PPAD-Membership for Problems with Exact Rational Solutions: A General Approach via Convex Optimization

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ABSTRACT

We introduce a general technique for proving membership of search problems with *exact rational* solutions in PPAD, one of the most well-known classes containing total search problems with polynomial-time verifiable solutions. In particular, we construct a "pseudogate", coined the *linear-OPT-gate*, which can be used as a "plug-and-play" component in a piecewise-linear (PL) arithmetic circuit, as an integral component of the "Linear-FIXP" equivalent definition of the class. The linear-OPT-gate can solve several convex optimization programs, including quadratic programs, which often appear organically in the simplest existence proofs for these problems. This effectively transforms existence proofs to PPAD-membership proofs, and consequently establishes the existence of solutions described by rational numbers.

Using the linear-OPT-gate, we are able to significantly simplify and generalize almost all known PPAD-membership proofs for finding exact solutions in the application domains of game theory, competitive markets, auto-bidding auctions, and fair division, as well as to obtain new PPAD-membership results for problems in these domains.

CCS CONCEPTS

• **Theory of computation → Problems, reductions and completeness**; **Algorithmic game theory**; *Exact and approximate computation of equilibria*; *Market equilibria.*

KEYWORDS

PPAD, exact solutions, Linear-FIXP

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1 INTRODUCTION

Total search problems, i.e., search problems for which a solution is always guaranteed to exist, have been studied extensively over the better part of the last century, in the intersection of mathematics, economics and computer science. Famous examples of such problems are finding Nash equilibria in games [55], competitive equilibria in markets [1] and envy-free divisions of resources [69]. While the classic works in mathematics and economics have been primarily concerned with establishing the existence as well as desirable properties of these solutions, the literature of computer science over the past 35 years has been instrumental in formulating and answering questions about the computational complexity of finding them.

More precisely, Megiddo and Papadimitriou [51] defined the class TFNP to include all total search problems for which a solution is verifiable in polynomial time. To capture the computational complexity of many problems including the aforementioned ones, several subclasses of TFNP were subsequently defined. Among those, one that has been extremely successful in this regard is the class PPAD of Papadimitriou [58], which was proven to characterize the complexity of computing Nash equilibria in games [14, 23], as well as competitive equilibria for several types of markets [16], among many others.

In reality, when making statements like the above, i.e., general statements of the form, "finding a Nash equilibrium is in PPAD", or similarly for a solution to some other total search problem, it is most often meant that what lies in the class is the problem of finding *approximate* solutions. For strategic games for example, that would mean strategy profiles which are *almost* Nash equilibria, up to some additive parameter ε. This is actually quite often necessary, as it has been shown that for many of these problems, there are cases where all of their solutions can only be described by irrational numbers, and hence we can not hope to compute them exactly on a computer.

Still, there is a large number of important variants of these domains for which *exact rational* solutions exist. For example, several strategic games always have equilibria in rational numbers, and so do certain markets for their competitive equilibria. There are also examples from fair division where rational partitions of the resources can be achieved. In all of those cases, PPAD-membership results for their approximate versions are unsatisfactory; we would like to place the *exact* problems in PPAD instead.

Indeed, coming up with proofs of existence that also guarantee rationality of solutions has been a topic of interest in the area since the very early days, way before the introduction of the relevant computational complexity classes, e.g., see [26, 40, 48, 49]. Driven by those classic results, a significant literature in computer science has attempted, and quite often has succeeded in placing the corresponding computational problems in PPAD, for several of the application domains mentioned above, including games [39, 45, 46, 52, 66], markets [31, 34, 35, 72], as well as the more recent domain of autobidding auctions [15].

While these PPAD-membership proofs typically do follow one of a few common approaches, in essence they are rather domain-specific and require reconstructing a set of arguments again for each application at hand (see Section 1.2.1 below for a detailed discussion). Instead, we would like to have *one general technique for proving PPAD-membership of problems with exact solutions*, and ideally one that arises "organically" as the computational equivalent of the standard proofs of existence. To do this, a very promising avenue seems to be via a characterization of PPAD, coined *Linear-FIXP*, due to Etessami and Yannakakis [27], which defines the class in terms of fixed points of problems represented by piecewise-linear arithmetic circuits. This is because a standard existence proof, e.g., via the Kakutani fixed point theorem [44] or via Brouwer's fixed point theorem [11], often obtains the solution as a fixed point of a set of local optimization problems, in which each agent or player is independently maximizing a piecewise utility/payoff function. If we could "embed" these optimization problems into a piecewise-linear circuit, that would essentially translate the existence proof into a PPAD-membership proof. This is crisply captured in the following quote from Vazirani and Yannakakis [72], in the context of proving PPAD-membership for competitive equilibria in certain markets:

> *"There are very few ways for showing membership in PPAD. A promising approach for our case is to use the characterization of PPAD of Etessami and Yannakakis [2010] as the class of exact fixed-point computation problems for piecewise-linear, polynomial time computable Brouwer functions.* [. . .] *Unfortunately, we do not see how to do this* [. . .] *it is not clear how to transfer the piecewise-linearity of the utility functions to the Brouwer function." [72].*

Recently, Filos-Ratsikas et al. [29] in fact developed a general technique along those lines: they designed an *optimization gate*, which can be used as part of a circuit to substitute the aforementioned optimization problems and obtain membership results. Crucially however, their membership results are *not* for the class PPAD, but rather for the class FIXP [27], a superclass of Linear-FIXP in which the main computational device is a (general) arithmetic circuit, not a piecewise-linear one. These circuits are particularly powerful and can capture solutions with irrationalities. Using their "OPT-gate for FIXP", Filos-Ratsikas et al. [29] showed the FIXP-membership of several very general problems related to strategic games, markets and fair division.

While FIXP is certainly a natural class, it has not enjoyed the same success as PPAD, even in the context of classifying problems with exact solutions. Besides, in the standard (Turing) model of computation, a FIXP-membership result can be interpreted as finding a point that is *close* to a solution (e.g., in the max norm). This is often a stronger guarantee than an approximate solution as described earlier, but it it still very much only an approximation. Again, this is unsatisfactory for those problems with exact rational solutions that should be in PPAD.

Could we hope to use Filos-Ratsikas et al.'s optimization gate to obtain PPAD-membership? This is actually practically impossible, for reasons which are deeply rooted in the definitions of the classes; we highlight those in Section 1.3 below. In short, the power of general arithmetic circuits over piecewise-linear ones lies in their capability to multiply and divide input variables, and this is of vital importance in the design of the OPT-gate for FIXP in [29]. What we really need is *a new gate*, one which avoids such multiplications/divisions and hence can be used in a piecewise-linear arithmetic circuit. Designing such a gate poses significant technical challenges, which we highlight in Section 1.3 and present in more detail in the full version of our paper. Additionally, clearly, the gate cannot capture the generality of applications that the OPT-gate for FIXP does, as, as we said earlier, problems with irrational solutions cannot be in PPAD. It should however be general enough to capture any problem for which exact rational solutions are possible.

This is the main technical contribution of our paper. We introduce the linear-OPT-gate,[1] which can be used as a general purpose tool for proving PPAD-membership of problems with exact rational solutions. We demonstrate its strength and generality on a host of different applications in game theory, markets, auctions and fair division. Via its use, we are able to *significantly simplify* or *generalize* virtually all of the PPAD-membership proofs for problems with exact solutions in the literature, as well as to prove *new* membership results for problems for which PPAD-membership was not known; we offer more details in the following subsection.

1.1 A Powerful Tool for PPAD-Membership: The linear-OPT-gate

We introduce the linear-OPT-gate for proving membership of problems in PPAD. The linear-OPT-gate can be used as a "plug-and-play" component in a PL arithmetic circuit, i.e., similarly to any of the other gates $\{+, -, \max, \min, \times\zeta\}$ of the circuit (see the full version of our paper for a formal definition). The gate is guaranteed to work correctly at a fixed point of the function that the circuit encodes, which, for the purposes of proving PPAD-membership of a problem, is equivalent to a standard gate.

The linear-OPT-gate allows us to compute solutions to optimization programs of a certain form, like those shown in the left-hand side of Figure 1. In particular, these are optimization programs with a non-empty and bounded feasible domain given by a set of linear inequalities, and the subgradient of the convex objective function (in the variables x) is given by a PL (piecewise-linear) arithmetic circuit. In particular, the linear-OPT-gate can compute the solution to any linear program, but also to more general convex programs, e.g., those with quadratic objective functions. The inherent strength

[1]The term "linear" here refers to piecewise-linear functions, as in the class Linear-FIXP.

Optimization Program C	Feasibility Program Q
$\min \quad f(x;c)$	$h_i(y) > 0 \implies a_i^\top x \le b_i$
$\text{s.t.} \quad Ax \le b$	$x \in [-R,R]^n$
$x \in [-R,R]^n$	

Figure 1: The optimization programs and feasibility programs that can be solved by the linear-OPT-gate.

of the technique lies in the fact that these types of optimization programs arise naturally in several of the applications in game theory, competitive markets and fair division. Now, for the purpose of showing membership in PPAD, they may effectively be substituted by linear-OPT-gates.

From the ability of the linear-OPT-gate to solve optimization programs of the form C of Figure 1, we can also derive *feasibility programs* with *conditional constraints*, like the program Q on the right-hand side of Figure 1. These feasibility programs also often arise naturally in the context of existence proofs, and can be also thought of as being solved in a black-box manner by a gate, which is constructed using the linear-OPT-gate.

Our linear-OPT-gate has a wealth of applications, which we discuss below.

1.2 Applications of the linear-OPT-gate

We apply our linear-OPT-gate to a plethora of different domains, and obtain PPAD-membership for finding solutions in several strategic games, competitive markets, auto-bidding auctions, as well as problems in fair division. We detail those applications in the corresponding sections below. Our results achieve the following three desired objectives simultaneously:

- Proofs of existence of solutions.
- Proofs of rationality of solutions.
- PPAD-membership of the corresponding problems.

For some of these domains, PPAD-membership results for the corresponding problems were known; still, the proofs to establish those were often rather involved. With the employment of our linear-OPT-gate, they become *conceptually and technically significantly simpler*. In essence, the linear-OPT-gate allows us to turn a simple existence proof into a PPAD-membership result. For some of our applications such simple existence proofs already existed, and are transformed to PPAD-membership proofs via the linear-OPT-gate. For others, developing these simpler existence proofs is also part of our contribution; we provide more details in the sections below. The linear-OPT-gate also allows us to *straightforwardly* obtain generalizations of some of the known PPAD-membership results, to cases beyond what was known in the literature. Finally, we also obtain the PPAD-membership of some problems whose complexity had not been studied in the literature before.

We summarize our results in Table 1, where we indicate which results were known in the literature before, which are generalizations, and which concern problems for which we did not know any results about their computational complexity.

Before we proceed with the applications, we present the main techniques that have been used in prior works for proving PPAD-membership results, and highlight the main technical challenges of using those techniques as opposed to the "plug-and-play" nature of our linear-OPT-gate.

1.2.1 Main Previous Approaches.

Linear Complementarity Programs & Lemke's Algorithm. The first main approach for establishing rationality of solutions and PPAD-membership is that of *linear complementarity programs (LCPs)* [19, 20]. Given an $n \times n$ matrix $\mathbf{M}$ and a vector $\mathbf{q}$, an LCP seeks to find two vectors $\mathbf{y}$ and $\mathbf{v}$ satisfying:

$$\mathbf{M} \cdot \mathbf{y} + \mathbf{v} = \mathbf{q}, \quad \mathbf{y} \ge 0, \quad \mathbf{v} \ge 0, \quad \text{and} \quad \mathbf{y}^\top \cdot \mathbf{v} = 0$$

The term "complementarity" stems from the fact that in a solution, we may have either $\mathbf{y}_i > 0$ or $\mathbf{v}_i > 0$, but not both. Lemke [48] designed an algorithm (based on the previously designed Lemke-Howson algorithm [49]) to solve LCPs via a series of *complementary pivoting* steps, i.e., steps in which when a variable enters the basis, a complementary variable exits. Interestingly, the algorithm was designed in the context of computing Nash equilibria in bimatrix games, long before the associated computational complexity classes were defined. LCP-based formulations of equilibria and other fixed point problems have in fact been a subject of study in classic works (e.g, see [26, 40]) as a means to obtain existence proofs that guarantee the rationality of solutions. PPAD membership can be obtained by pairing the algorithm with an appropriate local orientation of its complementarity paths [71].

Quite importantly, Lemke's algorithm terminates with either finding a solution to the LCP, or without finding a solution, in what is referred to as a *secondary ray*. This feature of the algorithm is well-documented (e.g., see [63] for an excellent exposition) and is known as *ray termination*. In terms of proving PPAD-membership, it seems almost inevitable that every PPAD-membership proof that uses this approach has to argue against ray termination. As Garg and Vazirani [35] pointedly remark, in the context of a succession of papers on equilibrium computation in competitive markets:

> *"In the progression of these three works, the LCPs have become more involved and proving the lack of secondary rays has become increasingly harder." [35].*

This is not particular to markets either. For example, in Hansen and Lund's generalization [39] of the results of Sørensen [66] from bimatrix to polymatrix games, those concerning ε-proper equilibria, a new LCP needs to be devised, together with a new argument against ray termination. Additionally, there are often significant challenges in even appropriately formulating the problems in question as LCPs. In some cases, the naive formulations may lead to inefficient representations, e.g., see [66]. In other cases, all known formulations lead to *nonstandard* LCPs, which cannot be handled by the "vanilla" version of Lemke's algorithm, and require variants of the algorithm to be devised, e.g., see [34, 52]. Finally in some cases, it is not known if the derived LCPs can be solved via any variant of Lemke's algorithm, thus leading to the development of entirely new pivoting algorithms [46]. These characteristics of the LCP approach make it somewhat insufficient as a general purpose PPAD-membership technique.

One advantage of LCP-based approaches is that they have been shown to perform well in practice, e.g., see [34] and references therein. However, for the purpose of proving PPAD-membership, we do not see any general advantage of the LCP method over our linear-OPT-gate.

Approximation and Rounding. The second general technique that has been used in several applications to prove the PPAD-membership of exact solutions is that of *approximation and rounding*. This generally consists of the two following main steps:

- consider an *approximation* or a *relaxation* of the solution (e.g., ε-approximate equilibria) and prove that the approximate version is in PPAD, and
- devise a rounding procedure to transform approximate solutions to exact solutions, while maintaining membership in the class.

This rather indirect approach certainly suffers in terms of elegance. More importantly however, it is very much domain-specific. First, showing the PPAD-membership for the approximate version typically still requires a non-trivial proof, often even a rather involved one, e.g., via some reduction to one of the well-known problems in PPAD, like End-of-Line or the computational version of Sperner's lemma [67]. Also, the rounding procedure itself may be rather complicated, and of an ad hoc nature. For certain applications, there is a general linear programming-based technique developed by Etessami and Yannakakis [27] to transform ε-approximate solutions to exact ones, for sufficiently small values of ε. Still, this does not apply to all problems, and it may need to be used in conjunction with other tailor-made rounding steps, e.g., see [15, 72].

The linear-OPT-gate as a "Plug-and-Play" Component. As we will explain in the following, and as it will be evident via inspection of our proofs throughout the paper, the linear-OPT-gate allows us to develop proofs which are very simple and streamlined, essentially mimicking the easiest proofs of existence. Clearly, most of the technical complications are "hidden" in the "inner workings" of the linear-OPT-gate. This is the advantage of having a "plug-and-play" component readily available for the proofs: one does not need to even be concerned about how the linear-OPT-gate works, but only to understand what kind of optimization programs it can solve. We consider this to be a significant advantage over the two aforementioned techniques, which require to devise application-specific arguments (be it arguments about ray termination or appropriate approximation and rounding). These arguments may be of a standard general nature, but they have to be devised anew for each application, as evidenced by all the different PPAD-membership results that employ these techniques.

1.2.2 Implicit Functions and Correspondences. As a final remark before we present our applications, we point out that, via machinery that we develop in the full version of our paper, our linear-OPT-gate can be used to show the PPAD-membership of problems for which the inputs (e.g., utilities or latency functions) are given *implicitly* in the input. In particular, we show how we can construct PL arithmetic circuits computing these functions, when those are inputted *succinctly* via Boolean circuits. In terms of the applications, this allows us to effectively consider functions of exponential size (in the size of the circuits), e.g., piecewise-linear utility functions with exponentially-many pieces. We provide details on how this capability of the linear-OPT-gate is used in each application in the corresponding sections below. We present applications for which the aforementioned techniques of Section 1.2.1 are *inherently insufficient* for obtaining PPAD-membership results for those implicit functions, when these results are in fact enabled by the use of the linear-OPT-gate.

1.2.3 PPAD-membership for Strategic Games. We start our discussion from the applications of the linear-OPT-gate to the problem of computing (exact) equilibria in strategic games. To provide some initial intuition, before the technical sections of the paper, we provide an informal example of the use of the linear-OPT-gate to compute mixed Nash equilibria in bimatrix games; this is exposed in more detail in the corresponding section in the full version of our paper.

An Example: Bimatrix Games. A bimatrix game is a game played between two players, in which the payoffs are given by two matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, one for each player, denoting the payoff of the players when they each choose certain actions. Each player chooses a *mixed strategy*, i.e., a probability distribution over actions in the game, aiming to maximize their expected payoff, against the choice of the opponent. A mixed Nash equilibrium is a pair of mixed strategies for which every player is *best responding*, i.e., she is maximizing her payoff, given the strategy of the other player. The existence of mixed Nash equilibria for bimatrix games follows from Nash's general existence theorem [55]. The proof of the theorem that employs the Kakutani fixed point theorem [44] constructs a fixed point of a function F from the domain of mixed strategies to itself, for which each coordinate F_i is a best response for player i in the game. These best responses can be captured by optimization programs of the form C in Figure 1 and in particular for the case of bimatrix games, these are linear programs in which the subgradients of the objective functions are linear functions. The existence proof then immediately yields a PPAD-membership proof if one substitutes those programs with linear-OPT-gates that compute them.

We remark that for bimatrix games, the original PPAD-membership proof of Papadimitriou [58] adopts the "LCP approach" that we mentioned earlier, i.e., it appeals to an alternative proof of Nash equilibrium existence due to Cottle and Dantzig [19] (see also [49]) that formulates the problem as an LCP. This is a good example of what we mentioned earlier; the linear-OPT-gate allows us to organically retrieve PPAD-membership from the standard, textbook existence proof of Nash [55].

Best Response Oracles, PL Concave Games and Generalized Equilibria.

PL Best Response Oracles. The approach that we highlighted above is not restricted to bimatrix games, but it actually captures a

Table 1: A summary of our PPAD-membership results - for other complementary results please see the respective sections/paragraphs in the introduction. Classes of domains that are within the same frame in the table (i.e., not separated by borders) are of increasing generality from top to bottom. Domains that appear in the same row of a frame are incomparable in terms of their generality. For the applications to game theory, all of the domains are special cases of PL concave games which in turn are a special case of PLBRO games. For those applications, the PPAD-membership extends to *generalized equilibria*. For all of the results in the table, regardless of whether we obtain entirely new results, generalizations, or simply results which were known in the literature, we obtain *significant simplifications* in the proofs.

Applications to Game Theory	
Games with PL Best Response Oracles (PLBRO)	**[Our Work]**
PL Concave Games	**[Our Work]**
Bilinear Games	[47], *implicitly*
General Threshold Games	**[Our Work]**
Bimatrix Games	[58] [19], *implicitly*
Polymatrix Games	[40], *implicitly*
PL Succinct Games	**[Our Work]**
Multi-class Congestion games with piecewise-linear latency functions	
Non-atomic Network Congestion Games	**[Our Work]** linear latencies [52]
Atomic Splittable Network Congestion Games	**[Our Work]** linear latencies [46]
Congestion Games with Malicious Players	**[Our Work]**
Other equilibrium notions	
ε-proper Equilibria in Bimatrix Games	[66]
ε-proper Equilibria in Polymatrix Games	[39]
ε-proper Equilibria in PL Succinct Games	**[Our Work]**
Personalized Equilibria	[45]
Applications to Competitive Markets	
Exchange Markets with Linear Utilities	[26], *implicitly*
Arrow-Debreu Markets with SPLC Utilities	[33]
Arrow-Debreu Markets with SPLC Utilities/Productions	[72] [35]
Arrow Debreu Markets with Leontief-free Utilities/Productions [34]	Arrow-Debreu Markets with *Succinct* SPLC Utilities/SPLC Productions **[Our Work]**
Applications to Auto-Bidding Auctions	
Pacing Equilibria in Second-Price Auctions with Budgets	[15]
Applications to Fair Division	
Envy-free Cake Cutting	[38], *implicitly*
Rental Harmony	**[Our Work]**

large class of strategic games. In the full version of our paper we provide a technical definition for a very general class of games, in which the best response of each agent is given by an oracle that can be computed by a PL arithmetic circuit. We refer to these games as *games with PL best response oracles (PLBRO games)*. An equilibrium of any PLBRO game can straightforwardly be formulated as a fixed point of a function like the function F above, where each coordinate F_i computes the best response of player i via the oracle. By using linear-OPT-gates as oracles, we immediately obtain PPAD-membership results for a wealth of different games.

PL Concave Games. The class of *concave games* is a very large class of games, studied notably by Rosen [62] and Debreu [24]. These are games with continuous strategy spaces, for which the existence of an equilibrium is guaranteed under certain continuity and concavity assumptions on the utility functions. This was proven by Rosen [62] but also earlier independently by Debreu [24], [28], and Glicksberg [36], and for that reason the existence result is often referred to as the Debreu-Fan-Glicksberg theorem for continuous games.

In the corresponding section of the full paper we prove that as long as the supergradient of the (concave) utility function can be computed by a PL arithmetic circuit, concave games are PLBRO games, and hence finding an equilibrium is in PPAD. We refer to those games as PL concave games, and emphasize again that the utility function does not have to be piecewise linear, but only its (super)gradient; in particular, it could for example be a quadratic function. Bimatrix games are PL concave games, and so are *polymatrix games* [40, 42], *bilinear games* [32], as well as generalizations of *(digraph) threshold games* [56], and thus we obtain membership of finding equilibria in all of these games in PPAD. The latter two games have continuous strategy spaces, and thus the equilibria that we compute are pure, whereas for polymatrix games (and as a result, for bimatrix games) we compute equilibria in mixed strategies.

PL Succinct Games. In fact, we define a large class of games, which generalize polymatrix games, one which we coin *PL succinct games*. In these games, the expected utility of a player, given a pure strategy j and a mixed strategy $\mathbf{x}_{-i}$ of the other players, can be computed by a PL arithmetic circuit. These are PL concave games, and the PPAD-membership of finding their mixed Nash equilibria is a corollary of the results mentioned above.

We draw parallels between PL succinct games and those defined in Daskalakis et al. [22] and Papadimitriou and Roughgarden [59]. Those works define classes of succinct games for which there is an oracle for computing the expected utility of the player. In [59], this oracle is referred to as the *polynomial expectation property* and is used to show that correlated equilibria [2] of games with this property can be computed in polynomial time. In [22], it is shown that if the oracle is given by a *bounded division free straight-line program of polynomial length*, then these games are in PPAD. Crucially, this latter result concerns *approximate equilibria*. One could view our result as a complement to those two results, one which concerns *exact* equilibria in *rational* numbers.

Our PPAD-membership result for PL concave games captures the limits of the class of concave games for which rational equilibria exist, and thus membership in PPAD is possible. The only other known complexity results for the general class of concave games are a FIXP-completeness result due to Filos-Ratsikas et al. [29], and a very recent PPAD-membership result for *approximate* equilibria due to [57].

Generalized Equilibrium. Debreu [24] did not only consider concave games, but in fact a more general equilibrium notion, one in which the strategy space of each player is dependent on the set of strategies chosen by the other players. This was coined a "social equilibrium" by Debreu [24] (see also Dasgupta and Maskin [21]) but over the years has been better known by the term *generalized equilibrium*. For our purposes, the dependence on other strategies can be embedded in the constraints of the optimization programs that we use as oracles in PLBRO games, in a way that can be handled by the linear-OPT-gate. As a corollary, we obtain all of the aforementioned PPAD-membership results for generalized equilibria (rather than standard equilibria) as well. To the best of our knowledge, these are the first PPAD-membership results for generalized equilibria in the literature.

Personalized Equilibria. The notion of *personalized equilibrium* was introduced by Kintali et al. [45] in the context of games played on hypergraphs, with an equivalent strategic form. Intuitively speaking, these equilibria allow players to "match" their strategies with those of their opponents, without obeying a product distribution. Kintali et al. [45] showed the PPAD-membership (and as a result, rationality of equilibria) of personalized equilibria via the "relaxation and rounding approach" (see Section 1.2.1). In particular, they first define an approximate version of the problem (the ε-personalized equilibrium), and reduce that problem to End-Of-Line, via a relatively involved construction. To obtain PPAD-membership for the exact problem (i.e., when $\varepsilon = 0$) Kintali et al. [45] construct an elaborate argument that appeals to linear programming compactness, by first showing that for sufficiently small ε, ε-personalized equilibria "almost satisfy" the constraints of the linear programs, and then carefully rounding the solution to obtain an exact equilibrium.

The use of the linear-OPT-gate allows us to obtain the PPAD-membership of the problem via an extremely simple argument. Essentially, each player computes their best response via a linear program which is computed by the linear-OPT-gate, which reduces the problem to finding an equilibrium of an PLBRO game.

ε-Proper Equilibria. We also consider an alternative equilibrium notion, that of *ε-proper equilibria*. This notion was introduced by Myerson [54] to refine the notion of ε-perfect equilibrium of Selten [65], and captures situations in which the players can make small mistakes ("trembles") in the choice of their mixed strategies. The PPAD-membership of computing ε-proper equilibria was known for bimatrix games due to Sørensen [66] and for polymatrix games due to Hansen and Lund [39]. Both of these works adopt the LCP approach, which means that they need to go through the hassle of establishing the properties of Lemke's algorithm, as discussed in Section 1.2.1 above. Additionally, formulating the problem as an LCP in this case is far from trivial, and requires an extended formulation of the generalized permutahedron due to Goemans [37], to make sure that the LCP has polynomially-many constraints.

The use of our linear-OPT-gate distinctly avoids all this labor. We formulate the problem of computing a best response for each player (where the best response is defined with respect to the ε-proper equilibrium notion) as a feasibility program of the form Q in Figure 1, which can be solved by the linear-OPT-gate. This essentially renders the game a PLBRO game, and the PPAD-membership follows simply as a corollary of our main theorem for PLBRO games.

Network Congestion Games. Our last application in the area of game theory is to multi-class congestion games. In particular, we will consider two models, *non-atomic congestion games* and *atomic splittable congestion games*. In the former case, there is a continuum of players who collectively form a class controlling a certain load allocation to different resources. In the latter case, each class is represented by a single (atomic) player, who controls the load and distributes it to the resources. For both of those settings, we will also consider the subclass of *network congestion games*, where the strategies can be represented more succinctly using flows over a directed network.

The existence of equilibria in those games was established in classic works, e.g., see [64] or [53], originally via the employment of the Debreu-Fan-Glicksberg theorem [24, 28, 36] for continuous games, assuming that the latencies on the resources are concave functions. Relevant to us are the works on their computational complexity, namely [52] (for non-atomic network congestion games) and [46] (for atomic splittable network congestion games). Both papers showed the PPAD-membership of finding pure equilibria in their respective settings, when the latency functions are *linear*. We remark that these games are different from atomic (non-splittable) congestion games, for which finding pure Nash equilibria is known to be in the class PLS defined by Johnson et al. [43].

Meunier and Pradeau obtain their PPAD-membership result via the "LCP approach" mentioned in Section 1.2.1. Interestingly, their LCP formulation turns out to not be amenable to the use of Lemke's algorithm, so they have to devise a "Lemke-like" complementary pivoting algorithm, tailored to their problem. As in the case of Lemke's algorithm, they argue explicitly against ray termination. Klimm and Warode note that in their case, the problem of finding an equilibrium can be formulated as an LCP, however, it is not known or clear whether this LCP can be solved using any known variant of Lemke's algorithm. For that, they devise a rather involved proof, based on a new homotopy method, essentially a new pivoting algorithm. Their algorithm solves the problem of finding a Nash equilibrium as a system of linear equations involving notions such as *excess flows*, *vertex potentials* and *block Laplacians*. At a very high level, the authors use the excess and potentials to define an undirected version of the End-of-Line graph, and the determinant of the block Laplacians to define a unique orientiation of the edges, effectively reducing the problem to End-of-Line.

The linear-OPT-gate allows us to avoid all of the technical complications of the proofs of Meunier and Pradeau [52] and [46] (which are rather involved, especially the latter), and essentially obtain the PPAD-membership for both of these problems as simple corollaries of our main results for PLBRO games or concave games. In fact, we obtain *generalizations* of those PPAD-membership results to games with more general latency functions, notably piecewise-linear latency functions (implicitly or explicitly represented). In exactly the same fashion, we can use the linear-OPT-gate to obtain the PPAD-membership of *congestion games with malicious players*, a setting studied by Babaioff et al. [3], for which computational complexity results had not been previously proven.

1.2.4 PPAD-Membership for Competitive Markets. We now move on to the application of our technique to the domain of competitive markets. The standard market model in the literature is that of the Arrow-Debreu market [1], where a set of consumers compete for goods endowed by them and other consumers and goods produced by a set of firms. A *competitive equilibrium* of the market is a set of allocations of goods to the consumers, a set of production quantities and a set of prices, such that at those prices, (a) all consumers maximize their individual utilities, (b) all firms produce optimal amounts, and (c) the market clears, i.e., supply equals demand. The existence of an equilibrium for the general market model was established by Arrow and Debreu [1] via the employment of Debreu's social equilibrium theorem [24], under some standard assumptions on the utilities of the consumers and the production sets of the firms.

Previous Results and Proofs. It has been well-known since the early works in the area [26] that in general Arrow-Debreu markets, competitive equilibria may be irrational. A significant literature, starting with the work of Eaves [26] aimed at identifying special cases of the Arrow-Debreu market for which *exact rational* solutions are always possible. When computer science took over in this quest, the related question of establishing the PPAD-membership of finding those exact solutions was also brought forward. Most of the PPAD-membership proofs that were developed through the years followed the "LCP approach", see Section 1.2.1. We present them here in succession:

- Eaves [26] considered the simplest case of exchange markets (no production) with linear utilities for the consumers and devised an LCP that can be solved by Lemke's algorithm. To establish the latter fact, he argued against ray termination, a characterstic of this approach that we emphasized in Section 1.2.1. A PPAD-membership proof is implicit in his result.[2]
- Garg et al. [33] considered exchange markets with *separable piecewise-linear concave (SPLC)* utilities, a generalization of linear utilities in which every agent has a piecewise linear concave utility for the amount of a good j that she receives, and her total utility for her bundle is additive over goods. The authors proved the PPAD-membership of finding competitive equilibria in those markets via devising an LCP that was "quite complex" [33], and naturally had to argue against ray termination, to establish that Lemke's algorithm will terminate on this LCP with a valid solution.
- Garg and Vazirani [35] considered Arrow-Debreu markets with SPLC utilities as well as SPLC production functions. This is in fact the work from which the quote of Section 1.2.1 is taken. The quote highlights the increasing challenge of developing these LCPs and establishing their successful termination. Indeed, for this LCP, Garg and Vazirani [35] devise a set of linear programs, and then use the complementary slackness and their feasibility

[2]Note that for exchange markets with linear utilities and no production the problem is in fact known to be polynomial-time solvable [41].

conditions to develop the LCP needed for production. The non-homogeneity of the resulting LCP for the equilibrium problem is dealt with in a manner which is different from previous works [26, 33] and, naturally, since the developed LCP is different, Garg and Vazirani again need to argue against ray termination.
- The most general class of utility/production functions for which a PPAD-membership of exact competitive equilibria was proven is that of *Leontief-free* functions [34], which generalize SPLC functions. For this, the authors devise yet another LCP formulation, which turns out to be even more complex than those of previous works. This is because it has to differentiate between "normal" and "abnormal" variables, the latter preventing the employment of Lemke's algorithm. To circumvent this, they exploit some additional structure of their *nonstandard* LCP, and then they also *modify* Lemke's algorithm, to account for the possibility of abnormal variables becoming zero. Finally, as they devise a new LCP, they also have to argue once again against ray termination.

Besides those works, the first work in computer science to prove PPAD-membership for markets with SPLC utilities/productions was [72]. The approach in that paper is not the "LCP approach" but the "approximation and rounding approach" (again, see Section 1.2.1). An issue with this method is that very small changes in the prices may result in drastic changes in the optimal bundles of the consumers, which makes the proof quite challenging. To deal with this, Vazirani and Yannakakis [72] devise a set of technical lemmas that allow them to "force" certain allocations over others.

Our Results. Our results in this section are twofold.
- **Simplified Proofs.** First, we employ the linear-OPT-gate to recover all of the aforementioned PPAD-membership results via proofs which are conceptually and technically quite simpler. In particular, we formulate the optimal consumption and the optimal production as linear programs similar to program C of Figure 1, which can be effectively substituted by linear-OPT-gates in a PL arithmetic circuit. We also apply a standard variable change which was first used by Eaves [26], and which we refer to as *Gale's substitution*, see the full version of our paper for more details. For the prices, we develop a feasibility program, similar to program Q of Figure 1. In a fixed point of the circuit, the optimality of consumption and production follows almost immediately by design. The main technical challenge of the proofs lies in arguing the market clearing of the outputted prices, which however still requires a relatively short proof.
  To introduce the reader gently to our proof technique, in the full version of our paper we first apply it to the simple setting of exchange markets with linear utilities, then to the setting of Arrow-Debreu markets with linear utilities and productions, and finally to the general case of Arrow-Debreu markets with Leontief-free utilities and productions.
- **PPAD-Membership for *Succinct* SPLC (SSPLC) Utilities.** In the full version of our paper we introduce a new class of utility functions, which we coin *succinct separable piecewise-linear (SSPLC) utilities*. These are SPLC utilities in which the different segments of the utility function need not be given explicitly in the input (as in the case of (explicit) SPLC utilities), but can be accessed implicitly via a boolean circuit. Effectively, this allows us to *succinctly* represent SPLC functions with exponentially many pieces, where the input size is the size of the given circuits. We remark that the "LCP-approach" developed in the aforementioned papers is inherently limited in providing PPAD-membership results for this class. Indeed, one could formulate the problem as a large LCP in exponentially-many variables, and that would establish the existence of rational solutions. However, this formulation would no longer be a polynomial time reduction (since now we do not have explicit input parameters u^i_{jk} for the utility of each piece) and hence it would not imply the PPAD-membership of the problem. In contrast, using our machinery we can make sure that our linear-OPT-gate can be used to obtain PPAD-membership for markets with SSPLC utilities as well. In our result we also add (explicit) SPLC production, which our technique clearly can handle. We provide a discussion on the challenges of extending our results to also capture SSPLC production functions at the end of the corresponding section in the full paper. Note that the SSPLC functions and the Leontief-free functions are of incomparable generality (and hence they appear on the same line of Table 1). Whether we can prove PPAD-membership for a class of "succinct Leontief-free functions", which would generalize both settings, is an interesting technical question.

1.2.5 PPAD-Membership for Auto-bidding Auctions.

Our next application is on the domain of auto-bidding auctions, which has received a lot of attention recently, due to its applicability in real-world scenarios [4–7, 9, 15, 17, 18, 50]. In particular, in the corresponding section of the full paper we consider the settings studied by Conitzer et al. [17, 18], Chen et al. [14] and Li and Tang [50], in which buyers participate in several parallel single-item auctions, via scaling their valuations by a chosen parameter α, called the *pacing multiplier*. The buyers do that while facing constraints on their feasible expenditure, typically provided by budgets or return-on-investment (ROI) thresholds. The objective is to find a *pacing equilibrium*, i.e., pacing multipliers and allocations for the buyers that are consistent with the format of the auction run (e.g, first-price or second-price) and satisfy the expenditure constraints of all the buyers simultaneously. Pacing equilibria have a similar flavor to the competitive equilibria discussed earlier, but are sufficiently different, and thus require separate handling.

Our Proof vs the Previous Approach. We prove that computing pacing equilibria in parallel second-price auctions with budgets is in PPAD. The problem was already known to be in PPAD (in fact, PPAD-complete) by the recent results of Chen et al. [15], building on the original existence result of Conitzer et al. [18]. Chen et al.'s proof rather heavily applies the "approximation and rounding" paradigm highlighted in Section 1.2.1. In particular, Chen et al. define a (δ, γ)-approximate variant of the pacing equilibrium, where $\delta, \gamma > 0$ are two approximation parameters. Intuitively, this equilibrium corresponds to an "almost equilibrium" (i.e., the expenditure constraints are "almost" satisfied) of an "almost second-price auction" (i.e., an auction in which the set of winners is those with "almost" the highest bid). The authors prove that finding these approximate equilibria is in PPAD, via a reduction to a computational version of Sperner's lemma [67], and then devise an intricate rounding procedure to convert (δ, γ)-equilibria into γ-equilibria. The final step in their proof applies the aforementioned technique of Etessami and

Yannakakis [27] (see Section 1.2.1) to further round these equilibria to pacing equilibria (i.e., where $\gamma = 0$).

Our proof employs the linear-OPT-gate and is conceptually and technically much simpler, without needing to use approximations. Instead, we again apply the standard variable change in Gale's substitution which we also used for the case of competitive markets, to work with the expenditures rather than the allocations directly. From there, we can formulate the task of finding the optimal expenditures as a set of linear programs (one for each buyer), and the pacing multipliers will be obtained as a fixed point solution of a single simple equation. These linear programs can be solved by linear-OPT-gates which essentially establishes the PPAD-membership of the problem.

ROI-Constrained Buyers. We observe that the existence proof underlying our PPAD-membership proof in this section can in fact almost straightforwardly be modified to yield the existence of pacing equilibria for a different setting in auto-bidding auctions, that of second-price auctions with average return-on-investment (ROI) constraints, studied by Li and Tang [50]. Li and Tang established the existence of pacing equilibria via a rather indirect proof, which first reduces the problem to a somewhat convoluted concave game and applies the Debreu-Fan-Glicksberg theorem [24] to obtain Nash equilibrium existence, and then recovers a pacing equilibrium as a limit point of such a Nash equilibrium. This proof in fact closely follows the original proof of Conitzer et al. [18] for the budgeted setting, and clearly does not have any implications on the computational complexity of the problem.

Our proof, besides its advantages in terms of simplicity, also allows us for the first time to obtain computational membership results for pacing equilibria in the ROI-constrained buyer case. It turns out that for this setting, all pacing equilibria may be irrational (see the example we provide in the full version of our paper), and hence membership in PPAD is not possible. Instead, we employ the OPT-gate for FIXP developed by Filos-Ratsikas et al. [29] to easily transform our existence proof into a FIXP-membership proof.

1.2.6 PPAD-Membership for Fair Division. The last applications of our linear-OPT-gate are related to the task of fairly partitioning a resource among a set of agents with different preferences over its parts. In particular, we show the PPAD-membership of computing *exact* envy-free solutions in two fundamental problems, namely *envy-free cake cutting* [30] and *rental harmony* [70], when the preferences of the agents ensure the existence of rational partitions.

Envy-Free Cake Cutting. The envy-free division of a continuous resource (metaphorically, a "cake") is one of the most fundamental and well-studied mathematical problems of the last century. The origins of the theory of the problem can be traced back to the pioneering work of [68], with different variants being studied over the years in a large body of literature in mathematics, economics, and computer science; see [10, 60, 61] for some excellent textbooks on the topic. The existence of an envy-free division was established in 1980 independently by Stromquist [69], by Woodall [73], and by Simmons (credited in [70]), even when the division is required to be *contiguous*, i.e., when each agent receives a single, connected piece of the resource. These proofs proceed by first establishing the existence of divisions that are *approximately* envy-free and then obtaining exact solutions as limit points of these approximations.

It is known that in general, envy-free divisions might be irrational (e.g., see [8], or the full version of our paper for a simpler example), and hence the problem of computing them cannot be in PPAD. Filos-Ratsikas et al. [29] showed that envy-free cake cutting is in the class FIXP, which, recall, is appropriate for capturing the complexity of such problems. Still, there are interesting cases for which rational divisions always exist. This is the case for example when the agents' preferences are captured by piecewise constant density functions [38], a class of functions which is general enough to capture many problems of interest. A FIXP-membership result for these variants is unsatisfactory, and we would like to obtain a PPAD-membership result instead.

Without the convenience of using our linear-OPT-gate, one can establish such a membership result via the "approximation and rounding" technique, see Section 1.2.1. Deng et al. [25] showed that *approximately* envy-free cake cutting is in PPAD, by transforming Simmons' proof into a computational reduction. Goldberg et al. [38] showed how to "round" the approximate solution to obtain an exact envy-free division for preferences captured by piecewise-constant densities, as long as ε is sufficiently small.

Luckily, the linear-OPT-gate allows us to avoid having to do that, and instead directly obtain a PPAD-membership result without any need for approximations. In particular, we revisit the FIXP-membership proof of Filos-Ratsikas et al. [29]; similarly to our approach in this paper, they essentially first construct an existence proof for the problem, one which involves a pair of optimization programs, and then substitute those programs with their OPT-gates for FIXP. One might wonder if, by simply following the steps of the proof and substituting those programs with linear-OPT-gates instead, we can recover the PPAD-membership of the problem, for those classes of preferences for which it is possible. This is almost true, apart from the fact that there is a step in their proof that cannot be done in a PL arithmetic circuit.

Still, we manage to substitute that part by a third optimization program, which is in fact a rather simple linear program, and can effectively be substituted by a linear-OPT-gate. This allows us to obtain the PPAD-membership of the problem for the general class of valuation functions (i.e., functions expressing the preferences via numerical values) that can be computed by a PL arithmetic circuit, capturing the aforementioned case of valuations with piecewise-constant densities.

Rental Harmony. The rental harmony problem, notably studied by Su [70], is concerned with the partition of rent among a set of tenants which have different preferences over combinations of rooms and rent partitions. In the generality studied by Su [70], this problem is in fact equivalent to that of finding an envy-free division of a *chore* among a set of agents. Su's existence proof is inspired by Simmons' proof for envy-free cake cutting, but employs a "dual Sperner labelling" [67]. Similarly to the proofs for cake-cutting, the proof also appeals to limits of approximate solutions. In contrast to cake-cutting however, computational complexity results about this general version of the problem were not known, not even for approximate partitions.

In the corresponding section of the full paper, we prove that the problem of finding a solution to rental harmony is in PPAD, as long as the valuations of the tenants for the rent partition are given by PL arithmetic circuits. Interestingly, this is established via very much the same approach as the proof for envy-free cake cutting, thus providing for the first time a unified proof of existence for those two problems. If one goes beyond the aforementioned valuation functions, all rental harmony solutions may be irrational, as we show in the full version of our paper. For those cases, we explain how the existence proof can be coupled with the OPT-gate for FIXP of Filos-Ratsikas et al. [29] to establish the FIXP-membership of the problem.

Computing Envy-Free and Pareto-Optimal Allocations. We remark that very recently Caragiannis et al. [12] used our linear-OPT-gates to establish that computing probabilistic envy-free and Pareto-optimal allocations of multiple divisible goods is in PPAD.

1.3 The linear-OPT-gate vs the OPT-gate for FIXP

As we mentioned in the introduction, Filos-Ratsikas et al. [29] were the first to develop an OPT-gate for the computational class FIXP [27]. FIXP is the class that captures the complexity of computing a fixed point of an arithmetic circuit, i.e., a circuit over the basis $\{+, -, \max, \min, \div, *\}$ with rational constants. FIXP is a larger class than Linear-FIXP, due to the fact that we can multiply and divide inside the circuit.

The tools that our linear-OPT-gate provides are conceptually very similar to those of the OPT-gate for FIXP of Filos-Ratsikas et al. [29], in that they can substitute convex optimization programs within existence proofs, when constructing a circuit whose fixed points are the solutions that we are looking for. However, the design of the gate itself is much more challenging.

The reason for this is the absence of the general multiplication gate $*$. While we can multiply any two circuit variables in a general arithmetic circuit, we can only multiply variables by constants in a PL arithmetic circuit. The construction of the OPT-gate for FIXP by Filos-Ratsikas et al. [29] makes extensive usage of the multiplication gate $*$ and can thus not directly be used for creating the linear-OPT-gate. In our case, the constraint matrix A is fixed (i.e., not an input to the linear-OPT-gate) and this does help to eliminate some of the general multiplication gates, but not all of them. At a high level, the construction of Filos-Ratsikas et al. [29] ensures that the output x of the gate satisfies

$$\mu_0 \cdot \partial f(x) + A^{\mathsf{T}}\mu = 0$$

where μ satisfies some standard KKT conditions. If x is feasible and if $\mu_0 > 0$, then it follows that x is an optimal solution by standard arguments (using the convexity of f). The term μ_0 is carefully constructed as a function of μ and x in order to ensure that x must be feasible and that $\mu_0 > 0$ when x is feasible. However, since both μ_0 and $\partial f(x)$ depend on x, in our case we cannot construct the term $\mu_0 \cdot \partial f(x)$, because that would entail multiplying two variables in the circuit. As a result, our construction instead ensures that the output x of the gate satisfies

$$\varepsilon \cdot \partial f(x) + A^{\mathsf{T}}\mu = 0$$

where μ again satisfies some standard KKT conditions, and where $\varepsilon > 0$ is some sufficiently small constant that is picked when constructing the gate. By standard arguments it still holds that if x is feasible, then it is an optimal solution. The challenge however is to ensure that x is indeed feasible. While the argument is relatively straightforward in the work of Filos-Ratsikas et al. [29], because μ_0 can depend on x, here μ_0 has been replaced by a constant ε. Our main technical contribution in the construction of the linear-OPT-gate is to show that there exists a sufficiently small $\varepsilon > 0$ that forces x to be feasible, and that such ε can be constructed efficiently given the parameters of the gate (but, importantly, not its inputs!). As a bonus, our modified construction and analysis allows us to obtain a linear-OPT-gate that does not require any constraint qualification, whereas the construction of Filos-Ratsikas et al. [29] required an explicit Slater condition (which of course, as they show, is necessary in the case where the matrix A is not fixed).

From the standpoint of applications, the linear-OPT-gate can be used in almost the same direct manner as the OPT-gate for FIXP of Filos-Ratsikas et al. [29]. In some cases, precisely because we cannot multiply within a PL arithmetic circuit, we may have to apply some standard variable changes, to "linearize" certain constraints. Still, the linear-OPT-gate can effectively substitute appropriate optimization programs in the same way that the OPT-gate for FIXP can. In a nutshell, one can view the linear-OPT-gate as a more powerful tool for those applications for which rational exact solutions exist.

We would like to emphasize that while the full version of our paper is very long, this is almost exclusively due to the fact that it covers so many applications, rather than due to the proofs that we develop for those applications, which in reality range from being very short to relatively short. For each of all of the domains that we consider, (a) we provide the appropriate definition and place the setting in context within the rest of the paper, (b) we discuss the related work and possibly the previous PPAD-membership results (if any), (c) we provide detailed comparisons with those previous proofs to demonstrate the effectiveness of our linear-OPT-gate as a general-purpose proof technique, and finally (d) we develop the proofs themselves. In some cases in fact, we first apply the technique to simpler settings for a gentle introduction, and then move on to study those settings in their full generality. We believe that all of our application sections are largely self-contained, and can be read almost in isolation, even after only reading the introduction of the paper, and by referring only to certain clearly referenced parts in other sections.

2 CONCLUSION AND FUTURE WORK

In this work, we developed the linear-OPT-gate, a powerful general-purpose tool for showing the PPAD-membership of problems that have *exact rational solutions*. We demonstrated its strength by applying it to a plethora of domains related to game theory, competitive markets, auto-bidding auctions and fair division. For those applications, we obtain new results and generalizations of the state-of-the-art complexity results, as well as *significant* simplifications in terms of the proof techniques.

There are some interesting open directions related to our work, mainly in the domain of competitive markets. First, it will be very interesting to see whether one could extend our implicit function machinery to also capture markets with SSPLC production sets; we discuss the challenges of this task in the full version of our paper. Similarly, it would be interesting to try to design a class of succinct utility and production functions that subsumes all the known classes for which rational solutions are known to exist, i.e., one that would generalize the Leontief-free class of functions. Finally, one application that we did not study in our work is that of *competitive markets for mixed manna*, where there are goods but also bads to be allocated to the consumers. Chaudhury et al. [13] studied these markets for SPLC utility functions and developed a complementary pivot algorithm, based on Lemke's algorithm, for computing an equilibrium. There does not seem to be any technical obstacle to applying our technique on those markets as well (and also possibly incorporating production functions as well); the details are still to be worked out.

Looking at the big picture, our linear-OPT-gate complements and refines the OPT-gate for FIXP of Filos-Ratsikas et al. [29] as a tool to proving computational membership of *exact* problems in the appropriate complexity classes. One interesting question is whether one could hope to develop a similar gate for *approximate* problems, i.e., an optimization gate that could be used in a very similar manner to those other two gates to establish PPAD-membership of more general problems (with irrational solutions), for their approximate versions. This certainly introduces new challenges and intriguing questions. One would have to work with approximate rather than exact fixed points. How should the gate be constructed to be useful in this regard? Should the gate work approximately as well? Applications domains like competitive markets, where the approximation in the competitive equilibrium notion comes from relaxing the clearing condition rather than the bundle optimality of the consumers, seem to suggest otherwise.

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