nu} = 1; \\ \\ d \max_{1 \leq i \leq d} \|S_i\|_A^4, \end{cases}
$$
$$
+ \begin{cases} \sum_{1 \leq i \neq j \leq d} \omega_A \left(S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i \right); \\ \\ [d(d-1)]^{\frac{1}{\alpha}} \left(\sum_{1 \leq i \neq j \leq d} \omega_A^{\beta} \left(S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i \right) \right)^{\frac{1}{\beta}}, \text{ where } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ d(d-1) \max_{1 \leq i \neq j \leq d} \omega_A \left(S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i \right). \end{cases} \tag{12}
$$

Proof. If we substitute $\theta_i = 1$ and $y_i = S_i^{\sharp_A} S_i x$ in Theorem 1, then

$$
\left(\sum_{i=1}^{d} \|S_i x\|_A^2 \right)^2 \leq \|x\|_A^2 \times \begin{cases} \sum_{i=1}^{d} \left\| S_i^{\sharp_A} S_i x \right\|_A^2; \\ \\ d^{\frac{1}{\mu}} \left(\sum_{i=1}^{d} \left\| S_i^{\sharp_A} S_i x \right\|_A^{2\nu} \right)^{\frac{1}{\nu}}, \text{ where } \mu > 1, \frac{1}{\mu} + \frac{1}{\nu} = 1; \\ \\ n \max_{1 \leq i \leq d} \left\| S_i^{\sharp_A} S_i x \right\|_A^2, \end{cases} \tag{13}
$$
$$
+ \|x\|_A^2 \times \begin{cases} \sum_{1 \leq i \neq j \leq d} \left| \left\langle S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i x, x \right\rangle_A \right|; \\ \\ [n(d-1)]^{\frac{1}{\alpha}} \left(\sum_{1 \leq i \neq j \leq d} \left| \left\langle S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i x, x \right\rangle_A \right|^{\beta} \right)^{\frac{1}{\beta}}, \\ \text{where } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ n(d-1) \max_{1 \leq i \neq j \leq d} \left| \left\langle S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i x, x \right\rangle_A \right| \end{cases}
$$

for $x \in \mathcal{F}$.

Observe that

$$
\sup_{x \in \mathbb{S}_{\mathcal{F}}^{A}} \left\| S_i^{\sharp_A} S_i x \right\|_A^2 = \left\| S_i^{\sharp_A} S_i \right\|_A^2 = \|S_i\|_A^4,
$$
$$
\sup_{x \in \mathbb{S}_{\mathcal{F}}^{A}} \left\| S_i^{\sharp_A} S_i x \right\|_A^{2\nu} = \|S_i\|_A^{4\nu}
$$

and

$$
\sup_{x \in \mathbb{S}_{\mathcal{F}}^{A}} \left| \left\langle S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i x, x \right\rangle_A \right| = \omega_A \left(S_j^{\sharp_A} S_j S_i^{\sharp_A} S_i \right).
$$

The desired result (12) can be obtained by taking the supremum in (13) over $x \in \mathbb{S}_{\mathcal{F}}^{A}$ and using the subadditivity property of the supremum. $\square$

5. Conclusions

In conclusion, this article presents new findings on Boas–Bellman-type inequalities in semi-Hilbert spaces, offering valuable insights into their properties and applications. These spaces are generated by semi-inner products induced by positive and positive semidefinite operators. By deriving novel inequalities relating to the joint A-numerical radius, joint operator A-seminorm, and Euclidean A-seminorm of tuples of semi-Hilbert space operators, we establish connections that enhance our understanding of these measures. Furthermore, assuming A as a non-zero positive operator adds applicability to our results.

This work not only contributes to the understanding of semi-Hilbert spaces and their implications in multivariable operator theory but also provides a starting point for future research. It opens up possibilities for exploring new results and other types of inequalities in semi-Hilbert spaces, which can advance the field and provide insights into functional analysis and operator theory. The findings presented here lay the foundation for further investigations and offer researchers the opportunity to delve into the intricacies of semi-Hilbert spaces, ultimately advancing the knowledge and application of these spaces.

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