Eur. Phys. J. C (2024) 84:274
https://doi.org/10.1140/epjc/s10052-024-12634-8

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

The Low soft-photon theorem again

V. S. Fadin[1,a], **V. A. Khoze**[2,b]

[1] Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia
[2] Institute for Particle Physics Phenomenology, Durham University, Durham DH1 3LE, UK

Received: 30 January 2024 / Accepted: 1 March 2024

Abstract It is shown that contrary to claims of Lebiedowicz et al. (Phys Rev D 105(1):014022, 2022) the formulated in the proper physical variables Low theorem (Low in Phys Rev 110(4):974–977, 1958) for soft photon emission does not require any modification. We also reject the criticism in Lebiedowicz et al. (2022) of the papers (Burnett and Kroll in Phys. Rev. Lett. 20:86–88,1968; Lipatov in Nucl Phys B 307:705–720, 1988). At the same time, we identify some inaccuracies in Burnett and Kroll (1968) in the presentation of the soft-photon theorem for the case of spin-one-half particles. We also point out shortcomings in consideration of the Low theorem in the classic textbooks (Berestetskii et al. in Quantum electrodynamics. Pergamon Press, Oxford, 1982; Lifshitz and Pitaevsky in Relativistic quantum theory, part 2, Fizmatlit, 2002).

1 Introduction

Nearly 65 years ago in his celebrated paper [1] Francis Low proved that the first two terms in the series expansion of the differential radiative cross-section in powers of photon energy k can be expressed via the corresponding non-radiative amplitude.

This result was extended to the cross-sections of unpolarized multiparticle processes involving charged spin 1/2 particles by Burnett and Kroll in [2] and generalized subsequently by Bell and Van Royen [3] to particles of arbitrary spin. The studies of soft photon radiation have a very long history and has continued to attract attention until now, see [4–9] and references therein. Recently the authors of [4] (see also [5,6]) have questioned the Low theorem, in what concerns the validity of the second term in the expansion over photon momentum. They also disputed some results of the basic works [2,10].

This paper aims to demonstrate that the critical comments of [4] concerning the consistency of the Low theorem and some results of [2,10] are unjustified. Some errors in interpretation of the Low theorem in [4] were also pointed out in [8,9]. While addressing this issue we found some inaccuracies in [2] and in the presentation of the Low theorem in the popular textbooks [11,12].

In the next section, we consider the soft-photon theorem for the case of two spin-zero particles exemplified similarly to [4] by the radiation in the $\pi^-\pi^0$ scattering. In Sect. 3 we address the Low theorem for the case of the process with spin-one-half emitter, and in Sect. 4 we consider the Burnett-Kroll extension of the soft photon theorem. Some inconsistencies in the presentation of the soft theorem in [11,12] are discussed in Sect. 5 and we conclude in Sect. 6.

2 The Low theorem for scalar particles

Let us use for particle momenta the same notation as in Ref. [4] and consider the processes

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1) + \pi^0(p_2) \tag{1}$$

with

$$p_a + p_b = p_1 + p_2, \quad p_a^2 = p_1^2 = m_a^2, \quad p_b^2 = p_2^2 = m_b^2, \tag{2}$$

and

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1') + \pi^0(p_2') + \gamma(k, e) \tag{3}$$

with

$$\begin{aligned} &p_a + p_b = p_1' + p_2' + k, \quad p_a^2 = p_1'^2 = m_a^2, \\ &p_b^2 = p_2'^2 = m_b^2, \quad k^2 = 0\,. \end{aligned} \tag{4}$$

[a] e-mail: v.s.fadin@inp.nsk.su
[b] e-mail: v.a.khoze@durham.ac.uk (corresponding author)

It is convenient to introduce notation

$$j_\mu = \frac{p'_{1\mu}}{(p'_1 k)} - \frac{p_{a\mu}}{(p_a k)}\ , \quad \mathcal{P}^\mu_f = t^{\mu\rho}_f p'_{2\rho}\ , \quad \mathcal{P}^\mu_i = t^{\mu\rho}_i p_{b\rho}\ , \tag{5}$$

where

$$t^{\mu\rho}_i = \frac{p^\mu_a}{(p_a k)} k^\rho - g^{\mu\rho}\ , \quad t^{\mu\rho}_f = \frac{p'^\mu_1}{(p'_1 k)} k^\rho - g^{\mu\rho}\ . \tag{6}$$

Then the original Low theorem [1] reads (see Eq. (2.16) in [1])

$$M_\mu = e\left[j_\mu + \left(\mathcal{P}_{i\mu} + \mathcal{P}_{f\mu}\right)\frac{\partial}{\partial\nu}\right] T(\nu,\Delta)\ , \tag{7}$$

where

$$\nu = (p_a p_b) + (p'_1 p'_2)\ , \quad \Delta = (p_b - p'_2)^2, \tag{8}$$

and $T(\nu,\Delta) = T(p'^2_1, p^2_a, ; \nu, \Delta)|_{p'^2_1 = p^2_a = m^2_a}$, where $T(p'^2_1, p^2_a, ; \nu, \Delta)$ is the off-mass-shell amplitude (amputated Green function) of elastic process. It is clearly stated in [1] (see Eq. (2.9) and a phrase below it) that this amplitude " conserves momentum and energy but not mass." Let us emphasize that the amplitude of the process (1) is a function of two invariant variables, $T(\nu_i, t_b)$, with $\nu_i = 2(p_a p_b)$ and $t_b = (p_b - p_2)^2$.

The important point here is that the Low theorem is formulated in terms of the physical momenta of the radiative process (3). The authors of [4] claim that such consideration is incorrect because these momenta cannot be the momenta of the non-radiative (elastic) process. Below Eq. (3.40) in [4] it is written "The term of order ω^0[1] in the expansion of the amplitude given in [1] corresponds to the fictitious process (3.31) which does not respect energy-momentum conservation." Here (3.31) means

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1) + \pi^0(p_2) + \gamma(k,e), \tag{9}$$

i.e. the inelastic process (3), but with momenta of the final pions corresponding to the non-radiative case (3). However, such an equation has never been presented in [1]. Actually, everywhere in [1] the physical momenta of the process (3) satisfying the energy-momentum conservation

(4) are used. In contrast, the momenta used in [4] correspond to the elastic process. It is true, that the momenta p_a, p_b, p'_1, p'_2 used in [1] cannot be momenta of the elastic process, because they do not satisfy energy-momentum conservation for this process, $p_a + p_b \neq p'_1 + p'_2$. But the arguments ν_i and t_b of the elastic scattering amplitude $T(\nu_i, t_b)$ are independent variables, so, certainly, they can be taken equal to $\nu = (p_a p_b) + (p'_1 p'_2)$ and $\Delta = t_b = (p_b - p'_2)^2$. Instead of using these variables the authors of [4] introduced artificial momenta p_1 and p_2 satisfying the elastic conservation law $p_a + p_b = p_1 + p_2$ and reformulate the theorem in terms of these momenta. The momenta p_1 and p_2 in [4] are written as

$$p_1 = p'_1 + l_1\ , \quad p_2 = p'_2 + l_2\ , \tag{10}$$

where the momenta l_1 and l_2 are small ($\mathcal{O}(k)$) and

$$l_1 + l_2 = k\ . \tag{11}$$

Since both p_1 and p'_1, as well as p_2 and p'_2 are on-shell, we get

$$2(p'_1 l_1) = -l^2_1, \quad 2(p'_2 l_2) = -l^2_2\ . \tag{12}$$

In Ref. [4] some particular representation for the momenta l_i is used, which is not essential for the discussion below. The amplitude of the radiative process is presented there (see Eqs. (A1) or (3.27) and (3.28) in [4]) at $k^2 = 0$, $(\epsilon^* k) = 0$, where ϵ^* is the photon polarization vector, as

$$\begin{aligned}
\mathcal{M}_\lambda &= e\mathcal{M}^{(0)}(s_L, t, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi) \\
&\quad\times\left[\frac{p_{a\lambda}}{(p_a k)} - \frac{p_{1\lambda}}{(p_1 k)} - \frac{p_{1\lambda}(l_1 k) - l_{1\lambda}(p_1 k)}{(p_1 k)^2}\right] \\
&\quad+2e\frac{\partial}{\partial s_L}\mathcal{M}^{(0)}(s_L, t, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi) \\
&\quad\times\left[p_{b\lambda} - p_{a\lambda}\frac{(p_b k)}{(p_a k)}\right] \\
&\quad-2e\frac{\partial}{\partial t}\mathcal{M}^{(0)}(s_L, t, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi) \\
&\quad\times\left[\frac{p_{a\lambda}}{(p_a k)} - \frac{p_{1\lambda}}{(p_1 k)}\right]\Big[((p_a - p_1)k) - (p_a l_1)\Big]\ .
\end{aligned} \tag{13}$$

Here $s_L = (p_a p_b) + (p_1 p_2)$, $t = (p_a - p_1)^2 = (p_b - p_2)^2$, $\mathcal{M}^{(0)}(s_L, t, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi)$ is the elastic scattering amplitude.

Note that $s_L \neq \nu$ and $t \neq \Delta$. With the accuracy to order k accounting for Eq. (12), we have

$$s_L = \nu + k(p'_2 + p'_1) = \nu + k(p_2 + p_1)\ , \quad t = \Delta - 2(l_2 p_b)\ .$$

Therefore, taking the first term in (13) the expansion in k

$$\begin{aligned}
&\mathcal{M}^{(0)}(s_L, t, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi) \\
&\quad= \mathcal{M}^{(0)}(\nu, \Delta, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi) \\
&\qquad+ (k(p_a + p_b)))\frac{\partial}{\partial\nu}\mathcal{M}^{(0)}(\nu, \Delta, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi) \\
&\qquad-2(l_2 p_b)\frac{\partial}{\partial\Delta}\mathcal{M}^{(0)}(\nu, \Delta, m^2_\pi, m^2_\pi, m^2_\pi, m^2_\pi)
\end{aligned} \tag{14}$$

and setting in the second and the third terms $s_L = \nu$, $t = \Delta$ (that is allowed because these terms are $\mathcal{O}(k)$), we arrive at the original formulation of the Low theorem (apart from the difference in common sign of the amplitudes in [4] and [1]). Thus, the formulation of the Low theorem proposed in

[1] Note that in the notation of Ref. [4] ω is the photon energy.

[4] fully agrees with the original one [1]. However, it uses artificially constructed variables that differ from the physical variables of the radiative process used in Ref. [1]). It is worth emphasizing here that such an agreement holds just for the first two terms in the expansion in the photon energy. Only these two terms obey the theorems for soft-photon radiation based on the gauge invariance. Let us note here that the choice in [1] of ν and Δ as two independent variables of the non-radiative scattering amplitude is a convenient, but obviously

not the only option (see, e.g. footnote 3 in [2]) Therefore, different formulations of the theorem are possible. For example, taking as the two independent variables $\nu_i = 2(p_a p_b)$ and Δ and using the relation

$$\nu = \nu_i - (k(p_a + p_b)) , \tag{15}$$

we obtain with the $\mathcal{O}(k)$ accuracy

$$T(\nu, \Delta) = -(k(p_a + p_b))\frac{\partial}{\partial \nu_i} T(\nu_i, \Delta) . \tag{16}$$

Inserting (16) in (7) with an account for momentum conservation and neglecting in the M_μ terms proportional to k_μ we get

$$M_\mu = e\left[j_\mu + 2P_{i\mu}\frac{\partial}{\partial \nu_i}\right] T(\nu_i, \Delta) . \tag{17}$$

Similarly, we can obtain

$$M_\mu = e\left[j_\mu + 2P_{f\mu}\frac{\partial}{\partial \nu_f}\right] T(\nu_f, \Delta) , \tag{18}$$

where $\nu_f = 2(p_1' p_2')$. In the expressions (7), (17), (18) the first arguments of the amplitude of the non-radiative process differ by the $\mathcal{O}(k)$ terms. It is always possible to set the second argument also being different by the terms of the same order. For example, we can write the singular term as $e j_\mu T(\nu_i, \Delta_p)$, where $\Delta_p = (p_a - p_1')^2 = \Delta + 2k(p_a - p_1')$. In this case, with the required accuracy

$$T(\nu_i, \Delta) = T(\nu_i, \Delta_p) - 2(k(p_a - p_1'))\frac{\partial}{\partial \Delta_p} T(\nu_i, \Delta_p) \tag{19}$$

and we obtain from (17)

$$\begin{aligned} M_\mu &= e\left[j_\mu + 2\mathcal{P}_{i\mu}\frac{\partial}{\partial \nu} - 2\big(\mathcal{R}_{i\mu} + \mathcal{R}_{f\mu}\big)\frac{\partial}{\partial \Delta_p}\right] \\ &\quad \times T(\nu_i, \Delta_p) , \end{aligned} \tag{20}$$

with

$$\mathcal{R}_f^\mu = t_f^{\mu\rho} p_{a\rho} , \quad \mathcal{R}_i^\mu = t_i^{\mu\rho} p'_{1\rho} , \tag{21}$$

where $t_{i,f}^{\mu\rho}$ are defined in (6).

Very often the soft radiation theorem is formulated using the partial derivatives with respect to momenta. Such an expression can be derived from the original one with the help of relations

$$\begin{aligned} \mathcal{P}_i^\mu \frac{\partial}{\partial \nu} T(\nu, \Delta) &= t_i^{\mu\rho}\frac{\partial}{\partial p_a^\rho} T(\nu, \Delta) , \\ \mathcal{P}_f^\mu \frac{\partial}{\partial \nu} T(\nu, \Delta) &= t_f^{\mu\rho}\frac{\partial}{\partial p_1^{\prime\rho}} T(\nu, \Delta) , \end{aligned} \tag{22}$$

where ν and Δ are defined in (8). Let us note that the amplitude $T(\nu, \Delta)$ is defined on the mass shell of all particle momenta, therefore, generally speaking, the partial derivatives over the components of the momenta are ill-defined because their definition requires exit from the mass shell. But due to the properties

$$t_f^{\mu\rho} p'_{1\rho} = 0, \quad t_a^{\mu\rho} p_{a\rho} = 0, \tag{23}$$

which follow from (6), the amplitudes in Eq. (22) remain on the mass shell, so that such a problem does not arise. Using the relations (22) we obtain from (7)

$$M_\mu = e\left[j_\mu(k) + \left(D_i^\mu + D_f^\mu \right)\right] T(\nu, \Delta) , \tag{24}$$

where

$$D_f^\mu = t_f^{\mu\rho}\frac{\partial}{\partial p_1^{\prime\rho}} , \quad D_i^\mu = t_i^{\mu\rho}\frac{\partial}{\partial p_a^\rho} , \tag{25}$$

Using the definitions of $\nu_i = 2(p_a p_b)$, $\nu_f = 2(p_1' p_2')$, $\nu = \frac{1}{2}(\nu_i + \nu_f)$, $\Delta = (p_b - p_2')^2$, $\Delta_p = (p_a - p_1')^2$ one can easily check that

$$\begin{aligned} \left(D_i^\mu + D_f^\mu \right) T(\nu_i, \Delta) &= \mathcal{P}_i^\mu \frac{\partial}{\partial \nu_i} T(\nu_i, \Delta) , \\ \left(D_i^\mu + D_f^\mu \right) T(\nu_f, \Delta) &= \mathcal{P}_f^\mu \frac{\partial}{\partial \nu_f} T(\nu_f, \Delta) , \end{aligned} \tag{26}$$

$$\begin{aligned} \left(D_i^\mu + D_f^\mu \right) T(\nu, \Delta_p) &= \left(\mathcal{P}_i^\mu + \mathcal{P}_f^\mu \right)\frac{\partial}{\partial \nu} \\ &\quad -2\left(\mathcal{R}_i^\mu + \mathcal{R}_f^\mu \right)\frac{\partial}{\partial \Delta_p}\Big] T(\nu, \Delta_p) . \end{aligned} \tag{27}$$

This means that all relations (7), (17), (18), (20) have the same differential in momenta form (24) with the corresponding arguments of amplitude T.

It is easy to see that the differential form (24) does not change if we take instead of ν and Δ any scalar variables, which are equal to them at $k = 0$, which means $a\nu_i + (1 - a)\nu_i$ and $b\Delta + (1 - b)\Delta_p$ with the coefficients $0 \le a, b \le 1$. Indeed, using relation $\nu_i - \nu_f = 2\big(k((p_a' + p_b'))\big) = 2\big(k((p_a + p_b))\big)$ we get

$$\begin{aligned} j^\mu(\nu_i - \nu_f) &= \frac{p_a^{\prime\mu}}{(kp_a')} 2\big(k((p_a' + p_b'))\big) \\ &\quad - \frac{p_a^\mu}{(kp_a)} 2\big(k((p_a + p_b))\big) = 2\mathcal{P}_f^\mu - 2\mathcal{P}_i^\mu , \end{aligned} \tag{28}$$

while

$$
\begin{aligned}
D^\mu(\nu_i-\nu_f) &= \big(D_i^\mu + D_f^\mu\big)\big(2(p_a p_b) - 2(p_a' p_b')\big) \\
&= 2\mathcal{P}_i^\mu - 2\mathcal{P}_f^\mu\,,
\end{aligned}
\tag{29}
$$

so that

$$
\Big(j^\mu + D^\mu\Big)(\nu_i-\nu_f) = 0\,. \tag{30}
$$

At first sight, the result (29) may look confusing, since $D^\mu \sim k^0$ and $(\nu_i - \nu_f) \sim k^1$, so the right-hand side of (29) should also be $\sim k^1$. The point is that when calculating derivatives, the vectors p_a, p_b, p_a', p_b' are considered as being independent (not related to k by momentum conservation). Similarly, we obtain

$$
\begin{aligned}
&\big(j^\mu + D^\mu\big)(\Delta - \Delta_p) = j^\mu 2\big(k((p_a' - p_a))\big) \\
&\quad + \big(D_i^\mu + D_f^\mu\big)\big((p_b - p_b')^2 - (p_a - p_a')^2\big) = 0\,.
\end{aligned}
\tag{31}
$$

Accounting for these remarks, we get for squared modulus of the matrix element $\epsilon^\mu M_\mu$ with the real photon polarisation vector ϵ^μ

$$
\begin{aligned}
|\epsilon^\mu M_\mu|^2 &= e^2(\epsilon^\mu j_\mu(k))\Big[(\epsilon^\mu j_\mu(k)) + \epsilon^\mu D_\mu\Big] \\
&\qquad \times |T(a\nu_i + (1-a)\nu_f, \\
b\Delta + (1-b)\Delta_p)|^2\,. &
\end{aligned}
\tag{32}
$$

3 The Low theorem for spin-one-half particles

In addition to Eq. (7) for the scalar particles, the original paper [1] contains also the derivation of a similar equation (Eq. (3.16) in [1]) for the case when the emitter is a spin-one-half fermion. For consideration of this case, it is convenient to use the same notation as in [1]). Then, for the process

$$
f(p_i) + \pi^0(q_i) \to f(p_f) + \pi^0(q_f) + \gamma(e,k)\,, \tag{33}
$$

where f is a spin-one-half fermion of charge e, anomalous magnetic moment λ and mass m, the result of [1] reads

$$
\begin{aligned}
M_\mu &= \bar u(p_f)\Big[(e\gamma_\mu + \frac{\lambda}{2}[\gamma_\mu,\hat k])\frac{1}{\hat p_f + \hat k - m}T \\
&\quad + T\frac{1}{\hat p_i - \hat k - m}(e\gamma_\mu + \frac{\lambda}{2}[\gamma_\mu,\hat k]) \\
&\quad + e\Big(\mathcal{P}_{f\mu} + \mathcal{P}_{i\mu}\Big)\left(\frac{\partial}{\partial\nu}T\right)\Big]u(p_a)\,,
\end{aligned}
\tag{34}
$$

where $\mathcal{P}_{f\mu}$ and $\mathcal{P}_{i\mu}$ are given by (5) and (6) with the replacement $p_a \to p_i$, $p_1' \to p_f$, and

$$
\begin{aligned}
T &= A(\nu,\Delta) + \frac{1}{2}(\hat q_i + \hat q_f)B(\nu,\Delta), \\
\nu &= (p_i q_i) + (p_f q_f),\ \ \Delta = (q_i - q_f)^2\,.
\end{aligned}
\tag{35}
$$

We recall that the Low theorem is written via the physical momenta of the process (33), and therefore, the criticism of [4] repeated in [6] is not justified.

The representation (34) can be rewritten with the $\mathcal{O}(k)$ accuracy as

$$
\begin{aligned}
M_\mu &= e\bar u(p_f)\Big[j_\mu T + \Big(\mathcal{P}_{f\mu} + \mathcal{P}_{i\mu}\Big)\left(\frac{\partial}{\partial\nu}T\right) \\
&\quad + \frac{[\gamma_\mu,\hat k])}{4(kp_f)}\Big(1 + \frac{\lambda}{e}(\hat p_f + m)\Big)T \\
&\quad - T\Big(1 + \frac{\lambda}{e}(\hat p_i + m)\Big)\frac{[\gamma_\mu,\hat k]}{4(kp_i)}\Big]u(p_i)\,,
\end{aligned}
\tag{36}
$$

where j_μ is given by (5) with the replacement $p_a \to p_i$, $p_1' \to p_f$. unpolarized, non-radiative In Ref. [1] the soft-photon theorem was considered on the amplitude level, and Eq. (34) is the final result of this paper. The extension of the Low theorem developed in [2] concerns its application to the radiative cross sections for unpolarized particles, and the main result is proof that "the first two terms of an expansion in the photon energy depend on the unpolarized, non-radiative cross-section only".

Let us obtain the square of the matrix element summed over fermion polarizations taking the original Low theorem in its differential in momenta form, that is using

$$
j_\mu T + \big(\mathcal{P}_{f\mu} + \mathcal{P}_{i\mu}\big)\left(\frac{\partial}{\partial\nu}T\right) = \Big(j_\mu + D_\mu\Big)T\,. \tag{37}
$$

Recall here that, as shown in the previous section, in the differential form (37) we can set in T (35) instead of ν and Δ any scalar variables, which are equal to them at $k = 0$. Using (36) and (37), we obtain for real photon polarisation vectors ϵ^μ

$$
\begin{aligned}
\sum_{spin}|\epsilon^\mu M_\mu|^2 &= e^2\Big[(\epsilon j)\Big((\epsilon j)Tr\Big((\hat p_i + m))T(\hat p_f + m)\overline{T} \\
&\quad + Tr\Big((\hat p_i + m)\big((\epsilon D)T\big)(\hat p_f + m)\overline{T}\Big) \\
&\quad + \Big((\hat p_i + m)T(\hat p_f + m)\big((\epsilon D)\overline{T}\big)\Big) \\
&\quad + \frac{(\epsilon j)}{4(kp_f)}Tr\Big((\hat p_f + m)T(\hat p_i + m) \\
&\quad \overline{T}\Big(1 + \frac{\lambda}{e}(\hat p_f + m)\Big)[\hat k,\hat e] \\
&\quad + (\hat p_f + m)[\hat e,\hat k]\Big(1 + \frac{\lambda}{e}(\hat p_f + m)\Big)T(\hat p_i + m)\overline{T}\Big) \\
&\quad - \frac{(\epsilon j)}{4(kp_i)}Tr\Big((\hat p_f + m)T(\hat p_i + m)[\hat k,\hat\epsilon]\Big(1 + \frac{\lambda}{e}(\hat p_i + m)\Big)\overline{T} \\
&\quad + (\hat p_f + m)T\Big(1 + \frac{\lambda}{e}(\hat p_i + m)\Big)[\hat\epsilon,\hat k](\hat p_i + m)\overline{T}\Big)\,,
\end{aligned}
\tag{38}
$$

where $\sum$ means summation over the fermion polarizations. It is easy to see from (38) that (as already observed in [2]) the terms with the anomalous magnetic moment have canceled.

Then using formulas

$$\frac{1}{4(kp_f)}\Big([\hat{k},\hat{\epsilon}](\hat{p}_f+m)+(\hat{p}_f+m)[\hat{k},\hat{\epsilon}]\Big)$$
$$=\epsilon_\mu t_f^{\mu\rho}\gamma_\rho=(\epsilon D)(\hat{p}_f+m),$$
$$\frac{1}{4(kp_i)}\Big([\hat{k},\hat{\epsilon}](\hat{p}_i+m)+(\hat{p}_i+m)[\hat{k},\hat{\epsilon}]\Big)$$
$$=\epsilon_\mu t_i^{\mu\rho}\gamma_\rho=(\epsilon D)(\hat{p}_i+m), \tag{39}$$

we get from (38)

$$\sum_{spin}|\epsilon^\mu M_\mu|^2=e^2(\epsilon j)\Big((\epsilon j)+(\epsilon D)\Big)\sum_{spin}|M|^2\,, \tag{40}$$

where

$$\sum_{spin}|M|^2=Tr\big((\hat{p}_i+m)T(\hat{p}_f+m)\overline{T}\big)$$
$$=\sum_{spin}|\bar{u}(p_f)Tu(p_i)|^2\,. \tag{41}$$

Recall that T here is expressed in terms of the momenta of the radiative process (33), and that we can take in T (35) instead of ν and Δ any scalar variables, that are equal to them at $k=0$.
Note that in Refs. [5,6] the soft-photon theorem for the pion-proton scattering was formulated following the same approach as in [4] for the pion-pion scattering. Derivation of the result presented in [6] "involved a lengthy and complex analysis" given in [5]. The result is also complicated, so that the proof of its equivalence to the original formulation is not presented here and will be considered elsewhere.

4 On the Burnett–Kroll extension of low theorem

In Ref. [1] the soft-photon theorem was considered on the amplitude level, and Eqs. (7) and (34) are the final results of paper [1] for the amplitudes of soft photon emission by spin-zero and spin-one-half particles respectively in scattering on a neutral spin-zero particle. Our Eqs. (32) and (40) were obtained by a direct application of the results of [1] to the amplitudes squared, summed over the fermion polarizations in the second case.

Note that our results seem to look like a confirmation (or repetition) of the conclusions of [2]. However, there is the essential difference between these results.

For illustration, let us reproduce the final result (Eq. (11)) of the paper [2]

$$\sum_{spins}|T_\gamma(\epsilon,k,p)|^2=\sum_a Q_a\frac{\epsilon\cdot p_a}{k\cdot p_a}\sum_b Q_b$$
$$\times\Big[\frac{\epsilon\cdot p_b}{k\cdot p_b}-\epsilon\cdot D_b(k)\Big]\sum_{spins}|T(p')|^2, \tag{42}$$

where $T_\gamma(\epsilon,k,p)$ and $T(p')$ are the radiative and non-radiative amplitudes, p_a and Q_a are the momentum and charge of the particle a, and k and ϵ are the momentum and polarisation vector of the photon. All particles except the photon are considered as incoming, so that $\sum_a p_a=k$, $p'_a=p_a-\xi_a(k)$, $\sum_a\xi_a(k)=k$, $p_a\cdot\xi_a=0$,

$$D_a(k)=\frac{p_a}{k\cdot p_a}k\cdot\frac{\partial}{\partial p_a}-\frac{\partial}{\partial p_a}\,. \tag{43}$$

Note, that Eqs. (32) and (40) (as well as Eqs. (7), (34)) are expressed through momenta of the radiative processes, while in Ref. [2] (see Eq. (42) above), the momenta of non-radiative processes are used. There is a large uncertainty in the choice of these momenta at the given momenta of the radiative process. An important restriction on this choice imposed in [2] is that "scalar invariants are the same to first order in k whether expressed in terms of p or p' ", i.e. momenta of the radiative or non-radiative processes. Of course, only the independent invariants are considered, because it is impossible to have simultaneously, for example, $(p'_iq'_i)=(p_iq_i)$ and $(p'_fq'_f)=(p_fq_f)$ since $(p'_iq'_i)=(p'_fq'_f)$, but $(p_iq_i)=(p_fq_f)+(k((p_i+q_i))$. It was assumed in (42) that $\sum_{spins}|T(p')|^2$ is expressed in invariants constructed from the momenta of the radiative process. Note that the imposed restriction leaves a choice of momenta of the non-radiative process far from being unique.

For the case of radiation by the scalar particles, formulations of the soft-photon theorem via momenta of the radiative and non-radiative processes coincide. Indeed, the form (32) is in a perfect agreement with Eq. (11) of [2] (see Eq. (42) above), because $|T(p')|^2$ expressed in invariants constructed from the momenta of the radiative process is nothing more than $|T(a\nu_i+(1-a)\nu_f,b\Delta+(1-b)\Delta_p)|^2$. The choice of different coefficients a and b corresponds to the choice of different scalar invariants which are the same to first order in k, whether expressed via p or p'.

The paper of Barnett and Kroll [2] was criticized in Ref.[4] (see the end of Appendix B) on the grounds that "their results contain derivatives of the non-radiative amplitudes with respect to one momentum keeping the other ones fixed". This is incorrect, as it follows from the sentence in [2] after Eq. (4) there: "The prescription then is that T is expressed in terms of the momenta p_a which are differentiated as independent variables, then evaluated with momenta p'_a". Note, that In Eq. (42) it is assumed that before differentiation $\sum_{spins}|T(p')|^2)$ is expressed in terms of the invariants constructed from the momenta of the radiative process.

We could see a weak point in the paper [4] regarding the partial derivatives $\frac{\partial}{\partial p_a}$, which are generally poorly defined, as it was already noted, because their definition requires exit from the mass shell. But these derivatives enter only in the combinations $\xi_a\cdot\frac{\partial}{\partial p_a}$ and $D_a(k)$, which do not require a shift from the mass shell, so the only reproach is that the authors of [4] did not explain this.

Therefore, in the case of radiation by scalar particles, the result of [4] is completely consistent with Eq. (32), and we have only minor comments regarding its derivation. For radiation by the spin-one-half particles, the situation is somewhat more complicated. In this case, the result of [4] written in the notation of Eq. (33) takes the form

$$\sum |\epsilon^\mu M_\mu|^2 = e^2(\epsilon j)\Big((\epsilon j) + (\epsilon D)\Big) \sum_{spin} |M'|^2 \ , \tag{44}$$

where prime labels momenta of the non-radiative process, and

$$j^\mu = \frac{p_f^\mu}{(p_f k)} - \frac{p_i^\mu}{(p_i k)} \ , \quad D^\mu = t_i^{\mu\rho}\frac{\partial}{\partial p_{i\rho}} + t_f^{\mu\rho}\frac{\partial}{\partial p_{f\rho}} \ ,$$
$$t_i^{\mu\rho} = \frac{p_i^\mu}{(p_i k)}k^\rho - g^{\mu\rho} \ , \quad t_f^{\mu\rho} = \frac{p_f^\mu}{(p_f k)}k^\rho - g^{\mu\rho} \ . \tag{45}$$

$$\sum_{spin} |M'|^2 = Tr\big((\hat{p}'_i + m)T(\hat{p}'_f + m)\overline{T}'\big)$$
$$= \sum_{spin} |\bar{u}(p'_f)T'u(p'_i)|^2 \ , \tag{46}$$

$$T' = A(\nu', \Delta') + \frac{1}{2}(\hat{q}'_i + \hat{q}'_2)B(\nu', \Delta') \ ,$$
$$\nu' = 2(p'_i q'_i) = 2(p'_f q'_f) \ ,$$
$$\Delta' = (q'_i - q'_f)^2 = (p'_i - p'_f)^2 \ . \tag{47}$$

The restriction that scalar invariants are the same to first order in k, whether expressed in terms of non-primed or primed momenta (i.e. momenta of radiative or non-radiative processes) is very important because it is needed for the definition of the derivatives in (44). Indeed, (44) contains the derivatives in non-primed momenta acting on a scalar function of the primed momenta, therefore, some prescription for the transformation of the scalar products of the primed momenta into the scalar products of the non-primed momenta is required.

Let us take $\nu'_i = 2(p'_i q'_i)$ and $\Delta'_p = (p'_i - p'_f)^2$ as the independent variables, so that

$$\nu'_i = \nu_i, \quad 2(p'_i q'_i) = 2(p_i q_i) \ , \quad \Delta'_p = \Delta_p,$$
$$(p'_i - p'_f)^2 = (p_i - p_f)^2 \ , \tag{48}$$

and take

$$p'_i = p_i, \ \ p'_f = p_f, \ \ q_i = q'_i + \eta_i, \ \ q_f = q'_f - \eta_f \ . \tag{49}$$

Therefore, T' becomes

$$T' = A(\nu_i, \Delta_p) + \frac{1}{2}(\hat{q}'_i + \hat{q}'_2)B(\nu_i, \Delta_p) \ , \quad \nu_i = 2(p_i q_i) \ ,$$
$$\Delta_p = (p_i - p_f)^2 \ . \tag{50}$$

The equalities (48) and the on-mass shell conditions for q_i, q_f lead to the relation

$$(\eta_i q_i) = 0, \quad (\eta_f q_f) = 0, \quad (\eta_i p_i) = 0 \ . \tag{51}$$

Using that from momentum conservation law it follows $\eta_i + \eta_f = k$, these conditions can be rewritten as

$$(\eta_i q_i) = 0, \quad (\eta_i q_f) = (k q_f), \quad (\eta_i p_i) = 0 \ . \tag{52}$$

It is easy to see that this system of equations has an infinite number of solutions except for the very special choice of kinematics.

Recall that in Eq. (40), in the expression for T (35) instead of ν and Δ we can take any variables, which are equal to ν and Δ at $k = 0$. Let us set these as ν_i and Δ_p, i.e. substitute in (41) T with $\mathcal{T}$,

$$\mathcal{T} = A(\nu_i, \Delta_p) + \frac{1}{2}(\hat{q}_i + \hat{q}_f)B(\nu_i, \Delta_p) \ , \tag{53}$$

so that

$$\mathcal{T} = T' + \frac{1}{2}(\hat{\eta}_i - \hat{\eta}_f)B(\nu_i, \Delta_p) \ . \tag{54}$$

Therefore, the difference between the factors in front of $e^2(\epsilon j)^2$ in (40) and (44) is (for compactness we drop the arguments of A and B)

$$\sum_{spin} |M|^2 - \sum_{spin} |M'|^2 = \frac{1}{2}Tr\Big((\hat{p}_f + m)(\hat{\eta}_i - \hat{\eta}_f)$$
$$\times B(\hat{p}_i + m)\big(A^* + \frac{1}{2}(\hat{q}'_i - \hat{q}'_f)B^*\big)$$
$$+\frac{1}{2}Tr\Big((\hat{p}_f + m)\big(A + \frac{1}{2}(\hat{q}'_i - \hat{q}'_f)B\big)(\hat{p}_i + m)$$
$$\times(\hat{\eta}_i - \hat{\eta}_f)B^*\Big)$$
$$= (AB^* + BA^*)2m\big((\eta_i - \eta_f)(p_i + p_f)\big)$$
$$+2BB^*\Big[\big(p_i(\eta_i - \eta_f)\big)\big(p_f(q'_i + q'_f)\big)$$
$$+\big(p_f(\eta_i - \eta_f)\big)\big(p_i(q'_i + q'_f)\big)$$
$$-\big((\eta_i - \eta_f)(q'_i + q'_f)\big)\big((p_i p_f) - m^2\big)\Big] \ , \tag{55}$$

which is not zero. Indeed, using $\eta_f = k - \eta_i$ and the relations (52) we obtain

$$\sum_{spin} |M|^2 - \sum_{spin} |M'|^2 = (AB^* + BA^*)$$
$$\times 2m\big(-(k(2q_f + p_f + p_i)\big)$$
$$+BB^*\big[2\nu_i\big(-(k(2q_f + p_f + p_i))\big)$$
$$-2\Delta_p\big((k(p_i + q_i))\big)\big] \ . \tag{56}$$

The difference $\sum_{spin} |M|^2 - \sum_{spin} |M'|^2$ (56) was presumed to be zero in [2] because it was written as

$$\frac{1}{2}\Big(\big(\eta_i \frac{\partial}{\partial q_i} - \eta_f \frac{\partial}{\partial q_i}\big)Tr$$

$$\times\Big((\hat{p}_f+m)T(\hat{p}_i++m)\overline{T}\Big)\Big|_{q_i=q_i',q_f=q_f'} . \quad (57)$$

Indeed, the latter expression nullifies if we take the derivatives after the calculation of traces with $q_i = q_i', q_f = q_f'$ and present the result in terms of ν_i and Δ_p. It was supposed in [2] that these operations are commutative, but this is untrue. Such an assumption was used in [2] where the term $\xi_b \frac{\partial}{\partial p_b} \sum_{spin} |T|^2$ in Eq. (10) was set to zero. This circumstance on its own would be sufficient to argue that the final result of [2], Eq. (11), is incorrect. But the term with derivatives in this equation is also erroneous because the same wrong assumption (commutability of taking derivatives and summation over the spins followed by the presentation of the result in terms of scalar variables expressed in momenta of the radiative process) was used in its derivation. Therefore, it is necessary to calculate the difference

$$D^\mu \sum_{spin} |M|^2 - D^\mu \sum_{spin} |M'|^2 , \quad (58)$$

where

$$\sum_{spin} |M|^2 = Tr\big[(\hat{p}_f+m)\big(A+\frac{1}{2}(\hat{q_i}+\hat{q_f})B\big) \times(\hat{p}_i+m)\big(A^*+\frac{1}{2}(\hat{q_i}+\hat{q_f})B^*\big)\big] , \quad (59)$$

and $\sum_{spin}|M'|^2$ is obtained from (59) by the replacement $q_i \to q_i', q_f \to q_f'$. Note that since in the difference (58) only $\mathcal{O}(k^0)$ terms must be kept, and invariant amplitudes A and B are the same in both terms, their derivatives do not contribute to the difference so that they will be considered in what follows as constants.

In the first term, the derivatives are taken before the calculation of the trace, so that

$$\begin{aligned}D^\mu \sum_{spin} |M|^2 &= t_f^{\mu\rho} Tr\big[\gamma_\rho\big(A+\frac{1}{2}(\hat{q_i'}+\hat{q_f'})B\big)\\ &\times(\hat{p}_i+m)\big(A^*+\frac{1}{2}(\hat{q_i'}+\hat{q_f'})B^*\big)\big]\\ &+t_i^{\mu\rho} Tr\big[(\hat{p}_f+m)\big(A+\frac{1}{2}(\hat{q_i'}+\hat{q_f'})B\big)\\ &\times\gamma_\rho\big(A^*+\frac{1}{2}(\hat{q_i'}+\hat{q_f'})B^*\big)\big] . \end{aligned}\quad (60)$$

Direct calculation gives

$$\begin{aligned}D^\mu \sum_{spin} |M|^2 &= 4AA^*\big(R_i^\mu+R_f^\mu\big)+2m(AB^*+BA^*)\\ &\times\big(2\mathcal{P}_i^\mu+2\mathcal{P}_f^\mu-R_i^\mu-R_f^\mu\big)\\ &+BB^*\Big[2\nu_i\big(2\mathcal{P}_i^\mu+2\mathcal{P}_f^\mu-R_i^\mu-R_f^\mu\big)\\ &+2\Delta_p\big(\mathcal{P}_i^\mu+\mathcal{P}_f^\mu\big)-4\mu^2\big(R_i^\mu+R_f^\mu\big)\Big] . \end{aligned}\quad (61)$$

where here μ is the mass of the scalar particle.

In the second term in (58) the derivatives D^μ act after calculation of the trace and the presentation of the result in terms of ν_i and $\Delta_p = (p_i - p_f)^2$, which gives

$$\begin{aligned}D^\mu \sum_{spin} |M'|^2 &= D^\mu\Big[2AA^*(4m^2-\Delta_p)+(AB^*+BA^*)\\ &\times 2m(2\nu_i+\Delta_p)+BB^*(2\nu_i^2+2\nu_i\Delta_p+2\mu^2\Delta_p)\Big]\\ &= 4AA^*\big(R_i^\mu+R_f^\mu\big)+2m(AB^*+BA^*)\\ &\times\big(4\mathcal{P}_i^\mu-2R_f^\mu-2R_i^\mu\big)\\ &+BB^*\Big[4\nu_i\big(2\mathcal{P}_i^\mu-R_i^\mu-R_f^\mu\big)+4\Delta_p\mathcal{P}_i^\mu\\ &-4\mu^2\big(R_i^\mu+R_f^\mu\big)\Big] . \end{aligned}\quad (62)$$

Using (61) and (62) we arrive at

$$\begin{aligned}D^\mu \sum_{spin} |M|^2 - D^\mu \sum_{spin} |M'|^2 &= 2m(AB^*+BA^*)\\ &\times\big(-2\mathcal{P}_i^\mu+2\mathcal{P}_f^\mu+R_f^\mu+2R_i^\mu\big)\\ &+BB^*\Big[2\nu_i\big(2\mathcal{P}_f^\mu-2\mathcal{P}_i^\mu+R_i^\mu+R_f^\mu\big)\\ &+2\Delta_p\big(\mathcal{P}_f^\mu-\mathcal{P}_i^\mu\big)\Big] . \end{aligned}\quad (63)$$

It is quite straightforward to see using (63), (56) and explicit expressions for j^μ, $\mathcal{P}_{i,f}^\mu$, $R_{i,f}^\mu$ that

$$\Big(j^\mu+D^\mu\Big)\sum_{spin}|M|^2-\Big(j^\mu+D^\mu\Big)\sum_{spin}|M'|^2=0, \quad (64)$$

which means, two errors made in [2] during the derivation of Eq. (42) as a result of the incorrect assumption of commutability of the two operations (taking derivatives and summation over the spins followed by the presentation of the result in terms of scalar variables written in momenta of the radiative process) cancel each other, as was shown above by the direct calculations. In reality, these calculations were not needed at all, because the cancellation can be proved using Eqs. (30) and (31).

The net outcome of our consideration is the proof of the validity of the result (44), obtained in [2]. However, in our opinion, it is much more convenient to use the form (40), which does not require an introduction of the artificial and ambiguously defined momenta of the non-radiative processes.

And finally, about the criticism in [4] concerning Ref.[10] (see Appendix A in [4]). It was based on the fact that the formula

$$\begin{aligned}\mathcal{M}_\lambda\Big|_{Lipatov} &= e\Big[\frac{p_{a\lambda}}{(p_a k)}-\frac{p'_{1\lambda}}{(p'_1 k)}\Big]\\ &\times\mathcal{M}^{(0)}(s_L,t,m_\pi^2,m_\pi^2,m_\pi^2,m_\pi^2)\\ &-e(p_a-p_1,k)\Big[\frac{p_{a\lambda}}{(p_a k)}-\frac{p'_{1\lambda}}{(p'_1 k)}\Big]\end{aligned}$$

$$\times \frac{\partial}{\partial t}\mathcal{M}^{(0)}(s_L, t, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2)\ , \tag{65}$$

obtained from the equations of [10] for the case of the process (3), is different from Eq. (13), due to the absence of the term with $\frac{\partial}{\partial s_L}\mathcal{M}^{(0)}$ in (65), and that the term proportional to $\frac{\partial}{\partial t}\mathcal{M}^{(0)}$ is different from the result of [4].

It is easy to see that this criticism has no basis. Ref.[10] considered the high energy process in multi-Regge kinematics, assuming the Regge behaviour of the inelastic amplitude. Therefore, the term with $\frac{\partial}{\partial s_L}\mathcal{M}^{(0)}$ was dropped in [10] because at large energies it falls down. As for the term proportional to $\frac{\partial}{\partial t}\mathcal{M}^{(0)}$, it has to be different from the result of [4], since in [10] it was used the original form (7) of the Low theorem, which is formulated in terms of momenta of the radiative process. There is no term with $\frac{\partial}{\partial t}\mathcal{M}^{(0)}$ in the original formula (7). In (65) such a term appeared because in this formula $t = \frac{1}{2}((p_a - p'_1)^2 + (p_b - p'_2)^2)$ was used instead of $\Delta = (p_b - p'_2)^2$ in (7). It is easy to see that the second term in (65) is the result of the transition in (7) from Δ to t.

5 The low theorem in the textbooks

As discussed above, a conclusion in Ref.[4] that the representation of soft photon emission amplitudes, given in [1] is incomplete, is based on the assumption that it is written in terms of momenta of the non-radiative processes that do not satisfy energy-momentum conservation. Unfortunately, such an erroneous assumption is promoted by some popular textbooks. In particular, this applies to the textbook [11](and its originals in Russian [12]). To clarify this issue, let us first note that the initial particle momenta are labeled in [11] as p_1 and p_2, whereas in [4] as p_a and p_b. Here for convenience, we continue using the same notation (1) and (3). Then we obtain from Eqs. (140.1), (140.9) and (140.10) of [11]

$$\begin{aligned} M &= (\epsilon^*)^\mu \left(M_\mu^{(-)} + M_\mu^{(0)}\right),\ M_\mu^{(-)} = e j_\mu M^{(el)},\ M_\mu^{(0)} \\ &= 2e\left(\mathcal{P}_{i\mu} + \mathcal{P}_{f\mu}\right)\frac{\partial}{\partial s}M^{(el)}, \end{aligned} \tag{66}$$

so that

$$\begin{aligned} M_\mu &= M_\mu^{(-)} + M_\mu^{(0)} \\ &= e\left[j_\mu + 2\left(\mathcal{P}_{i\mu} + \mathcal{P}_{f\mu}\right)\frac{\partial}{\partial s}\right]M^{(el)})\ , \end{aligned} \tag{67}$$

where j_μ and $\mathcal{P}_{i,f\mu}$ are defined in (5) and (6) and $M^{(el)}$ is defined in [11] as elastic scattering amplitudeite. Equation(67) is analogous to the original Low equation (7), but it does not coincide with it. In particular,(67) contains an extra factor 2 in front of $(\mathcal{P}_{i\mu} + \mathcal{P}_{f\mu})$. The appearance of this factor is caused by the contradictory definitions of $M^{(el)}$. As it follows from the unnumbered equation after Eq. (140.7) (as well as from Eqs. (140.8), (140.10) and the unnumbered equation after it), $M^{(el)}$ is considered a function of s and t. There is no problem with t defined in Eq. (140.6) as

$$t = (p_b - p'_2)^2\ , \tag{68}$$

since both t (Eq. (68) in [11]) and Δ (Eq. (8) in [1]) are expressed in terms of the same momenta, so that they just coincide, $\Delta \equiv t$. It is not so neither for the amplitudes $M^{(el)}(s,t)$ and $T(\nu,\Delta)$, nor for their arguments s and ν. The point is that the amplitude $M^{(el)}(s,t)$ is not defined in [11] so clearly, as $T(\nu,\Delta)$ in [1]. Indeed, $M^{(el)}(s,t)$ is defined in [11] by the diagram (140.5) as the amplitude of the elastic scattering process, but with the same particle momenta, as for the inelastic process (3). Moreover, there is the equality in Eq. (140.6) in [11] which in the accepted notation [4] reads as

$$s = (p_a + p_b)^2 = (p'_1 + p'_2)^2\ . \tag{69}$$

This is correct for the momenta of the non-radiative process, but not for the momenta of the process (3). Thus, the momenta of radiative and non-radiative processes are confused in [11], and this leads to an error in the presentation of the theorem.

The book [11] also contains a differential form of the equation (67), which is claimed to be derived from (67), but this derivation is questionable. It is based on the equation following (140.10), which reads as

$$2p_{b\nu}\frac{\partial}{\partial s}\bigg|_t = \frac{\partial}{\partial p_a^\nu}\bigg|_{p_b, p'_1, p'_2} \tag{70}$$

and an analogous equation for $\frac{\partial}{\partial p'_{1'}}$. We stress again that these equations themselves are incorrect, since the partial derivatives over components of the momenta are ill-defined, as was already discussed in Sect. 2 because their definition requires exit from the mass shell. It should be noted that this circumstance is not essential, since in the final formula for the differential form in [11], which can be written using expressions (25) as

$$M_\mu = e\big(j_\mu + D_{i\mu} + D_{f\mu}\big)M^{(el)}, \tag{71}$$

the derivatives appear in the combinations $D_{(i,f)\mu}$ which keep momenta on the mass shell. But the disputed equations also have another, much more serious drawback. These equations are confusing because the elastic scattering momenta are related by the conservation law, and if three of them are fixed, the fourth one is also fixed, so taking a derivative with respect to it makes no sense. Actually, it is always possible to set a partial derivative with respect to a certain momentum to zero expressing the amplitude via other momenta.

Due to these inaccuracies in the derivation, the equation (71) turns out to be unequivalent to the original one (67). Indeed, if we assume that $M^{(el)}$ in Eq. (71) is $M^{(el)}((p_a +$

$p_b)^2, t)$ then we obtain (67) without the term $2\mathcal{P}_{f\mu}$, and if we assume that $M^{(el)}$ in Eq. (71) is $M^{(el)}((p_1' + p_2')^2, t)$ then we obtain (67) without the term $2\mathcal{P}_{i\mu}$. Note that in the first case, we actually get the formulation (17), and in the second (18). This means that the formulation of the Low theorem in the differential form (71) can be considered correct if we assume that $M^{(el)}$ in it is $M^{(el)}(s, t)$, where s and t are expressed in terms of the momenta of the radiative process. But, in our opinion, the formulation (67) could not be justified.

6 Conclusion

Recently there has been a renewal of interest in the physics of soft photon emission and the status of the celebrated Low theorem [1], see for instance [4–9]. In particular, the authors of [4] (see also [5,6]) questioned the validity of the Low theorem and claimed that the term of the order of k^0 in the expansion of the radiative amplitude in photon energy k needs modification. In this paper, we demonstrated that contrary to this claim, the Low theorem when formulated in terms of the final state momenta, it does not require revision. It is shown here that the formulation of the soft-photon theorem proposed in [4] is in complete agreement with the original result of Low [1]. Note that though here we considered explicitly only the processes with spin zero and spin-one-half emitters, similar arguments should hold for the soft photon emission in other scattering processes with an arbitrary number of external charged particles (as already mentioned in Refs. [1,2]). We also address the criticism in [4] of the papers [2] and [10] and explain why it is unsubstantiated.

It is worth emphasizing that the agreement of the formulations does not mean their equivalence, because different forms of presentation could differ by the values of the omitted terms. In some sense, the situation here reminds the expansion in terms of running the coupling constant in perturbation theory. It is well known there that when a finite number of terms in the perturbative expansion is taken, the definition of the coupling constant at different scales leads to different results. A lot of work has been devoted to this problem in QCD, where running is really important, and many recipes have been proposed for the choice of the best scale (that is, how to get the finite number of terms closer to the exact result), see for instance, the book [13]. However, it seems to be impossible to suggest a prescription that is universally applicable to all cases. In the case of the soft-photon theorem, the situation is even worse, since we are talking not about the choice of one parameter, but about the choice of the amplitude in front of the expression which is singular in photon energy (classical current), depending on many parameters. Finally, in this paper, we address some inaccuracies in the discussion of the Low theorem in Ref. [2] and in the classic textbooks [11,12].

Data availability statement This manuscript has no associated data or the data will not be deposited. [Authors' comment:This is a purely theoretical paper.]

Funded by SCOAP^3.

References

1. F.E. Low, Bremsstrahlung of Very Low-Energy Quanta in Elementary Particle Collisions. Phys. Rev. **110**(4), 974–977 (1958)
2. T.H. Burnett, N.M. Kroll, Extension of the low soft photon theorem. Phys. Rev. Lett. **20**, 86 (1968)
3. J.S. Bell, R. Van Royen, On the low-burnett-Kroll theorem for soft-photon emission. Nuovo Cim. A **60**, 62–68 (1969)
4. P. Lebiedowicz, O. Nachtmann, A. Szczurek, High-energy $\pi\pi$ scattering without and with photon radiation. Phys. Rev. D **105**(1), 014022 (2022). arXiv:2107.10829 [hep-ph]
5. P. Lebiedowicz, O. Nachtmann, A. Szczurek, Soft-photon theorem for pion-proton elastic scattering revisited. arXiv:2307.12673 [hep-ph]
6. P. Lebiedowicz, O. Nachtmann, A. Szczurek, Soft-Photon Theorem for Pion-Proton Scattering: Next to Leading Term. arXiv:2307.13291 [hep-ph]
7. T. Engel, A. Signer, Y. Ulrich, Universal structure of radiative QED amplitudes at one loop. JHEP **04**, 097 (2022). arXiv:2112.07570 [hep-ph]
8. R. Balsach, D. Bonocore, A. Kulesza, Soft-photon spectra and the LBK theorem. arXiv:2312.11386 [hep-ph]
9. R. Balsach, D. Bonocore, A. Kulesza, The emission of soft-photons and the LBK theorem, revisited. arXiv:2401.01820 [hep-ph]
10. L.N. Lipatov, Massless Particle Bremsstrahlung Theorems for High-energy Hadron Interactions. Nucl. Phys. B **307**, 705–720 (1988)
11. V.B. Berestetskii, E.M. Lifshitz, L.P. Pitaevskii, *Quantum electrodynamics*, Course of Theoretical Physics, vol. 4. (Pergamon Press, Oxford, 1982)
12. E.M. Lifshitz, L.P. Pitaevsky, Relativistic quantum theory, part 2, Fizmatlit, Nauka, Moscow,1971; V.B. Berestetsky, E.M. Lifshitz and L.P. Pitaevsky, "Quantum Electrodynamics," Fizmatlit (2002)
13. B. L. Ioffe, V. S. Fadin, L. N. Lipatov, Quantum chromodynamics: Perturbative and nonperturbative aspects. Cambridge Univ. Press, 2010, ISBN 978-1-107-42475-3, 978-0-521-63148-8, 978-0-511-71744-4