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Crank-Nicolson Finite Difference Method with Sobolev Space Energy Estimate Theorem for Capital Market Prices

By

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ABSTRACT

In this study, we have Black-Scholes analytic formula and Crank- Nicolson (CN) finite difference method for valuation of European put option which has earned the interests of researchers for determining both analytic and approximate solutions to Partial Differential Equations (PDEs) with Sobolev space energy estimate theorem. The simulations of analytical and numerical were effectively carried out. The results showed as follows: increase in volatility increases the values of option for both BS and CN prices, there are significant difference between BS and CN due to the changes of stock volatility, a little increase in the initial stock prices significantly increases the value of put option, when the strike price is greater than the initial stock price it increases put option values, Sobolev space energy estimates were used as asset value function to estimate asset prices at different maturity periods. Finally, the graphical solutions and comparisons of other parameters were discussed all in this paper which is informative to investors for the proper investment plans.

KEYWORDS:

Stock prices, Crank-Nicolson, Option pricing, BS PDE and Put Option.

Introduction

Financial derivatives, also known as the Principal Asset from which the worth of an Option is derived, is a kind of derivative having anything in general with mathematical meaning of derivative. Specifically, an Option on a basic asset is an investment involving people who agree to come together to either buy and sell a basic at a fixed strike price in the future for an established price. The underlying asset is the basis upon which the cost of the Option is sustained, which is typically a currency or an index, commodity or stock. The holder possesses legitimate right, nevertheless, cannot be obligated to buy, for Call Option where European Put Option consists of the capability of selling an asset for specific charge at a predetermined date in the future Wokoma et al.(2020).

Derivatives or Options are termed "in-the-money", "at-the-money", or "out-of-the-money". Suppose S is a stock price and K is the strike price, a Call Option is said to be "in-the-money" if $S \succ K$, also, a Call Option is "at-the-money" if $S = K$, and it is "Out-of-the-money" when $S < K$. But a Put Option is said to be "in-the-money" if $S < K$, it is "at-the-money" if $S = K$ and it is "Out-of-the-money" when $S > K$.

Evidently, an Option or derivative is practiced exclusively when it is "in-the-money". An "in-the-money" Option is usually practiced in the absence of transaction costs, on the date of expiration, given that it has not been previously practiced, Hull (2012).

All the same, significance and applicability of Option valuation or estimation was displayed first by Black-Scholes (1973) when Option encountered complexities in estimation and evaluation of Options at the end. No-arbitrage argument was used in explaining a partial differential equation which controls the growth of the Option price with respect to the expiration and cost of the underlying asset. The Black-Scholes equation has been extensively utilized in several financial applications. Several researchers as follows, have carried out studies on this area, Macbeth and Mervile (1979),Hull(2003),Razali (2006),Rinalini(2006) and Nwobi et al.(2019) etc.

Solving partial differential equations, then applying the initial and boundary conditions, gives the Option values. One of the well-known mathematical tools applied in solving partial differential equation is the Finite difference method. Brennan and Schwartz(1978) While valuing financial derivatives, was the first to examine these methods. The Finite difference method consists of the following methods: The Explicit method, implicit method and the Crank-Nicolson method. And all these methods are applicable in solving the Black-Scholes partial differential equations. They are approximately closed to each other although different in stability and accuracy. The Crank-Nicolson (CN) method is considered here in because, it enables us to obtain the value of the option at different lines, such as time zero in a single repetition and of Combining Explicit and Implicit Methods; it has the optimum numeration approximation Wokoma et al.(2020) and Fadugba and Ayebusi (2020).

This paper is arranged as follows: Section 2 presents mathematical frame work, results and discussion are seen in Section 3, while the paper is concluded in Section 4.

Research Problem

Most often, Solving Stochastic Partial Differential Equations (PDEs) are non-trivial; When solved with numerical methods, it becomes Complex. It becomes more difficult when the analysis of the problem is needed. The difficulty in the analysis of this stochastic PDE is as a result of the numerical formulation, assumption, coding of solution for accurate simulations and interpretation of real-life phenomena that cannot be comprehended easily, especially when relating to its property. In the past, researchers have examined the same problems, however, the modified Black Scholes partial differential equation for solving societal problem and need were not considered. Specifically, various researches such as Nwobi et al.(2019),Amadi et al.(2020) and Fadugba et al.(2012) considered exclusively on the major Black-Scholes Partial Differential equation for solving societal problems and needs were not considered. However, none has demonstrated the relevance of "in -the -money," and "out- of -the -money" respectively to proposing an improved knowledge to option traders or investors on the routine activities of option pricing.

Nevertheless, we consider Sobolev space energy estimate theorem to satisfactorily optimize good estimate asset prices; partial differential equations exist in appropriate Sobolev spaces despite the fact that strong solutions do not exist in the spaces but its derivatives are understood in the classical sense. Hence, Sobolev spaces is a Banach spaces of smooth functions of one and several variables with a condition imposed in the first partial derivatives.

Objectives of the Study

1. To use the notion of European option to value put options and other stock market variations
2. To estimate asset values at different maturity days using Sobolev space energy theorem

Significant of the Study

In this paper, we shall consider Crank-Nicolson (CN) finite difference method for valuation of European call option which is exercised only on the expiration date. Furthermore, this paper describes the analysis of the following: closed form solution of BS formula for call , CN numerical approximation of put option; due to the increase in volatility and other comparisons such as effects of initial stock price on the value of put option at different levels, Sobolev energy estimate theorem arising in financial market. The above concepts stand as an advantage over previous efforts in this dynamic area of mathematical finance.

Literature Review

Dremkova and Ehrhardt (2011) than presented compact finite difference schemes to solve nonlinear Black-Scholes equations for American option with a non linear volatility function. Since a compact scheme cannot be applied on the American type options, the study used a fixed-domain transformation around the same time, Song and Wang. (2013) applied symbolic calculation software to provide a numerical solution using the implicit scheme of the finite difference method. This study combined the time- fractional Black-Scholes equation with the conditions satisfied by the standard put options. Two years later, Uddin et al.(2015) presented the numerical result of semi-discrete and full discrete schemes for European call option and put option by Finite Difference Method and Finite Element Method. In a recent study, Zhang et al.(2016) used the Tempered fractional derivative to price a European-double-knock-out barrier option. This study analysed characteristics of three fractional Black-Scholes models through comparison with the classical Black-Scholes model. Much earlier, Cortes et al.(2005) incorporated an important aspect of errors into the numerical solution. This study applied the Mellin transformation and proposed that the errors of composite Simpson's rule or Euler's method can be avoided whilst pricing the Black-Scholes equation in real-world financial derivatives. Company et al. (2008) also applied the Mellin transform and a delta- defining sequence of the involved generalized Dirac delta function to provide a numerical solution of the modified Black-Scholes equation Other studies on numerical solution literature considered different aspects. For instance, Company et al. (2008) applied the semi-discretization technique to deal with the issues arising as a result of a non-linear case of interest modelling option pricing with transaction cost. Ankudinova and Ehrhardt. (2008) analysed the Crank-Nicolson and the R3C scheme are the most accurate techniques to price the European call option. This study, incorporated different volatility problem in stock price, option price and its derivates. Recently, the two-dimensional Black-Scholes equations was explored by Cerna et al.(2016) using cubic spline wavelets and multi-wavelet bases.

On the contrary,a lot of authors have written extensively on Sobolev spaces such as in references Amadiet al.(2020), Amadi et al.(2022), Osu and Amadi (2022), Osu et al.(2022) and Diperna and Lions (1989), Osu and Olunkwa(2014) etc.

Our interest in this study is the Crank- Nicolson (CN) finite difference method for valuation of European put option which has earned the interests of scholars for determining approximate solutions to PDEs and this enthusiasm is borne out of the desire for application to needs in the society. Additionally, in this paper, the following analyses are considered: closed form solution of BS formula for call, CN numerical approximation of put option, the effects of initial stock prices, Sobolev space energy estimate theorem arising in financial market. This research is inspired and induced by the concepts above, thus, it may be instrumental to Government, decision makers, mathematicians, investors, Option traders, etc.

Methods

Here we present some basic definitions that cut across the area of study:

Stochastic Processes

Definition 1 Stochastic process: A stochastic process $X(t)$ is a relation of random variables $\{X_t(\gamma), t \in T, \gamma \in \Omega\}$, i.e, for each t in the index set T, $X(t)$ is a random variable. Now we understand t as time and call $X(t)$ the state of the procedure at time t. In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors. It can also be seen as a statistical event that evolves time in accordance to probabilistic laws. Mathematically, a stochastic process may be defined as a collection of random variables which are ordered in time and defines at a set of time points which may be continuous or discrete.

Definition 2.A stochastic process whose finite dimensional probability distributions are all Gaussian.(Normal distribution).

Definition 3. Random Walk: There are different methods to which we can state a stochastic process. Then relating the process in terms of movement of a particle which moves in discrete steps with probabilities from a point $x = a$ to a point $x = b$. A random walk is a stochastic sequence $\{S_n\}$ with $S_0 = 0$, defined by

$$S_n = \sum_{k-1}^{n} X_k \tag{1}$$

where X_k are independent and identically distributed random variables

Definition 4: A Stochastic Differential Equation (SDE) is integration of differential equation with stochastic terms. So ,in considering the Geometric Brownian Motion (GBM) which govern price dynamics of a non-dividend paying stock as:

$$dS(t) = \mu S(t)dt + \sigma S(t)dz(t) \ , \tag{2}$$

Where S denotes the asset value, μ is the stock rate of return (drift) which is also known as the average rate of the growth of asset price and σ denotes the volatility otherwise called standard deviation of the returns. The $dz(t)$ is a Brownian motion or Wiener process which is defined on probability space $(\Omega, \mathrm{F}, \wp)$, Osu (2010). However, stock price follows the Ito's process and the drift rate is stated as follows:

$$\mu = \left(\frac{\partial f}{\partial S_t} a_t + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S_T^2} b_t^2 \right), \tag{3}$$

$$\sigma^2 = \frac{\partial^2 f}{\partial S_t^2} b_t^2$$

Definition 5: A standard Brownian motion is simply a stochastic process $\{B_t\}_{t\in\tau}$ with the following properties:

i) With probability 1, $B_0 = 0.$

ii) For all $0 \le t_1 \le t_2 \le \cdots \le t_n$, the increments $B_{t2-}B_{t1}, B_{t3-}B_{t2}, B_{t2}, \cdots, B_{tn} - B_{tn-1}$ are independent.

iii) For $t \ge s \ge 0,\ B_t - B_s \sim N(0, t-s)$.

With probability 1, the function $\rightarrow B_i$ is continuous.

Definition 5: Ito's process is a stochastic process $\{X_t, t \ge 0\}$ known as Ito's process which follows:

$$X_t = X_0 + \int_0^t (\mathrm{t}, \varpi) d\tau + \int_0^t b(t, \varpi) dz_t \ .$$

Where $a(t,\varpi)$ and $b(t,\varpi)$ are adapted random function, George and Kenneth (2019).

Definition 6:(Ito's lemma). Let $f(S,t)$ be a twice continuous differential function on $[0,\infty) \times \mathrm{A}$ and let S_t denotes an Ito's process

$$dS_t = a_t dt + b_t dz(t), t \ge 0 \ ,$$

Applying Taylor series expansion of F gives:

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}(dS_t)^2 + \text{higer order terms } (h.\mathrm{o,t}) \ ,$$

So, ignoring h.o.t and substituting for dS_t we obtain

$$dF_t = \frac{\partial F}{\partial S_t}(a_t dt + bdz(t)) + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}(a_t dt + bdz(t))^2$$

$$= \frac{\partial F}{\partial S_t}(a_t dt + bdz(t)) + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2} b_t^2 dt,$$

$$= \left(\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2} b_t^2\right) dt + \frac{\partial F}{\partial S_t} b_t dz(t)$$

More so, given the variable $S(\mathrm{t})$ denotes stock price, then following GBM implies (2) and hence, the function $F(S,t)$,Ito's lemma gives:

$$dF = \left(\mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) dt + \sigma S \frac{\partial F}{\partial S} dz(t)$$

However, our interest in this paper is the parabolic financial PDE which is governed with the dynamics of option pricing; hence we have the following:

$$\frac{\partial V}{\partial t}+\frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.\ t>0. \tag{5}$$

Where, r represents interest rate, σ represents volatility of the underlying assets and t represents time of maturity. The details of the above option model can be expressly found in the following papers: Wokoma et al.(2020), Fadugba and Ayebusi (2020), Amadi et al.(2020), Fadugbu et al.(2012), Fadugba and Nwozo (2013)and Amadi et al.(2022) etc. However, Black-Scholes model is based on seven assumptions: The asset price follows a Brownian motion with μ and σ as constants, there are no transaction costs or taxes, all securities are perfectly divisible, there is no dividend during the life of the derivatives, there are no riskless arbitrage opportunities, the security trading is continuous. The analytic formula for the prices of European put option is given as:

$$\left.\begin{array}{c} P=SN\left(d_1\right)-Ke^{-rt}N\left(d_2\right) \\ \dfrac{ln\left(\dfrac{S}{K}\right)+\left(\dfrac{r+\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2=d_1-\sigma\sqrt{T} \end{array}\right\} \tag{6}$$

where P is Price of a put option, S is price of underlying asset, K is the strike price, r is the riskless rate, T is time to maturity, σ^2 is variance of underlying asset, σ is standard deviation of the(generally referred to as volatility) underlying asset, and N is the cumulative normal distribution.

European Options

In the work of Black-Scholes (1973) they obtained a mathematical structure for finding the reasonable price of European options by the use of no-arbitrage principle to describe a PDE which governs the growth of the option price that evolves time to expiration. The details of this options can be found in the following books: Heston (1993), Hull(1993),Hull (2012),Hull,(2013) etc.

European Put Option

The BS PDE for European Put Options with value $P(S,t)$ is given in the following equations:

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 P}{\partial S^2}+rS\frac{\partial P}{\partial S}-rP=0 \tag{7}$$

With the following initial and boundary conditions:

$$\left.\begin{array}{c} P(0,t)=Ke^{-rt} \\ P(S,t)=0\ \ when\ \ S\to\infty \\ P(S,T)=\max(K-S,0) \end{array}\right\} \tag{8}$$

The Numerical Scheme and Analysis

The Crank-Nicolson finite difference method is to conquer the stability short-comings by applying the stability and convergence restrictions of the explicit finite difference methods. It is essentially an average of the implicit and explicit methods. However, to implement Crank-Nicolson approximation scheme on Black-Scholes partial differential equation, there must be a price time mesh in order to enhance efficiency as solution exits, the vertical axis in the mesh denotes the stock prices, while the horizontal axis denotes time. Therefore, every grid point in the mesh denotes a horizontal index i and a vertical index j such that every point in the mesh is the option price for a distinct time and a distinct stock price. At every time in the mesh $j\Delta s$ is equivalent to the stock price, and $i\Delta t$ is equivalent to the time. There exist boundary conditions which help in the numerical calculations; by means of the pay-off function. The maturity period, $t = T$ and the option are well computed for all the different initial stock prices using boundary conditions for uniqueness of solution. To get the prices at $t = 0$, the model solves backwards for every time step from $t = T$,Yueng (2012),Wokoma et al.(2020).

Formulation of the Scheme

One of the normal ways of approximating the solution of partial differential equations is applying Crank-Nicolson finite difference method which we shall use our proposed model to transform into the scheme. Hence , we have the price time mesh below.

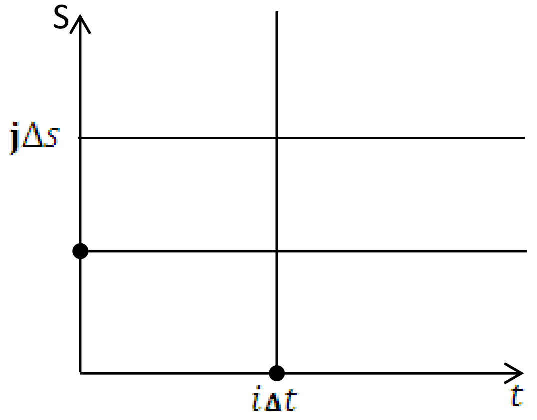

Figure 1: An illustration of Price time mesh.

Recall that the Black-Scholes partial differential equation (2). Let a function $V(S,t)$ in two dimensional grid points, that is to say i and j stands for the index for stock price, S and time, t respectively. The function $V(S,t) = V_i^j$ can be stated as follows in the subsequent difference scheme.

$$Z_i^j = \frac{1}{2}\sigma^2 S^2 DSS + rS_i DS - rV_i^j \tag{9}$$

where,

$S = i\Delta s$, for $0 \le i \le m$, $t = j\Delta t$ for $0 \le j \le i$

$$DSS = \frac{V^{j}_{i+1} - 2V^{j}_{i} + V^{j}_{i-1}}{\Delta^2} \tag{10}$$

$$DS = \frac{V^{j}_{i+1} - V^{j}_{i-1}}{2\Delta S} \tag{11}$$

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below.

If we use a forward difference approximation to the time partial derivative, we obtain explicit scheme

$$\frac{V_i^{j+1} - V_i^{j}}{\Delta t} + Z_i^{j} = 0 \tag{12}$$

and similarly we obtain the implicit scheme

$$\frac{V_i^{j+1} - V_i^{j}}{\Delta t} + Z_i^{j+1} = 0 \tag{13}$$

The averages of equations (12) and (13) yields Crank-Nicolson method of approximation

$$\frac{V_i^{j+1} - V_i^{j}}{\Delta t} + \frac{1}{2}\left(Z_i^{j} + Z_j^{j+1}\right) = 0 \tag{14}$$

From equation (14)

$$V_i^{j} - \frac{\Delta t}{2} Z_i^{j} = V_i^{j+1} + \frac{\Delta t}{2} Z_i^{j+1} \tag{15}$$

$$\frac{\Delta t}{2} Z_i^{j} = V_i^{j+1} - V_i^{j} + \frac{\Delta t}{2} Z_i^{j+1}$$

$$\therefore Z_i^{j} = \frac{2}{\Delta t}\left(u_i^{j+1} - u_i^{j}\right) - Z_i^{j+1} \tag{16}$$

Substituting (8) in (16), gives in view of (15) and (16) we obtain after collecting like term in V_{i-1} ,

$$\frac{\sigma^2 S^2}{2}\left[\frac{V^{j}_{i+1} - 2V^{j}_{i} + V^{j}_{i-1}}{(\Delta S)^2}\right] + rS_i\left[\frac{V^{j}_{i+1} - V^{j}_{i-1}}{2\Delta S}\right] - rV_i^{j} = \frac{2}{\Delta t}\left(V_i^{j+1} - V_i^{j}\right) - Z_i^{j+1}$$

That is

$$\frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[V^{j}_{i+1} - 2V^{j}_{i} + V^{j}_{i-1}\right] + \frac{rS_i}{2\Delta S}\left[V^{j}_{i+1} - V^{j}_{i-1}\right] - rV_i^{j} = \frac{2}{\Delta t}\left(V_i^{j+1} - V_i^{j}\right) + Z_i^{j+1}$$

Collecting like terms in of V_{i-1}, V_i *and* V_{i+1} and simplifying gives

$$Z_i^{j} = V^{j}_{i-1}\left[\frac{\Delta t \sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t S_i}{2\Delta S}\right] + V_i^{j}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t \sigma^2 S_i^2}{(\Delta S)^2} - r\Delta t\right] + V^{j}_{i+1}\left[\frac{\Delta t \sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t S_i}{2\Delta S}\right]$$

and

$$Z_i^{j+1} = V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t S_i}{2\Delta S}\right] + V_i^{j+1}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t\sigma^2 S_i^2}{(\Delta S)^2} - r\Delta t\right] + V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t S_i}{2\Delta S}\right]$$

Using (8) in (14) solving simultaneously and taking the average of these two equations we obtain

$$\left.\begin{aligned} &V_{i-1}^{j}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{rS_i\Delta t S_i}{4\Delta S}\right] + V_i^{j}\left[1 - \left(\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right)\right] + V_{i+1}^{j}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{rS_i\Delta t S_i}{4\Delta S}\right] \\ &= V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{rS_i\Delta t S_i}{4\Delta S}\right] + V_i^{j+1}\left[1 - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] + V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{rS_i\Delta t S_i}{4\Delta S}\right]\end{aligned}\right\} \tag{17}$$

The expressions inside the square brackets will be replaced with the coefficients a, b, c. The following equations obtained.

$$aV_{i-1}^{j} + bV_i^{j} + cV_{i+1}^{j} = aV_{i-1}^{j+1} + bV_i^{j+1} + cV_{i+1}^{j+1} \tag{18}$$

Where

$a_i = \frac{\Delta t}{4}\left((\sigma^2 S_i^2) - rS_i\right)$, $b_i = -\frac{\Delta t}{2}\left(\sigma^2 S_i^2 - r\right)$ and $c_i = -\frac{\Delta t}{4}\left((\sigma^2 S_i^2) - rS_i\right)$

a_i, b_i, c_i are random variables; $i = 0, 1, \ldots, M$.

Equation (18) can now be represented in matrix form as follows

$$XV^j = YV^{j+1},\ j = 0, 1, 2, \ldots$$

$$\Rightarrow u^j = X^{-1}YV^{j+1}$$

where $V^j = (V_{1,i}, V_{2,i}, V_{3,i}, \ldots, V_{m,i})^T$, ($V^i$ is an $m \times n$)

$$\begin{bmatrix} b & c & 0 & \ldots & 0 \\ -a & b & c & \ldots & 0 \\ 0 & -a & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & -a_{M-1} & b_{M-1} \end{bmatrix}\begin{bmatrix} V_1^j \\ V_2^j \\ V_3^j \\ \vdots \\ V_{M-1}^j \end{bmatrix} = \begin{bmatrix} X_1^j \\ X_2^j \\ X_3^j \\ \vdots \\ X_{M-1}^j \end{bmatrix}$$

$$\begin{bmatrix} b & c & 0 & \ldots & 0 \\ a & b & c & \ldots & 0 \\ 0 & a & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} \end{bmatrix}\begin{bmatrix} V_1^{j+1} \\ V_2^{j+1} \\ V_3^{j+1} \\ \vdots \\ V_{M-1}^{j+1} \end{bmatrix} = \begin{bmatrix} Y_1^{j+1} \\ Y_2^{j+1} \\ Y_3^{j+1} \\ \vdots \\ Y_{M-1}^{j+1} \end{bmatrix}$$

Theorem 1.1. (**Energy estimate**). There is a constant C , which depends only on ϕ, T and its coefficients is of L ,such that

$$\max_{0\le t\le T} \| u_m(t) \| L^2(\phi) + \| u_m \| L^2\left(0,T;H_0'(\phi)\right) + \| u_m' \| L^2\left(0,T,H^{-1}(\phi)\right) \\ \le C\left(\| f \| L^2(0,T;L^2(\phi)\right) + \| g \| L^2(\phi)), \tag{19}$$

For $m = 1, 2, ...,$.

The details of this proof can be seen in Amadi et al.(2022), Osu et al.(2022),Osu and Amadi (2022), Osu et al.(2022) etc.

Results and Discussions

Here we present simulation results obtained using equations (17) viamatlab codes for Black-Scholes exact values and Crank-Nicolson numerical solutions for Put option.

Table 1: Comparing the performance of the Black-Scholes analytic values and Crank-Nicolson finite difference method for European Put Option when initial stock prices are 40 and 50 with K = 100, r = 0.2 and T = 1

Sigma Sigma	S_o = 40, K = 100			S_o = 50, K = 100,		
	BS Exact values	CN	Error	BS Exact Values	CN	Error
0.3	41.9211	41.8977	5.5819E-04	32.2717	32.1011	5.2864E-03
0.4	42.2025	42.0283	4.1277E-03	33.1966	32.5633	0.01908
0.5	42.8277	42.2898	0.01256	34.5553	33.1562	0.04049
0.6	43.7659	42.6422	0.02568	36.1951	33.7805	0.06671
0.7	44.9429	43.0389	0.04236	38.0079	34.3849	0.09532
0.8	46.2893	43.4474	0.06139	39.9226	34.9476	0.1246
0.9	47.7501	43.8478	0.08172	41.8921	35.4616	0.1535
1.0	49.2837	44.2297	0.10255	43.8839	35.9265	0.1813

In Table 2 , a little increase in the volatility of stock also increases the close form prices of BS and CN through an initial stock prices of 40 and 50 respectively. This remark is reasonable in the aspect of an investor whose primary aim is to maximize profit; because the investor is only obliged to sell; besides it is reasonable when the strike price is greater than the initial prices for put options.

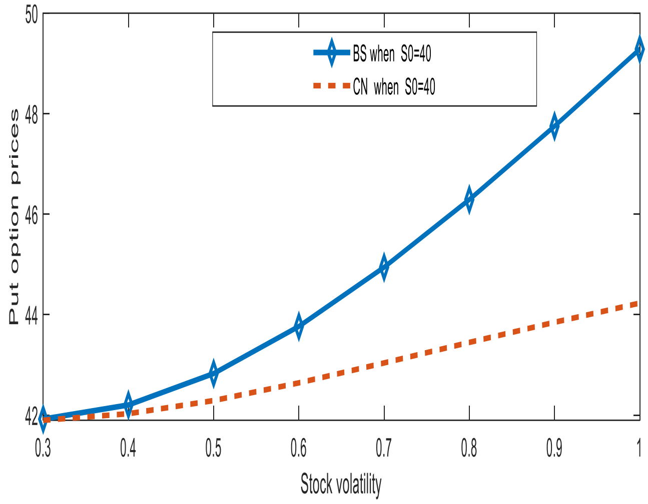

Figure 2: Plots of Black-Scholes analytic values and Crank-Nicolson Numerical approximations when initial stock prices is40 and 50 for Put option.

Figure 2defines the nature of option sales between BS and CN. The two option plots of the investments grew more likely to exponentially trends and well profiting. The CN sales grew with the shortest optimum level. That is to say in this kind of scenario BS plot is encouraging in terms of profit maximizing hence it moves exponentially.

Table 2 : The effects of initial stock prices on the value of Put option

Strike price	Initial stock	Put option
20.00	10.00	4.1482
	15.00	0.7410
	20.00	0.0448
	25.00	0.0016
30.00	10.00	11.1409
	15.00	6.2120
	20.00	2.2681
	25.00	0.4996
40.00	10.00	18.1875
	15.00	13.1891
	20.00	8.2776
	25.00	4.0831
50.00	10.00	25.2344
	15.00	20.2344
	20.00	15.2391
	25.00	10.3440

In Table 2, it can be observed that, increase in strike and stock prices increases the value of Put option prices; which implies that the strike and stock prices has significant effects in pricing of an option. Also when the strike prices is greater than the initial stock prices ($K>S0$) the value of put option improves and investors are the liberty in making large profits; On the contrary, when $K<S0$ for call option indicates that the option has some intrinsic value which is beneficial to exercise the options. It is also known as in-the-money because the stock price is above the strike price at expiration. The call option owner can exercise the option, putting up cash or buy the stock at the strike price or the owner

can simply sell the option at its fair market value to another buyer before it expires. This remark is quite profit maximizing to sustain the future growth of the investments.

it shows that the option will have intrinsic value; that is to say that put option with strike price higher than the current price will be in-the money since one can sell the stock higher than the market price and then buy it back for a guaranteed profit. This is realistic because the higher the strike price of a put option the higher the price it can sell the underlying asset. Clearly that this type of option is suitable and profitable to only investors who want to sell its asset which may be indexed in millions of naira throughout the trading days.

Empirical Illustrations of Sobolev Space Energy Estimate

Here we present some empirical illustration of Sobolev space energy estimate theorem and analyzed based on financial market variables using Matlab programming software. This is enables us observe the behavior of some stock variables or quantities on value of asset at different maturity days worth of the investments. Therefore, the Energy estimate is assumed to be asset value function where the left hand side of the estimate where constrained and the right hand side were used as a function $u(t)$ which gave the following: Hence we have the following :

$$u(t) = C\left(\| f \| L^2(0,T;L^2(\phi)\right) + \| g \| L^2(\phi)).$$

Hence the following parameter values of the BS PDE were used in the simulation study:

$$L^2(\theta) = r = 0.2,\ f = S = 13.45,\ C = \sigma = 1,\ 2,\ L^2 = k = 5, \text{and } g = 0.25.$$

Table 3. The estimates of asset value at different Maturity days

T	Volatility(σ)	$U(t)$	Volatility(σ)	$U(t)$
0	1.0000	0.05	2.0000	0.100
2	1.0000	20.95	2.0000	53.9
4	1.0000	53.85	2.0000	107.7
6	1.0000	80.75	2.0000	161.5
8	1.0000	107.65	2.0000	215.3
10	1.0000	134.55	2.0000	269.1
12	1.0000	161.45	2.0000	322.9
14	1.0000	188.35	2.0000	376.7
16	1.0000	215.25	2.0000	430.5
18	1.0000	242.15	2.0000	484.3

Table 3describes Sobolev space energy estimate theorem of BS PDE which is understood in suitable weak sense; here is used as an asset value function which, estimates asset values at different maturity days. It can be seen that increase in maturity days increases the value of asset when volatility is made constant 1.0000 and 2.000 through columns 2 and 4 respectively. Also volatility significantly affects the value of asset pricing which implies increasing volatility of the underlying asset increases the value of asset. This is quite informative to every investor based on decision making.

Conclusion And Recommendations

This paper considered European put option for both analytic formula and numerical approximation prices. The simulations of analytical and numerical were effectively carried out using MATLAB programming software. The results showed as follows: increase in volatility increases the values of option for both BS and CN prices, there are significant difference between BS and CN due to the changes of stock volatility, a little increase in the initial stock prices significantly increases the value of put option, when the strike price is greater than the initial stock price it increases put option values, Sobolev space energy estimate were used as asset value function to estimate asset prices at different maturity periods. To this end, combining delay and control parameters into the BS PDE will be an interesting study to explore.

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