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Abstract We give a simple combinatorial proof of the toric version of Mori's theorem that the only smooth projective varieties with ample tangent bundle are the projective spaces $\mathbb{P}^n$.

1. Introduction

It is a well-known theorem that the only smooth projective varieties (over an algebraically closed field k) with ample tangent bundles are the projective spaces $\mathbb{P}^n_k$. This is first conjectured by Hartshorne [5, Problem 2.3] and later proved by Mori [8] using the full force of his now-celebrated "bend and break" technique. Here we say that a vector bundle $\mathcal{E}$ is ample (resp. nef) if the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ on the projectivized bundle $\mathbb{P}\mathcal{E}$ is ample (resp. nef).

In this paper, we consider a toric version of this theorem and show that it admits a simple combinatorial proof.

Theorem 1.1. *Let X be an n-dimensional smooth projective toric variety (over an algebraically closed field k) with ample tangent bundle $\mathcal{T}_X$. Then X is isomorphic to $\mathbb{P}^n_k$.*

In the proof we fix an ample divisor on X and consider the corresponding polytope $P \subseteq \mathbb{R}^n$. The key observation we make is that the ampleness of $\mathcal{T}_X$ implies that the sum of any pair of two adjacent angles on a 2-dimensional face of P is smaller than π. It follows that P has to be an n-simplex, and hence X is isomorphic to $\mathbb{P}^n$.(1)

2. Preliminaries

Here we list out some definitions and facts regarding toric varieties and toric vector bundles that we will use in this article. One may refer to [4, 1] for more details about toric varieties, and [9, 2] for more details about toric vector bundles.

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(1)After this work was complete the author learned of the sources [7, 3] that contain different proofs of results in this paper.

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2.1. Toric varieties. We work throughout over an algebraically closed field k. By a toric variety, we mean an irreducible and normal algebraic variety X containing a torus $T \cong (k^*)^n$ as a Zariski open subset such that the action of T on itself (by multiplication) extends to an algebraic action of T on X.

Let M be the group of the characters of T, and N the group of the 1-parameter subgroups of T. Both M and N are lattices of rank n (equal to the dimension of T), i.e. isomorphic to $\mathbb{Z}^n$. They are dual to each other in the sense that there is a natural pairing of M and N denoted by $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$.

Every toric variety X is associated to a fan Σ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \, (\cong \mathbb{R}^n)$. A fan Σ is said to be *complete* if it supports on the whole $N_{\mathbb{R}}$, and is said to be *smooth* if every cone in Σ is generated by a subset of a $\mathbb{Z}$-basis of N. A toric variety X is complete if and only if its associated fan Σ is complete, and X is smooth if and only if Σ is smooth.

There is an inclusion-reversing bijection between the cones $\sigma \in \Sigma$ and the T-orbits in X. Let $O_\sigma \subseteq X$ be the orbit corresponding to σ. The codimension of O_σ in X is equal to the dimension of σ. Each cone $\sigma \in \Sigma$ also corresponds to a T-invariant open affine set $U_\sigma \in X$, which is equal to the union of all the orbits O_τ corresponding to cones τ contained in σ. Given a 1-dimensional cone $\rho \in \Sigma$, the closure of O_ρ is a T-invariant Weil divisor, denoted by D_ρ. The class group of X is generated by the classes of the divisors D_ρ corresponding to the 1-dimensional cones in Σ.

2.2. Polytopes and toric varieties. Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. A lattice polytope P in $M_{\mathbb{R}}$ is the convex hull of finitely many points in M. The dimension of P is defined to be the dimension of the affine span of P. When $\dim P = \dim M_{\mathbb{R}}$, we say that P is full dimensional.

Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope, and let $P_1, \ldots, P_m$ be the *facets* of P, i.e. codimension 1 faces of P. For each facet P_k, there exists a unique primitive lattice point $v_k \in N$ and a unique integer $c_k \in \mathbb{Z}$ with $P_k = \{u \in P \mid \langle u, v_k \rangle = -c_k\}$ and $\langle u, v_k \rangle \geqslant -c_k$ for all $u \in P$.

Let Σ_P be the (inner) normal fan of P. The toric variety X_{Σ_P} associated to Σ_P is called the toric variety of P, and denoted by X_P. Denote by D_k the divisor corresponding to the 1-dimensional cone generated by v_k. Then we may define a divisor on X_P by $D_P := \sum_{k=1}^m c_k D_k$. Such a divisor D_P is necessarily ample.

This process is reversible. Given an ample T-invariant divisor D on X, we have $D = \sum_{k=1}^m c'_k D_k$ for some integers $c'_k \in \mathbb{Z}$. Then, the polytope $P = P_{(X,D)}$ corresponding to X and D may be defined by

$$P_{(X,D)} := \{u \in M_{\mathbb{R}} \mid \langle u, v_k \rangle \geqslant -c'_k \text{ for all } k\}.$$

This gives a 1-to-1 correspondence between full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ and a pair (X, D) of a complete toric variety X together with an ample T-invariant divisor D on X.

We say that P is *smooth* if given a vertex $u \in P$, u is contained in exactly n edges (i.e. 1-dimensional faces), and $\{u_1 - u, \ldots, u_n - u\}$ is a $\mathbb{Z}$-basis of M, where $u_1, \ldots, u_n$ are the next lattice points on the n edges. The toric variety X_P is smooth if and only if P is smooth.

2.3. Toric vector bundles. A vector bundle $\pi : \mathcal{E} \to X$ over a toric variety $X = X_\Sigma$ is said to be toric (or equivariant) if there is a T-action on $\mathcal{E}$ that is linear on each fiber and satisfies $t \circ \pi = \pi \circ t$ for all $t \in T$.

Given a cone $\sigma \in \Sigma$ and $u \in M$, define $\mathcal{L}_u|_{U_\sigma}$ to be the line bundle $\mathcal{O}_{U_\sigma}(\operatorname{div} \chi_u)$ over U_σ. Explicitly, $\mathcal{L}_u|_{U_\sigma}$ is the trivial line bundle $U_\sigma \times k$ equipped with the T-action given by $t.(x, z) := (t.x, \chi^u(t) \cdot z)$. If $u, u' \in M$ satisfy $u - u' \in \sigma^\perp$, then $\chi^{u-u'}$ is a non-vanishing regular function on U_σ which gives an isomorphism $\mathcal{L}_u|_{U_\sigma} \cong \mathcal{L}_{u'}|_{U_\sigma}$. In fact, the group of toric line bundles on U_σ is isomorphic to $M_\sigma := M/(M \cap \sigma^\perp)$. Therefore, we also write $\mathcal{L}_{[u]}|_{U_\sigma}$, where $[u] \in M_\sigma$ is the class of u.

Let $\mathcal{E} \to X$ be a toric vector bundle of rank r. Its restriction to an invariant open affine set U_σ splits into a direct sum of toric line bundles with trivial underlying line bundles [9, Proposition 2.2]; i.e. we have $\mathcal{E}|_{U_\sigma} \cong \bigoplus_{i=1}^r \mathcal{L}_{[u_i]}|_{U_\sigma}$ for some $[u_i] \in M_\sigma$. Define the *associated characters* of $\mathcal{E}$ on σ to be the multiset $\mathbf{u}_{\mathcal{E}}(\sigma) \subset M_\sigma$ of size r that contains the $[u_i]$ showing up in the splitting.

Example 2.1 (Associated characters of tangent bundles). Let $X = X_\Sigma$ be an n-dimensional smooth projective toric variety, and consider its tangent bundle $\mathcal{T}_X$. Fix a maximal cone $\sigma \in \Sigma$. Since X is smooth, the dual cone $\check{\sigma}$ of σ is generated by some $u_1, \ldots, u_n \in M$ that form a $\mathbb{Z}$-basis of M. Denote by $x_1, \ldots, x_n \in \Gamma(U_\sigma, \mathcal{O}_X)$ the coordinates on $U_\sigma \cong k^n$ corresponding to $u_1, \ldots, u_n$. Then $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ is a local frame of $\mathcal{T}_X$ on U_σ. Each non-vanishing section $\frac{\partial}{\partial x_i} \in \Gamma(U_\sigma, \mathcal{T}_X)$ induces a map from the trivial line bundle $U_\sigma \times k$ to $\mathcal{T}_X|_{U_\sigma}$, the image of which is a toric line subbundle of $\mathcal{T}_X|_{U_\sigma}$ isomorphic to $\mathcal{L}_{u_i}|_{U_\sigma}$. We have $\mathcal{T}_X|_{U_\sigma} \cong \bigoplus_{i=1}^n \mathcal{L}_{u_i}|_{U_\sigma}$, and hence the associated characters of $\mathcal{T}_X$ on σ are $\mathbf{u}_{\mathcal{T}_X}(\sigma) = \{u_1, \ldots, u_n\}$.

2.4. Positivity of toric vector bundles.

Let $X = X_\Sigma$ be a complete toric variety. By an *invariant curve* on X, we mean a complete irreducible 1-dimensional subvariety that is invariant under the T-action. Via the cone-orbit correspondence, there is a one-to-one correspondence between the invariant curves and the codimension-1 cones; every invariant curve is the closure of an 1-dimensional orbit, which corresponds to a codimension-1 cone in Σ. For each codimension-1 cone $\tau \in \Sigma$, denote the corresponding invariant curve by C_τ.

The positivity of toric vector bundles can be checked on invariant curves according to the following result in [6].

Theorem 2.2. [6, Theorem 2.1] *A toric vector bundle on a complete toric variety is ample (resp. nef) if and only if its restriction to every invariant curve is ample (resp. nef).*

Note that every invariant curve is a $\mathbb{P}^1$. By Birkhoff–Grothendieck theorem, every vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles. Hence, the positivity of vector bundles on $\mathbb{P}^1$ is well understood, namely $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ is ample (resp. nef) if and only if every a_i is positive (resp. non-negative). It is common to call the r-tuple (or multiset) $(a_i)_{i=1}^r$ the *splitting type* of the vector bundle.

Fix a codimension-1 cone τ, and let σ, σ' be the two maximal cones containing τ. Given $u, u' \in M$ satisfying $u - u' \in \tau^\perp$, define a toric line bundle $\mathcal{L}_{u,u'}$ on $U_\sigma \cup U_{\sigma'}$ by glueing the toric line bundles $\mathcal{L}_u|_{U_\sigma}$ and $\mathcal{L}_{u'}|_{U_{\sigma'}}$ with the transition function $\chi^{u'-u}$. Since the invariant curve C_τ is contained in $U_\sigma \cup U_{\sigma'}$, we may restrict $\mathcal{L}_{u,u'}$ to get a toric line bundle $\mathcal{L}_{u,u'}|_{C_\tau}$ on C_τ.

Proposition 2.3. [6, Corollary 5.5 and 5.10] *Let X be a complete toric variety. Any toric vector bundle $\mathcal{E}|_{C_\tau}$ on the invariant curve C_τ splits equivariantly as a sum of line bundles*

$$\mathcal{E}|_{C_\tau} = \bigoplus_{i=1}^{r} \mathcal{L}_{u_i,u_i'}|_{C_\tau}.$$

The splitting is unique up to reordering.

Combining this with the following lemma that computes the underlying line bundle of $\mathcal{L}_{u,u'}|_{C_\tau}$, one gets the splitting type of $\mathcal{E}|_{C_\tau}$.

Lemma 2.4. [6, Example 5.1] *Let u_0 be the generator of $M \cap \tau^\perp \cong \mathbb{Z}$ that is positive on σ, and let m be the integer such that $u - u' = mu_0$. Then, the underlying line bundle of $\mathcal{L}_{u,u'}|_{C_\tau}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(m)$.*

3. Restricting $\mathcal{T}_X$ to invariant curves

Let $X = X_\Sigma$ be a smooth complete toric variety of dimension n. In this section, we consider the restrictions of the tangent bundle $\mathcal{T}_X$ to the invariant curves. The goal is to get the splitting types in terms of the combinatorial data of the fan Σ of X. This has in fact been done in [2, Example 5.1 and 5.2] and [10, Theorem 2]. We repeat the calculation for the convenience of the readers.

Fix an $(n-1)$-dimensional cone $\tau \in \Sigma$. Let $\sigma, \sigma' \in \Sigma(n)$ be the two maximal cones containing τ. Let $v_1, \dots, v_{n-1}, v_n, v'_n \in N$ be primitive vectors such that τ is generated by $\{v_1, \dots, v_{n-1}\}$, σ is generated by $\{v_1, \dots, v_{n-1}, v_n\}$, and σ' is generated by $\{v_1, \dots, v_{n-1}, v'_n\}$. There are unique $u_i, u'_i \in M$ $(i = 1, \dots, n)$ such that $\langle u_i, v_i\rangle = \langle u'_i, v'_i\rangle = 1$ for all i and $\langle u_i, v_j\rangle = \langle u'_i, v'_j\rangle = 0$ for all $i \neq j$, where we define $v'_i = v_i$ for $i = 1, \dots, n-1$. The dual cones $\check{\sigma}$ and $\check{\sigma}'$ are generated by $\{u_1, \dots, u_n\}$ and $\{u'_1, \dots, u'_n\}$, respectively.

By Example 2.1, the associated characters of $\mathcal{T}_X$ on σ and σ' are given by

$$\mathbf{u}_{\mathcal{T}_X}(\sigma) = \{u_1, \dots, u_n\}\ ,\ \mathbf{u}_{\mathcal{T}_X}(\sigma') = \{u'_1, \dots, u'_n\}\ .$$

Following Section 2.4, let C_τ be the invariant curve corresponding to τ. The splitting of $\mathcal{T}_X|_{C_\tau}$ as in Proposition 2.3 is easy to get by the following fact.

Lemma 3.1. *The associated characters u_i, u'_i satisfy $u_i - u'_i \in \tau^\perp$ for all $i = 1, \dots, n$, and $u_i - u'_j \notin \tau^\perp$ for all $i \neq j$.*

Proof. Note that $u \in M$ is contained in $\tau^\perp$ if and only if $\langle u, v_\ell\rangle = 0$ for all $\ell = 1, \dots, n-1$. The first part of the lemma follows from the fact that $\langle u_i - u'_i, v_\ell\rangle = 0$ for all $\ell = 1, \dots, n-1$, and the second part of the lemma follows from $\langle u_i - u'_j, v_i\rangle = -\langle u_i - u'_j, v_j\rangle = 1$, where at least one of i, j is not n. ☐

Definition 3.2. *Define $a_i \in \mathbb{Z}$ (for $i = 1, \dots, n$) to be the integers satisfying $u_i = u'_i + a_i u_n$. Such integers exist since u_n is a primitive generator of $\tau^\perp \cap M \cong \mathbb{Z}$. Note that $u'_n = -u_n$ so that $a_n = 2$.*

Proposition 3.3. *On the invariant curve C_τ, the restriction $\mathcal{T}_X|_{C_\tau}$ of the tangent bundle (as a toric vector bundle) splits into the following direct sum of toric line bundles*

$$\mathcal{T}_X|_{C_\tau} \cong \bigoplus_{i=1}^n \mathcal{L}_{u_i, u'_i}|_{C_\tau}\,.$$

In particular, we have the following splitting of $\mathcal{T}_X|_{C_\tau}$ as a vector bundle

$$\mathcal{T}_X|_{C_\tau} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)\,.$$

Proof. By Proposition 2.3, we have that $\mathcal{T}_X|_{C_\tau}$ splits into a direct sum of toric line bundles of the form $\mathcal{L}_{u,u'}|_{C_\tau}$. This gives a bijection $\iota : \mathbf{u}_{\mathcal{E}}(\sigma) \to \mathbf{u}_{\mathcal{E}}(\sigma')$ mapping u to u' whenever $\mathcal{L}_{u,u'}|_{C_\tau}$ shows up in the splitting. Note that $u_i - \iota(u_i) \in \tau^\perp$ by the definition of $\mathcal{L}_{u,u'}$. Then Lemma 3.1 implies that we must have $\iota(u_i) = u'_i$ for all i, hence the splitting in the first part.

The second part follows directly from the first part together with Lemma 2.4. ☐

Remark 3.4. The integers a_i are the same as the integers b_i that show up in the "wall relation"

$$b_1 v_1 + \dots + b_{n-1} v_{n-1} + v_n + v'_n = 0\,,$$

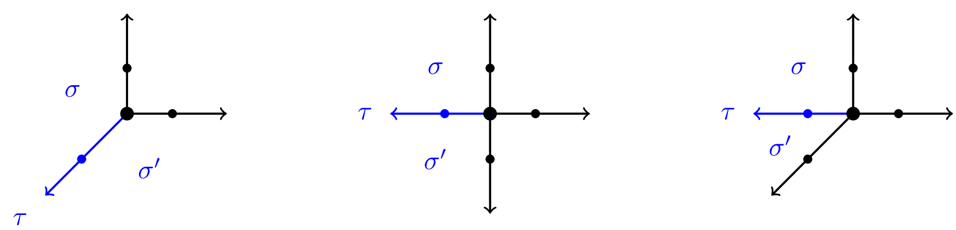


FIGURE 1. Fans of toric surfaces

mentioned in [10] and [2]. Indeed we have $b_i = -\langle u_i, v'_n \rangle = a_i$ for all $i = 1, \dots, n-1$.

EXAMPLE 3.5. For each of the following toric surfaces X, we fix a 1-dimensional cone τ in its fan as shown in Figure 1 and compute the splitting type of $\mathcal{T}_X|_{C_\tau}$.

(1) $X = \mathbb{P}^2$. The dual cones of the maximal cones containing τ are given by $\check{\sigma} = \mathrm{Cone}\{(-1,0),(-1,1)\}$ and $\check{\sigma}' = \mathrm{Cone}\{(0,-1),(1,-1)\}$. Therefore we get $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. In fact, the restrictions of $\mathcal{T}_X$ to the other two invariant curves have the same splitting type, so $\mathcal{T}_X$ is ample by Proposition 2.2.

(2) $X = \mathbb{P}^1 \times \mathbb{P}^1$. The dual cones of the maximal cones containing τ are given by $\check{\sigma} = \mathrm{Cone}\{(-1,0),(0,1)\}$ and $\check{\sigma}' = \mathrm{Cone}\{(-1,0),(0,-1)\}$. Therefore we get $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. In fact, the restrictions of $\mathcal{T}_X$ to the other three invariant curves have the same splitting type, so $\mathcal{T}_X$ is nef but not ample by Proposition 2.2.

(3) Let X be the Hirzebruch surface $\mathbb{F}_1$, which is isomorphic to $\mathbb{P}^2$ blown up at one point. The dual cones of the maximal cones containing τ are given by $\check{\sigma} = \mathrm{Cone}\{(-1,0),(0,1)\}$ and $\check{\sigma}' = \mathrm{Cone}\{(-1,1),(0,-1)\}$. Therefore we get $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, and hence $\mathcal{T}_X$ is not nef by Proposition 2.2.

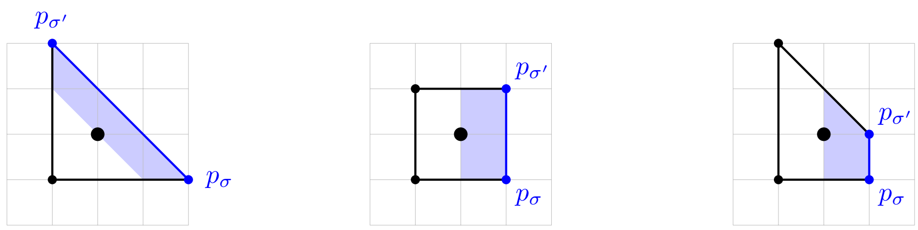
4. POLYTOPES AND AMPLENESS OF THE TANGENT BUNDLE

Let $X = X_\Sigma$, $\mathcal{T}_X$, τ, σ, σ', u_i, u'_i, a_i be as in the previous section.

Fix an ample T-invariant divisor D on X, and let $P = P_{(X,D)}$ be the corresponding polytope in the sense of Section 2.2. Note that X and Σ are simplicial as they are smooth; in particular, every maximal cone in Σ has exactly n faces of dimension $(n-1)$, and every $(n-1)$-dimensional cone has exactly $(n-1)$ faces of dimension $(n-2)$. This implies that there are exactly n edges emanating from every vertex of P and that every edge of P is contained in exactly $(n-1)$ faces of dimension 2.

Let $p_\sigma \in P$ be the vertex corresponding to the maximal cone σ. Let $P - p_\sigma$ denote the translation of P by $-p_\sigma$. Then the cone generated by $P - p_\sigma$ is given by $\{u \in M_\mathbb{R} \mid \langle u, v_i \rangle \geqslant 0 \text{ for all } i = 1, \dots, n\}$, which is exactly the dual cone $\check{\sigma}$ of σ. Thus, each of the n edges of P emanating from p_σ contains exactly one of $p_\sigma + u_1, \dots, p_\sigma + u_n$. Similarly, each of the n edges emanating from the vertex $p_{\sigma'}$ corresponding to σ' contains exactly one of $p_{\sigma'} + u'_1, \dots, p_{\sigma'} + u'_n$.

Recall that the u_i and u'_i satisfy $u'_i = u_i - a_i u_n$ for all $i = 1, \dots, n-1$ and $u'_n = -u_n$. Since σ and σ' contain the $(n-1)$-dimensional cone τ as a common face, the convex hull of $\overline{p_\sigma, p_{\sigma'}}$ of p_σ and $p_{\sigma'}$ is an edge of P; it corresponds to τ and contains $p_\sigma + u_n$ and $p_{\sigma'} + u'_n$. Fix $j \in \{1, \dots, n-1\}$. Consider the points $p_\sigma + u_j, p_{\sigma'} + u'_j \in M$. The point $p_\sigma + u_j$ is on an edge emanating from p_σ, and $p_{\sigma'} + u'_j$ is on an edge emanating from $p_{\sigma'}$. In addition, since $(p_\sigma + u_j) - (p_{\sigma'} + u'_j) = (p_\sigma - p_{\sigma'}) + a_j u_n$, $\overline{p_\sigma + u_j, p_{\sigma'} + u'_j}$



(1) $X = \mathbb{P}^2$ (2) $X = \mathbb{P}^1 \times \mathbb{P}^1$ (3) $X = \mathbb{F}_1$

Figure 2. Polytopes $P(X, -K_X)$ of toric surfaces

is parallel to $\overline{p_\sigma, p_{\sigma'}}$. Thus, the four points $p_\sigma, p_{\sigma'}, p_\sigma + u_j, p_{\sigma'} + u'_j$ are contained in a common 2-dimensional face $A_j \subseteq P$. In fact, A_j is the 2-dimensional face of P corresponding to the $(n-2)$-dimensional cone $\tau \cap (u_j)^\perp = \tau \cap (u'_j)^\perp$.

Denote the angles at p_σ and $p_{\sigma'}$ on A_j by $\theta(p_\sigma, A_j)$ and $\theta(p_{\sigma'}, A_j)$, respectively. Their sum is related to the integer a_j in the following way.

Proposition 4.1. *The sum $\theta(p_\sigma, A_j) + \theta(p_{\sigma'}, A_j)$ is smaller than π if and only if $a_j > 0$, equal to π if and only if $a_j = 0$, and greater than π if and only if $a_j < 0$.*

Proof. Suppose $a_j > 0$. Consider the convex hull of the four points $p_\sigma, p_{\sigma'}, p_{\sigma'} + u'_j, p_\sigma + u_j \in M$, which is either a triangle (if $p_{\sigma'} + u'_j = p_\sigma + u_j$) or a trapezoid with the edges $\overline{p_\sigma + u_j, p_{\sigma'} + u'_j}$ and $\overline{p_\sigma, p_{\sigma'}}$ parallel to each other. See Figure 2(1) for an example of this trapezoid. If the convex hull is a triangle, then it is clear that $\theta(p_\sigma, A_j) + \theta(p_{\sigma'}, A_j) < \pi$. If the convex hull is a trapezoid, since

$$\big((p_{\sigma'} + u'_j) - (p_\sigma + u_j)\big) - (p_{\sigma'} - p_\sigma) = -a_j u_n,$$

the edge $\overline{p_\sigma + u_i, p_{\sigma'} + u'_i}$ is shorter than $\overline{p_\sigma, p_{\sigma'}}$, implying $\theta(p_\sigma, A_j) + \theta(p_{\sigma'}, A_j) < \pi$.

Similarly, if $a_j < 0$, then the edge $\overline{p_\sigma + u_i, p_{\sigma'} + u'_i}$ is longer than $\overline{p_\sigma, p_{\sigma'}}$ and hence $\theta(p_\sigma, A_j) + \theta(p_{\sigma'}, A_j) > \pi$. (See Figure 2(3).)

If $a_j = 0$, then the edges $\overline{p_\sigma + u_i, p_{\sigma'} + u'_i}$ and $\overline{p_\sigma, p_{\sigma'}}$ have the same length, i.e. the trapezoid is in fact a parallelogram. Therefore, we have $\theta(p_\sigma, A_j) + \theta(p_{\sigma'}, A_j) = \pi$. (See Figure 2(2).) □

Remark 4.2. Although the angles $\theta(p_\sigma, A_j), \theta(p_{\sigma'}, A_j)$ themselves are not invariant under a change of bases of M, whether their sum is smaller than, equal to, or greater than π is.

Example 4.3. In Figure 2 are polytopes $P(X, -K_X)$ corresponding to the toric surfaces X in Example 3.5 together with their anticanonical line bundles $-K_X$, The cones τ, σ, σ' are the same as in Example 3.5, and the shaded area in each picture is the convex hull of $p_\sigma, p_{\sigma'}, p_{\sigma'} + u'_j, p_\sigma + u_j$ in the proof of Proposition 4.1

(1) $X = \mathbb{P}^2$. Recall $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = 1 > 0$. Here we see that $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) < \pi$.
(2) $X = \mathbb{P}^1 \times \mathbb{P}^1$. Recall $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = 0$. Here we see that $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) = \pi$.
(3) $X = \mathbb{F}_1$. Recall $\mathcal{T}_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = -1 < 0$. Here we see that $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) > \pi$.

5. Proof of Theorem 1.1

Proof of Theorem 1.1. As in Section 4, fix an ample T-invariant divisor D on X, and let $P = P_{(X,D)}$ be the corresponding polytope. We will show that P is an n-simplex.

Let A be a 2-dimensional face of P. Let m be the number of vertices of A, and let $p_1, \dots, p_m$ be the vertices of A, ordered so that p_k is adjacent to p_{k+1} for all $k = 1, \dots, m$, where $p_{m+1} := p_1$. Since $\mathcal{T}_X$ is ample, its restriction to every invariant curve is ample. Then, by Proposition 3.3 and Proposition 4.1, $\theta(p_k, A) + \theta(p_{k+1}, A) < \pi$ for all k. This implies

$$m\pi > \sum_{k=1}^{m} (\theta(p_k, A) + \theta(p_{k+1}, A)) = 2\sum_{k=1}^{m} \theta(p_k, A) = 2(m-2)\pi.$$

We get $m < 4$, so A is a triangle. The same is true for all 2-dimensional faces of P.

Now, we start with a vertex q_0 of P. Note that P is smooth since X is smooth. Thus, q_0 is contained in exactly n edges, and if $w_1, \dots, w_n$ are the next lattice points on the n edges, then $\{w_1 - q_0, \dots, w_n - q_n\}$ is a $\mathbb{Z}$-basis of M. This implies that q_0 is adjacent to exactly n vertices and that every two edges containing q_0 is contained in a 2-dimensional face of P. Let $q_1, \dots, q_n$ be the n vertices adjacent to q_0. For each $1 < j \leqslant n$, let A_j be the 2-dimensional face containing the edges $\overline{q_0 q_1}$ and $\overline{q_0 q_j}$. Since A_j is in fact a triangle, q_1 is also adjacent to q_j. Thus q_1 is adjacent to $q_0, q_2, \dots, q_n$. Similarly, every q_j is adjacent to exactly $q_0, \dots, \widehat{q_j}, \dots, q_n$. Consequently, $q_0, q_1, \dots, q_n$ are the only vertices of P, and hence P is the n-simplex with vertices $q_0, q_1, \dots, q_n$. ☐

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