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Ramanujan's Tau-Function and Convolution Sums

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Abstract:

We study certain type of convolution sums involving an arbitrary arithmetic function *f*, which it is applied to Ramanujan's tau function when *f* coincides with the sum of divisors function.

Keywords: *Convolution sums, Sum of divisors function, Ramanujan's function* $\tau(n)$,*Niebur's formula, Lanphier's identity.*

Introduction

We shall study convolution sums with the structure:

$$S_g^{(r)}(n) = \sum_{k=0}^{n} k^r\, g(k)\, g(n-k), \qquad r, n \geq 0, \tag{1}$$

where g is an arbitrary arithmetic function, in particular:

$$S_g^{(0)}(n) = \sum_{k=0}^{n} g(k)\, g(n-k), \qquad S_g^{(1)}(n) = \sum_{k=0}^{n} k\; g(k)\, g(n-k), \tag{2}$$

then with the Z-transform (Grove, 1991; Patra, B. (2018), it is easy to prove the identity $n\, S_g^{(0)}(n) = 2\, S_g^{(1)}(n)$, that is:

$$\sum_{k=0}^{n} (n-2k)\, g(k)\, g(n-k) = 0\,. \tag{3}$$

In Sec. 2 we use (3) to obtain the Lanphier's identity involving convolution sums of the form (1), and we show how to generalize his identity. In Sec. 3 we employ the mentioned Lanphier's expression to simplify the Niebur's formula for the Ramanujan's tau function.

Lanphier's Identity

The arithmetic function g is arbitrary, then in (3) we can employ $g(n) = n\, f(n)$, to obtain:

$$\sum_{k=0}^{n} (n-2k)\, k\, (n-k)\, f(k)\, f(n-k) = \sum_{k=0}^{n} (2k^3 - 3n\, k^2 + n^2\, k)\, f(k)\, f(n-k) = 0, \quad (4)$$

that is:

$$2\, S_f^{(3)}(n) - 3n\, S_f^{(2)}(n) + n^2 S_f^{(1)}(n) = 0, \quad (5)$$

in complete agreement with Lanphier (Lanphier, 2004; Gallardo, 2010).

Similary, in (3) we use $g(n) = n^2\, f(n)$ to deduce the property:

$$2\, S_f^{(5)}(n) - 5n\, S_f^{(4)}(n) + 4n^2 S_f^{(3)}(n) - n^3 S_f^{(2)}(n) = 0, \quad (6)$$

and for the general case $g(n) = n^r\, f(n)$:

$$\sum_{j=r}^{2r} (-1)^j \binom{r}{j-r} n^{2r-j} \left[2\, S_f^{(j+1)}(n) - n\, S_f^{(j)}(n) \right] = 0, \qquad r \geq 0, \quad (7)$$

which is equivalent to (1) of Gallardo(2010).

Niebur's Relation for $\tau(n)$

If f is the sum of divisors function, then the Niebur's formula (Gallardo, 2010; Ewell, 1984; Wikipedia, n.a.) for the Ramanujan's tau function (Ramanujan, 1916; Roy, 2017):

$$\tau(n) = n^4\, \sigma(n) - 24 \sum_{k=0}^{n} k^2\, (35\, k^2 - 52\, k\, n + 18\, n^2)\, \sigma(k)\, \sigma(n-k), \qquad n \geq 0, \quad (8)$$

can be written in the form:

$$\tau(n) = n^4\, \sigma(n) - 24 \left[35\, S_\sigma^{(4)}(n) - 52\, n\, S_\sigma^{(3)}(n) + 18\, n^2 S_\sigma^{(2)}(n) \right], \quad (9)$$

where it is possible to employ the identity (5) obtained by Lanphier (2004) to deduce:

$$\tau(n) = n^4\, \sigma(n) - 24 \left[35\, S_\sigma^{(4)}(n) - 60\, n^2 S_\sigma^{(2)}(n) + 26\, n^3 S_\sigma^{(1)}(n) \right], \quad (10)$$

$$= n^4\,\sigma(n) - 24\,\sum_{k=0}^{n} k\,(35\,k^3 - 60\,n^2 k + 26n^3)\,\sigma(k)\,\sigma(n-k). \tag{11}$$

Remark.-From (8) are immediate the following congruences of Ramanujan (1920:

$$\tau(j\,n\,) \equiv 0\ \ (mod\ j), \qquad j = 2, 3, 5, 7, \tag{12}$$

and the result of Ewell (1984):

$$\tau(n) \equiv \begin{cases} 0 & (mod\ 8),\ \ n\ even, \\ \sigma(n)(mod\ 8), & n\ odd. \end{cases} \tag{13}$$

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