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Profile Likelihood in Directed Graphical Models from BUGS Output

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Introduction

Approximate profile likelihood of parameters/functions of parameters in directed graphical models with incomplete data from posterior samples from software BUGS (Bayesian inference Using Gibbs Sampling).

Directed Graphical Model

Directed graphical model defined by

- directed acyclic graph, $\mathcal{G} = (V, E)$
- joint probability distribution of $\boldsymbol{v} = (v_v)_{v \in V}$ that is directed Markov w.r.t. to $\mathcal{G}$

i.e. admit *recursive factorization property*

$$p(\boldsymbol{v}) = \prod_{v \in V} p(v_v | \boldsymbol{v}_{\mathrm{pa}(v)})$$

Assume

$$V = X \cup Y \cup \Theta \cup C$$

	random (single-edged)	constant (double-edged)
observed (rectangle)	X: observed data, $\boldsymbol{x} = (x_v)_{v \in X}$	C: covariates, $\boldsymbol{c} = (c_v)_{v \in C}$
unobserved (circle)	Y: missing data/ latent variables, $\boldsymbol{y} = (y_v)_{v \in Y}$	Θ: parameters, $\boldsymbol{\theta} = (\theta_v)_{v \in \Theta}$

As Spiegelhalter (1998) consider constants as random variables with priors on, but condition on them.

Let

$$z_v = \begin{cases} x_v & \text{for } v \in X \\ y_v & \text{for } v \in Y \end{cases}$$

$$\boldsymbol{z}_{\mathrm{pa}(v)} = (\boldsymbol{x}_{\mathrm{pa}(v)}, \boldsymbol{y}_{\mathrm{pa}(v)})$$

Then

$$p(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\theta}, \boldsymbol{c}) = \prod_{v \in X \cup Y} p(z_v | \boldsymbol{z}_{\mathrm{pa}(v)}, \boldsymbol{\theta}_{\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)}) \times \prod_{v \in \Theta} p(\theta_v) \prod_{v \in C} p(c_v)$$

Since $\boldsymbol{\theta}$ and $\boldsymbol{c}$ mutually independent

$$p(\boldsymbol{x}, \boldsymbol{y} | \boldsymbol{\theta}, \boldsymbol{c}) = \prod_{v \in X \cup Y} p(z_v | \boldsymbol{z}_{\mathrm{pa}(v)}, \boldsymbol{\theta}_{\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})$$

Likelihood of $\boldsymbol{\theta}$

$$L(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c}) = \int p(\boldsymbol{x}, \boldsymbol{y} | \boldsymbol{\theta}, \boldsymbol{c}) d\boldsymbol{y}$$

Profile likelihood of θ_i, $i \in \Theta$,

$$\hat{L}(\theta_i | \boldsymbol{x}, \boldsymbol{c}) = \sup_{\boldsymbol{\theta}_{\setminus i}} L(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c})$$

where $\boldsymbol{\theta}_{\setminus i} = \boldsymbol{\theta}_{\Theta \setminus \{i\}}$.

Profile likelihood of $\phi = g(\boldsymbol{\theta})$

$$\hat{L}(\phi | \boldsymbol{x}, \boldsymbol{c}) = \sup_{\boldsymbol{\theta} \in g^{-1}(\phi)} L(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c}), \quad \phi \in \Phi$$

where Φ image of parameter space of $\boldsymbol{\theta}$ under g.

Integrated likelihood of θ_i, $i \in \Theta$,

$$\begin{aligned} \check{L}(\theta_i | \boldsymbol{x}, \boldsymbol{c}) &= \int L(\theta_i, \boldsymbol{\theta}_{\setminus i} | \boldsymbol{x}, \boldsymbol{c}) p(\boldsymbol{\theta}_{\setminus i} | \theta_i) d\boldsymbol{\theta}_{\setminus i} \\ &= \int p(\boldsymbol{x}, \boldsymbol{w} | \theta_i, \boldsymbol{c}) d\boldsymbol{w} \end{aligned}$$

where $\boldsymbol{w} = (\boldsymbol{y}, \boldsymbol{\theta}_{\setminus i})$.

Method

Version 1

$\boldsymbol{\theta}_0$: fixed value of $\boldsymbol{\theta}$. As Geyer & Thompson (1992) consider

$$\frac{p(\boldsymbol{x} | \boldsymbol{\theta}, \boldsymbol{c})}{p(\boldsymbol{x} | \boldsymbol{\theta}_0, \boldsymbol{c})} = \frac{1}{\alpha} L(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c})$$

Rewrite likelihood function

$$L(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c}) \propto \int \prod_{v \in \mathrm{ch}(\Theta)} \frac{p(z_v | \boldsymbol{z}_{\mathrm{pa}(v)}, \boldsymbol{\theta}_{\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})}{p(z_v | \boldsymbol{z}_{\mathrm{pa}(v)}, \boldsymbol{\theta}_{0\,\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})} p(\boldsymbol{y} | \boldsymbol{x}, \boldsymbol{\theta}_0, \boldsymbol{c}) d\boldsymbol{y}$$

Draw $\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \ldots, \boldsymbol{y}^{(N)}$ from $p(\boldsymbol{y} | \boldsymbol{x}, \boldsymbol{\theta}_0, \boldsymbol{c})$.

Approximate likelihood function

$$\tilde{L}(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c}) \propto \sum_{j=1}^{N} \prod_{v \in \mathrm{ch}(\Theta)} \frac{p(z_v^{(j)} | \boldsymbol{z}_{\mathrm{pa}(v)}^{(j)}, \boldsymbol{\theta}_{\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})}{p(z_v^{(j)} | \boldsymbol{z}_{\mathrm{pa}(v)}^{(j)}, \boldsymbol{\theta}_{0\,\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})}$$

where $z_v^{(j)} = x_v$, for $v \in X$, $z_v^{(j)} = y_v^{(j)}$, for $v \in Y$, and $\boldsymbol{z}_{\mathrm{pa}(v)}^{(j)} = (\boldsymbol{x}_{\mathrm{pa}(v)}, \boldsymbol{y}_{\mathrm{pa}(v)}^{(j)})$.

Sampling from $p(\boldsymbol{y} | \boldsymbol{x}, \boldsymbol{\theta}_0, \boldsymbol{c})$: in BUGS with $\boldsymbol{\theta}$ fixed as $\boldsymbol{\theta}_0$.

$\boldsymbol{\theta}_0$: posterior mean of initial Gibbs sampler with priors on $\boldsymbol{\theta}$.

Compute $\log \tilde{L}(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c})$ in grid formed by quantiles of initial Gibbs sampler.

Maximize over grid to approximate profile log-likelihood.

Version 2

$\boldsymbol{\psi}$: another generic symbol of $\boldsymbol{\theta}$. Rewrite likelihood function

$$L(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c}) \propto \iint \prod_{v \in \mathrm{ch}(\Theta)} \frac{p(z_v | \boldsymbol{z}_{\mathrm{pa}(v)}, \boldsymbol{\theta}_{\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})}{p(z_v | \boldsymbol{z}_{\mathrm{pa}(v)}, \boldsymbol{\psi}_{\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})} p(\boldsymbol{y}, \boldsymbol{\psi} | \boldsymbol{x}, \boldsymbol{c}) d\boldsymbol{y} d\boldsymbol{\psi}$$

Draw $(\boldsymbol{y}^{(1)}, \boldsymbol{\psi}^{(1)}), (\boldsymbol{y}^{(2)}, \boldsymbol{\psi}^{(2)}), \ldots, (\boldsymbol{y}^{(N)}, \boldsymbol{\psi}^{(N)})$ from $p(\boldsymbol{y}, \boldsymbol{\psi} | \boldsymbol{x}, \boldsymbol{c})$.

Approximate likelihood function

$$\tilde{\tilde{L}}(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c}) \propto \sum_{j=1}^{N} \prod_{v \in \mathrm{ch}(\Theta)} \frac{p(z_v^{(j)} | \boldsymbol{z}_{\mathrm{pa}(v)}^{(j)}, \boldsymbol{\theta}_{\mathrm{pa}(v)}, \boldsymbol{c}_{\mathrm{pa}(v)})}{p(z_v^{(j)} | \boldsymbol{z}_{\mathrm{pa}(v)}^{(j)}, \boldsymbol{\psi}_{\mathrm{pa}(v)}^{(j)}, \boldsymbol{c}_{\mathrm{pa}(v)})}$$

Sampling from $p(\boldsymbol{y}, \boldsymbol{\psi} | \boldsymbol{x}, \boldsymbol{c})$: in BUGS with priors on $\boldsymbol{\theta}$.

Compute $\log \tilde{\tilde{L}}(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c})$ in grid formed by quantiles of Gibbs sampler in this version.

Maximize over grid to approximate profile log-likelihood.

Profile likelihood of a function

$\tilde{\nabla}$: grid points already considered. Form pairs

$$(g(\boldsymbol{\theta}), \tilde{L}(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c})), \quad \boldsymbol{\theta} \in \tilde{\nabla}$$

Need to calculate $g(\boldsymbol{\theta})$, already have $\tilde{L}(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c})$.

Order pairs in increasing order w.r.t. $\phi = g(\boldsymbol{\theta})$, partition into bins.

Find pair with maximum value of $\tilde{L}(\boldsymbol{\theta} | \boldsymbol{x}, \boldsymbol{c})$ for each bin.

Combine maximum pairs to approximate profile log-likelihood of $g(\boldsymbol{\theta})$.

Integrated likelihood of θ_i

Consider $\Theta \setminus \{i\}$ as latent variables in method.

Get a 1-dimensional grid. Marginalisation done by summation instead of maximalisation.

Comments

Priors: no significant impact, once Gibbs sampler converged.

Method: hybrid of Bayesian and likelihood - prior used to compute credible region, approximation done there.

Complementary tool to BUGS.

Example

From Spiegelhalter, Thomas, Best & Gilks (1996b). Measure for ten power plant pumps

c_i : operation time

x_i : number of failures

Model: $i = 1, 2, \ldots, 10$

$$x_i | y_i, c_i \sim \mathrm{Po}(y_i c_i)$$

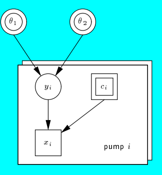
$$y_i | \theta_1, \theta_2 \sim \Gamma(\theta_1, \theta_2)$$

where

y_i : failure rate of pump i

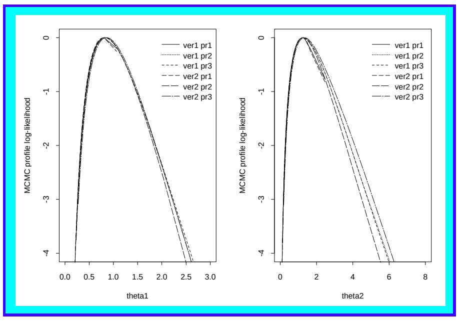
θ_1 : shape parameter

θ_2 : scale parameter

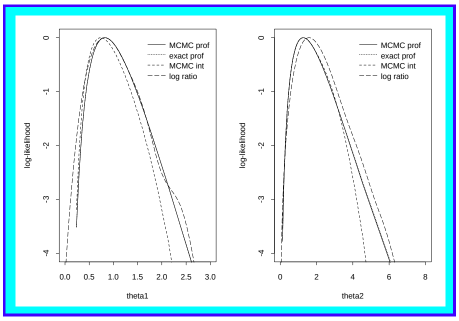


PRIORS	Prior 1	Prior 2	Prior 3
θ_1	$\mathrm{Exp}(10)$	$\mathrm{Exp}(1.0)$	$\mathrm{Exp}(0.01)$
θ_2	$\Gamma(0.1, 1.0)$	$\Gamma(0.01, 0.1)$	$\Gamma(0.001, 0.001)$

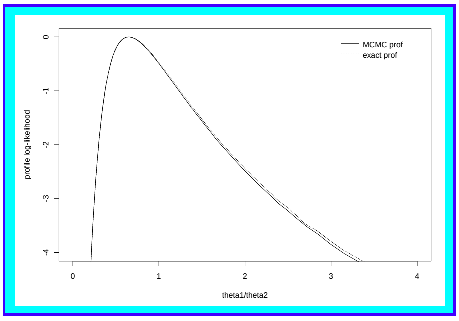
Sample size: $N = 5.000$ Grid size: 200×200



Priors no significant impact



Good approximation



Good approximation

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